Not Really a Knot Talk (Part 2)

Johnny L. Fonseca

GARTS

January 23, 2021

Ribbon Categories

A (strict) ribbon category is a (strict) monoidal category with a braiding, a twisting, and duals compatible with the twisting i.e. $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta, \theta)$ where the natural transformations satisfy some diagrams.

Ribbon Categories

A (strict) ribbon category is a (strict) monoidal category with a braiding, a twisting, and duals compatible with the twisting i.e. $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta, \theta)$ where the natural transformations satisfy some diagrams.

Think about the category of vector spaces, or of modules for your favorite Lie algebra or group.

Graphical Notation for Morphisms

Let $\mathscr{V}$ be a strict ribbon category.

Graphical Notation for Morphisms

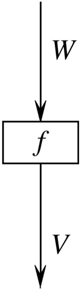
Let $\mathscr{V}$ be a strict ribbon category. We will present morphisms in $\mathscr{V}$ by plane diagrams.

Graphical Notation for Morphisms

Let $\mathscr{V}$ be a strict ribbon category. We will present morphisms in $\mathscr{V}$ by plane diagrams. Specifically, a morphism $f : V \to W$ will be represented by a coupon with two vertical arrows oriented downwards as in the following figure:

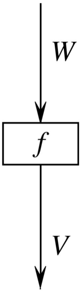
Graphical Notation for Morphisms

Let $\mathscr{V}$ be a strict ribbon category. We will present morphisms in $\mathscr{V}$ by plane diagrams. Specifically, a morphism $f : V \to W$ will be represented by a coupon with two vertical arrows oriented downwards as in the following figure:



Graphical Notation for Morphisms

Let $\mathscr{V}$ be a strict ribbon category. We will present morphisms in $\mathscr{V}$ by plane diagrams. Specifically, a morphism $f : V \to W$ will be represented by a coupon with two vertical arrows oriented downwards as in the following figure:



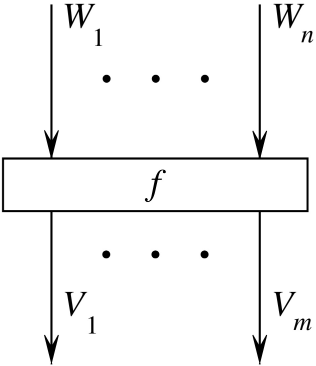
We call V and W the colors of the arrows, and f is the color of the coupon.

Graphical Notation for Morphisms (cont.)

More generally, a morphism $f : V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be represented by the following figure

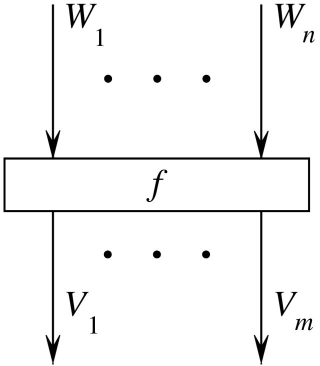
Graphical Notation for Morphisms (cont.)

More generally, a morphism $f : V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be represented by the following figure



Graphical Notation for Morphisms (cont.)

More generally, a morphism $f : V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ will be represented by the following figure



Note: If $m, n = 0$, then one interprets the corresponding empty tensor product as equal to the unit object $\mathbb{1}$ and no arrows are drawn.

Graphical Notation for Morphisms (cont.)

We will also use vertical arrows oriented upwards with the convention that the coloring of the arrow is in fact the dual object.

Graphical Notation for Morphisms (cont.)

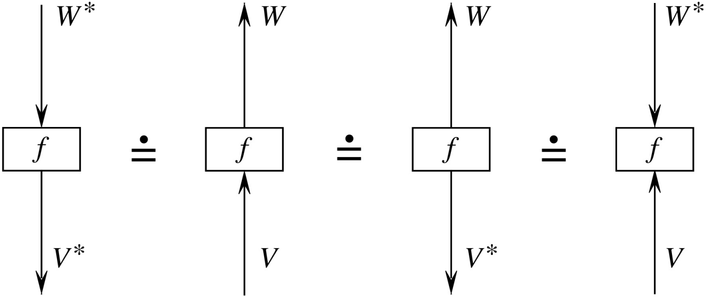
We will also use vertical arrows oriented upwards with the convention that the coloring of the arrow is in fact the dual object.

That is, the morphism $f : V^* \to W^*$ may be drawn in four equivalent ways

Graphical Notation for Morphisms (cont.)

We will also use vertical arrows oriented upwards with the convention that the coloring of the arrow is in fact the dual object.

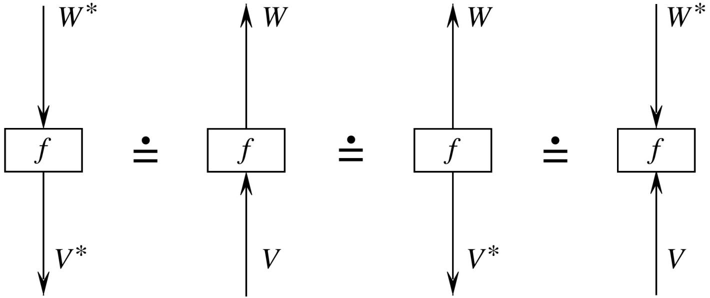
That is, the morphism $f : V^* \to W^*$ may be drawn in four equivalent ways



Graphical Notation for Morphisms (cont.)

We will also use vertical arrows oriented upwards with the convention that the coloring of the arrow is in fact the dual object.

That is, the morphism $f : V^* \to W^*$ may be drawn in four equivalent ways



Where the symbol $\doteq$ denotes equality of the corresponding morphisms in $\mathscr{V}$.

