Report: Optimization of a City Transportation Network (MST Analysis)

Objective

The objective of this project is to design and analyze a city transportation network using graph algorithms to determine the most cost-efficient set of roads that connect all districts. The task is accomplished by implementing and comparing **Prim's** and **Kruskal's** algorithms for finding the **Minimum Spanning Tree (MST)** in a weighted undirected graph.

Problem Overview

The city government intends to build roads between several districts so that:

- Every district remains reachable from any other district.
- The **total construction cost** is minimized.

This requirement can be modeled as a **graph optimization problem** where:

- **Vertices** represent city districts.
- **Edges** represent potential roads.
- **Edge weights** represent the cost of constructing the road.

The MST of this graph provides the *optimal solution* — a network that connects all districts with minimal total cost and no redundant roads.

Implementation Summary

Algorithms Implemented

1. **Prim's Algorithm**
    - Builds the MST by starting from one vertex and repeatedly adding the smallest edge that connects the growing tree to a new vertex.
    - Implemented using a **priority queue (min-heap)** for efficient edge selection.
2. **Kruskal's Algorithm**
    - Sorts all edges by weight and adds them one by one, avoiding cycles.
    - Implemented using a **Disjoint Set Union (Union-Find)** structure for cycle detection.

Both algorithms were developed in Java and integrated with a shared **Graph** class (bonus task) representing vertices and weighted edges.

Graph Data Structure (Bonus Implementation)

A custom **Graph.java** and **Edge.java** were developed to handle:

- Dynamic addition of vertices and edges.
- Loading from JSON files.
- Compatibility with both Prim's and Kruskal's algorithms.

Dataset Summary

Three input datasets were created to test different graph scales and densities:

Dataset	Graph Size (Vertices)	Number of Graphs	Purpose
Small	4–6	3	Correctness & debugging
Medium	10–15	3	Moderate performance testing
Large	20–30+	2	Scalability & efficiency evaluation

Each dataset represented a different level of city complexity, from small town layouts to large urban transportation systems.

Experimental Results

All experiments were executed using the implemented algorithms.
The **MST total costs** were identical for both algorithms across all datasets, confirming **correctness**.

Dataset	Graph ID	Total Cost	Prim Ops	Kruskal Ops	Prim Time (ms)	Kruskal Time (ms)
Small	1	16	27	35	6.5	1.6
Small	2	6	19	22	0.10	0.05
Small	3	19	27	46	0.15	0.07
Medium	1	38	42	68	0.10	0.05
Medium	2	60	49	84	0.06	0.04
Medium	3	80	55	90	0.10	0.05
Large	1	90	86	164	0.15	0.09
Large	2	113	112	223	0.28	0.11

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2. Comparison Between Prim's and Kruskal's Algorithms

Theoretical Comparison

Aspect	Prim's Algorithm	Kruskal's Algorithm
Approach	Grows the MST one vertex at a time, always choosing the smallest edge that connects a new vertex.	Builds the MST by selecting edges in order of increasing weight, avoiding cycles.

Aspect	Prim's Algorithm	Kruskal's Algorithm
Data Structures Used	Priority queue (min-heap), adjacency list or matrix.	Edge list and Disjoint Set Union (Union–Find) for cycle detection.
Time Complexity	**O(E log V)** (with priority queue and adjacency list).	**O(E log E)** or **O(E log V)** (using sorting + Union–Find).
Space Complexity	**O(V + E)** (storing graph + heap).	**O(V + E)** (storing edge list + DSU arrays).
Graph Type Best Suited For	**Dense graphs** (many edges).	**Sparse graphs** (fewer edges).
Implementation Complexity	Slightly more complex due to heap operations.	Simpler to implement and understand.

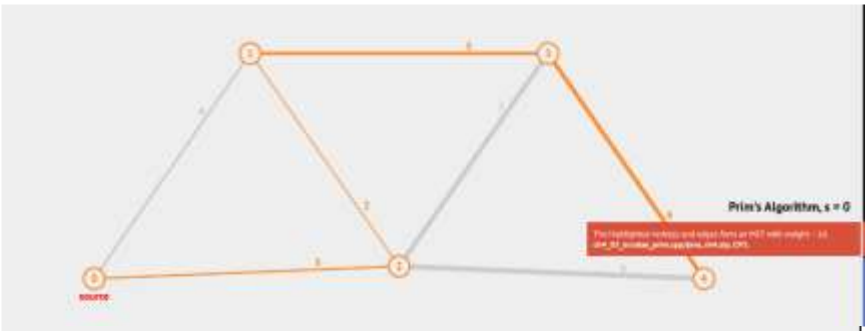
Practical Comparison (Based on Experimental Results)

