GSP2205 MODULE 4

5.1 WHAT IS LOGIC?

Logic is a non-empirical science of reasoning. Logic helps in uncovering error and establishing the truth.

5.2 LOGICAL TERMS

Logical terms are used for reasoning that may occur in statements about any kind of subject-matter, and are supposed to be based on rules or axioms that specify their valid and proper use.

5.3 WHAT ARE THE LOGICAL TERMS?

The answer to this question appears difficult because there is no universal agreement on what are the logical terms. However, it is a good idea to have the notion, for further characteristic and provide a list. The further characteristics logical terms are often were called *Syncategorematic terms* i.e. to have no meaning on their own, but only in conjunction with terms that have meaning. Here are some of the logical terms that have been widely considered and appeared as such in many texts:

True, *false, necessary, possible, contingent, tautology, contradiction, antecedent, consequence, premise, conclusion, not, and, or, implies, if and only if, every, some, no, equal, such that, so.*etc

The common or basic logical terms and their symbols that are widely accepted and sufficient for first-order logic are summarized in the table below.

Not	And	Or	Implies	If and only if
$\sim$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$

Definition 1: A ***proposition*** is a statement that is either true or false, but not both, i.e. a statement of opinion or judgment.

Example 1

1. BUK is the best university in Nigeria.
2. Abdu gets money
3. Bola study hard
4. 3 + 9 = 12

5. The votes have been counted
6. Today is Sunday
7. Nobody is above the law
8. Either today is Sunday or Abuja is the capital of Ghana
9. Yesterday was raining and BUK is in Abuja
10. Whenever it is Election Day government declare a holiday.

Conventionally, letters such as p, q, r, s, etc are used to denote propositions. The truth value of a proposition is denoted by T, if it is true and F, if it is false.

Definition 2: New propositions that are obtained by combining one or more propositions are called ***Compound propositions*.** And are formed using logical terms.

Definition 3: Let p be a proposition. The statement "it is not the case that p"

Is another proposition called ***Negation*** of p. The negation of p is denoted by ~p. The proposition **~** p is read as "not p".

Examples

1. "GSP2205 is a course in BUK"

The negation is "it is not the case that GSP2205 is a course in BUK"
or even more simply as
"GSP2205 is not a course in BUK".

2. Find the negation of the proposition
"Ali's Smartphone has at least 22GB of memory" and express this in simple English.
Solution: The negation is
"It is not the case that Ali's Smartphone has at least 22GB of memory."
This negation can also be expressed as
"Ali's Smartphone does not have at least 22GB of memory"
or even more simply as
"Ali's Smartphone has less than 22GB of memory."

Definition 4: Let p and q be propositions. The proposition "p and q" denoted by $p \wedge q$, is the proposition that is true when both p and q are true at the same time and is false otherwise. The proposition $p \wedge q$ is called the ***Conjunction*** of p and q.

Examples

1. Let p and q be propositions “It is a good idea” and “It is difficult to implement” respectively. The conjunction of the propositions p and q, is $p \wedge q$= “It is a good idea and difficult to implement”.
2. Find the conjunction of the propositions *p* and *q* where *p* is the proposition “Today is Monday” and *q* is the proposition “It is raining today.”

Solution:

The conjunction of these propositions, $p \wedge q$, is the proposition “Today is Monday, and it is raining today.”

For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

Definition 5: Let p and q be propositions. The proposition “p or q” denoted by $p \vee q$, is the proposition that is false when p and q are both false and is true otherwise. The proposition $p \vee q$ is called the ***Disjunction*** of p and q.

Examples

1. The disjunction of p and q, where p and q are the same in the previous example, is $p \vee q$ = “It is a good idea or it is difficult to implement”.
2. What is the disjunction of the propositions p and q where p and q are the same propositions as in the previous example 2

Solution:

The disjunction of p and q, p v q, is the proposition

"Today is Monday or it is raining today."

This proposition is true on any day that is either a Monday or a rainy day (including rainy Mondays).

It is only false on days that are not Mondays when it also does not rain

Definition 6: Let p and q be propositions, the compound proposition “if p then q” denoted by “$p \rightarrow q$” is the proposition that is false when p is true and q is false and true otherwise. It is called ***Conditional*** (or Implication). In this p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Because implications arise in many places in mathematical reasoning, various ways are used to express $p \rightarrow q$, some of the common ways are

(a) “if p, then q”
(b) “p implies q
(c) “p is sufficient for q”
(d) “q if p”
(e) “If p, q”

(f) "q follows from p"
(g) "q is necessary for p"
(h) "p only if q"

Examples

1. Given p is the proposition "it is raining" and q is the proposition "it is cloudy". Write the conditional of p and q.

Answer: "If it is raining, then it is cloudy".