Graphical Notation for Morphisms (cont.)

The identity endomorphism will be represented by a vertical arrow directed downwards and colored with V with no coupon.

Graphical Notation for Morphisms (cont.)

The identity endomorphism will be represented by a vertical arrow directed downwards and colored with V with no coupon.

In particular, the identity endomorphism of a dual object may be drawn in any of the following equivalent ways

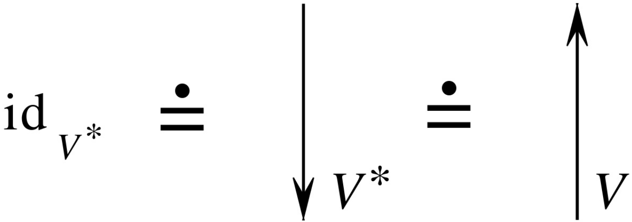
The identity endomorphism will be represented by a vertical arrow directed downwards and colored with V with no coupon.

In particular, the identity endomorphism of a dual object may be drawn in any of the following equivalent ways

$$\mathrm{id}_{V^*} \doteq \Big\downarrow V^* \doteq \Big\uparrow V$$

The identity endomorphism will be represented by a vertical arrow directed downwards and colored with V with no coupon.

In particular, the identity endomorphism of a dual object may be drawn in any of the following equivalent ways



By convention, the empty picture is the identity endomorphism of $\mathbb{1}$.

Graphical Notation for Morphisms (cont.)

- The tensor product of two morphisms is given by horizontal concatenation i.e. just place a picture of the left tensor factor morphism to the left of a picture of the right tensor factor morphism.

Graphical Notation for Morphisms (cont.)

- The tensor product of two morphisms is given by horizontal concatenation i.e. just place a picture of the left tensor factor morphism to the left of a picture of the right tensor factor morphism.
- The composition of two morphisms, say $f \circ g$, is obtained by vertically stacking a picture of f on top of a picture of g.

Graphical Notation for Morphisms (cont.)

For example, for morphisms $f : V \to W$ and $g : V' \to W'$, the identities

$$(f \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_V \otimes g) = f \otimes g = (\mathrm{id}_{V'} \otimes g) \circ (f \otimes \mathrm{id}_W)$$

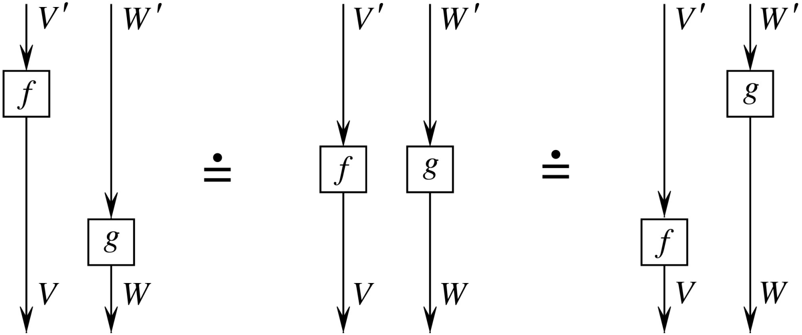
are captured graphically by the following figure

Graphical Notation for Morphisms (cont.)

For example, for morphisms $f : V \to W$ and $g : V' \to W'$, the identities

$$(f \otimes \mathrm{id}_{W'}) \circ (\mathrm{id}_V \otimes g) = f \otimes g = (\mathrm{id}_{V'} \otimes g) \circ (f \otimes \mathrm{id}_W)$$

are captured graphically by the following figure

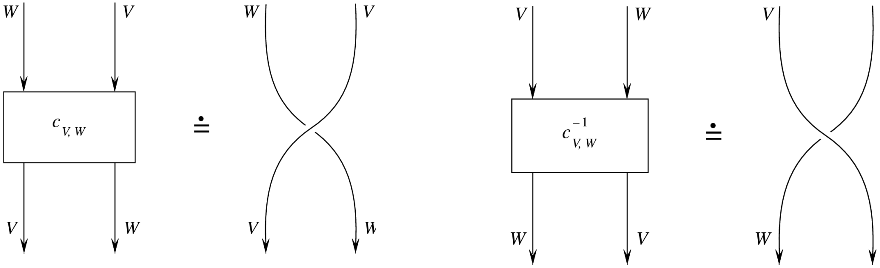


Graphical Notation for Morphisms (cont.)

The braiding morphism $c_{V,W} : V \otimes W \to W \otimes V$ and its inverse are represented by the following figures

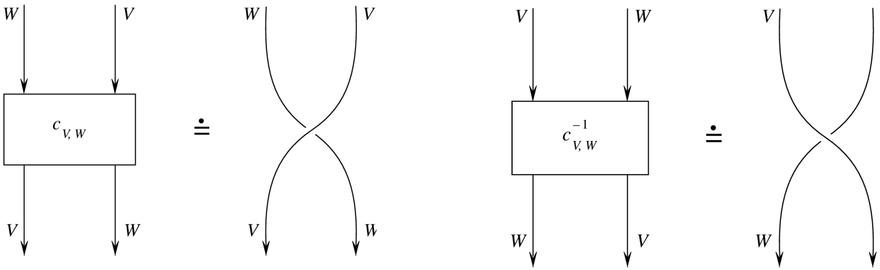
Graphical Notation for Morphisms (cont.)

The braiding morphism $c_{V,W} : V \otimes W \to W \otimes V$ and its inverse are represented by the following figures



Graphical Notation for Morphisms (cont.)

