Data Structures and Algorithms Spring 2024 — Problem Sets

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Week 4. Problem set

1. Compute asymptotic worst case time complexity of the `solve` procedure:

   (i) Express the running time of the `solve` procedure $T(n)$ as a recurrence relation.

   (ii) Find the asymptotic complexity of $T(n)$ using the master theorem.

   (iii) Specify which case of the master theorem applies (if any).

   (iv) Write down conditions for this case that need to be checked (write down specialized version for a particular recurrence).

   (v) Prove that the conditions are satisfied. For asymptotic notation, each property used in a proof **must** be either proven explicitly or properly referenced (e.g. by citing [Cormen] or a particular Lecture/Tutorial slide).

```
/* A is a 0-indexed array
 * n is the number of elements in A */
solve(A, n):
  helper(A, 0, n - 1)

helper(A, l, r):
  if l > r
    return
  else
    k := ⌈(r - l + 1) / 4⌉
    for j from 0 to 3 /* inclusive */
      helper(A, l + j * k, min(r, l + (j + 1) * k))
    for a from l to r
      m := i
      for b from l to r
        if A[b] ≥ A[m]:
          m = b
      exchange A[m] with A[a]
```

2. For each of the following recurrences, apply the master theorem [Cormen, Theorem 4.1] yielding a closed form formula for the asymptotic complexity of $T(n)$. Assume that $T(n) = 1$ when $n < 10$ for all recurrences below. You must specify the following:

   - Can the theorem be applied to the recurrence?
   - If yes, then
     (i) Which case of the master theorem applies?
     (ii) Which conditions for this case that need to be checked (write down specialized version for a particular recurrence).
     (iii) Prove that the condition is satisfied. For asymptotic notation, each property used in a proof **must** be either proven explicitly or properly referenced (e.g. by citing [Cormen] or a particular Lecture/Tutorial slide).
     (iv) Provide asymptotic complexity for $T(n)$ using Θ-notation.
   - Otherwise, provide an explicit justification, explaining why the theorem cannot be applied.

   (a) $T(n) = 2T(n/3) + \log_2 n$
   (b) $T(n) = 7T(n/49) + \sqrt{n} \cdot \log_2^2 n$
   (c) $T(n) = 4T(n/3) + n^2$
   (d) $T(n) = 2T(\sqrt{n}) + n$
   (e) $T(n) = \frac{1}{2}T(2n) + n \log_2 n$

3. **(+1% extra credit)** For each of the following recurrences, apply the master theorem [Cormen, Theorem 4.1] yielding a closed form formula for the asymptotic complexity of $T(n)$. Assume that $T(n) = 1$ when $n < 10$ for all recurrences below. You must specify the following:

   - Can the theorem be applied to the recurrence?
   - If yes, then
     (i) Which case of the master theorem applies?
     (ii) Which conditions for this case that need to be checked (write down specialized version for a particular recurrence).
     (iii) Prove that the condition is satisfied. For asymptotic notation, each property used in a proof **must** be either proven explicitly or properly referenced (e.g. by citing [Cormen] or a particular Lecture/Tutorial slide).
     (iv) Provide asymptotic complexity for $T(n)$ using Θ-notation.
   - Otherwise, provide an explicit justification, explaining why the theorem cannot be applied.

   (a) $T(n) = 9T(n/2) + n^3 \log_2 n$
   (b) $T(n) = 8T(n/2) + 2^n \log_2 n$
   (c) $T(n) = T(n/2) + n(2 + \sin(\frac{n\pi}{2}))$
   (d) $T(n) = 2T(n/2) + \frac{n}{\log_2 n}$
   (e) $T(n) = 3T(n/3) + 3n \log_2(\log_2 n)$

References

[Cormen] T. H. Cormen, C. E. Leiserson, R. L. Rivest and C. Stein. *Introduction to Algorithms, Fourth Edition.* The MIT Press 2022