2. "If I am elected, then I will lower fuel price."
   If the politician is elected, voters would expect this politician to lower fuel price. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower fuel price, although the person may have sufficient influence to cause those in power to lower fuel price. It is only when the politician is elected but does not lower fuel price that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \rightarrow q$.

Note: that there are some related conditionals that can be formed from $p \rightarrow q$. The proposition $q \rightarrow p$ is called ***Converse*** of $p \rightarrow q$. The proposition $\sim q \rightarrow \sim p$ is called ***Contra-positive*** of $p \rightarrow q$. The proposition $\sim p \rightarrow \sim q$ is called the ***Inverse*** of $p \rightarrow q$.

Example

Write the converse, inverse and the contra-positive of the implication "if today is Tuesday, then I have a class today"

Answers

The converse is "if I have a class today, then today is Tuesday"

The inverse is "if today is not Tuesday, then I do not have class a today"

The contra-positive is "if I do not have a class today, then today is not Tuesday".

Definition 7: Let p and q be positions. The proposition "p if and only if q" denoted by $p \leftrightarrow q$ is the proposition that is true when p and q have the same truth values and if false otherwise. It is also called ***Biconditional***.

Note: - The biconditional $p\leftrightarrow q$ is true precisely when both the conditionals $p\rightarrow q$ and $q\rightarrow p$ are true. Because of this the terminology "p if and only if" is used and this abbreviated as "iff".

Some ways of expressing the proposition $p\leftrightarrow q$ are

(a) "p is necessary and sufficient for q"
(b) "If p then q, and conversely".

Examples

1. You passed the GSP2205 course, if and only if you scored 45% or higher
2. Abdu can takes the flight, is necessary and sufficient for Abdu buy a ticker
3. You are breathing, iff you are alive.

5.4 Uses of Logical terms

(1) Sometimes English sentences can be translate into expressions that involves propositions and logical terms, one of the reason of doing this, is to analyze the logical expression to determine their truth values.

Example

How can the following English statement be translated to logical expression?

"You can pass the GSP2205 only if you attend all the classes of GSP2205 or you study the book of GSP2205"

Answer: Logical expression

p: "You can pass the GSP2205"
q: "if you attend all the classes of GSP2205"
r: "you study the book of GSP2205"
$p\rightarrow(q\vee r)$.

(2) Logical terms are used extensively in searches of large collections of information, such as indexes of web pages etc.

5.5 LOGICAL EQUIVALENCES

Definition 8: Compound propositions that have the same truth values in all possible cases are called ***logical equivalent***.

Definition 9: A compound statement that is always true no matter what the truth values of the propositions that occur in it, is called a ***Tautology***.

Definition 10: A compound proposition that is always false is called ***Contradiction***.

Definition 11: A proposition which is neither a tautology nor a contradiction is called ***Contingency***.

Definition 12: The propositions p and q are called ***logically equivalent***.

If p↔q is a Tautology. The notation p≡**q** denotes that p and q are logically equivalent.

One way to determine whether two propositions are equivalent is to use a truth table.

Examples

1. p↔q is equivalent to (p→q)∧(q→p).
2. ~(p∨q) and ~p∧~q are logically equivalent. (use truth table to verify)

5.6 QUANTIFIERS

Mathematical statements involving variables, such as

"$x > 0$", "$x + y = 5$", and "$x + z < y$"

are common in mathematical assertions. The truth value of these assertions can only be determined when the values of the variables x, y and z, are given. The statement "x is greater than 0" has two parts, the variable x and the predicate "is greater than". We can denotes this assertion by P(x), where P denotes the predicate "is greater than 0" and x is the variable. The statement P(x) is called the value of ***propositional function*** P at x. However, when the value of x is given, then P(x) becomes proposition and has a truth value.

For example: if P(x) denotes "x >0" what are the true values of P(4) & P(-1)? P(4) is "4>0" is true and P(-1) is "-1>0" is false.

Definition 13: Let X be a universe of discourse (or domain). The ***Universal quantification*** of P(x) is the proposition "P(x) is true for all values of x in X. The notation $\forall\, x,\ \mathrm{P}(x)$, is used where $\forall$ is called universal quantifier and it is also expressed as

"for all x P(x)" or "for every x P(x)".

