

Data 405 Assignment 3

Recall the joint pdf:

$$f(x, y) = \begin{cases} xe^{-x(1+y)}, & x \geq 0, y \geq 0, [4pt] 0, & \text{otherwise.} \end{cases}$$

Q1:

(P(X>1,Y>1))

The probability that 1 component survives more than 1 unit of time is given by the integral of the density from 1 to infinity, thus:

$$\begin{aligned} P(X > 1, Y > 1) &= \int_{x=1}^{\infty} \int_{y=1}^{\infty} xe^{-x(1+y)} dy dx, \\ P(X > 1, Y > 1) &= [x=1 \Rightarrow y=1 \Rightarrow x-(1+y), dy, dx]. \end{aligned}$$

Time to solve it:

Step 1: inner integral (over y):

$$\int_{y=1}^{\infty} e^{-x(1+y)} dy = e^{-x} \int_1^{\infty} e^{-xy} dy = e^{-x} \left[\frac{e^{-xy}}{-x} \right]_1^{\infty} = e^{-x} \cdot \frac{e^{-x}}{x} = \frac{e^{-2x}}{x},$$

$$[y=1 \Rightarrow x-(1+y), dy = e-x \cdot [1 \Rightarrow e-xy, dy = e-x \cdot [-xe-xy] \Rightarrow 1 \Rightarrow e-x \cdot xe-x = xe-2x].$$

multiply by the factor (x) from the integrand:

$$x \cdot \frac{e^{-2x}}{x} = e^{-2x}.$$

$$x \cdot xe-2x = e-2x.$$

Now we integrate x from 1 to (inf) - that's our outer integral:

$$\begin{aligned} P(X > 1, Y > 1) &= \int_1^{\infty} e^{-2x} dx = \left[-\frac{1}{2} e^{-2x} \right]_1^{\infty} = \frac{1}{2} e^{-2}. \\ P(X > 1, Y > 1) &= [1 \Rightarrow e-2x, dx = [-2e-2x] \Rightarrow 2e-2]. \end{aligned}$$

So the probability that both components last at least 1 unit of time is

$$P(X > 1, Y > 1) = \frac{e^{-2}}{2},$$

$$P(X > 1, Y > 1) = 2e-2.$$

Q2:

Lets compute the following:

$$f_X(x) = \int_0^{\infty} f(x, y) dy = \int_0^{\infty} xe^{-x(1+y)} dy.$$

$$f_X(x) = [0 \Rightarrow f(x, y), dy = [0 \Rightarrow xe-x-(1+y), dy.$$

First lets factor (x e^{-x} x):

$$f_X(x) = xe^{-x} \int_0^{\infty} e^{-xy} dy = xe^{-x} \cdot \frac{1}{x} = e^{-x}, \quad x \geq 0.$$

$$f_X(x) = xe-x \cdot [0 \Rightarrow e-xy, dy = xe-x \cdot x1 = e-x, \quad x \geq 0,$$

so our answer should end up as:

$$f_X(x) = e-x, \quad x \geq 0,$$

$$f_X(x) = e-x, \quad x \geq 0.$$

which is an

$$(Exp(1))$$

(Exp(1)) density function

Thusly, the marginal density of X is:

$$f_X(x) = \int_0^{\infty} xe^{-x(1+y)} dy.$$

$$f_X(x) = [0 \Rightarrow xe-x-(1+y), dy.$$

Q3:

By definition,

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xe^{-x(1+y)}}{e^{-x}} = xe^{-xy}, \quad y \geq 0.$$

$$f_{Y|X}(y|x) = f_X(x)f(y) = e-xe-x-(1+y) = xe-xy, \quad y \geq 0.$$

So

$$f_{Y|X}(y|x) = xe^{-xy}, \quad y \geq 0.$$

$$f_{Y|X}(y|x) = xe-x-y, \quad y \geq 0.$$

Now we compute the conditional expectation:

$$\begin{aligned} E[Y|X=x] &= \int_0^{\infty} y xe^{-xy} dy = x \int_0^{\infty} ye^{-xy} dy, \\ E[Y|X=x] &= [0 \Rightarrow ye-ay, dy = a21 \Rightarrow (a>0). Here (a=x), so \end{aligned}$$

Use the known integral

$$\left(\int_0^{\infty} ye^{-ay} dy = \frac{1}{a^2} \right)$$

$$[0 \Rightarrow ye-ay, dy = a21 \Rightarrow (a>0). Here (a=x), so$$

$$E[Y|X=x] = x \cdot \frac{1}{x^2} = \frac{1}{x}$$

Thus

$$E[Y|X=x] = \frac{1}{x}$$

$$E[Y|X=x] = x1$$

Q4:

Lets compute the following:

$$f_Y(y) = \int_0^{\infty} xe^{-x(1+y)} dx.$$

$$f_Y(y) = [0 \Rightarrow xe-x-(1+y), dx.$$

If ($a=x>0$) -- our integral becomes

$$\int_0^{\infty} xe^{-ax} dx.$$

0 < xe-ax, dx = a21.

