

To start, we will use the Taylor series expansions for sine and cosine:

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

We can then differentiate these series to obtain:

$$\sin'(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots = \cos(\theta)$$

$$\cos'(\theta) = -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \dots = -\sin(\theta)$$

We can also use the fact that we are assuming small oscillations, i.e., that  $\theta$  is small. We can therefore write:

$$\sin(\theta) \approx \theta + o(\theta^2)$$

$$\cos(\theta) \approx 1 + o(\theta^2)$$

where the notation  $o(\theta^2)$  means “smaller than or equal to  $\theta^2$ ”.

Using these Taylor series expansions and the small-o notation, we can rewrite the two given equations as:

$$-(\theta - o(\theta^2))\theta' + (1 + o(\theta^2))\theta'' = (-g/l)(\theta + o(\theta^2))(1 + o(\theta^2))$$

$$(1 + o(\theta^2))\theta' + (\theta - o(\theta^2))\theta'' = (-g/l)(\theta + o(\theta^2))^2$$

We can then simplify these equations by neglecting terms that are smaller than  $\theta^2$ :

$$-\theta\theta' + \theta'' = (-g/l)\theta + o(\theta^2)$$

$$\theta' + \theta'' = (-g/l)\theta + o(\theta^2)$$

Finally, we can add these two equations to eliminate  $\theta'$ :

$$\theta'' + (g/l)\theta = o(\theta^2)$$

Since the small-o notation means that the right-hand side is smaller than or equal to  $\theta^2$ , we can neglect this term for small  $\theta$ , and we obtain the simple harmonic oscillator equation:

$$\theta'' + (g/l)\theta = 0.$$