

# Data 405 Assignment 3

Recall the joint pdf:

$$f(x,y) = \begin{cases} xe^{-x(1+y)}, & x \geq 0, y \geq 0, \{1\}d[0], \text{ otherwise.} \\ f(x,y) = (xe-x(1+y), x \geq 0, y \geq 0, \{1\}d[0], \text{ otherwise.} \end{cases}$$

## Q1:

$$(P(X>1,Y>1))$$

The probability that 1 component survives more then 1 unit of time is given by the integral of the density from 1 to infinity, thus:

$$P(X>1,Y>1)=\int_{x=1}^{\infty}\int_{y=1}^{\infty}xe^{-x(1+y)}dy,dx.$$

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Time to solve it:

Step 1: inner integral (over (y)):

$$\int_{y=1}^{\infty}e^{-x(1+y)}dy=e^{-x}\int_1^{\infty}e^{-xy}dy=e^{-x}\left[\frac{e^{-xy}}{-x}\right]_{y=1}^{y=\infty}=e^{-x}\cdot\frac{e^{-x}}{x}=\frac{e^{-2x}}{x}.$$

$$\int_{y=1}^{\infty}e^{-x(1+y)}dy=e^{-x}\int_1^{\infty}e^{-xy}dy=e^{-x}\left[\frac{e^{-xy}}{-x}\right]_{y=1}^{y=\infty}=e^{-x}\cdot\frac{e^{-x}}{x}=\frac{e^{-2x}}{x}.$$

multiply by the factor (x) from the integrand:

$$x\cdot\frac{e^{-2x}}{x}=e^{-2x},$$

$$x\cdot xe^{-2x}=e^{-2x}.$$

Now we integrate x from 1 to (inf) -- thats our outer integral:

$$P(X>1,Y>1)=\int_1^{\infty}e^{-2x}dx=\left[-\frac{1}{2}e^{-2x}\right]_1^{\infty}=\frac{1}{2}e^{-2}.$$

$$P(X>1,Y>1)=\int_1^{\infty}\int_1^{\infty}xe^{-2x}dx=\left[-\frac{1}{2}e^{-2x}\right]_1^{\infty}=\frac{1}{2}e^{-2}.$$

So the probability that both components last at least 1 unit of time is

$$P(X>1,Y>1)=\frac{e^{-2}}{2}.$$

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## Q2:

Lets compute the following:

$$f_X(x)=\int_0^{\infty}f(x,y)dy=\int_0^{\infty}xe^{-x(1+y)}dy,$$

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First lets factor (x e^{-(x)}):

$$f_X(x)=xe^{-x}\int_0^{\infty}e^{-xy}dy=xe^{-x}\cdot\frac{1}{x}=e^{-x},\quad x\geq 0.$$

$$f_X(x)=xe^{-x}\int_0^{\infty}e^{-xy}dy=xe^{-x}\cdot\frac{1}{x}=e^{-x},\quad x\geq 0.$$

so our answer should end up as:

$$f_X(x)=e^{-x},x\geq 0,$$

$$f_X(x)=e^{-x},x\geq 0,$$

which is an

$$(Exp(1))$$

(Exp(1)) density function

Thusly, the marginal density of X is:

$$f_X(x)=\int_0^{\infty}xe^{-x(1+y)}dy,$$

$$f_X(x)=\int_0^{\infty}xe^{-x(1+y)}dy,$$

## Q3:

By definition,

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\frac{xe^{-x(1+y)}}{e^{-x}}=xe^{-xy},\quad y\geq 0.$$

$$f_{Y|X}(y|x)=f_X(x)f(x,y)=e^{-x}xe^{-x(1+y)}=xe^{-xy},\quad y\geq 0.$$

So

$$f_{Y|X}(y|x)=xe^{-xy},y\geq 0.$$

$$f_{Y|X}(y|x)=xe^{-xy},y\geq 0,$$

Now we compute the conditional expectation:

$$E[Y|X=x]=\int_0^{\infty}y\cdot xe^{-xy}dy=x\int_0^{\infty}ye^{-xy}dy,$$

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Use the known integral

$$\int_0^{\infty}ye^{-ay}dy=\frac{1}{a^2}$$

$$\int_0^{\infty}ye^{-ay}dy=\frac{1}{a^2}\text{ for (a>0). Here (a=x), so}$$

$$E[Y|X=x]=x\cdot\frac{1}{x^2}=\frac{1}{x}$$

$$E[Y|X=x]=x\cdot\frac{1}{x^2}=\frac{1}{x}$$

Thus

$$E[Y|X=x]=\frac{1}{x}$$

$$E[Y|X=x]=\frac{1}{x}$$

## Q4:

Lets compute the following:

$$f_Y(y)=\int_0^{\infty}xe^{-x(1+y)}dx,$$

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If (a=1+y>0) -- our integral becomes

