HW4: Solutions

№1

$$\hat{x}^2 = \frac{x_0^2}{2}(\hat{a}+\hat{a}^+)(\hat{a}+\hat{a}^+) = \frac{x_0^2}{2}\Big(\hat{a}\hat{a} + \underbrace{\hat{a}\hat{a}^+}_{\hat{N}+1} + \underbrace{\hat{a}^+\hat{a}}_{\hat{N}} + \hat{a}^+\hat{a}^+\Big) =$$

$$= \frac{x_0^2}{2}\Big(\hat{a}\hat{a} + \hat{a}^+\hat{a}^+ + 2\hat{N}+1\Big)$$

$$[\hat{a},\hat{a}^+] = 1 = \hat{a}\hat{a}^+ - \underbrace{\hat{a}^+\hat{a}}_{\hat{N}}$$

$$\langle n|\hat{x}^2|n\rangle = \frac{x_0^2}{2}\Big[\underbrace{\cancel{\langle n|\hat{a}\hat{a}|n\rangle}^{0}}_{\propto\langle n|n-2\rangle=0} + \underbrace{\cancel{\langle n|\hat{a}^+\hat{a}^+|n\rangle}^{0}}_{\propto\langle n|n+2\rangle=0} + \langle n|2\hat{N}+1|n\rangle\Big] =$$

$$= \frac{x_0^2}{2}(2n+1);$$

$$\hat{N}|n\rangle = n|n\rangle$$

$$\hat{p}^2 = -\frac{p_0^2}{2}(\hat{a}^+-\hat{a})(\hat{a}^+-\hat{a}) = -\frac{p_0^2}{2}\Big(\hat{a}^+\hat{a}^+ - \hat{a}^+\hat{a} - \hat{a}\hat{a}^+ + \hat{a}\hat{a}\Big) =$$

$$= -\frac{p_0^2}{2}\Big(\hat{a}^+\hat{a}^+ + \hat{a}\hat{a} - (2\hat{N}+1)\Big)$$

Analogously to $\hat{x}^2$:

$$\langle n|\hat{p}^2|n\rangle = +\frac{p_0^2}{2}\Big[\langle n|2\hat{N}+1|n\rangle\Big] = \frac{p_0^2}{2}(2n+1)$$

For $\hat{x}^4 = \hat{x}^2\hat{x}^2 = (\hat{x}^2)^+\hat{x}^2$:

$$\langle n|\hat{x}^4|n\rangle = \langle n|(\hat{x}^2)^+\hat{x}^2|n\rangle = \langle \hat{x}^2 n|\hat{x}^2 n\rangle = \|\hat{x}^2|n\rangle\|^2; \quad (*)$$

$$\hat{x}^2|n\rangle \; \cancel{\frac{x_0^2}{2}\Big[\sqrt{n(n-1)} + \sqrt{(n+1)(n+2)} +}$$

$$\| \quad \frac{x_0^2}{2}\Big[\sqrt{n(n-1)}\,|n-2\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle + (2n+1)|n\rangle\Big] \quad (**)$$

Now we substitute $(**)$ to $(*)$. Due to $\langle k|n\rangle = \delta_{kn}$, cross-terms will vanish and only diagonal terms survive:

$$\|\hat{x}^2|n\rangle\|^2 = \frac{x_0^4}{4}\Big[n(n-1)\underbrace{\langle n-2|n-2\rangle}_{1} + (n+1)(n+2)\underbrace{\langle n+2|n+2\rangle}_{1} +$$

$$+ (2n+1)^2 \underset{\overset{\shortparallel}{1}}{\langle n|n\rangle}\Big] = \frac{x_0^4}{4}(6n^2+6n+3)$$

(„Diagonal" means having the same quantum number).

$\langle n|\hat{x}^{2k+1}|n\rangle = ?$

For $k=0$ we proved in class $\langle n|\hat{x}|n\rangle = 0$.

Let's consider $k=1 \longrightarrow \hat{x}^3 = \frac{x_0^3}{2\sqrt{2}}(\hat{a}+\hat{a}^+)^3 = \frac{x_0^3}{2\sqrt{2}}(\hat{a}\hat{a}\hat{a} + \hat{a}\hat{a}\hat{a}^+ + \hat{a}\hat{a}^+\hat{a} + \hat{a}\hat{a}^+\hat{a}^+ + \hat{a}^+\hat{a}\hat{a} + \hat{a}^+\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}^+\hat{a} + \hat{a}^+\hat{a}^+\hat{a}^+)$. (Remark: if you use Newton's binomial, that would lead to the wrong result! $[\hat{a}, \hat{a}^+] \neq 0$!)

