Generic two integer variable equation solver

Alpertron › Programs › Generic two integer variable equation solver

$ax^2 + bxy + cy^2 + dx + ey + f = 0$

a	2
b	0
c	-1
d	-2
e	1
f	0

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$2x^2 - y^2 - 2x + y = 0$

The discriminant is $b^2 - 4ac = 8$

Let D be the discriminant. We apply the transformation of Legendre $Dx = X + \alpha$, $Dy = Y + \beta$, and we obtain:

$\alpha = 2cd - be = 4$

$\beta = 2ae - bd = 4$

$2X^2 - Y^2 = 16$ **(1)**

where the right hand side equals $-D(ae^2 - bed + cd^2 + fD)$

The algorithm requires that the coefficient of X^2 and the right hand side are coprime. This does not happen, so we have to find a value of m such that applying one of the unimodular transformations

- $X = mU + (m-1)V$, $Y = U + V$
- $X = U + V$, $Y = (m-1)U + mV$

the coefficient of U^2 and the right hand side are coprime. This coefficient equals $2m^2 - 1$ in the first case and $-(m-1)^2 + 2$ in the second case.

We will use the first unimodular transformation with $m = 1$: $X = U$, $Y = U + V$ **(2)**

Using **(2)**, the equation **(1)** converts to:

$U^2 - 2UV - V^2 = 16$ **(3)**

We will have to solve several quadratic modular equations. To do this we have to factor the modulus and find the solution modulo the powers of the prime factors. Then we combine them by using the Chinese Remainder Theorem.

The different moduli are divisors of the right hand side, so we only have to factor it once.

$16 = 2^4$

Searching for solutions U and V coprime.

We have to solve: $T^2 - 2T - 1 \equiv 0 \pmod{16 = 2^4}$

There are no solutions modulo 2^4, so the modular equation does not have any solution.

Let $2U' = U$ and $2V' = V$. Searching for solutions U' and V' coprime.

From equation **(3)** we obtain $U'^2 - 2U'V' - V'^2 = 16 / 2^2 = 4$

We have to solve: $T^2 - 2T - 1 \equiv 0 \pmod{4 = 2^2}$

There are no solutions modulo 2^2, so the modular equation does not have any solution.

Let $4U' = U$ and $4V' = V$. Searching for solutions U' and V' coprime.

From equation **(3)** we obtain $U'^2 - 2U'V' - V'^2 = 16 / 4^2 = 1$

We have to solve: $T^2 - 2T - 1 \equiv 0 \pmod{1 = 1}$

1. $T = 0$

   The transformation $U' = -k$ **(4)** converts $U'^2 - 2U'V' - V'^2 = 1$ to $PV'^2 + QV'k + Rk^2 = 1$ **(5)**

   where: $P = (aT^2 + bT + c) / n = -1$, $Q = -(2aT + b) = 2$, $R = an = 1$

   The continued fraction expansion of $(Q + \sqrt{D/4}) / R = (1 + \sqrt{2}) / 1$ is:

   2**+ //** **2**// **(6)**

   Solution of **(5)** found using the convergent $V' / (-k) = 0 / 1$ of **(6)**

   From **(4)**: $U' = 1$, $V' = 0$

   $U = 4$, $V = 0$

   From **(2)**:

   $X = 4$, $Y = 4$

   $X + \alpha = 8$, $Y + \beta = 8$

   Dividing these numbers by $D = 8$:

   x = 1
   y = 1

   $U = -4$, $V = 0$

   From **(2)**:

   $X = -4$, $Y = -4$

$X + \alpha = 0$, $Y + \beta = 0$

Dividing these numbers by $D = 8$:

x = 0
y = 0

The continued fraction expansion of $(-Q + \sqrt{D / 4}) / (-R) = (-1 + \sqrt{2}) / (-1)$ is:

-1+ // 1, 1**, 2**// **(7)**

Solution of **(5)** found using the convergent $V' / (-k) = 0 / 1$ of **(7)**

From **(4)**: $U' = 1$, $V' = 0$

$U = 4$, $V = 0$

From **(2)**:

$X = 4$, $Y = 4$

$X + \alpha = 8$, $Y + \beta = 8$

Dividing these numbers by $D = 8$:

x = 1
y = 1

$U = -4$, $V = 0$

From **(2)**:

$X = -4$, $Y = -4$

$X + \alpha = 0$, $Y + \beta = 0$

Dividing these numbers by $D = 8$:

x = 0
y = 0

x = 1
y = 1

x = 0
y = 0

Recursive solutions:

$x_{n+1} = 3\,x_n + 2\,y_n - 2$
$y_{n+1} = 4\,x_n + 3\,y_n - 3$

and also:

$x_{n+1} = 3 x_n - 2 y_n$
$y_{n+1} = - 4 x_n + 3 y_n + 1$

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