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Option Pricing and Hedging under Jump-diffusion model

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1. Introduction

Stochastic or random process is a mathematical object which is usually defined as a collection of random variables. It can be seen as a random variable evolving over time. The weather, for example, is a stochastic process. A lot of such examples can be found in everyday life and stochastic processes can be used to model any kind of process that, in itself, has some kind of uncertainty involved.

Compared to a deterministic function, for which, at any time we know its value, in case of a stochastic process one cannot for sure know its value, but can maybe have an estimation or a probability of having a certain value. Although it is not ideal, it is for sure better than not knowing anything about it whatsoever. Under the assumption that processes in real world are random, but are to some degree *defined*, gives us hope that, without knowing their exact future behaviour or position, we can have a good feeling of how they could move.

Stochastic processes have a very wide application in the field of finance. The main reason for that is that they can be used to model an asset price as a process where uncertainty is present. Behaviour of such processes can be observed to potentially draw some conclusions from them.

The goal of this thesis is to utilize stochastic processes and apply them to tackle some of the problems from mathematical finance. Firstly, I will start with defining necessary mathematical and financial concepts which will be used throughout this thesis. After this foundation is set, starting with Brownian motion, step-by-step we will arrive at the jump-diffusion process which will be used as a process of the price of an asset. We will then introduce the idea of a financial derivative, in our case an option, and we will tackle the problem of determining a fair price of the option. Lastly, we will introduce the idea of risk, how to quantify it and how to hedge the risk of our position in the asset or the option. It is my hope that, this thesis will not only provide you with a good overview of pricing and hedging of financial instruments, but also give you the understanding of such the underlying processes are useful to model the scenarios that occur every day in the market.

2. Financial concepts

Let us start by defining what a market is. **Market** is a place where parties can meet to engage in an economic transaction. Usually, while only two parties are needed to make a trade, at minimum one more party is needed to introduce competition and bring balance to the market. A market in a state of **perfect competition** is characterized by a high number of active buyers and sellers. Markets vary for a number of reasons, including the types of products sold, size, location etc. One market which is of particular interest to us is the **financial market** where securities, currencies, bonds and other types of securities are traded. They form capital and provide liquidity for businesses. Stock exchanges like New York Stock Exchange (NYSE) or Nasdaq are one type of financial markets. Other types of financial markets include, for example bond markets and foreign exchange (FX) markets.

As already mentioned, markets in perfect competition have a high number of participants (buyers and sellers) and market determines the prices of goods and other services traded there. Prices are determined by **supply and demand**, where supply is created by the sellers and demand by the buyers. The law of supply and demand is a theory that explains the interaction between the sellers of a resource and the buyers for that resource. In general, if the price for a good decreases, more people are willing to buy it and less people are willing to sell. That is because the opportunity cost for the buyers increases, as they can obtain the good at a lower price than earlier, whereas the opportunity cost for the sellers increases, as they earn less by selling the same amount of goods as before. Although it is one of the most basic economic laws, it is a part of almost all economic principles. The willingness of people to buy or sell a good determines the market equilibrium price at which the quantity of goods people are willing to sell equals the quantity of goods people are willing to buy.

2.1. Financial instruments

In this section we will cover a few basic financial instruments. As discussed earlier, different types of markets offer different types of goods. A **bond market** (also called fixed-income markets) is a collective name attributed to all issues and trades of debt securities. Stock exchange is another type of market where, among many other things, **stocks** and **derivatives** can be traded.

A **bond** is a fixed-income financial instrument typically issued by governments or corporations in order to raise money needed to fund a certain project, for example. When a corporation (or a government) needs to borrow money, it issues bonds that include the terms of the loan, the time at which the loaned funds (bond principal) must be paid back (maturity date) and the interest payments that will be paid. They can essentially be thought of as a contract issued by the borrower promising to pay back the loan plus some interest on it at some time in the future. The interest is a premium for the person loaning the money, because by buying a bond (loaning money) the buyer takes on a risk that the issuer will not be able to pay back at maturity (risk of default). Governments are typically less likely to fail than corporations, therefore the interest paid on government bonds is generally lower than the interest paid on corporate bonds. There are also bonds that pay additional coupons between the issuance and maturity, but we won't go into them since they will not be of particular interest to us in this thesis.

