Data 405 Assignment 3

Recall the joint pdf:

$$f(x, y) = \{ \; xe^{-x(1+y)}, \quad x \geq 0, \; y \geq 0, [4pt]0, \quad \text{otherwise.}$$

f(x, y) = {xe−x(1+y),x ≥ 0, y ≥ 0, [4pt]0,otherwise.

Q1:

(P(X>1,,Y>1))

The probability that 1 component survives more then 1 unit of time is given by the integral of the density from 1 to infinity, thus:

$$P(X > 1, Y > 1) = \int_{x=1}^{\infty} \int_{y=1}^{\infty} xe^{-x(1+y)}, dy, dx.$$

P (X > 1, Y > 1) = ∫x=1∞ ∫y=1∞ xe−x(1+y), dy, dx.

Time to solve it:

Step 1: inner integral (over (y)):

$$\int_{y=1}^{\infty} e^{-x(1+y)}, dy = e^{-x} \int_1^{\infty} e^{-xy}, dy = e^{-x} \left[\frac{e^{-xy}}{-x} \right]_{y=1}^{y=\infty} = e^{-x} \cdot \frac{e^{-x}}{x} = \frac{e^{-2x}}{x}.$$

∫y=1∞ e−x(1+y), dy = e−x ∫1∞ e−xy, dy = e−x [−xe−xy]y=1y=∞ = e−x · xe−x = xe−2x.

multiply by the factor (x) from the integrand:

$$x \cdot \frac{e^{-2x}}{x} = e^{-2x}.$$

x · xe−2x = e−2x.

Now we integrate x from 1 to (inf) -- thats our outer integral:

$$P(X > 1, Y > 1) = \int_1^{\infty} e^{-2x}, dx = \left[-\tfrac{1}{2} e^{-2x} \right]_1^{\infty} = \frac{1}{2} e^{-2}.$$

P (X > 1, Y > 1) = ∫1∞ e−2x, dx = [−21e−2x]1∞ = 21e−2.

So the probability that both components last at least 1 unit of time is

$$P(X > 1, Y > 1) = \frac{e^{-2}}{2}.$$

P (X > 1, Y > 1) = 2e−2.

Q2:

Lets compute the following:

$$f_X(x) = \int_0^{\infty} f(x, y), dy = \int_0^{\infty} xe^{-x(1+y)}, dy.$$

fX (x) = ∫0∞ f (x, y), dy = ∫0∞ xe−x(1+y), dy.

First lets factor (x e^{-x}):

$$f_X(x) = xe^{-x} \int_0^{\infty} e^{-xy}, dy = xe^{-x} \cdot \frac{1}{x} = e^{-x}, \qquad x \geq 0.$$

fX (x) = xe−x ∫0∞ e−xy, dy = xe−x · x1 = e−x,   x ≥ 0.

so our answer should end up as:

$$f_X(x) = e^{-x}, ; x \geq 0,$$

fX (x) = e−x,; x ≥ 0,

which is an

$$(\mathrm{Exp}(1))$$

(Exp(1)) density function

Thusly, the marginal density of X is:

$$f_X(x) = \int_0^{\infty} xe^{-x(1+y)}, dy.$$

fX (x) = ∫0∞ xe−x(1+y), dy.

Q3:

By definition,

$$f_{Y|X}(y \mid x) = \frac{f(x, y)}{f_X(x)} = \frac{xe^{-x(1+y)}}{e^{-x}} = xe^{-xy}, \qquad y \geq 0.$$

fY |X (y | x) = fX (x)f (x, y) = e−xxe−x(1+y) = xe−xy,   y ≥ 0.

So

$$f_{Y|X}(y \mid x) = xe^{-xy}, ; y \geq 0.$$

fY |X (y | x) = xe−xy,; y ≥ 0.

Now we compute the conditional expectation:

$$E[Y \mid X = x] = \int_0^{\infty} y, xe^{-xy}, dy = x \int_0^{\infty} ye^{-xy}, dy.$$

E[Y | X = x] = ∫0∞ y, xe−xy, dy = x ∫0∞ ye−xy, dy.