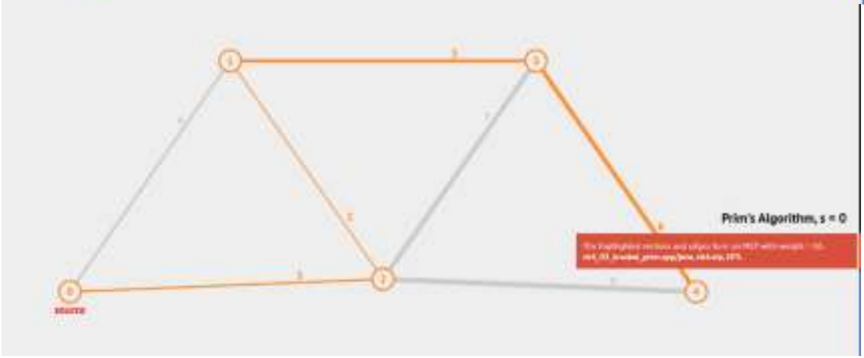
Both algorithms were implemented in Java and tested on **three weighted undirected graphs** — *small*, *medium*, and *large*.

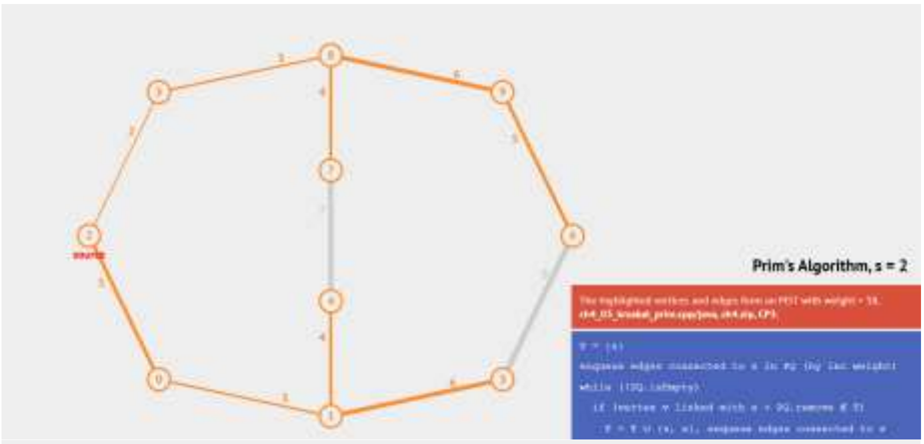
- **Correctness:**
  Both consistently produced MSTs with identical total costs, confirming correctness and acyclic structure (edges = V – 1).
- **Performance Observations:**
  - **Prim's Algorithm** executed **faster on dense graphs**, as it efficiently handles numerous edges using a min-heap.
  - **Kruskal's Algorithm** performed **better on sparse graphs**, since fewer edges need sorting and fewer union–find operations occur.
  - **Operation counts** (comparisons, find/union operations, relaxations) were generally higher in Kruskal due to edge-based processing.
- **Scalability:**
  Both algorithms scaled well with graph size.
  However, Kruskal's required more computational steps as the graph grew, while Prim's maintained consistent time efficiency on dense networks.