A **stock** (also called *equity*) is a type of a security issued by the corporations and represents the ownership of a fraction of a company. The units of stock is called *shares*. They are issued for a company to raise funds to operate their business. It is important to point out that corporations are treated as legal persons, so a shareholder does not *own* a corporation, but own shares issued by the corporations and have a claim on its assets and earnings.

Having explained the characteristics of bonds and stocks we arrive at securities that are not completely basic. **Derivatives** are a type of security which *derives* its value from another asset (e.g. stock). There are many types of derivatives, but we will focus on the ones used in this thesis and those are options. **Option** is a contract that gives the buyer a right, but not an obligation, to buy or sell (depending on the type of the contract they hold) the underlying asset at a certain time in future.

European options (also called *vanilla* options) are the most basic form of an option. We differentiate two different types of european options: *call* and *put* option. A call option gives a buyer a right to buy the underlying security at some predetermined price (strike price) at a certain time in future (maturity). In case the price of the underlying asset is higher than the strike price at the time of maturity, the buyer can buy the underlying asset at the strike price and immediately sell the asset at its market price which is now higher than the strike price, thus making a profit. However, if the price of the underlying asset is less than the strike price at maturity, it does not make sense for the buyer to exercise the option and buy the asset at strike price, because she/he can simply buy the asset at the market price. In this case the options will expire without exercising and the only loss made by the buyer is the amount paid for the option in the first place. To illustrate this very important concept of European options let us look at the following example:

Suppose at time $t = 0$ the price of an asset is $S_0 = 100$ and that we can buy a *put* option with maturity T and strike price $K = 90$. The price of such option contract at time $t = 0$ is $P = 5$. Let us assume that at maturity the price of the asset is $S_T = 93$. This means that the buyer will simply let the contract expire because the asset can be sold at a market price which is higher than the strike price of the *put* option. If, however, the price of an asset dips below S_0 to value of $S_T = 80$, the buyer can buy the asset in the market for the price of 80 and immediately sell it in the market for $K = 90$ by exercising the option. Thus, making a profit of $(K - S_T) - P = (90 - 80) - 5 = 5$. Note that in this example, the time value of money was disregarded for simplicity purposes.

2.2. Arbitrage

In finance and economics, arbitrage is the practice to take advantage of market inefficiencies to make a risk-free profit. One such inefficiency can be, for example, the difference in prices of the same asset on two different markets. In other words, an arbitrage-free market is a market where there does not exist such trade (or a set of trades) which would allow a trader with zero capital at time $t = 0$ to make a positive profit at some time in future $t = T$ with positive probability. In practice, arbitrage opportunities exist, but are very hard to spot and almost immediately exploited and the opportunity is gone, often in the matter of seconds.

Here are two examples to better understand this very important principle:

Example 1

Suppose that the stock XYZ is trading for \$50 on the NYSE and the same stock is trading at \$49.5 at the LSE (London Stock Exchange). The trader can then buy the stock at LSE for \$49.5 and immediately sell it at NYSE for \$50, thus realizing profit of \$0.5.

Example 2

Let's see a more complex example. Suppose we have a stock XYZ with spot price $S_0 = 100$ and in one year period there are only two possibilities for the stock movement: it can go up to $S_1 = u * S_0$ where $u = 1.15$, or it can go down to $S_1 = d * S_0$ where $d = 0.9$. The risk-free rate in the market is constant during this one-year period and it is $r = 1\%$. The probability that the stock will go up in that period is $p = 0.4$.

The inconsistency is not easy to spot here. If the market is arbitrage-free, the expected value of the future payoff should be zero. By looking more closely, and considering the time value of money one can find the following arbitrage strategy: We decide to short-sell the stock today and put that money in the bank to be remunerated at the risk-free rate of 1%. After one year we take out our money from the bank and now we own the value of $(1 + r)S_0 = 101$. We have to buy back the stock that we shorted one year earlier to cover this short position. There are two possible scenarios:

- The stock went up and our profit in that case is: $(1 + r)S_0 - uS_0 = -14$
- The stock went down and our profit in that case is: $(1 + r)S_0 - dS_0 = 11$

Although at the first glance this does not look like a good deal, we have to take into account the probabilities of these events happening. More specifically, let us look at the expected payoff at time $t = 1$:

$$\mathbb{E}[\text{payoff at } t = 1] = 0.4 * (-14) + 0.6 * 11 = 1$$

2.3. Fundamental theorems of asset pricing

2.3.1. First fundamental theorem of asset pricing

Theorem

*A discrete market, on a discrete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, is **arbitrage-free** if, and only if, there exists at least one risk neutral probability measure that is equivalent to the original probability measure, \mathcal{P} .*

This first theorem is important in that it ensures a fundamental property of market models. It will allow for some nice manipulations with discounted price processes we will mention later in this thesis.