Use the known integral

$$\left(\int_0^{\infty} ye^{-ay}, dy = \frac{1}{a^2} \right)$$

(∫0∞ ye−ay, dy = a21) for (a>0). Here (a=x), so

$$E[Y \mid X = x] = x \cdot \frac{1}{x^2} = \frac{1}{x}$$

E[Y | X = x] = x · x21 = x1

Thus

$$E[Y \mid X = x] = \frac{1}{x}.$$

E[Y | X = x] = x1.

Q4:

Lets compute the following:

$$f_Y(y) = \int_0^{\infty} xe^{-x(1+y)}, dx.$$

fY (y) = ∫0∞ xe−x(1+y), dx.

If (a=1+y>0) -- -- our integral becomes

$$\int_0^{\infty} xe^{-ax}, dx.$$

∫0∞ xe−ax, dx.

A standard integral (or integrate by parts) gives

$$\int_0^{\infty} xe^{-ax}, dx = \frac{1}{a^2}.$$

∫0∞ xe−ax, dx = a21.

Thus with (a=1+y),

$$f_Y(y) = \frac{1}{(1+y)^2}, \qquad y \geq 0.$$

fY (y) = (1 + y)21,   y ≥ 0.

So the correct marginal df is

$$f_Y(y) = \frac{1}{(1+y)^2}, ; y \geq 0.$$

fY (y) = (1 + y)21,; y ≥ 0.

Q5:

Recall the joint pdf

$$f(x, y) = xe^{-x(1+y)}, \qquad x \geq 0, \; y \geq 0.$$

f (x, y) = xe−x(1+y),   x ≥ 0, y ≥ 0.

If (X) and (Y) were independent - we would have

$$f(x, y) = f_X(x), f_Y(y)$$

f (x, y) = fX (x), fY (y)

But from earlier work we found

$$f_X(x) = e^{-x}, \quad f_Y(y) = \frac{1}{(1+y)^2}.$$

fX (x) = e−x,   fY (y) = (1 + y)21.

So the product would be

$$f_X(x) f_Y(y) = e^{-x} \frac{1}{(1+y)^2}$$

fX (x)fY (y) = e−x(1 + y)21

which is not equal to

$$(xe^{-x(1+y)})$$

(xe−x(1+y)) in general

Since (e^{-2}) is not equal (e^{-1}/4), the equality fails

so (X) and (Y) cannot be independent

thus: proven: X & Y are **not independent**.

Q6:

Lets use this R code:

```r
set.seed(2025)
n <- 100

# simulate X ~ Exp(1)
X <- rexp(n, rate = 1)

# simulate Y | X ~ Exp(rate = X)
Y <- numeric(n)
for (i in seq_len(n)) {
  Y[i] <- rexp(1, rate = X[i])
}

# show first 10 pairs:
head(cbind(X, Y), 10)
```

R output (first 10 pairs):

```
                X           Y
 [1,] 0.465240310   1.1200714
 [2,] 1.038634094   1.6087953
 [3,] 0.560569047   0.5113417
 [4,] 0.008504318 183.8800281
 [5,] 0.088620726  30.4930303
 [6,] 0.139062109   4.9391748
 [7,] 1.146713257   1.1622734
 [8,] 3.991217797   0.1174850
 [9,] 0.946187682   0.1338989
[10,] 1.155446981   0.3215125
```

there exist a couple very large Y values occasionally when X is very small — that's expected because (E[Y| X=x] == 1/x) - thus such behavior is explianed

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Q7:

Lets use my R code again:

```r
# Setup from above:
set.seed(2025)
n <- 100
X <- rexp(n, rate = 1)
Y <- numeric(n)
for (i in seq_len(n)) { Y[i] <- rexp(1, rate = X[i]) }
head(cbind(X, Y), 10)

# Assuming X and Y from above
plot(X, Y, pch = 19, cex = 0.6,
     xlab = "X", ylab = "Y",
     main = "Scatterplot of Y vs X with conditional mean y = 1/x")
xgrid <- seq(0.01, quantile(X, 0.99) + 1, length.out = 400)
lines(xgrid, 1 / xgrid, lwd = 2)
```

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> The scatterplot shows the cloud of points roughly centered on the curve (y= 1/x). This matches our earlier calculation of the conditional expectation (E[Y| X=x] = 1/x).