The braiding morphism $c_{V,W} : V \otimes W \to W \otimes V$ and its inverse are represented by the following figures



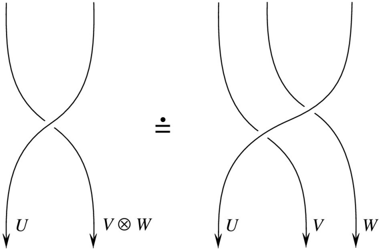
Note: Colors of arrows do not change when going through a crossing. They may change only when arrows hit coupons.

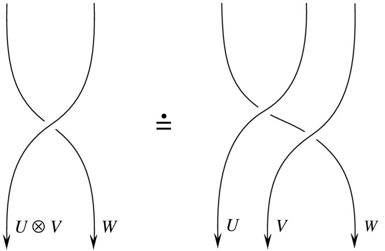
Graphical Notation for Morphisms (cont.)

With this, the following figures illustrate the commutativity of the hexagon axioms:

Graphical Notation for Morphisms (cont.)

With this, the following figures illustrate the commutativity of the hexagon axioms:





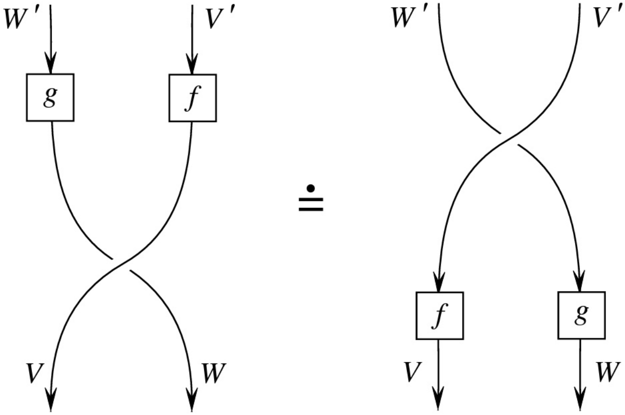
$\beta_{U,V\otimes W} = (\mathrm{id}_V \otimes \beta_{U,W}) \circ (\beta_{U,V} \otimes \mathrm{id}_W)$, $\quad \beta_{U\otimes V,W} = (\beta_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \beta_{V,W})$

Graphical Notation for Morphisms (cont.)

And the following demonstrates the naturality of the braiding with a pair of morphisms $f : V \to V'$ and $g : W \to W'$:

Graphical Notation for Morphisms (cont.)

And the following demonstrates the naturality of the braiding with a pair of morphisms $f : V \to V'$ and $g : W \to W'$:



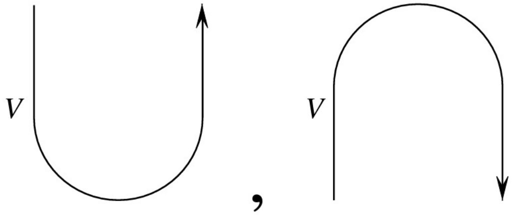
$$(g \otimes f) \circ \beta_{V,W} = \beta_{V',W'} \circ (f \otimes g)$$

Graphical Notation for Morphisms (cont.)

For every object V, the maps $b_V : \mathbb{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbb{1}$ will be represented by the following right-oriented cup and cap diagrams, respectively:

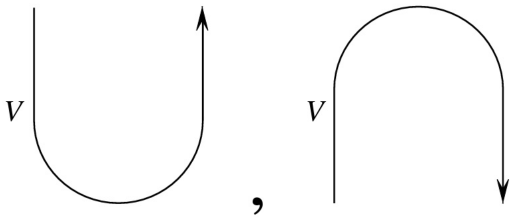
Graphical Notation for Morphisms (cont.)

For every object V, the maps $b_V : \mathbb{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbb{1}$ will be represented by the following right-oriented cup and cap diagrams, respectively:



Graphical Notation for Morphisms (cont.)

For every object V, the maps $b_V : \mathbb{1} \to V \otimes V^*$ and $d_V : V^* \otimes V \to \mathbb{1}$ will be represented by the following right-oriented cup and cap diagrams, respectively:



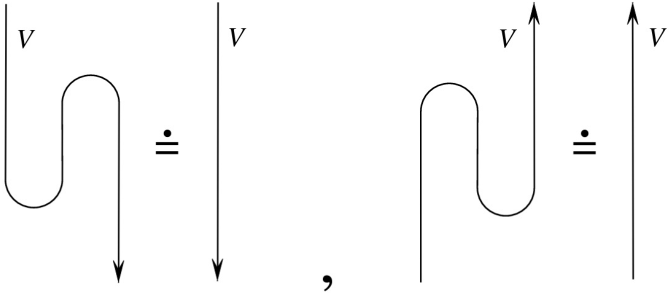
Where recall that the unit object has no arrows drawn to it or from it, and that upward oriented arrows represent duals.

Graphical Notation for Morphisms (cont.)

With this pictorial notation for the (co)evaluation maps, the following diagram depicts the required equality in the definition of being a dual object:

Graphical Notation for Morphisms (cont.)

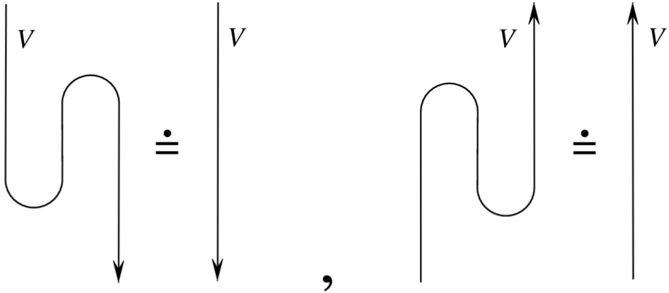
With this pictorial notation for the (co)evaluation maps, the following diagram depicts the required equality in the definition of being a dual object:



$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \quad (d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

Graphical Notation for Morphisms (cont.)