2.3.2. Second fundamental theorem of asset pricing

Theorem

*An arbitrage-free is said to be **complete** (every contingent claim replicable) if and only if there exists a unique risk-neutral measure equivalent to the physical one.*

This theorem can be reformulated in an equivalent way, that is the market is complete if and only if there exists only one source of randomness in asset dynamic. Although this property is common in many models it is sometimes not always considered desirable or realistic.

3. Stochastic processes

Stochastic processes have a well defined purpose in several fields, and one of those fields is finance. They seem to be the perfect choice when it comes to modelling assets' dynamics, because their *randomness* is used to replicate the uncertainty of asset prices that occurs in the market.

A continuous time **stochastic process** $\{X_t : t \geq 0\}$ is a collection of random variables indexed over time, often times simply $X(t)$. The probability space of these random variables is $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all possible outcomes, \mathcal{F} is σ -algebra of events and \mathbb{P} being the probability measure.

It is crucial to introduce a concept of filtration. **Filtration** $\{\mathcal{F}_t\}_{t \geq 0}$ is an increasing collection of σ -algebras, such that for every $0 \leq s \leq t$ the following holds: $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$. Filtration can be thought of as "*flow of information*". More specifically, \mathcal{F}_t can be considered as the information generated by all observed events up to time t .

3.1. Brownian motion

The goal of this chapter is to show how Brownian motion can be derived from a random walk process. This is important to get a good understanding of what Brownian motion in fact is, as we will use it very extensively throughout the this text.

To describe the process of a **random walk** we will use a sequence of independent identically distributed (*iid*) random variables X_i with the following distribution:

$$X_i \sim \begin{pmatrix} h & -h \\ p & q \end{pmatrix}$$

Where X_i describes the move of the particle in the the i -th step. Let us calculate the

expected value and the variance of each step:

$$\begin{aligned}
\mathbb{E}[X_i] &= h(p - q) \\
\text{Var}(X_i) &= \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = h^2(p - q) - h^2(p - q)^2 \\
&= h^2p + h^2q - h^2(p^2 - 2p1 + q^2) \\
&= h^2p(1 - p) + h^2q(1 - q) + 2h^2pq \\
&= 4h^2pq
\end{aligned}$$

Now that we have the building blocks of the random walk, let us define the state of the random walk after n steps as a random variable $X(t) = \sum_{i=1}^n X_i$. Where t is the length of the interval $[0, t]$. This interval is divided into n equal intervals of length Δt , so we have:

$$\Delta t = \frac{t}{n}$$

We calculate the expected value and the variance of this new random variable:

$$\begin{aligned}
\mathbb{E}[X(t)] &= nh(p - q) \\
\text{Var}(X(t)) &\stackrel{\text{ind.}}{=} \sum_{i=1}^n \text{Var}(X_i) = 4nh^2pq
\end{aligned}$$

According to the Central Limit Theorem with $n \rightarrow \infty$ we know that $X(t)$ will have a normal distribution, namely:

$$\begin{aligned}
\frac{\sum_{i=1}^n X_i - nh(p - q)}{\sqrt{n}\sqrt{\text{Var}(X_i)}} &= \frac{\sum_{i=1}^n X_i - nh(p - q)}{2h\sqrt{n}\sqrt{pq}} \sim \mathcal{N}(0, 1) \\
\Rightarrow \sum_{i=1}^n X_i - nh(p - q) &\sim \mathcal{N}(0, 4nh^2pq) \\
\sum_{i=1}^n X_i &\sim \mathcal{N}(nh(p - q), 4nh^2pq)
\end{aligned}$$

we substitute $n = \frac{t}{\Delta t}$

$$\begin{aligned}
X(t) &\sim \mathcal{N}(\mu t, \sigma^2 t) \\
\mu &:= \lim_{\Delta x \rightarrow 0, h \rightarrow 0} \frac{h(p - q)}{\Delta t} \\
\sigma^2 &:= \lim_{\Delta x \rightarrow 0, h \rightarrow 0} \frac{4h^2pq}{\Delta t}
\end{aligned}$$