A standard integral (or integrate by parts) gives

$$\int_0^{\infty} xe^{-ax} dx = \frac{1}{a^2}.$$

$$[0 \Rightarrow xe-ax, dx = a21.$$

Thus with ($a=1+y$),

$$f_Y(y) = \frac{1}{(1+y)^2}, \quad y \geq 0.$$

$$f_Y(y) = (1+y)21, \quad y \geq 0.$$

So the correct marginal of is

$$f_Y(y) = \frac{1}{(1+y)^2}; \quad y \geq 0.$$

$$f_Y(y) = (1+y)21; \quad y \geq 0.$$

Q5:

Recall the joint pdf

$$f(x, y) = xe^{-x(1+y)}, \quad x \geq 0, y \geq 0.$$

If (X) and (Y) were independent - we would have

$$f(x, y) = f_X(x), f_Y(y)$$

f(x, y) = xe-x, f(Y) = (1+y)21.

But from earlier work we found

$$f_X(x) = e^{-x}, \quad f_Y(y) = \frac{1}{(1+y)^2}.$$

$$f_X(x)f_Y(y) = e-x(1+y)21$$

So the product would be

$$f_X(x)f_Y(y) = e^{-x} \frac{1}{(1+y)^2}$$

$$f_X(x)f_Y(y) = e-x(1+y)21$$

which is not equal to

$$(xe-x(1+y))$$

(xe-x(1+y)) in general

Since ($e^{-x(1+y)}$) is not equal ($e^{-x(1+y)}$), the equality fails

so (X) and (Y) cannot be independent

thus: proven: X & Y are not independent.

Q6:

Lets use this R code:

```
set.seed(2025)
n <- 100
```

```
# simulate X ~ Exp(1)
X <- rexp(n, rate = 1)
```

```
# simulate Y | X ~ Exp(1)
Y <- numeric(n)
for (i in seq_len(n)) {
  Y[i] <- rexp(1, rate = X[i])
}
```

```
# show first 10 pairs:
head(cbind(X, Y), 10)
```

R output (first 10 pairs):

```
x           y
[1,] 0.465240310 1.1286714
[2,] 1.038634894 1.6807953
[3,] 0.569569847 0.5113417
[4,] 0.088584318 183.3880281
[5,] 0.088862072 30.4939303
[6,] 1.146713257 1.1622734
[7,] 3.991211779 0.1174850
[8,] 0.946187682 0.1338989
[9,] 1.155446981 0.3215125
```

there exist a couple very large Y values occasionally when X is very small — that's expected because ($E[Y|X=x] = 1/x$) - thus such behavior is explained

Q7:

Lets use my R code again:

```
# Setup from above:
set.seed(2025)
n <- 100
```

```
X <- rexp(n, rate = 1)
Y <- numeric(n)
```

```
for (i in seq_len(n)) { Y[i] <- rexp(1, rate = X[i]) }
```

```
head(cbind(X, Y), 10)
```

Assuming X and Y from above

```
plot(X, Y, pch = 19, cex = 0.6,
     xlab = "x", ylab = "y",
     main = "Scatterplot of Y vs X with conditional mean y = 1/x")
```

```
xgrid <- seq(0.1, quantile(X, 0.99) + 1, length.out = 400)
lines(xgrid, 1 / xgrid, lwd = 2)
```

lines(xgrid, 1 / xgrid, lwd = 2)

Scatterplot shows the cloud of points roughly centered on the curve ($y = 1/x$). This matches our earlier

calculation of the conditional expectation ($E[Y|X=x] = 1/x$).

Q8:

Q8a: Lets get a formula to get beta's MLE given alpha

The density (scale parametrization) is

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha), \beta^{\alpha}}, \quad x > 0.$$

$$f(x; \alpha, \beta) = \Gamma(\alpha), \beta^{\alpha-1} e^{-x/\beta} x^{\alpha-1}$$

Log-likelihood for a sample (x_1, \dots, x_n) with known (α alpha):

$$\ell(\beta) = \sum_{i=1}^n \left[(\alpha-1) \ln x_i - \ln \Gamma(\alpha) - \ln \beta - \frac{x_i}{\beta} \right] = -n \alpha \ln \beta - \frac{1}{\beta} \sum_{i=1}^n x_i + \text{const.}$$

$$\ell(\beta) = i=1 \sum n (\alpha-1) \ln x_i - \ln \Gamma(\alpha) - \alpha \ln \beta - \beta \sum_{i=1}^n x_i + \text{const.}$$

Differentiate w.r.t. (β beta) and set to zero:

$$\frac{d\ell}{d\beta} = -\frac{n\alpha}{\beta} + \frac{\sum x_i}{\beta^2} = 0 \Rightarrow \sum_{i=1}^n x_i = n\alpha\beta$$

$$\beta^{\alpha} = n\alpha \sum_{i=1}^n x_i = \alpha X$$

so:

$$\hat{\beta} = X/\alpha = 7/2 = 3.5.$$

Q9:

Using my R code:

```
set.seed(2025)
```

```
# known alpha
alpha <- 2
```

```
# observed sample
obs <- (9, 5, 7)
```

```
n <- length(obs)
```

```
# observed mle
beta_hat_obs <- mean(obs) / alpha # 3.5
```

```
# simulate nsim samples of size n from Gamma(alpha, beta_hat_obs)
```

```
nsim <- 1000
```

```
beta_hats <- numeric(nsim)
```

```
for (i in 1:nsim) {
```

```
  samp <- rgamma(n, shape = alpha, scale = beta_hat_obs)
```

```
  beta_hats[i] <- mean(samp) / alpha
```

```
}
```

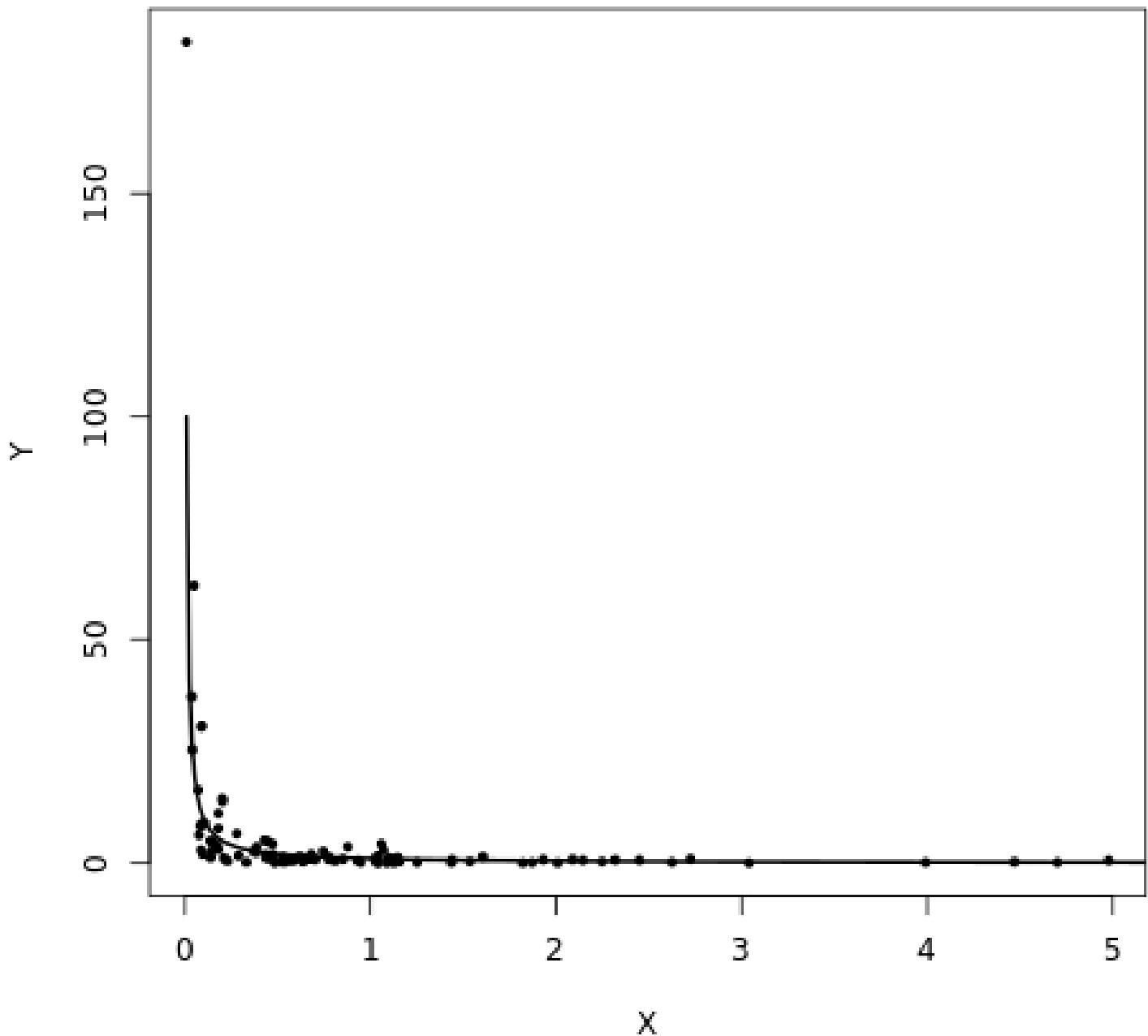
```
# estimate probability P(beta_hat > observed)
```

```
p_ge <- mean(beta_hats > obs)
```

```
# summary
```

```
cat("Observed beta_hat:", beta_hat_obs, "\n")
```

Scatterplot of Y vs X with conditional mean $y = 1/x$



Simulated beta_hats