With this pictorial notation for the (co)evaluation maps, the following diagram depicts the required equality in the definition of being a dual object:

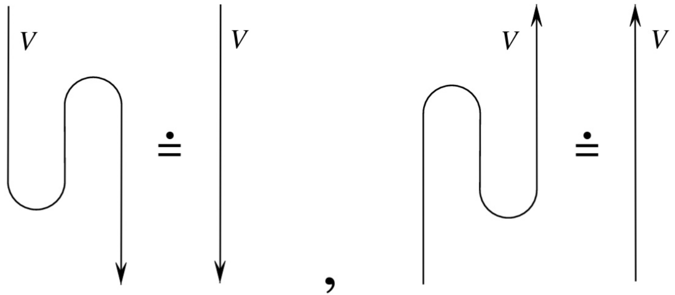


$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \quad (d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

Note: The orientation of the cups and caps matters i.e. we do not have a notion of left-oriented cups or caps...

Graphical Notation for Morphisms (cont.)

With this pictorial notation for the (co)evaluation maps, the following diagram depicts the required equality in the definition of being a dual object:



$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \quad (d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}$$

Note: The orientation of the cups and caps matters i.e. we do not have a notion of left-oriented cups or caps...yet!

Graphical Notation for Morphisms (cont.)

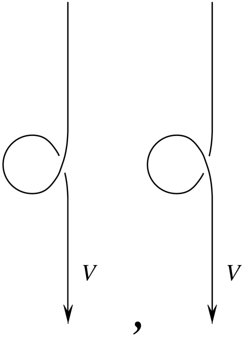
As for the depiction of twistings...

Graphical Notation for Morphisms (cont.)

As for the depiction of twistings...we will have to wait for ~~next week's talk~~ later for the motivation, but it is given by the following figure

Graphical Notation for Morphisms (cont.)

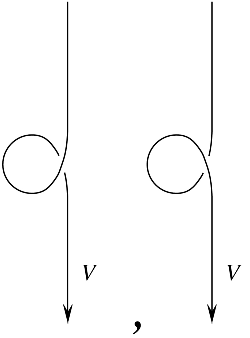
As for the depiction of twistings...we will have to wait for ~~next week's talk~~ later for the motivation, but it is given by the following figure



θ_V θ_V^{-1}

Graphical Notation for Morphisms (cont.)

As for the depiction of twistings...we will have to wait for ~~next week's talk~~ later for the motivation, but it is given by the following figure



θ_V θ_V^{-1}

These will be called the positive and negative twists, respectively.

Ribbon Graphs

In short, a ribbon graph is an oriented compact surface in $\mathbb{R}^3$ decomposed into elementary pieces called bands, annuli, and coupons.

Ribbon Graphs (cont.)

- Coupons are rectangles with a distinguished top and bottom that do not touch the aforementioned lines.

Ribbon Graphs (cont.)

- Coupons are rectangles with a distinguished top and bottom that do not touch the aforementioned lines.
- Bands are flexible ribbons that are connected to the aforementioned "top" and "bottom" lines, or to the "top" or "bottom" of a coupon. Moreover, they may have full twists in them as well as cross over other bands.

Ribbon Graphs (cont.)

- Coupons are rectangles with a distinguished top and bottom that do not touch the aforementioned lines.
- Bands are flexible ribbons that are connected to the aforementioned "top" and "bottom" lines, or to the "top" or "bottom" of a coupon. Moreover, they may have full twists in them as well as cross over other bands.
- Annuli are just (stretched) cylinders, which may also contain full twists.

Ribbon Graphs (cont.)

- Coupons are rectangles with a distinguished top and bottom that do not touch the aforementioned lines.
- Bands are flexible ribbons that are connected to the aforementioned "top" and "bottom" lines, or to the "top" or "bottom" of a coupon. Moreover, they may have full twists in them as well as cross over other bands.
- Annuli are just (stretched) cylinders, which may also contain full twists.
- The only components that may intersect are bands and coupons as mentioned above. Otherwise the components are disjoint.

Ribbon Graphs (cont.)

- Coupons are rectangles with a distinguished top and bottom that do not touch the aforementioned lines.
- Bands are flexible ribbons that are connected to the aforementioned "top" and "bottom" lines, or to the "top" or "bottom" of a coupon. Moreover, they may have full twists in them as well as cross over other bands.
- Annuli are just (stretched) cylinders, which may also contain full twists.
- The only components that may intersect are bands and coupons as mentioned above. Otherwise the components are disjoint.
- The orientation of a ribbon graph is essentially picking a preferred side of all of the components.

Ribbon Graphs (cont.)

- Coupons are rectangles with a distinguished top and bottom that do not touch the aforementioned lines.
- Bands are flexible ribbons that are connected to the aforementioned "top" and "bottom" lines, or to the "top" or "bottom" of a coupon. Moreover, they may have full twists in them as well as cross over other bands.
- Annuli are just (stretched) cylinders, which may also contain full twists.
- The only components that may intersect are bands and coupons as mentioned above. Otherwise the components are disjoint.
- The orientation of a ribbon graph is essentially picking a preferred side of all of the components.
- Furthermore, the "central line" of a band and annulus are directed.

Ribbon Graphs in Standard Position

- The ribbon graph lies almost parallel and very close to the aforementioned plane containing the top and bottom lines.