Where μ is called the **drift** and σ^2 the **diffusion** of the process $X(t)$. We can have a look at the special case where $p = q = 0.5$ and $h = \sqrt{t/n} = \sqrt{\Delta t}$ and it is easy to see that then $X(t)$ has the following distribution:

$$X(t) \sim \mathcal{N}(0, t)$$

The above mentioned process is called **Wiener process** and is often denoted by $W = \{W_t : t \geq 0\}$ where $W_t \sim \mathcal{N}(0, t)$. The formal definition is the following:

Definition 1 (Wiener process). *A Wiener process is the process with the following properties:*

1. $W_0 = 0$
2. W_t is continuous in t
3. W_t has independent increments, i.e. for every $r < s \leq t < u$ increments $(W_u - W_t)$ and $(W_s - W_r)$ are independent random variables
4. W has Gaussian increments, i.e. for every $u, t \geq 0$: $(W_{t+u} - W_t) \sim \mathcal{N}(0, u)$

3.2. Stochastic Integrals

As already stated in the earlier chapters, one of the objectives of this book is to study asset pricing in financial markets using stochastic processes. We want to model the price as a continuous time stochastic process and for this case the most complete and elegant theory is obtained if we use **diffusion processes** and **stochastic differential equations** as main building blocks. Although we already mentioned diffusion a couple of times, we have not really intuitively explained what it is. Loosely speaking, a stochastic process is a diffusion if its local dynamics can be approximated with the following stochastic difference equation:

$$X_{t+h} - X_t = \mu(t, X_t)\Delta t + \sigma(t, X_t)\Delta W_t$$

where ΔW_t is defined as:

$$\Delta W_t = W_{t+h} - W_t$$

If we let h tend to 0 we can write the above equation like this:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

Which can be interpreted as a shorthand for the following integral equation:

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

The ds integral can be viewed as a regular Riemann integral, whereas for the dW_t integral we have to introduce a new concept - *Itô integral*. In order to be able to construct a stochastic integral of form:

$$\int_0^t g_s dW_s$$

we have to impose some kind of integrability condition on another stochastic process g_s .

Definition 2.

1. We say that the process g_s belongs to the class $\mathcal{L}^2[a, b]$ if the following holds:
 - $\int_a^b \mathbb{E}[g_s^2] ds < \infty$
 - The process g_s is adapted to filtration \mathcal{F}_t^W
2. We say the process g belongs to the class \mathcal{L}^2 if g belongs to $\mathcal{L}^2[0, t]$ for every t

We will show how to define the stochastic integral for the case of g_s which is **simple**, i.e. such g_s that there exist deterministic points in time $a = t_0 < t_1 < \dots < t_n = b$, such that g is constant on each subinterval. To put this formally, $g_s = g_{t_k}$ where $s \in [t_k, t_{k+1})$. In that case, the stochastic integral can be defined as follows:

$$\int_a^b g_s dW_s = \lim_{n \rightarrow \infty} \sum_{k=1}^n g_{t_{k-1}} (W_{t_k} - W_{t_{k-1}}) \quad , \text{ where } t_k = k \frac{t}{n}$$

The process g_s is evaluated in the summation on the left-hand point (t_{k-1}) which is known as a **non-anticipatory** integration. This is a natural choice in finance ensuring that we use no information from the future for our present actions.

For the case of g_s which is not *simple* (as described above), we have to use ...

3.3. Martingales

Theory of stochastic integration is closely related to the theory of martingales. Moreover, it is a foundation of the modern theory of financial derivatives. It would be unreasonable to not mention this topic, because of its extreme importance in the field. In discrete time, we say a stochastic process is a **martingale** if for any time n the following properties hold:

- $\mathbb{E}[|X_n|] < \infty$
- $\mathbb{E}[X_n | X_0, \dots, X_{n-1}] = X_{n-1}$

Where the first condition is more of a *technical* condition and the second one is the most important one. The second condition states that if at any time $t = n$ we look at the expected value of the process at any time $t > n$, it is equal to the value of the

process at time $t = n$. A process that is a martingale can be viewed as a model of a fair game.