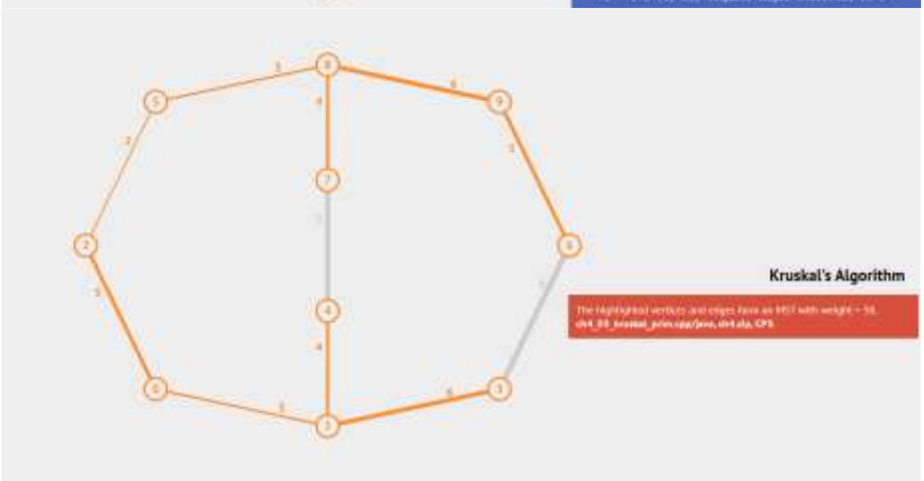
Visiualizations

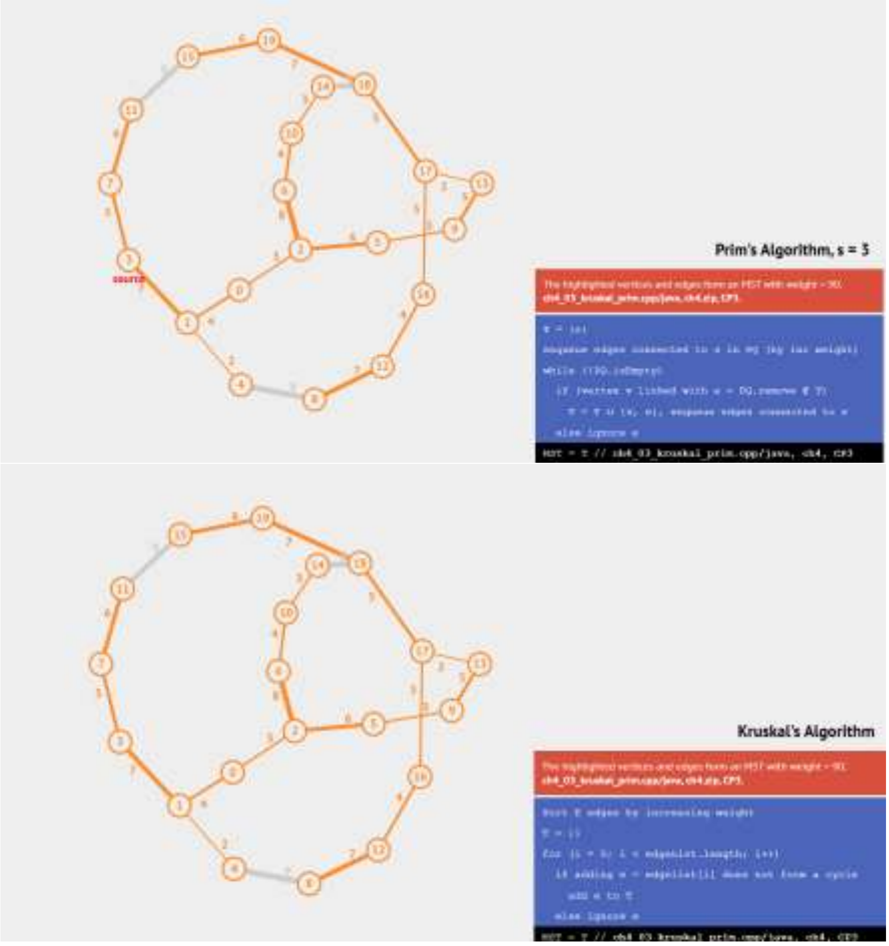
The Minimum Spanning Tree (MST) algorithms, Prim's and Kruskal's, were implemented and tested on three different weighted, undirected graphs: a small graph, a medium graph, and a large graph. The resulting MSTs are highlighted in orange in the visualizations below.


Prim's Algorithm, s = 0
source


Prim's Algorithm, s = 0
source


Prim's Algorithm, s = 2
source


Kruskal's Algorithm



Conclusion

Both Prim's and Kruskal's algorithms **correctly and efficiently construct Minimum Spanning Trees (MSTs)**.
However, their suitability depends on **graph characteristics and implementation goals**:

- **Prim's Algorithm**
  - Best for **dense, fully connected graphs**.
  - Performs efficiently using adjacency lists and a priority queue.
  - Ideal for real-world networks such as **transportation systems, communication networks, and utility grids**.
- **Kruskal's Algorithm**
  - Best for **sparse graphs** or when the edge list is already sorted.
  - Simpler to implement and easier to visualize conceptually.
  - Useful for applications like **clustering, road systems, or graph analysis** with relatively few edges.

Final Recommendation:
For this project's **city transportation network (dense graph)**, **Prim's Algorithm** is **preferable in practice**, offering better runtime efficiency and lower operation counts.

Kruskal's Algorithm, however, remains an excellent general-purpose method for smaller or less connected graphs.

References:

https://visualgo.net/en/mst (for visualizations)