Ribbon Graphs in Standard Position

- The ribbon graph lies almost parallel and very close to the aforementioned plane containing the top and bottom lines.
- The bases of coupons should be parallel to the bottom line, and the top of a coupon should be higher than the bottom one.

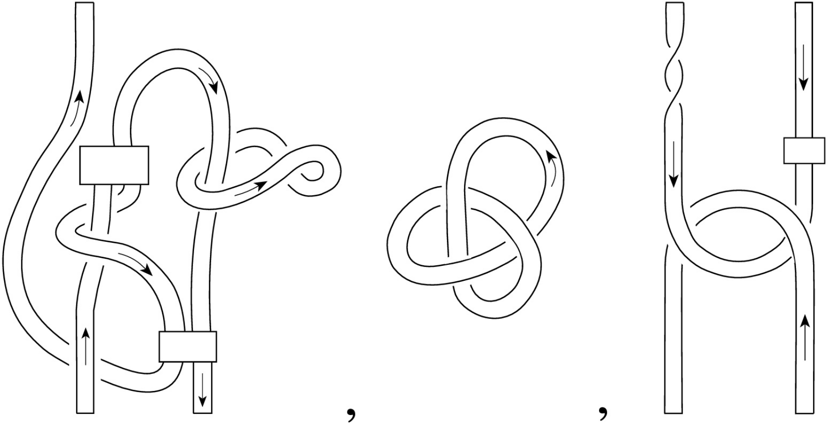
Ribbon Graphs in Standard Position

- The ribbon graph lies almost parallel and very close to the aforementioned plane containing the top and bottom lines.
- The bases of coupons should be parallel to the bottom line, and the top of a coupon should be higher than the bottom one.
- The preferred side is the side facing the viewer.

Ribbon Graphs in Standard Position

- The ribbon graph lies almost parallel and very close to the aforementioned plane containing the top and bottom lines.
- The bases of coupons should be parallel to the bottom line, and the top of a coupon should be higher than the bottom one.
- The preferred side is the side facing the viewer.
- The projections of the cores of bands and annuli onto the aforementioned plane should have only double crossings and not overlap with the projections of coupons.

Ribbon Graphs in Standard Position (cont.)



(a) (b) (c)

Isotopic Ribbon Graphs

- Our isotopies preserve bottoms, tops, splitting into components, orientation of the surface, and direction of the cores.

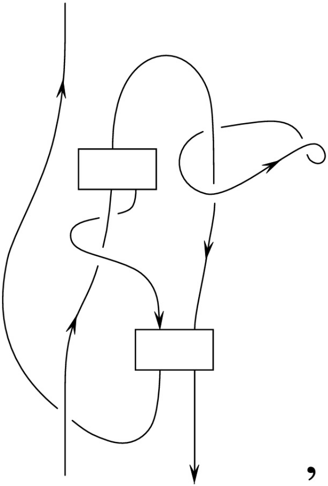
Isotopic Ribbon Graphs

- Our isotopies preserve bottoms, tops, splitting into components, orientation of the surface, and direction of the cores.
- Every ribbon graph is then isotopic to a ribbon graph in standard position.

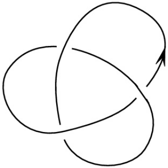
Isotopic Ribbon Graphs

- Our isotopies preserve bottoms, tops, splitting into components, orientation of the surface, and direction of the cores.
- Every ribbon graph is then isotopic to a ribbon graph in standard position.
- Ribbon graphs in standard position may be presented by a "graph diagram," which is essentially contracting along the cores of bands and annuli.

Ribbon Graphs in Standard Position



,

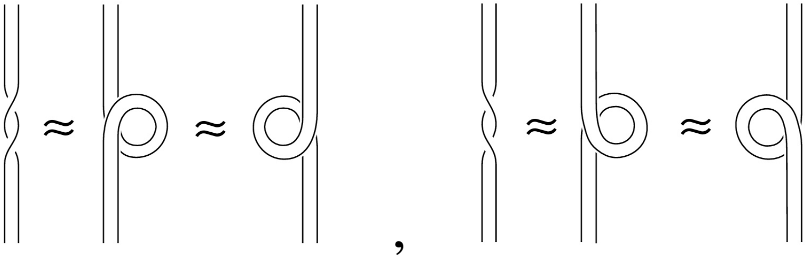


Ribbon Graphs (cont.)

Positive and negative twists in a band are isotopic to curls which go "parallel" to the plane.

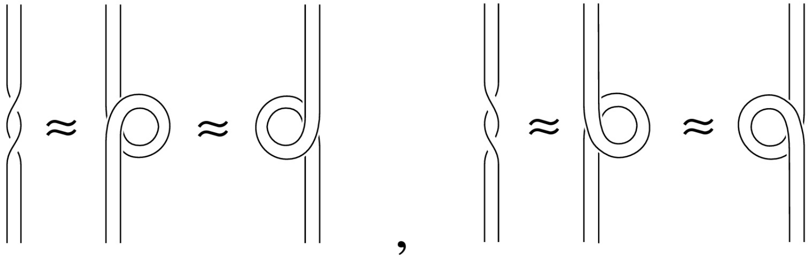
Ribbon Graphs (cont.)

Positive and negative twists in a band are isotopic to curls which go "parallel" to the plane.



Ribbon Graphs (cont.)

Positive and negative twists in a band are isotopic to curls which go "parallel" to the plane.



In particular, whenever one constructs the graph diagram for a ribbon graph, one must deform the twists in a band or annulus accordingly.

Ribbon Graphs over $\mathscr{V}$ (Informally)

For any strict monoidal category with duals $\mathscr{V}$, we will label, or color, our ribbon graphs by objects and morphisms from $\mathscr{V}$.