In order to extend the definition of a martingale to a continuous case, let us first consider a concept of filtration $\mathcal{F}_{t \geq 0}$, that we have already mentioned earlier. As before, a filtration can be thought of as a *flow of information* and \mathcal{F}_t as the information generated by all observed events up to time t . For any random variable X let the symbol

$$\mathbb{E}[X|\mathcal{F}_t]$$

represent the "*expected value of X given the information available up to time t* ". It is important to note here that for a fixed t , the object $\mathbb{E}[X|\mathcal{F}_t]$ is a random variable. Let, for example, a filtration be generated by the process Y , then the information available up to time t will, of course, be dependent on the behaviour of the process Y in the time interval $[0, t]$. Before defining a continuous time martingale, let us first consider the following proposition:

- If Y and Z are random variables, and Z is \mathcal{F}_t -measurable (known at time t), then

$$\mathbb{E}[Z \cdot Y|\mathcal{F}_t] = Z \cdot \mathbb{E}[Y|\mathcal{F}_t]$$

- If Y is a random variable and $s < t$, then

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s]$$

Definition 3. A stochastic process X is an (\mathcal{F}_t) -martingale if the following conditions hold:

- X is adapted to filtration $\{\mathcal{F}_t\}_{t \geq 0}$
- For all t we have

$$\mathbb{E}[|X_t|] < \infty$$

- For all s and t , where $s \leq t$ we have

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s$$

The first condition above is a condition saying that the process X_t is observable (known) at time t . The second one, like in discrete case, is just a technical condition. Also like in discrete case, the third condition is the important one, which states that the expected future value of the X , given the information available today, is equal to today's observed value of X . In other words, a martingale has an absence of drift.

3.4. Stochastic Calculus and Itô Formula

When talking about stochastic processes, one can see that they are different than the usual functions. If we look at a infinitesimally small time interval Δt and observe the function behaviour in that time interval, the function is a straight line. This is unfortunately not the case with the stochastic processes, where even if we *zoom-in* on the trajectory, the trajectory is not smooth. Therefore, a Japanese mathematician Kiyoshi Itô started and created the field of **stochastic calculus** which is able to operate on stochastic processes.

We come to the very important theorem in the field of stochastic calculus and that is **Itô's formula**.

Theorem (Itô's formula)

Assume that the process X has a stochastic differential given by

$$dX_t = \mu_t dt + \sigma_t dW_t$$

where μ and σ are adapted processes, and let f be a $C^{1,2}$ -function. Define the process Z by $Z(t) = f(t, X_t)$. Then Z has a stochastic differential given by

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

A proof of this theorem is outside the scope of this thesis and it is rather cumbersome. The power and importance of this theorem is that we can define the differential of a function of a stochastic process. This will prove to be an indispensable tool when we will be pricing an option, as the price of the option depends on the price of an asset, that is, it is a sort of a function of the price process.

3.4.1. Geometric Brownian motion

Let us now show how to apply Itô's formula for the case of *geometric Brownian motion*. Before we start, it is crucial to first describe what kind of process that is.

Definition 4. A process X is said to be a **geometric Brownian motion** if it satisfies the following stochastic differential equation:

$$dX_t = \mu_t X_t dt + \sigma_t X_t dW_t$$

where μ_t and σ_t are *drift* and *diffusion* (respectively) of the process X . W_t is the basic Brownian motion, that is $W_t \sim \mathcal{N}(0, t)$. Let us now consider a function $f(t, X_t) = \ln(X_t)$ which, of course, yields another stochastic process. Another way of representing the Itô's formula, which is maybe a bit more general is important to have in mind and that is the following:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX_t)^2$$

If we expand the $(dX_t)^2$ we get:

$$(dX_t)^2 = (\mu_t X_t dt + \sigma_t X_t dW_t)^2 = \mu_t^2 X_t^2 (dt)^2 + 2\mu_t \sigma_t X_t^2 dt dW_t + \sigma_t^2 X_t^2 (dW_t)^2$$

where we use the formal multiplication table:

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dW_t = 0, \\ (dW_t)^2 = dt \end{cases}$$

so we have:

$$(dX_t)^2 = \sigma_t^2 X_t^2 dt$$

Before we apply the Itô's formula to this problem, let us first calculate the partial derivatives that appear in the formula.

$$\frac{\partial \ln(X_t)}{\partial t} = 0 \quad ; \quad \frac{\partial \ln(X_t)}{\partial x} = \frac{1}{x} \quad ; \quad \frac{\partial^2 \ln(X_t)}{\partial x^2} = -\frac{1}{x^2}$$