From this example it's obvious that $\forall k$ we get all possible combinations of $\hat{a}^+$ and $\hat{a}$, and in every term $\hat{a}^+$ and $\hat{a}$ are present unequally (because $2k+1$ is odd).

If we act $\hat{x}^3|n\rangle$, this would give sum of $|n-3\rangle$, $|n-1\rangle$, $|n+1\rangle$ and $|n+3\rangle$ with some coefficients. Then, it's clear that

$\underline{\langle n|\hat{x}^3|n\rangle = 0}$

The same logic is valid for $\forall\, 2k+1$, the vector $\hat{x}^{2k+1}|n\rangle$ does not contain $|n\rangle$ because $\hat{a}$ and $\hat{a}^+$ are imbalanced.

Therefore, $\underline{\langle n|\hat{x}^{2k+1}|n\rangle = 0}$

$\boxed{\text{Alternative argument:}}$ $\langle n|\hat{x}^{2k+1}|n\rangle = \int_{-\infty}^{+\infty} x^{2k+1}|\psi_n(x)|^2 dx$.

From general form of solution, $|\psi_n(x)|^2$ is an even function. Then, we have integral of odd function, which is <u>zero</u>.

Remark: actually, there is no need to know the form of $\psi_n(x)$ to prove this. Oscillator has even potential $U(x) = \frac{1}{2}m\omega^2 x^2$, and there is a theorem, which says that in this case all solutions of corresponding Schrödinger eq. are even or odd, i.e.: $\psi(-x)=\psi(x)$ or $\psi(-x)=-\psi(x)$.

Knowing this general property, it's obvious that $|\psi_n(x)|^2$ is even (without knowing the explicit form).

You can read e.g. § 3.6 of Gasiorowicz.

Variances: $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{x_0^2}{2}(2n+1)$;

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2 = \frac{p_0^2}{2}(2n+1);$$

$$\Delta x \cdot \Delta p = \sqrt{\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle} = \frac{x_0 p_0}{2}(2n+1) = \frac{\hbar}{2}(2n+1)$$

Minimal value is $\hbar/2$ for $n=0$, which is minimal possible value, according to Heisenberg principle. //

№ 2.

From the general form:

$$\psi_2(x) = \frac{1}{\sqrt{2^2\cdot 2!\cdot x_0\sqrt{\pi}}}\underbrace{H_2\left(\frac{x}{x_0}\right)}_{H_2(\xi)=4\xi^2-2} e^{-\frac{x^2}{2x_0^2}} = \frac{1}{\cancel{2}\sqrt{2x_0\sqrt{\pi}}}\left(\cancel{4}^{2}\left(\frac{x}{x_0}\right)^2-\cancel{2}^{1}\right)e^{-\frac{x^2}{2x_0^2}} \quad (1)$$

Now applying $\hat{a}^+$. In coordinate representation:

$$\hat{a}^+ = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - x_0\frac{d}{dx}\right)$$

In class we derived: $\hat{a}^+|0\rangle = |1\rangle$.