Ribbon Graphs over $\mathscr{V}$ (Informally)

For any strict monoidal category with duals $\mathscr{V}$, we will label, or color, our ribbon graphs by objects and morphisms from $\mathscr{V}$.

Briefly, the bands and annuli will be colored by objects while coupons will be colored by morphisms between the corresponding objects of the bands.

Ribbon Graphs over $\mathscr{V}$ (Formally)

We say a ribbon graph is colored if an object of $\mathscr{V}$ is associated to each band and annulus.

Ribbon Graphs over $\mathscr{V}$ (Formally)

We say a ribbon graph is colored if an object of $\mathscr{V}$ is associated to each band and annulus. We then say a ribbon graph is *v*-colored if each coupon is associated with a morphism in the following way:

Coloring a Coupon

- Let Q be a coupon in a ribbon graph Ω.

Coloring a Coupon

- Let Q be a coupon in a ribbon graph Ω.
- Denote by $V_1, \ldots, V_m$ (resp. $W_1, \ldots, W_n$) the colors of the bands incident to the bottom (resp. top) base of Q read left-to-right.

Coloring a Coupon

- Let Q be a coupon in a ribbon graph Ω.
- Denote by $V_1, \ldots, V_m$ (resp. $W_1, \ldots, W_n$) the colors of the bands incident to the bottom (resp. top) base of Q read left-to-right.
- Let $\varepsilon_i, \nu_j \in \{\pm 1\}$ be numbers determined by the directions of these bands in the following way

Coloring a Coupon

- Let Q be a coupon in a ribbon graph Ω.
- Denote by $V_1, \ldots, V_m$ (resp. $W_1, \ldots, W_n$) the colors of the bands incident to the bottom (resp. top) base of Q read left-to-right.
- Let $\varepsilon_i, \nu_j \in \{\pm 1\}$ be numbers determined by the directions of these bands in the following way
  - $\varepsilon_i = 1$ (resp. $\nu_j = -1$)if the band is directed out of the coupon, and $\varepsilon_i = -1$ (resp. $\nu_j = 1$) if the band is directed into the coupon.

Coloring a Coupon

- Let Q be a coupon in a ribbon graph Ω.
- Denote by $V_1, \ldots, V_m$ (resp. $W_1, \ldots, W_n$) the colors of the bands incident to the bottom (resp. top) base of Q read left-to-right.
- Let $\varepsilon_i, \nu_j \in \{\pm 1\}$ be numbers determined by the directions of these bands in the following way
  - $\varepsilon_i = 1$ (resp. $\nu_j = -1$)if the band is directed out of the coupon, and $\varepsilon_i = -1$ (resp. $\nu_j = 1$) if the band is directed into the coupon.

Then a color of the coupon Q is an arbitrary morphism

$$f : V_1^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n}$$

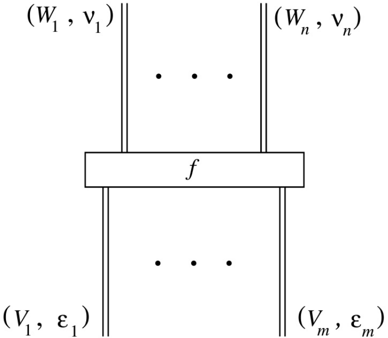
where for an object V of $\mathscr{V}$ we set $V^1 := V$ and $V^{-1} := V^*$.

A Colored Coupon

An example of such a coloring of a coupon is given by

A Colored Coupon

An example of such a coloring of a coupon is given by



If Ω consists solely of such a colored coupon Q, then we call this ribbon graph an elementary v-colored ribbon graph.

Category of Ribbon Graphs over $\mathscr{V}$

- Let $\mathscr{V}$ be a strict monoial category with duals.

Category of Ribbon Graphs over $\mathscr{V}$

- Let $\mathscr{V}$ be a strict monoial category with duals.
- We construct a category denoted by $\text{Rib}_{\mathscr{V}}$ with objects being finite sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$ where the V_i's are objects of $\mathscr{V}$ and $\varepsilon_i \in \{\pm 1\}$.

Category of Ribbon Graphs over $\mathscr{V}$

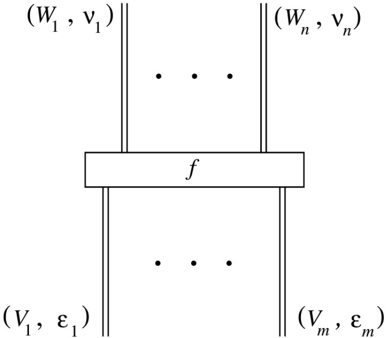
- Let $\mathscr{V}$ be a strict monoial category with duals.
- We construct a category denoted by $\mathrm{Rib}_{\mathscr{V}}$ with objects being finite sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$ where the V_i's are objects of $\mathscr{V}$ and $\varepsilon_i \in \{\pm 1\}$.
- The empty sequence is also considered as an object.

Category of Ribbon Graphs over $\mathscr{V}$

- Let $\mathscr{V}$ be a strict monoial category with duals.
- We construct a category denoted by $\text{Rib}_{\mathscr{V}}$ with objects being finite sequences $((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m))$ where the V_i's are objects of $\mathscr{V}$ and $\varepsilon_i \in \{\pm 1\}$.
- The empty sequence is also considered as an object.
- A morphism $\eta \to \eta'$ is an isotopy type of a v-colored ribbon graph such that η (resp. η') is the sequence of colors and directions of those bands which hit the bottom (resp. top) boundary intervals.

Category of Ribbon Graphs over $\mathscr{V}$ (cont.)