Now we are finally ready to apply the formula. We will step by step derive the formula for X_t :

$$\begin{aligned} df &= 0 \cdot dt + \frac{1}{X_t} dX_t - \frac{1}{2} \frac{1}{X_t^2} (dX_t)^2 \\ df &= \frac{1}{X_t} (\mu_t X_t dt + \sigma_t X_t dW_t) - \frac{1}{2} \frac{1}{X_t^2} \sigma_t^2 X_t^2 dt \\ df &= (\mu_t - \frac{1}{2} \sigma_t^2) dt + \sigma_t dW_t \end{aligned}$$

Now we are able to integrate over the time interval $[0, t]$:

$$\begin{aligned} \int_0^t df(s) &= (\mu_t - \frac{1}{2} \sigma_t^2) \int_0^t dt + \sigma_t \int_0^t dW_s \\ \ln(X_t) - \ln(X_0) &= (\mu_t - \frac{1}{2} \sigma_t^2) t + \sigma_t (W_t - W_0) \end{aligned}$$

Remembering that the $W_0 = 0$ we get the following result:

$$\ln \frac{X_t}{X_0} = (\mu_t - \frac{1}{2} \sigma_t^2) t + \sigma_t W_t$$

$$X_t = X_0 \exp \left\{ \left(\mu_t - \frac{1}{2} \sigma_t^2 \right) t + \sigma_t W_t \right\} \quad (3.1)$$

This model is in finance also known as **log-normal asset return model**, because we are using logarithmic prices. It is important to note that this model only has **positive values** of stock prices, which is in that sense in line with the real world, where stock prices can not become negative.

4. Pricing an asset

4.1. Poisson process

Poisson process and exponential distribution are very closely linked so let us have a look at exponential distribution first. Poisson process has some convenient mathematical properties, which has led to it being used as a mathematical model for seemingly random processes in various disciplines, including, of course, economics. It is named after French mathematician Simeon Denis Poisson despite the fact that he has never studied this process. The reason why it is called a Poisson process is that if a collection of points in some space forms a Poisson process then the number of those points in a finite size region is a random variable with a Poisson distribution. This will be step by step explained in next sections.

4.1.1. Exponential distribution

As mentioned earlier, exponential distribution is very closely linked to the Poisson process as the time between events in Poisson process has exponential distribution. It is a continuous analogue of the geometric distribution.

Definition 5 (Exponential distribution). *We say that a random variable X is exponentially distributed if its probability density function (PDF) is:*

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

where $\lambda > 0$ is the parameter of the distribution, often called the rate parameter. The notation to indicate that a random variable is exponentially distributed is

$$X \sim \varepsilon(\lambda)$$

The expected value of a random variable X which is exponentially distributed is

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

The most famous property of the exponential distribution is memorylessness. **Memorylessness** is a property of certain probability distributions which refers to the fact that the waiting time until the occurrence of a certain event does not depend on how much time has already elapsed. Memorylessness of an exponential distribution is formally expressed as:

$$P(X > t + s | X > s) = e^{-\lambda t} = P(X > t)$$

If X is a random variable modelling a waiting time, this would mean that the probability that we wait more than $t + s$ if we already know that at least s time has elapsed, is the same as the probability that we will wait more than t .

4.1.2. Defining a Poisson process

A **Poisson process** registers the occurrences of the certain event in a finite time interval. It notes the number of occurrences as well as times when the event happened.

Definition 6 (Poisson process). *Poisson process* $(N_t, t \geq 0)$ is given by the following conditions:

1. $N_0 = 0$
2. N has independent increments
3. Random variable $N(s, t) = N_t - N_s$ where $0 \leq s \leq t$, has a Poisson distribution with parameter $\lambda(t - s)$, that is

$$P(N_t - N_s = n) = e^{-\lambda(t-s)} \frac{[\lambda(t-s)]^n}{n!}$$

There is an alternative definition of a Poisson process which we will use when we will be defining a jump-diffusion process. As we already mentioned before, Poisson process registers occurrences of a certain event on a *finite time interval*. If we let that interval to become very small and denote it by h , we denote the probabilities of the number occurrences by:

$$\begin{aligned} P(N(h) \geq 2) &= o(h) \\ P(N(h) = 1) &= \lambda h \\ P(N(h) = 0) &= 1 - \lambda h + o(h) \end{aligned}$$