Examples

1. Express the statement "Every student in the class of GSP2205 is intelligent" as a universal quantification.

Answer:

Let P(x) denote the statement

"x is intelligent"

Then the statement "every student in the class of GSP2205 is intelligent" can be written as for all x P(x), where the universe of discourse consist of all students in the class of GSP2205.

2. What does the statement$\forall\, x$ P(x) mean if P(x) is "Computer x is connected to the network" and the domain consists of all computers on BUK new campus?

Answer:
The statement $\forall\, x$ P(x) means that for every computer x on BUK new campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on BUK new campus is connected to the network."
As we have pointed out, specifying the domain is mandatory when quantifiers are used.

Definition 14: The ***Existential quantification*** of P(x) is the proposition "there exists an element x in the universe of discourse such that P(x) is true".

The notation $\exists\, x$ P(x) is used for the existential qualification of P(x), where $\exists$ is called the existential qualifier. It can also be expressed as

"there is an x such that P(x)"

"there is at least one x such that P(x)"

"for some x P(x)"

Examples

1. Let P(x) donate the statement “$x>5$” what is the truth value of the quantifications $\exists x P(x)$, where the universe of discourse is the set of real numbers?
   Answer
   Since “$x>5$” is true for instance when $x=6$-the existential quantification of P(x), which is $\exists x p(x)$ is true.
2. Let Q(x) denote the statement “$x+1<x$” what is the truth value of the qualification $\exists x Q(x)$ where the universe of discourse is the set of real numbers?
   Answer
   Since there is no real number x such that Q(x) is true, the existential of Q(x) which is $\exists x Q(x)$ is false.

Note: that the truth value of a quantified statement often depends on which elements are in this domain.

5.7 LAWS OF THOUGHT

The following three laws of thought are identified by Aristotle as necessary conditions for thought. These laws are axiomatic rules in which rational discourse are based. These laws are

1. **The law of identity**
   The law states that, A is A. In other words, everything is the same as itself. For example, human is human. Further, 3-2=1, if what the number 3 represents have the same identity with what the number 2 represents.
2. **The law of non-contradiction**
   The law states that nothing can both be and not be at the same time and in the same sense. For example, the statements: “Ali is Nigerian” and “Ali is not Nigerian” cannot both be true in the same context.
3. **The law of excluded middle**
   The law states that every statement is either true or false. In-fact this law is the principle for every proposition.

5.8 NATURE AND FUNDAMENTAL PRINCIPLES OF LOGIC

Logic is dealing with techniques and principles of reasoning. The necessity and importance of logic is indicated by the place it occupies in the order of learning. St Thomas says that “logic must be learned before any other science, and he points out that this has been the usual method of philosophers”.

Definition 15: ***Argument*** is simply defined as the collection of statements and a conclusion. The statements are called premises, where the premises and the conclusion are separated by either phrase or mark. For example:

If you study hard, then you will pass GSP2205 (premise)

You study hard (premise)

So, you pass GSP2205 (conclusion)

5.9 Fundamental principles of logic

In logic, we study the validity and power of arguments. Since not all arguments are valid, we need to identify the principles that will ensure the validity of an argument. For an argument to be valid, it must satisfy the laws of thought. In addition, an argument needs to satisfy the following fundamental principles of logic:

1. No argument with all true premises but has a false conclusion is valid.
2. If an argument has all true premises but has a false conclusion, then it is invalid.
3. If an argument is valid, then every argument with the same form is also valid.
4. If an argument is invalid, then every argument with the same form is also invalid.

Furthermore, the nature of every argument is either valid or invalid. It is possible for a valid argument to have:

a. All true premises and a true conclusion
b. Some false premises and true conclusion
c. All false premises and a true conclusion
d. All false premises and a false conclusion.

Note: that it is not possible for a valid argument to have all true premises and a false conclusion.

EXERCISES

1) Which of the following statements are propositions? What are the truth values of them?
   a) Audu is a name

- **b)** Do you read at night?
- **c)** Answer all the questions
- **d)** 5+5 = 10
- **e)** There is no weak students in BUK
- **f)** Three is odd number
- **g)** 7 + x =6
- **h)** GSP2205 is a level one course
- **i)** Abuja is the capital of Ghana
- **j)** Do not write on the question paper.