An example of an object in this category is given by the following figure:



Note: By definition, isotopic v-colored ribbon graphs denote the same morphism in $\mathrm{Rib}_{\mathscr{V}}$.

Category of Ribbon Graphs over $\mathscr{V}$ (cont.)

- The composition of two morphisms $f : \eta \to \eta'$ and $g : \eta' \to \eta''$ is obtained by putting a v-colored ribbon graph representing g on top of one representing f.

Category of Ribbon Graphs over $\mathscr{V}$ (cont.)

- The composition of two morphisms $f : \eta \to \eta'$ and $g : \eta' \to \eta''$ is obtained by putting a v-colored ribbon graph representing g on top of one representing f.
- The identity morphisms are represented by ribbon graphs which have no annuli, no coupons, and consists of untwisted unlinked vertical bands.

Category of Ribbon Graphs over $\mathscr{V}$ (cont.)

- The composition of two morphisms $f : \eta \to \eta'$ and $g : \eta' \to \eta''$ is obtained by putting a v-colored ribbon graph representing g on top of one representing f.
- The identity morphisms are represented by ribbon graphs which have no annuli, no coupons, and consists of untwisted unlinked vertical bands.
- The identity endomorphism of the empty sequence is represented by the empty ribbon graph.

Category of Ribbon Graphs over $\mathscr{V}$ (cont.)

For two sequences η and η', we define their tensor product as concatenation or juxtaposition i.e.

$$\begin{aligned}((V_1,\varepsilon_1),\dots,(V_m,\varepsilon_m))\otimes((V_1',\varepsilon_1'),\dots,(V_n',\varepsilon_n')) :=\\ ((V_1,\varepsilon_1),\dots,(V_m,\varepsilon_m),(V_1',\varepsilon_1'),\dots,(V_n',\varepsilon_n'))\end{aligned}$$

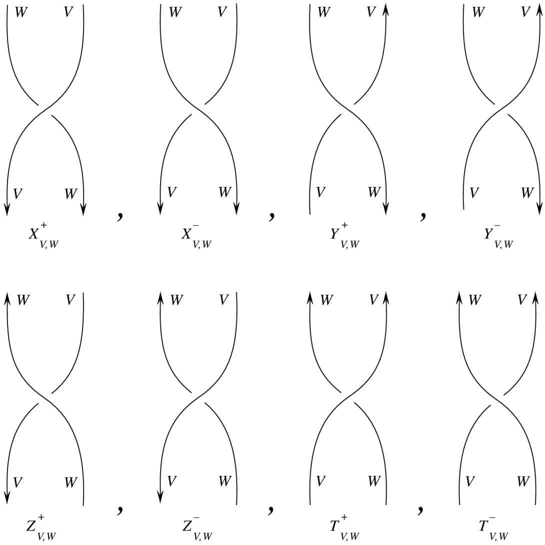
Category of Ribbon Graphs over $\mathscr{V}$ (cont.)

For two sequences η and η', we define their tensor product as concatenation or juxtaposition i.e.

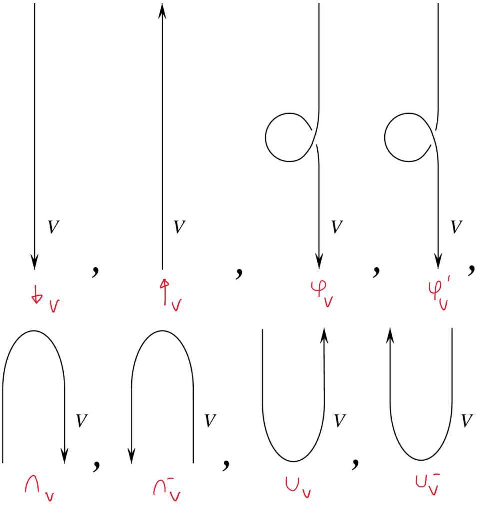
$$((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m)) \otimes ((V'_1, \varepsilon'_1), \ldots, (V'_n, \varepsilon'_n)) := \\ ((V_1, \varepsilon_1), \ldots, (V_m, \varepsilon_m), (V'_1, \varepsilon'_1), \ldots, (V'_n, \varepsilon'_n))$$

The tensor product of morphisms is given by placing corresponding v-colored ribbon graphs aside one another.

Category of Ribbon Graphs over $\mathscr{V}$ (cont.)



Category of Ribbon Graphs over $\mathscr{V}$ (cont.)



Subcategory of Ribbon Tangles

A ribbon graph over $\mathscr{V}$ which has no coupons is called a ribbon tangle over $\mathscr{V}$. We may then consider the subcategory of $\mathrm{Rib}_{\mathscr{V}}$ which has the same objects and this restriction in morphisms.

The Operator Invariant F

Theorem

Let $\mathscr{V}$ be a strict ribbon category with braiding β, twist θ, and compatible duality b, d. Then there exists a unique covariant functor $F = F_{\mathscr{V}} : Rib_{\mathscr{V}} \to \mathscr{V}$ preserving the tensor product and satisfying the following conditions

The Operator Invariant F

Theorem

Let $\mathscr{V}$ be a strict ribbon category with braiding β, twist θ, and compatible duality b, d. Then there exists a unique covariant functor $F = F_{\mathscr{V}} : Rib_{\mathscr{V}} \to \mathscr{V}$ preserving the tensor product and satisfying the following conditions

1. *F maps $(V, 1)$ (resp. $(V, -1)$) to V (resp. V^*).*

The Operator Invariant F

Theorem

Let $\mathscr{V}$ be a strict ribbon category with braiding β, twist θ, and compatible duality b, d. Then there exists a unique covariant functor $F = F_{\mathscr{V}} : Rib_{\mathscr{V}} \to \mathscr{V}$ preserving the tensor product and satisfying the following conditions