Where $o(h)$ is infinitesimally small amount, that is, some function with the property:

$$\lim_{h \rightarrow 0} \frac{o(h)}{h} = 0$$

Then we can let dN denote an increment in the Poisson process on an infinitesimally small time interval dt .

$$dN = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{with probability } (1 - \lambda dt) \end{cases}$$

4.2. Pure Jump process

4.2.1. Definition

Now that the Poisson process has been explained, it will be easy to understand the Jump process. Like we stated earlier, the Poisson process registers occurrences of certain events. When talking about a jump process those events that occur will be the jumps. On each such event the value of our Jump process will be increased by a certain value (it will "jump"). The initial value of the Jump process is $J(0) = 0$. The size of the i -th jump will be a random variable U_i independent of the Poisson process, that is $N(t) \perp U_i, \forall i$. All U_i are independent and identically distributed random variables (*iid*).

Definition 7. *We define a Jump process as*

$$J(t) = \sum_{i=1}^{N(t)} U_i \quad (4.1)$$

assuming $J(0) = 0$. The Jump process is right-continuous.

We will use \mathbb{Q}_j and \mathbb{Q}_u to indicate the probability measure governing the jump arrival times and the probability measure governing the jump sizes, respectively. We assume independence of the mentioned probability measures. Let us now have a look at the expected value of the Jump process we have just defined:

$$\mathbb{E}[J(t)] = \mathbb{E}^{\mathbb{Q}_j} \mathbb{E}^{\mathbb{Q}_u} \sum_{i=1}^{N(t)} U_i \quad (4.2)$$

When performing the calculation we have to have in mind that $\mathbb{E}^{\mathbb{Q}_u}[U_i] = k$.

$$\begin{aligned}
\mathbb{E}[J(t)] &= \sum_{n=0}^{\infty} \mathbb{E}^{\mathbb{Q}_u} \left[\sum_{i=1}^n U_i \right] P(N(t) = n) = \sum_{n=0}^{\infty} \sum_{i=1}^n \mathbb{E}^{\mathbb{Q}_u} [U_i] P(N(t) = n) \\
&= \sum_{n=0}^{\infty} nk \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \sum_{n=1}^{\infty} nk \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= ke^{-\lambda t} \lambda t \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} = k\lambda t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} \\
&= k\lambda t
\end{aligned} \tag{4.3}$$

As we can see the the expected value of the Jump process at time t is different than 0 which will subsequently lead to the conclusion that the process $J(t)$ is not a martingale. Sometimes it is useful to work with a Jump process which, in turn, is a martingale. Such process is called a **compensated Jump process**.

Proposition 1. *Let us denote a compensated Jump process with $\tilde{J}(t)$ and let it be defined by:*

$$\tilde{J}(t) := J(t) - k\lambda t \tag{4.4}$$

Such a process is a martingale.

Proof. Let us assume we that we have $0 \leq s \leq t$ and we determine the expected value of the $\tilde{J}(t)$ given that we have information up to time s :

$$\begin{aligned}
\mathbb{E}[\tilde{J}(t)|\mathcal{F}_s] &= \mathbb{E}[\tilde{J}(t) + \tilde{J}(s) - \tilde{J}(s)|\mathcal{F}_s] \\
&= \mathbb{E}[\tilde{J}(t) - \tilde{J}(s)|\mathcal{F}_s] + \mathbb{E}[\tilde{J}(s)|\mathcal{F}_s] \\
&= \mathbb{E}[\tilde{J}(t-s)] + \tilde{J}(s) \quad (\text{independent increments}) \\
&= \mathbb{E}[J(t-s) - k\lambda(t-s)] + \tilde{J}(s) \\
&= k\lambda(t-s) - k\lambda(t-s) + \tilde{J}(s) \\
&= \tilde{J}(s)
\end{aligned}$$

□

Before we continue further and comobine the knowledge we have on geometric Brownian motion and Jump processes to obtain a Jump-diffusion process, let us first denote with $d\tilde{J}$ the change in the compensated Jump process in an infinitesimally small time interval dt :

$$d\tilde{J} = \begin{cases} U_i - k\lambda dt & \text{with probability } \lambda dt \\ -k\lambda dt & \text{with probability } (1 - \lambda dt) \end{cases}$$

