07 – Partial Differential Equations

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Basic Definitions

$$\frac{\partial f}{\partial x}(x_0, y_0) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

The shorthand notation: $f = f(t,x)$ $\quad f_t = \frac{\partial f}{\partial t}(t,x)$ $\quad f_x = \frac{\partial f}{\partial x}(t,x)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}(t,x) \qquad f_{xt} = \frac{\partial^2 f}{\partial t\, \partial x}(t,x) \qquad f_{tt} = \frac{\partial^2 f}{\partial t^2}(t,x)$$

The example calculation:

$$f(t,x) = t^2 \sin x$$

$$f_t = 2t \sin x \qquad f_{tt} = 2 \sin x$$

$$f_x = t^2 \cos x \qquad f_{xt} = 2t \cos x$$

$$f_{xx} = -t^2 \sin x$$

The physical examples

Maxwell equations:

$$\begin{cases} \operatorname{div} \boldsymbol{D} = 4\pi\rho \\ \operatorname{div} \boldsymbol{B} = 0 \\ \operatorname{rot} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\ \operatorname{rot} \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} \end{cases}$$

Wave equation:

$$u_{tt} = c^2 \Delta u$$

Heat equation:

$$u_t = \kappa \Delta u$$

Poisson equation:

$$\Delta \phi = -4\pi\rho$$

2^{nd} Order PDE: General Classification

The general linear second-order PDE in two independent variables has the form:

$$A(x,y)u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + \cdots \text{(lower order terms)} = 0$$

1. $B^2 - AC < 0$ – **elliptic PDE.** The example: **Laplace equation:**

$$u_{xx} + u_{yy} = 0 \ (A = C = 1, B = 0)$$

2. $B^2 - AC = 0$ – **parabolic PDE.** The example: **heat (or diffusion) equation:**

$$u_t = u_{xx} \ (A = 1, B = C = 0)$$

3. $B^2 - AC > 0$ – **hyperbolic PDE.** The example: **wave equation:**

$$\frac{1}{c^2}u_{tt} = u_{xx} \ (A = 1, B = 0, C = -1/c^2)$$

Understanding the Problem

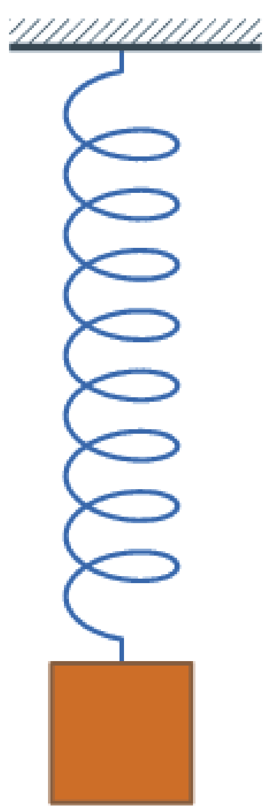
We will consider the 1D wave equation as the example:

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le a; \;\; 0 \le t \le t_{max}$$

Possible physical problem: vibrations of the string

To solve this, we need the initial and boundary conditions

Harmonic oscillator



The equation to solve:

$$m\ddot{x} = -kx$$

The initial conditions:

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = v_0 \end{cases}$$

String vibrations

The equation to solve:

$$u_{tt} = c^2 u_{xx}$$

The initial conditions:

$$\begin{cases} u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$$

$g(x)$ – the analogue of the initial velocity

The boundary conditions:

$$\begin{cases} u(t,0) = \phi(t) = 0 \\ u(t,a) = \psi(t) = 0 \end{cases}$$

The string is fixed at its ends

Types of Initial & Boundary Conditions

Boundary conditions put constraints on the values of $u(t, x)$ at $x = 0$ and $x = a$.

They can be specified in different ways:

1. Dirichlet boundary conditions: $\begin{cases} u(t,0) = \phi(t) \\ u(t,a) = \psi(t) \end{cases}$ **Put on the solved function itself**

2. Neumann boundary conditions: $\begin{cases} u_x(t,0) = \phi(t) \\ u_x(t,a) = \psi(t) \end{cases}$ **Put on the derivative of the solved function**