Q8:

Q8a: Lets get a formula to get beta's MLE given alpha

The density (scale parametrization) is

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha), \beta^{\alpha}}, \qquad x > 0.$$

f (x; α, β) = Γ(α), βαxα−1e−x/β,   x > 0.

Log-likelihood for a sample (x_1,\dots,x_n) with known (\alpha):

$$\ell(\beta) = \sum_{i=1}^{n} \left[(\alpha - 1) \ln x_i - \ln \Gamma(\alpha) - \alpha \ln \beta - \frac{x_i}{\beta} \right] = -n\alpha \ln \beta - \frac{1}{\beta} \sum_{i=1}^{n} x_i + \text{const.}$$

ℓ(β) = i=1∑n [(α − 1) ln xi − ln Γ(α) − α ln β − βxi] = −nα ln β − β1 i=1∑n xi + const.

Differentiate w.r.t. (beta) and set to zero:

$$\frac{d\ell}{d\beta} = -\frac{n\alpha}{\beta} + \frac{\sum x_i}{\beta^2} = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i = n\alpha\widehat{\beta}$$

dβdℓ = −βnα + β2∑ xi = 0   ⇒   i=1∑n xi = nαβ^

So the MLE is

$$\widehat{\beta} = \frac{1}{n\alpha} \sum_{i=1}^{n} x_i = \frac{\bar{X}}{\alpha}.$$

β^ = nα1 i=1∑n xi = αX¯.

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Q8b: Lets estimate beta given sample (9,5,7)

$$\bar{X} = \frac{9+5+7}{3} = 7, \qquad \alpha = 2$$

X¯ = 39 + 5 + 7 = 7,   α = 2

so:

$$\widehat{\beta} = \bar{X}/\alpha = 7/2 = 3.5.$$

β^ = X¯/α = 7/2 = 3.5.

Q9:

Using my R code:

```r
set.seed(2025)

# known alpha
alpha <- 2
# observed sample
obs <- c(9,5,7)
n <- length(obs)

# observed mle
beta_hat_obs <- mean(obs) / alpha  # 3.5

# simulate nsim samples of size n from Gamma(alpha, beta_hat_obs)
nsim <- 1000
beta_hats <- numeric(nsim)
for (i in 1:nsim) {
  samp <- rgamma(n, shape = alpha, scale = beta_hat_obs)
  beta_hats[i] <- mean(samp) / alpha
}

# estimate probability P(beta_hat >= observed)
p_ge <- mean(beta_hats >= beta_hat_obs)

# summary
cat("Observed beta_hat:", beta_hat_obs, "\n")
cat("Mean simulated beta_hats:", mean(beta_hats), "\n")
cat("SD simulated beta_hats:", sd(beta_hats), "\n")
cat("P(beta_hat >= observed) approx:", p_ge, "\n")

# Optional: quick histogram of simulated beta_hats
hist(beta_hats, breaks = 30, col = "lightblue", main = "Simulated beta_hats",
     xlab = expression(hat(beta)))
abline(v = beta_hat_obs, col = "red", lwd = 2)
```

R Output:

```
Observed beta_hat: 3.5
Mean simulated beta_hats: 3.467059
SD simulated beta_hats: 1.421013
P(beta_hat >= observed) approx: 0.447

[Execution complete with exit code 0]
```

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The estimated probability ~= 0.47 - which indicates the observed MLE (3.5) is not unusually large under (a=2, beta=3.5)

This was built with A MathJax extension for VS Code in a MarkDown doccument, Math was done specifically in MathJax format. This doccument was exported to HTML & then to PDF prior to submission.