Chapter 7.

Matrices, linear systems.

Sec 7.1:

[1] **Matrix**: a matrix is a rectangular array of elements (objects) arrangend in rows and columns enclose in brackets [ ] or ( ).

* We use the letters A, B, C, ... to denote a matrix.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 5 & 0 \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} x & x+1 \\ 2y & 3y \end{bmatrix}$

[2] Size of a matrix = no. of rows × no. of columns

Ex: $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & -4 \end{bmatrix}_{2\times 3}$ size = 2×3.

[3] An $m \times n$ matrix A: (the general form)

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix},$$

a_{ij} = entry of i^{th} row and j^{th} column.

5 $A_{n \times n}$ is an $n \times n$ matrix = square matrix.

A is an $m \times n$ matrix = rectangular matrix.

6 A is an $n \times n$ matrix. $\boxed{A = [a_{ij}]_{n \times n}}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$ main diagonal

$$A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$$ diagonal matrix.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$$ upper-triangular matrix.

$$A = \begin{bmatrix} a_{11} & & 0 \\ \vdots & \ddots & a_{nn} \\ a_{n1} & \cdots & \end{bmatrix}$$ Lower-triangular matrix.

$$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$ = identity matrix.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} & \end{bmatrix}$.

I identity of size 2

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$\boxed{7}$ Zero matrix: O, need not a square matrix.

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{8}$ ✓ Row matrix: A has 1-row.

$$A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}_{1 \times n} = \text{row vector.}$$

✓ Column matrix = has 1-column.

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1} = \text{column vector.}$$

$\boxed{9}$ Equality of matrices:

$$A = B \begin{cases} \text{i) size } A = \text{size } B. \\ a_{ij} = b_{ij} \quad \forall i, j. \end{cases}$$

$[a_{ij}]$ $[b_{ij}]$

Ex: if $\begin{bmatrix} a^2-1 & 3 \\ 0 & b+2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}$, Find a, b.

$\boxed{9a}$ Addition and subtraction ✓ A, B of the same size.

$$A \mp B = [a_{ij} \mp b_{ij}]$$

Ex: If $\begin{bmatrix} x^2 & 1 \\ 3 & 1 \end{bmatrix} + 2 \begin{bmatrix} x & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$

Find x.

$x^2 + 2x + 1 = 0$

Rules for addition

$\boxed{10}$ (i) $A + B = B + A$

(ii) $A + (B + C) = (A + B) + C$.

$\boxed{11}$ Scalar multiplications:

Let c be a scalar, $A = [a_{ij}]_{m \times n}$.

then $cA = [ca_{ij}]$,

<u>note</u> A and cA are of the same size.

<u>Ex:</u> Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 4 \end{bmatrix}$

1) Find a) $2A - 3B$.

b) $A + 2B$

2) Find the matrix C such that

$2A + 4C = B$

$\boxed{12}$ Rules for scalar multiplications.

$c(A+B) = cA + cB$

$(c+k)A = cA + kA$

$c(kA) = (ck)A$.

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Sec 7.2

1 Matrix multiplication:

$A_{n\times q}$, $B_{q\times m}$ then

$$AB = [C]_{n\times m} = [c_{ij}].$$

$$c_{ij} = a_{i1} b_{1j} + \cdots + a_{iq} b_{qj}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}_{3\times 2}$.

$$AB = \begin{bmatrix} \boxed{1\times 2 + 2\times(-1) + 3\times 0} & \boxed{} \\ \boxed{} & \boxed{} \end{bmatrix}_{2\times 2}$$

$$BA = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times 3}$$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A I_2 = A$, $I_2 A = A$.

Note: ① $AB \neq BA$.

② if A square, then $AI = IA$

Ex: $A = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}$

Find $A(B+C)$, $AB + AC$.

Ex2 $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, Find AB, BA.

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. but $A \neq 0$, $B \neq 0$.

$AB = 0 \not\Rightarrow$ A, B need not be zero matrices.

Ex3: If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, Find A^2.

Find A^{100}.

$A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

A

$A^{100} = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$ □.

Problems:

① Is $(A+B)^2 = A^2 + 2AB + B^2$ <u>No</u>

Sol: $(A+B)^2 = (A+B)(A+B)$

$$= A^2 + AB + BA + B^2$$

we Know $AB \neq BA$ in general.

② If $AB = AD$ then $B = D$ <u>No</u>

$AB = AD \Rightarrow AB - AD = 0$

$$\Rightarrow A(B-D) = 0$$

$B-D$ need not be zero,

$\therefore B \neq D$ in general

Ex: $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

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1 Transpose of the matrix:

$$A = [a_{ij}]_{n\times m} \Rightarrow A^T = [a_{ji}]_{m\times n}.$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 0 & 5 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & 5 \end{bmatrix}$.

properties of Transpose:

a) $(A+B)^T = A^T + B^T$.

b) $(AB)^T = B^T A^T$.

Examples ...

c) $(A^T)^T = A$.

(d) If $A^T = A$, then A is symmetric matrix.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(e) $A^T = -A$ then A is skew-symmetric.

Ex: $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $A^T = -A$

(f) $A^2 = A$ idempotent matrix:

Ex: $A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$