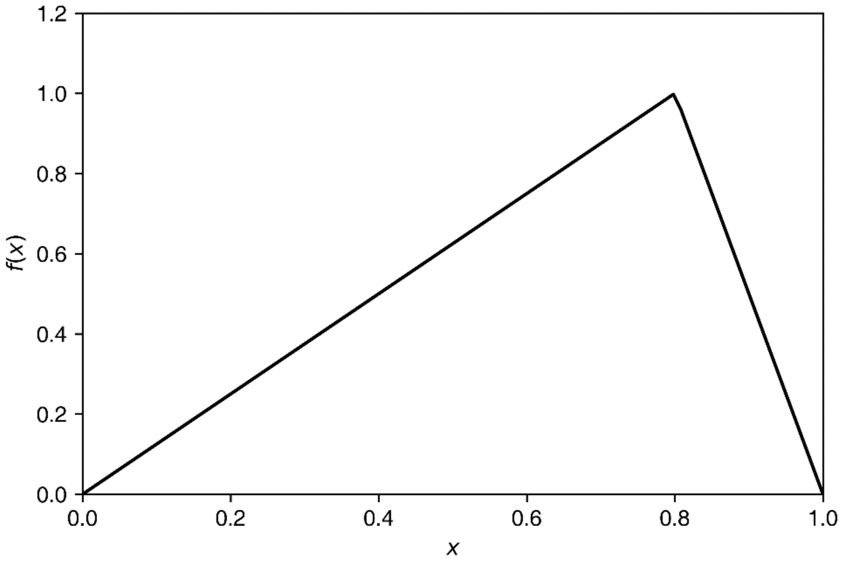
3. Mixed boundary conditions: $\begin{cases} u(t,0) = \phi(t) \\ u_x(t,a) = \psi(t) \end{cases}$ **Just a combination of the two above**

The Formulation and the Solution

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le a;\ \ 0 \le t \le t_{max} \tag{1}$$

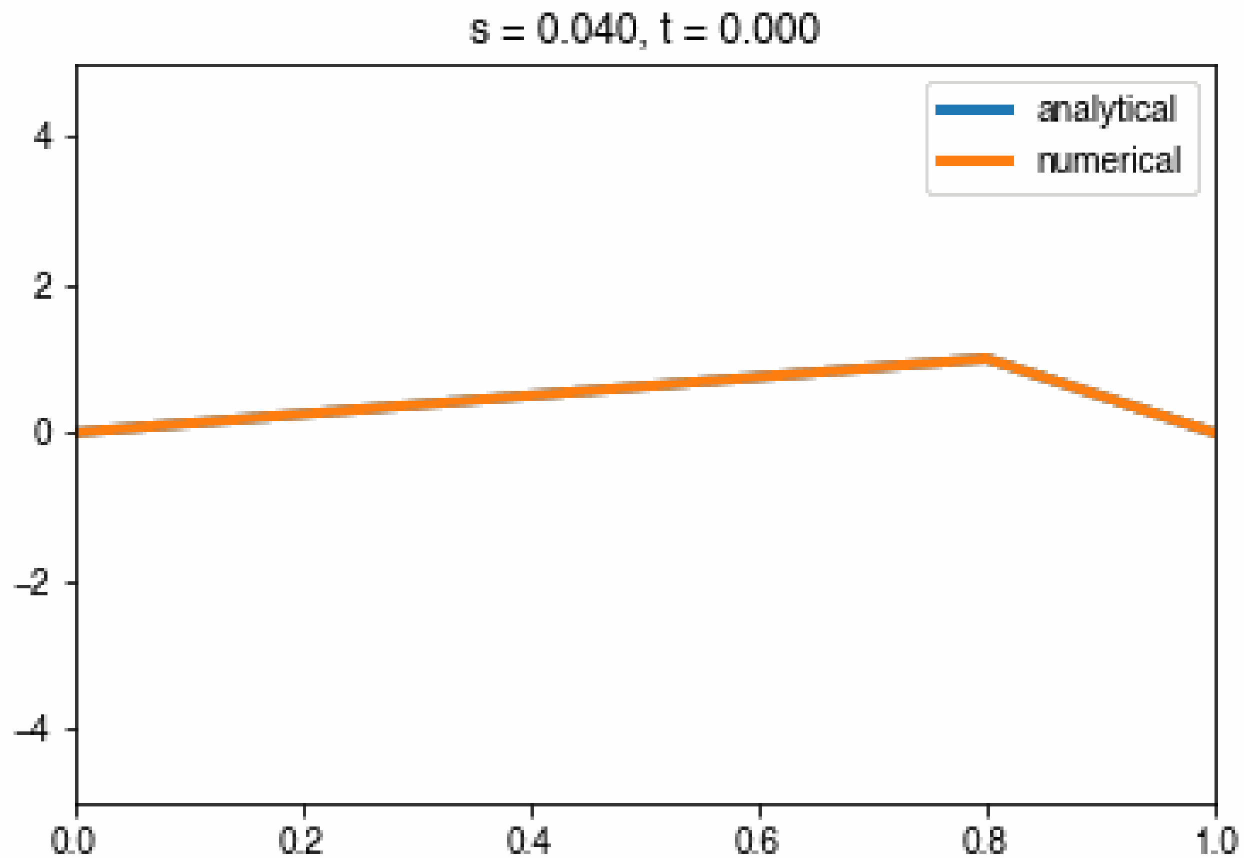
The initial conditions:

$$\begin{cases} u(0,x) = f(x) \\ u_t(0,x) = g(x) = 0 \end{cases} \qquad f(x) = \begin{cases} 1.25\dfrac{x}{a}, & x \le 0.8a \\ 5\left(1 - \dfrac{x}{a}\right), & x > 0.8a \end{cases}$$



The boundary conditions:

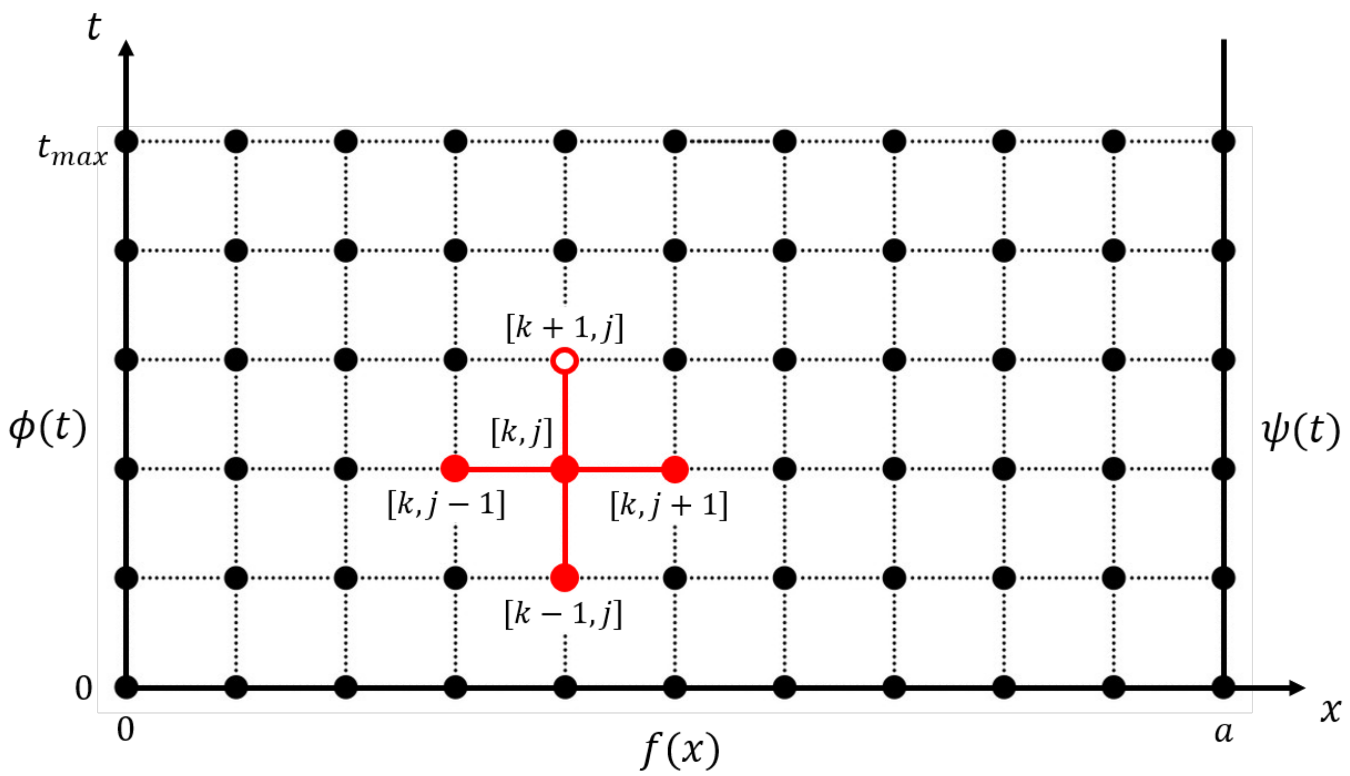
$$\begin{cases} u(t,0) = \phi(t) = 0 \\ u(t,a) = \psi(t) = 0 \end{cases}$$



