1. The ground state of 1D quantum harmonic oscillator of mass m and frequency ω is perturbed by the potential

$$\hat{V} = \frac{1}{2}\alpha m\omega^2 x^2,\ \alpha \ll 1.$$

In other words, this corresponds to slightly changing the spring constant $k = m\omega^2$ of oscillator. Solve this problem using perturbation theory. To be short, please use bra-/ket-notation throughout the solution, no need to write the wave functions explicitly. Below is the standard algorithm for that:

Algorithm 1 Perturbation theory (non-degenerate case)

1: Calculate the perturbation matrix elements V_{k0} using bra-/ket-notation
2: Find the first-order energy correction $E_0^{(1)}$ to the ground state
3: Find the first-order correction $|0^{(1)}\rangle$ to the ground-state ket
4: Find the second-order energy correction $E_0^{(2)}$

The full Hamiltonian of the problem $\hat{H} = \hat{H}_0 + \hat{V}$ also admits the exact solution. Identify how to get it (easily) from the known solution for $\hat{H}_0$ (do it only for energies). Compare the exact and perturbative solutions for the energy. What can you say?