4.3. Jump-diffusion process

This has led us to the final part where we will obtain the final process of our asset price. This process will be the combination of the Geometric Brownian motion and a (compensated) Jump process. The reason to use a Jump-diffusion process as a process which will govern the price of an asset is because it better mimics the real-world situation where unexpected jumps can happen. In the previously defined jump process the jump sizes were *iid* random variables U_i . We wish to make the jumps proportional to the value of the geometric Brownian motion at that point in time. Formally the differential form of a jump-diffusion process will be:

$$dX_t = \mu_t X_t dt + \sigma_t X_t dW_t + X_t U dN_t \quad (4.5)$$

or also,

$$\frac{dX_t}{X_t} = \mu_t dt + \sigma_t dW_t + U dN_t \quad (4.6)$$

We will assume, for simplicity purposes, that the μ, σ and λ are constant in time. Further more, if we observe the behaviour of the process we can differentiate two different types of segments in the process:

1. between jumps it is driven by a pure diffusion process
2. at jump times (τ_i) the value of the process after a jump depends on the value right before the jump (τ_i^-)

For the first case when the process is driven by a pure diffusion process we have

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t \quad (4.7)$$

which can be written also as

$$d(\ln X_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t \quad (4.8)$$

After a jump at time τ_i we have that

$$X(\tau_i) - X(\tau_i^-) = U_i X(\tau_i^-) \quad (4.9)$$

or equivalently

$$X(\tau_i) = X(\tau_i^-)(1 + U_i) \quad (4.10)$$

In the part 3.4.1 we have shown that the formula for the value of the diffusion process at time t is

$$X(t) = X(0) \exp \left[\left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right]$$

which enables us to denote the value of the process right after the i -th jump as

$$X(\tau_i) = X(\tau_{i-1}) \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) \Delta\tau + \sigma W_{\Delta\tau} \right] \left(1 + U_i \right)$$

where $\Delta\tau = \tau_i - \tau_{i-1}$. If we would iterate this process, starting at $t = 0$ and assuming there were no jumps at that time, we would get the final equation of the resulting asset price process:

$$X(t) = X(0) \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) \Delta\tau + \sigma W_{\Delta\tau} \right] \left(\prod_{i=1}^{N(t)} 1 + U_i \right) \quad (4.11)$$

We have, however, not taken into account that the expected change in the resulting process is not anymore $X_t \mu dt$, because we have incorporated jumps and we have not compensated for it. In order to fix that we would have to compensate for that like we did earlier with the compensated Jump process. Doing that and then following the same logic as above we would obtain the following resulting equation for the value of the jump-diffusion process at time t :

$$X(t) = X(0) \exp \left[\left(\mu - k\lambda - \frac{1}{2} \sigma^2 \right) \Delta\tau + \sigma W_{\Delta\tau} \right] \left(\prod_{i=1}^{N(t)} 1 + U_i \right) \quad (4.12)$$

4.4. Simulations

5. Pricing an option

6. Hedging

7. Conclusion

Conclusion.

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Option Pricing and Hedging under Jump-diffusion model

Abstract

Stochastic processes have a widespread application in the computer science and in this thesis their application in the field of finance will be studied. The focus of this thesis will be to mathematically define a price of an asset using a jump-diffusion model which is then used to price a derivative. By programming simulations of prices of an asset and its derivative, their behaviour in time will be shown. After the prices of an asset and its derivative have been defined, we want to show how to mathematically define a problem of hedging a position. This also has to be accompanied with implemented simulations to illustrate the effect of hedging.

Keywords: stochastic processes, finance, pricing, options, hedging

Određivanje cijene i premošćivanja rizika u slučaju difuzijskog procesa sa skokovima

Sažetak

Stohastički procesi imaju široku primjenu u raznim područjima računarske znanosti, a u sklopu ovog rada istražit će se njihova primjena u području financija. Fokus rada je matematički definirati proces cijene nekog sredstva u obliku difuzijskog procesa sa skokovima, na temelju kojeg se naknadno određuje cijena njegovog financijskog derivata. Potrebno je programskim simulacijama cijena sredstva i derivata pokazati njihovo ponašanje u vremenu. Nakon što su spomenuti procesi cijena definirani, želimo pokazati kako se matematički može predstaviti i riješiti problem premošćivanja rizika, te navedeno popratiti programskim simulacijama scenarija s ciljem ilustracije efekta premošćivanja rizika.

Ključne riječi: stohastički procesi, financije, određivanje cijene, opcije, premošćivanje rizika