Explicit Forward (Euler) Method

We are going to discretize our equation on the 2D grid in space and time:

Also known as the "cross" scheme



We approximate the second derivatives using central differences:

$$u_{tt} = \frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{\Delta t^2}, \qquad u_{xx} = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{\Delta x^2}$$

Using Eq. (1) and $s = \frac{c\Delta t}{\Delta x}$, we solve for u_j^{k+1} at time step $k+1$:

$$u_j^{k+1} = -u_j^{k-1} + 2u_j^k(1 - s^2) + s^2\left(u_{j+1}^k + u_{j-1}^k\right) \tag{2}$$

For Dirichlet-type conditions, everything is straightforward:

$$u(0,x) = f(x) \quad \longrightarrow \quad u_j^0 = f\left(x_j\right), \forall\, j = 1, \dots, n_x$$

However, Neumann-type conditions (which are put on the derivatives) require additional numerical approximation. Imagine we have the following initial condition:

$$u_t(0,x) = g(x)$$

To maintain this condition, we have to discretize it:

$$u_t(0,x) \approx \frac{u_j^1 - u_j^{-1}}{2\Delta t} = g_j \equiv g(x_j) \qquad (3)$$

In other words, we introduce a 'ghost', fictitious layer with $k = -1$. This layer is required **only** to evaluate **the first timestep**. The only purpose of it is to simulate the correct value of the derivative at the beginning. Afterwards we can forget about this 'ghost' layer.

Dealing with Initial / Boundary Conditions

Solving the Eq. (3) we obtain:

$$u_j^{-1} = u_j^1 + 2\Delta t g(x_j)$$

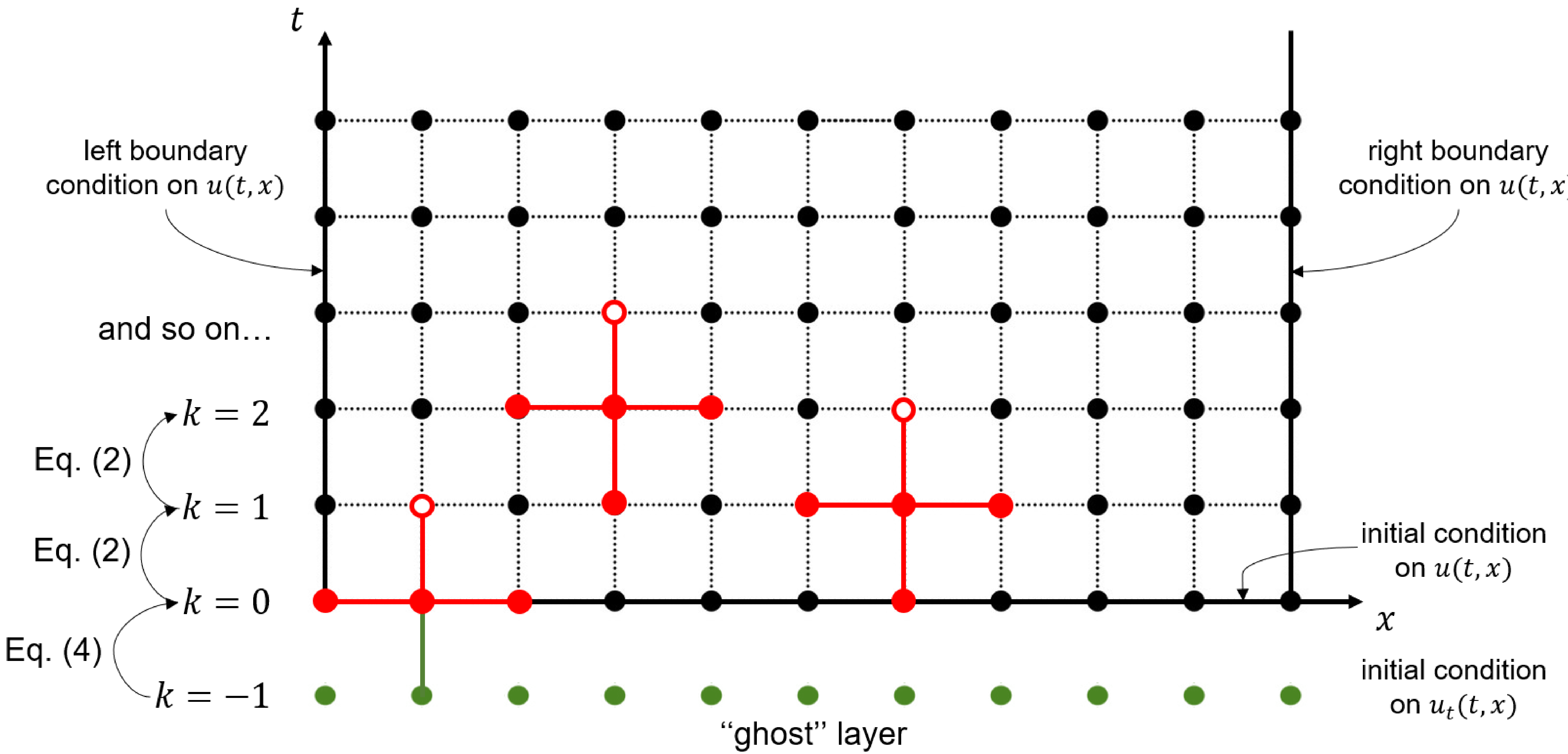
In the special case when $g(x) = 0$, this means $u_j^{-1} = u_j^1$. Then, our finite difference expression will change a bit at the first timestep:

