Transformation and Kinematics Problems

Almansur Kakimov

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Theoretical Background

In robotics and kinematics, transformations are used to describe the movement and orientation of objects in space. Transformations can be represented using matrices that capture rotations, translations, and scaling effects. Some key concepts include:

Homogeneous Transformation Matrices

A homogeneous transformation matrix represents rotation and translation in a single matrix:

$$T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}$$

where R is a 3×3 rotation matrix and t is a 3×1 translation vector.

Rotation Matrices

Rotations are performed around coordinate axes using specific matrices: - Rotation about the x-axis:

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation about the y-axis:

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

- Rotation about the z-axis:

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler Angles and Kinematics

Euler angles describe rotations using three sequential rotations about different axes. They are commonly used in robotic kinematics to define the orientation of a frame with respect to a reference.

Forward and Inverse Kinematics

- Forward kinematics: Given joint parameters, find the position and orientation of the end effector using:

$$T = T_1 T_2 \dots T_n$$

- Inverse kinematics: Given a desired end-effector position, find joint angles by solving:

$$\theta = \text{ATAN2}(y, x)$$

and other trigonometric relations.

2.6 For frame F, find the values of the missing elements and complete the matrix representation of the frame.

$$F = \begin{bmatrix} ? & 0 & -1 & 5 \\ ? & 0 & 0 & 3 \\ ? & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

For frame F, find the values of the missing elements and complete the matrix representation of the frame.

$$F = \begin{bmatrix} n_x & 0 & -1 & 5 \\ n_y & 0 & 0 & 3 \\ n_z & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution

Given that:

$$\begin{bmatrix} i & j & k \\ n_x & n_y & n_z \\ 0 & 0 & -1 \end{bmatrix}$$

From $\mathbf{n} \times \mathbf{o} = \mathbf{a}$, we get:

$$i(-n_y) - j(-n_x) + k(0) = -i$$

which simplifies to:

$$n_y = 1, \quad n_x = 0, \quad n_z = 0$$

Thus, the completed matrix is:

$$F = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.9. Derive the matrix that represents a pure rotation about the z-axis of the reference frame.

Solution

$$p_x = p_n \cos\theta - p_o \sin\theta$$

$$p_y = p_n \sin\theta + p_o \cos\theta$$

$$p_z = p_a$$

Expressed in matrix form:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_n \\ p_o \\ p_a \end{bmatrix}$$

where $C = \cos\theta$ and $S = \sin\theta$.

2.13. Find the new location of point P1; 2; 3T relative to the reference frame after a rotation of 30 about the z-axis followed by a rotation of 60 about the y-axis.

Solution

$$UP = \mathrm{Rot}(y, 60) \cdot \mathrm{Rot}(z, 30) \cdot P$$

$$\begin{bmatrix} 0.5 & 0 & 0.866 & 0 \\ 0 & 1 & 0 & 0 \\ -0.866 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.531 \\ 2.232 \\ 1.616 \\ 1 \end{bmatrix}$$

2.21. Write the correct sequence of movements that must be made in order to restore the original orientation of the spherical coordinates and make it parallel to the reference frame. About what axes are these rotations supposed to be?

Solution

$$T(r,\beta,\gamma)\cdot \mathrm{Rot}(o,-\beta)\cdot \mathrm{Rot}(a,-\gamma)=\begin{bmatrix}1&0&0&rS_\beta C_\gamma\\0&1&0&rS_\beta S_\gamma\\0&0&1&rC_\beta\\0&0&0&1\end{bmatrix}$$

As shown, the rotations are relative to the o- and a-axes in the sequence given. These rotations must be relative to the current moving frame in order to prevent changing the position of the frame.