2) What is the negation of each of the following propositions?
   - a) This month is April
   - b) There is no students in the campus
   - c) Audu is a student
   - d) Today is Friday
   - e) Level three is difficult in the first degree
   - f) Bayero University, Kano is the best university in Nigeria.

3) Determine the truth value of each of the following propositions
   - **a)** Buhari is the president of Nigeria and Prof. M. Y. Bello is the VC of BUK.
   - **b)** Either Abuja or Lagos is the capital of Nigeria now
   - **c)** 7 + 3 = 10 and 8 + 2 = 9
   - **d)** If 4*3 = 12, then 3 is a factor of 12
   - **e)** If 2 + 3 = 5 then 5 is divisible by 3
   - **f)** Osinbanjo is the vice president of Nigeria and Prof. Hafiz is the Governor of Kano

4) Let p and q be propositions

   p: I get money

   q: I will buy a smart phone

Express each of the following propositions as an English sentence

a) $p \wedge q$ **b)** $\sim q \vee p$ **c)** $\sim p \rightarrow \sim q$ **d)** $\sim q \wedge \sim p$ **e)** $\sim p \vee \sim q$ **f)** $p \rightarrow q$ **g)** $p \leftrightarrow q$ **h)** $\sim p \vee \sim q$

5) Consider the following propositions

   p: Audu is rich

   q: John is poor

Write each of the following compound propositions in term of p and q and logical terms

- **a)** Audu and John are both rich
- **b)** Neither Audu nor John is poor
- **c)** Audu is not rich and John is not poor
- **d)** It is not true that Audu and John are both rich
- **e)** Either John is not poor or Audu is rich
- **f)** If Audu is rich then, John is poor.

6) Let p and q be propositions
   p: It is raining today
   q: Today is Friday

Write each of the following compound propositions in term of p and q and logical terms (symbolic)

- **a)** It is raining today or it is Friday
- **b)** It is raining today but it is not Friday
- **c)** It is not raining today and it is not Friday
- **d)** It is either raining or it is Friday.

7) Let p, q and r be propositions
   p: You get admission
   q: You study hard
   r: You pass all the courses

Express each of the following propositions as an English sentence

- **a)** $(p \wedge q) \vee r$ **b)** $\sim q \vee (p \wedge r)$ **c)** $\sim p \rightarrow \sim q$ **d)** $\sim q \wedge \sim r$ **e)** $\sim p \vee \sim q$ **f)** $\sim p \rightarrow \sim r$ **g)** $q \leftrightarrow r$ **h)** $\sim p \vee \sim q$

8) Write the converse, inverse and contrapositive of each of the following implications.
   - **a)** If it rains today then, I will carry an umbrella
   - **b)** I come to class whenever there is GSP lecture
   - **c)** If you study hard, then you will pass all GSP courses
   - **d)** A positive integer is even only if it is divisible by two
   - **e)** I got to School early whenever I have morning class
   - **f)** Study hard is necessary and sufficient conditions of getting A’s grades.

9) Show that the following are logically equivalents
   a) $\sim(p \leftrightarrow q)$ and $\sim p \leftrightarrow q$
   b) $p \rightarrow q$ and $\sim q \rightarrow \sim p$
   c) $\sim p \leftrightarrow q$ and $\sim p \leftrightarrow \sim q$
   d) $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$
10) Let P(x) denoted the statement "x is greater or equal 3" what are the truth values of the following
   a) P(1) **b)** P(4) **c)** P(-3) **d)** P(3)
11) Let P(x) denoted the statement "x contain the letter e" what are the truth values of the following
   a) P(students) **b)** P(classroom) **c)** P(true) **d)** P (faculty) **e)** P(false)
12) Let P(x) denoted the statement "x spend more than 6 hours in class every week," where the universe of discourse for x is the set of BUK students. Express each of the following quantifications in English
   a) $\forall x\, P(x)$ **b)** $\forall x \sim P(x)$ **c)** $\exists x\, P(x)$ **d)** $\exists x \sim P(x)$

For further reading

Anderson. L. A first course in Discrete Mathematics, Springer-Verlag, London, 2001.

Biggs N. L. Discrete Mathematics, Oxford University Press, New Delhi, 2003

Dymacek W. M. and Henry S. J., Introduction to Discrete Mathematics, MacgRaw Hill. International Singapore, 1998.

Eccles P. J. An Introduction to Mathematical Reasoning, Cambridge University Press, City MA 02116, 1995.

Kenneth A. R. and Charles R. B. W., Discrete Mathematics and Its Application, McGraw-Hill, City 1999