$$u_j^1 = -u_j^{-1} + 2u_j^0(1 - s^2) + s^2\left(u_{j+1}^0 + u_{j-1}^0\right)$$

$$\Big\downarrow \; u_j^{-1} = u_j^1$$

$$\boxed{u_j^1 = u_j^0(1 - s^2) + \frac{1}{2}s^2\left(u_{j+1}^0 + u_{j-1}^0\right)} \qquad (4)$$

Scheme of the Solution Algorithm



Stability of the Explicit Forward Method

There is a single dimensionless parameters in our problem which defines its scale:

$$s = \frac{c\Delta t}{\Delta x}$$

It appears that this parameter called **Courant number** plays the key role in determining the stability of numerical schemes for hyperbolic equations.

If the errors made at one timestep of the calculation do not cause the errors to be magnified as the computations are continued, then the scheme is **stable**.

If, on the contrary, the errors grow with time the numerical scheme is said to be **unstable**.

It can be shown that Explicit Forward Method considered here is stable, if the following condition is fulfilled:

$$s = \frac{c\Delta t}{\Delta x} \leq 1$$

The Analytical Solution

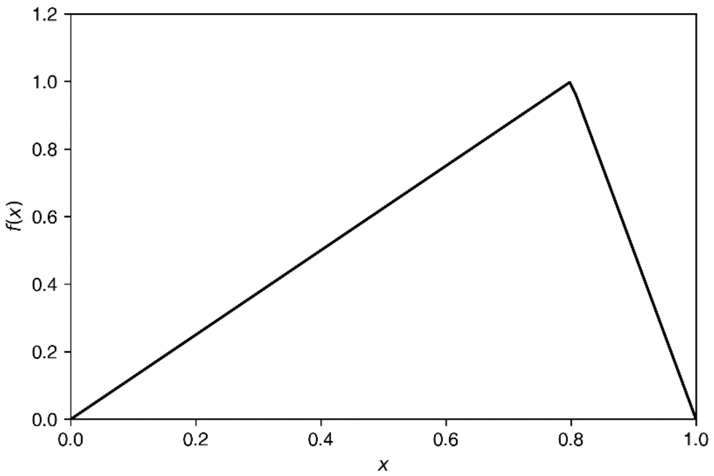
$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le a; \;\; 0 \le t \le t_{max}$$

The initial conditions:

$$\begin{cases} u(0,x) = f(x) \\ u_t(0,x) = g(x) = 0 \end{cases}$$

The boundary conditions:

$$\begin{cases} u(t,0) = \phi(t) = 0 \\ u(t,a) = \psi(t) = 0 \end{cases}$$



The general solution of this equation:

$$u(t,x) = \frac{f(x-ct) + f(x+ct)}{2} \qquad (5)$$

We should define $f(x)$ for arbitrary x. To satisfy boundary conditions, we can do:

$$f(-x) = -f(x), \qquad f(x+2a) = f(x)$$

This extended function $f(x)$ should be used to calculate the analytical solution according to the Eq. (5).