$$\Big\downarrow \text{coord. representation}$$

$$\hat{a}^+\psi_0(x) = \psi_1(x)$$

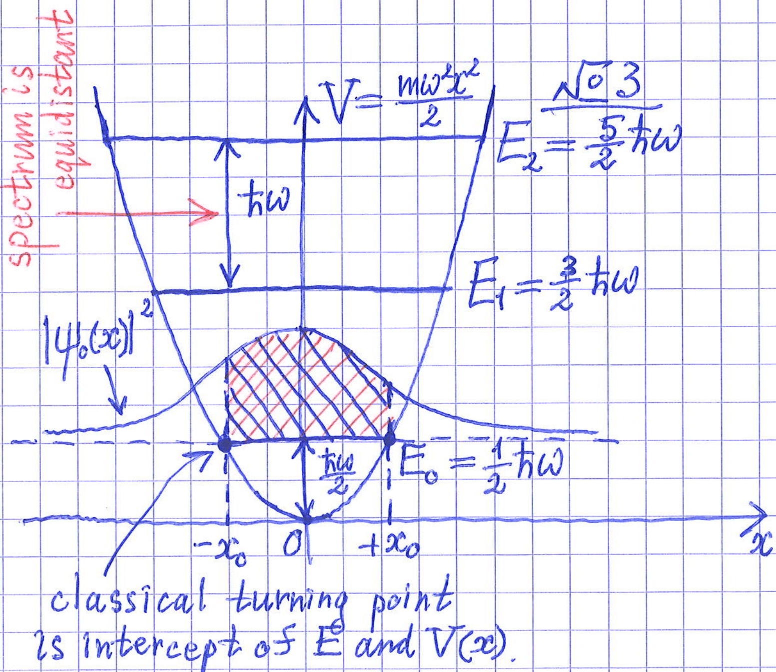
$$\psi_1(x) = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - x_0\frac{d}{dx}\right)\left(\frac{1}{\sqrt{x_0\sqrt{\pi}}}e^{-\frac{x^2}{2x_0^2}}\right) = \frac{1}{\sqrt{2x_0\sqrt{\pi}}}\cdot\left(\frac{x}{x_0}e^{-\frac{x^2}{2x_0^2}} - x_0\cdot\left(-\frac{\cancel{2}x}{\cancel{2}x_0^2}\cdot e^{-\frac{x^2}{2x_0^2}}\right)\right) = \frac{1}{\sqrt{2x_0\sqrt{\pi}}}\cdot 2\left(\frac{x}{x_0}\right)\cdot e^{-\frac{x^2}{2x_0^2}};$$

From seminar: $|2\rangle = \frac{1}{\sqrt{2}}\hat{a}^{+}|1\rangle \xrightarrow{\text{coord. rep.}} \psi_2(x) = \frac{1}{\sqrt{2}}\hat{a}^{+}\psi_1(x)$.

$$\psi_2(x) = \frac{1}{\cancel{\sqrt{2}}}\frac{1}{\cancel{\sqrt{2}}}\left(\frac{x}{x_0} - x_0\frac{d}{dx}\right)\left(\frac{1}{\sqrt{2x_0\sqrt{\pi}}}\cancel{2}\left(\frac{x}{x_0}\right)e^{-\frac{x^2}{2x_0^2}}\right) = \frac{1}{\sqrt{2x_0\sqrt{\pi}}}\cdot$$

$$\cdot\left(\left(\frac{x}{x_0}\right)^2 - \cancel{x_0}\left(\frac{1}{\cancel{x_0}} - \frac{x}{\cancel{x_0}}\cdot\frac{\cancel{2}x}{\cancel{2}x_0^2}\right)\right)e^{-\frac{x^2}{2x_0^2}} = \frac{1}{\sqrt{2x_0\sqrt{\pi}}}\left(2\left(\frac{x}{x_0}\right)^2 - 1\right)e^{-\frac{x^2}{2x_0^2}} \quad (2$$

$\Rightarrow$ (1) and (2) are the same. //



Schematics of QHO.

Dashed area = probability to be <u>inside</u> clasically allowed region.

It's given by:

$$\mathbb{P}(|x| \leqslant x_0) = \int_{-x_0}^{x_0} |\psi_0(x)|^2 dx.$$

We need probability <u>outside</u> the region, which is obviously:

$$\mathbb{P}(|x| > x_0) = 1 - \mathbb{P}(|x| \leqslant x_0) = 1 - \int_{-x_0}^{x_0} \frac{1}{x_0\sqrt{\pi}}\cdot e^{-\frac{x^2}{x_0^2}}dx =$$

$$= \left| \begin{array}{l} y = x/x_0; \\ dx = x_0 dy; \end{array} \right| = 1 - \int_{-1}^{1}\frac{1}{\sqrt{\pi}}\underbrace{e^{-y^2}}_{\text{even}}dy = 1 - \underbrace{\frac{2}{\sqrt{\pi}}\int_0^1 e^{-y^2}dy}_{\text{erf}(1)\text{ by definition}} \ominus$$

↑ symmetric interval

$\ominus$ $1 - \text{erf}(1) \approx 0.157$.

This illustrates that quantum particle is very different: there is 15% chance to find it outside classical region. //