2.24 Suppose that a robot is made of a Cartesian and Euler combination of joints. Find the necessary Euler angles to achieve the following transformation:

$$T=\begin{bmatrix}0.527&-0.574&0.628&4\\0.369&0.819&0.439&6\\-0.766&0&0.643&9\\0&0&0&1\end{bmatrix}$$

Solution

$$\phi=\mathrm{ATAN2}(a_y,a_x)\quad\text{or}\quad\phi=\mathrm{ATAN2}(-a_y,-a_x)$$

$$\begin{cases}\phi=\mathrm{ATAN2}(0.439,0.628)=35^\circ\\\phi=\mathrm{ATAN2}(-0.439,-0.628)=215^\circ\end{cases}$$

$$\psi=\mathrm{ATAN2}\left(\frac{-n_xS\phi+n_yC\phi}{-o_xS\phi+o_yC\phi}\right)$$

$$\begin{cases}\psi=\mathrm{ATAN2}(0,1)=0^\circ\\\psi=\mathrm{ATAN2}(0,-1)=180^\circ\end{cases}$$

$$\theta=\mathrm{ATAN2}\left(\frac{a_xC\phi+a_yS\phi}{a_z}\right)$$

$$\begin{cases}\theta=\mathrm{ATAN2}[(0.628\times 0.819+0.439\times 0.573),0.643]=50^\circ\\\theta=\mathrm{ATAN2}[(0.628\times(-0.819)+0.439\times(-0.573)),0.643]=-50^\circ\end{cases}$$

Thus, the possible Euler angles are:

$$\begin{cases}\phi=35^\circ,\quad\psi=0^\circ,\\\theta=50^\circ\\\phi=215^\circ,\quad\psi=180^\circ,\\\theta=-50^\circ\end{cases}$$

2.27. A frame UF was moved along its own n-axis a distance of 5 units, then rotated about its o-axis an angle of60,followedbyarotationof60

aboutthez-axis,thentranslatedaboutitsa-axisfor3units,and finally rotated 45 about the x-axis.

Solution

$$UT = \mathrm{Rot}(x,45)\cdot\mathrm{Rot}(z,60)\cdot\mathrm{Trans}(5,0,0)\cdot\mathrm{Rot}(o,60)\cdot\mathrm{Trans}(0,0,3)$$

$$B = \begin{bmatrix} 0.25 & -0.866 & 0.433 & 3.8 \\ 0.918 & 0.354 & 0.177 & 3.59 \\ -0.306 & 0.354 & 0.884 & 5.71 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For Cartesian coordinates:

$$p_x = 3.8, \quad p_y = 3.59, \quad p_z = 5.71$$

For RPY angles:

$$\phi = \mathrm{ATAN2}(a_{yx}, a_{xx}) = \mathrm{ATAN2}(0.918, 0.25) = 74.8^\circ$$

or

$$\phi = \mathrm{ATAN2}(-a_{yx}, -a_{xx}) = 254.8^\circ$$

$$\theta = \mathrm{ATAN2}(-a_{zx}, (a_{yx}C\phi + a_{xx}S\phi)) = \mathrm{ATAN2}(0.306, 0.951) = 17.8^\circ$$

or

$$\theta = \mathrm{ATAN2}(0.306, -0.951) = 162.2^\circ$$

$$\psi = \mathrm{ATAN2}((-a_yC\theta + a_xS\theta), (o_yC\theta - o_xS\theta)) = \mathrm{ATAN2}(0.372, 0.928) = 21.8^\circ$$

or

$$\psi = \mathrm{ATAN2}(-0.372, -0.928) = 201.8^\circ$$

2.30 In the 2-DOF robot shown, the transformation matrix 0TH is given in symbolic form, as well as in numerical form for a specific location. The length of each link l1 and l2 is 1 ft. Calculate the values of u1 and u2 for the given location.

$$\theta_1 = \mathrm{ATAN2}\left(\frac{p_y - l_2S_{12}}{l_1}, \frac{p_x - l_2C_{12}}{l_1}\right)$$

$$l_2S_{12} + l_1S_1 = p_y$$

Solution:

$$\theta_{12} = \mathrm{ATAN2}(S_{12}, C_{12}) = \mathrm{ATAN2}(0.9563, -0.2924) = 107^\circ$$

$$\begin{cases} l_2 C_{12} + l_1 C_1 = p_x \\ l_2 S_{12} + l_1 S_1 = p_y \end{cases}$$

Solving for θ_1:

$$C_1 = \frac{p_x - l_2 C_{12}}{l_1}, \quad S_1 = \frac{p_y - l_2 S_{12}}{l_1}$$

$$\theta_1 = \mathrm{ATAN2}\left(\frac{0.8172 - 0.9563}{0.6978 + 0.2924}\right) = \mathrm{ATAN2}(-0.1391, 0.9902) = -8^\circ$$

$$\theta_2 = 107^\circ - (-8^\circ) = 115^\circ$$