1. *F maps $(V, 1)$ (resp. $(V, -1)$) to V (resp. V^*).*
2. *For any objects V, W of $\mathscr{V}$, we have*

$$F(X^+_{V,W}) = \beta_{V,W}, \quad F(\varphi_V) = \theta_V, \quad F(\cup_V) = b_V, \quad F(\cap_V) = d_V;$$

The Operator Invariant F

Theorem

Let $\mathscr{V}$ be a strict ribbon category with braiding β, twist θ, and compatible duality b, d. Then there exists a unique covariant functor $F = F_{\mathscr{V}} : Rib_{\mathscr{V}} \to \mathscr{V}$ preserving the tensor product and satisfying the following conditions

1. *F maps $(V, 1)$ (resp. $(V, -1)$) to V (resp. V^*).*
2. *For any objects V, W of $\mathscr{V}$, we have*

$$F(X^+_{V,W}) = \beta_{V,W}, \quad F(\varphi_V) = \theta_V, \quad F(\cup_V) = b_V, \quad F(\cap_V) = d_V;$$

3. *For any elementary v-colored ribbon graph Γ, we have $F(\Gamma) = f$ where f is the color of the only coupon of Γ.*

The Operator Invariant F (cont.)

Moreover, the functor F has the following properties

$$F(\varphi'_V) = \theta_V^{-1}, \quad F(X^-_{V,W}) = \beta^{-1}_{W,V}, \quad F(Y^+_{V,W}) = \beta^{-1}_{W,V^*},$$

$$F(Y^-_{V,W}) = \beta_{V^*,W}, \quad F(Z^-_{V,W}) = \beta_{V,W^*}, \quad F(Z^+_{V,W}) = \beta^{-1}_{W^*,V},$$

$$F(T^+_{V,W}) = \beta_{V^*,W^*}, \quad F(T^-_{V,W}) = \beta^{-1}_{W^*,V^*}$$

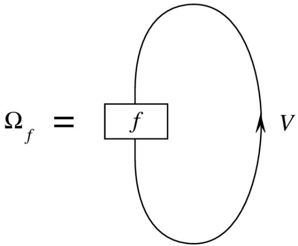
Well-Definedness of F

- How is F actually defined on a ribbon graph?
  - It turns out that every ribbon graph can be put in a "generic position" in which it can be viewed as tensor products and compositions of the previously shown ribbon graphs, for which we already have the value of F on.
- What if we represent a ribbon graph by one obtained through an isotopy?
  - Two (colored) ribbon graphs are in fact isotopic if and only if one can be obtained from the other by a series of type 2, type 3, and modified type 1 Reidemeister moves. It thus suffices to prove that F is invariant for such local changes in the ribbon graph.

An Application of F

Corollary

Let f be an endomorphism of an object V of $\mathscr{V}$. Let Ω_f be the ribbon $(0,0)$-graph consisting of one f-colored coupon and one V-colored band and presented by the following figure:



Then $F(\Omega_f) = tr(f)$.

Amazing Examples

- If we consider $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$, and color every band and annulus with the "fundamental vector representation" V (an irreducible 2-dimensional module), then on tangles we obtain the Jones polynomial!...

Amazing Examples

- If we consider $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$, and color every band and annulus with the "fundamental vector representation" V (an irreducible 2-dimensional module), then on tangles we obtain the Jones polynomial!...up to reparametrization and a normalization.

Amazing Examples

- If we consider $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$, and color every band and annulus with the "fundamental vector representation" V (an irreducible 2-dimensional module), then on tangles we obtain the Jones polynomial!...up to reparametrization and a normalization.
- If we consider $\mathrm{Rep}(U_q(\mathfrak{sl}_n))$, and color every band and annulus with the "fundamental vector representation" V (an irreducible n-dimensional module), then on tangles we obtain the HOMFLYPT polynomial!...

Amazing Examples

- If we consider $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$, and color every band and annulus with the "fundamental vector representation" V (an irreducible 2-dimensional module), then on tangles we obtain the Jones polynomial!...up to reparametrization and a normalization.
- If we consider $\mathrm{Rep}(U_q(\mathfrak{sl}_n))$, and color every band and annulus with the "fundamental vector representation" V (an irreducible n-dimensional module), then on tangles we obtain the HOMFLYPT polynomial!...up to reparametrization and a normalization.

References

- *Quantum Invariants of Knots and 3-Manifolds* by V.G. Turaev (2010).

References

- *Quantum Invariants of Knots and 3-Manifolds* by V.G. Turaev (2010).
- *An Introduction to Quantum and Vassiliev Knot Invariants* by D. M. Jackson and I. Moffatt (2019)

References

- *Quantum Invariants of Knots and 3-Manifolds* by V.G. Turaev (2010).
- *An Introduction to Quantum and Vassiliev Knot Invariants* by D. M. Jackson and I. Moffatt (2019)
- **Ribbon Graphs and Their Invariants Derived From Quantum Groups** by N. Yu. Reshetikhin and V.G. Turaev (1990)

References

- *Quantum Invariants of Knots and 3-Manifolds* by V.G. Turaev (2010).
- *An Introduction to Quantum and Vassiliev Knot Invariants* by D. M. Jackson and I. Moffatt (2019)
- **Ribbon Graphs and Their Invariants Derived From Quantum Groups** by N. Yu. Reshetikhin and V.G. Turaev (1990)
- And the dozens of other papers of theirs, and that of their co-authors.

Thank you!

Thank you! **applause**