$$\int_0^{\infty}xe^{-ax}dx,$$

$$\int_0^{\infty}xe^{-ax}dx,$$

A standard integral (or integrate by parts) gives

$$\int_0^{\infty}xe^{-ax}dx=\frac{1}{a^2}$$

$$\int_0^{\infty}xe^{-ax}dx=\frac{1}{a^2}.$$

Thus with (a=1+y),

$$f_Y(y)=\frac{1}{(1+y)^2},\quad y\geq 0.$$

$$f_Y(y)=\frac{1}{(1+y)^2},\quad y\geq 0.$$

So the correct marginal df is

$$f_Y(y)=\frac{1}{(1+y)^2},y\geq 0.$$

$$f_Y(y)=\frac{1}{(1+y)^2},y\geq 0,$$

## Q5:

Recall the joint pdf

$$f(x,y)=xe^{-x(1+y)},\quad x\geq 0,y\geq 0.$$

$$f(x,y)=xe^{-x(1+y)},\quad x\geq 0,y\geq 0.$$

If (X) and (Y) were independent - we would have

$$f(x,y)=f_X(x)\cdot f_Y(y)$$

$$f(x,y)=f_X(x)\cdot f_Y(y)$$

But from earlier work we found

$$f_X(x)=e^{-x},\quad f_Y(y)=\frac{1}{(1+y)^2}.$$

$$f_X(x)=e^{-x},\quad f_Y(y)=\frac{1}{(1+y)^2}.$$

So the product would be

$$f_X(x)f_Y(y)=e^{-x}\cdot\frac{1}{(1+y)^2}$$

$$f_X(x)f_Y(y)=e^{-x}\cdot\frac{1}{(1+y)^2}$$

which is not equal to

$$(xe^{-x(1+y)})$$

$$(xe^{-x(1+y)})$$

Since (e^{-(2)}) is not equal (e^{-(1/4)}), the equality fails

so (X) and (Y) cannot be independent

thus: proven: X & Y are **not independent**.

## Q6:

Lets use this R code:

```
set.seed(2025)
n <- 100

# simulate X ~ Exp(1)
X <- rexp(n, rate = 1)

# simulate Y | X ~ Exp(rate = X)
Y <- numeric(n)
for (i in seq_len(n)) {
  Y[i] <- rexp(1, rate = X[i])
}

# show first 10 pairs:
head(cbind(X, Y), 10)
```

R output (first 10 pairs):

	X	Y
[1,]	0.465240310	1.1200714
[2,]	1.038634094	1.6087953
[3,]	0.560660047	0.5113417
[4,]	0.008604318	183.8800281
[5,]	0.888620726	30.4930303
[6,]	0.139062109	4.9391748
[7,]	1.14013257	1.1622734
[8,]	3.991217797	0.1174850
[9,]	0.946187682	0.1338989
[10,]	1.155446081	0.3215125

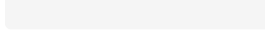
there exist a couple very large Y values occasionally when X is very small — that's expected because (E[Y|X=x] = 1/x) - thus such behavior is explained

## Q7:

Lets use my R code again:

```
# Setup from above:
set.seed(2025)
n <- 100
X <- rexp(n, rate = 1)
Y <- numeric(n)
for (i in seq_len(n)) { Y[i] <- rexp(1, rate = X[i]) }
head(cbind(X, Y), 10)

# Assuming X and Y from above
plot(X, Y, pch = 19, cex = 0.6,
      xlab = "X", ylab = "Y",
      main = "Scatterplot of Y vs X with conditional mean y = 1/x")
xgrid <- seq(0.01, quantile(X, 0.99) + 1, length.out = 400)
lines(xgrid, 1 / xgrid, lwd = 2)
```



The scatterplot shows the cloud of points roughly centered on the curve (y=1/x ). This matches our earlier calculation of the conditional expectation (E[Y|X=x] = 1/x).

## Q8:

Q8a: Lets get a formula to get beta's MLE given alpha

The density (scale parametrization) is

$$f(x;\alpha,\beta)=\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}},\quad x>0.$$

$$f(x;\alpha,\beta)=\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^{\alpha}},\quad x>0.$$

Log-likelihood for a sample (x\_1,...,x\_n) with known (alpha):

$$\ell(\beta)=\sum_{i=1}^n\left[(\alpha-1)\ln x_i-\ln\Gamma(\alpha)-\alpha\ln\beta-\frac{x_i}{\beta}\right]=-n\alpha\ln\beta-\frac{1}{\beta}\sum_{i=1}^nx_i+\text{const.}$$

$$\ell(\beta)=1\sum_{i=1}^n[(\alpha-1)\ln x_i-\ln\Gamma(\alpha)-\alpha\ln\beta-\beta x_i]=-n\alpha\ln\beta-\beta^{-1}\sum_{i=1}^nx_i+\text{const.}$$

Differentiate w.r.t. (beta) and set to zero:

$$\frac{d\ell}{d\beta}=-\frac{n\alpha}{\beta}+\sum_{i=1}^n\frac{x_i}{\beta^2}=0\Rightarrow\sum_{i=1}^nx_i=n\alpha\beta$$

$$d\beta/d\alpha=-\beta n\alpha+\beta^2\sum_{i=1}^nx_i=0\Rightarrow\beta^{-1}\sum_{i=1}^nx_i=n\alpha\beta$$

So the MLE is

$$\hat{\beta}=\frac{1}{n\alpha}\sum_{i=1}^nx_i=\frac{\bar{X}}{\alpha}$$

$$\hat{\beta}^{\wedge}=n\alpha^{-1}\sum_{i=1}^nx_i=\alpha\bar{X}^{-1}$$

.

Q8b: Lets estimate beta given sample (9,5,7)

$$\bar{X}=\frac{9+5+7}{3}=7,\quad\alpha=2$$

$$\bar{X}=30+5+7=7,\quad\alpha=2$$

so:

$$\hat{\beta}=\bar{X}/\alpha=7/2=3.5.$$

$$\hat{\beta}^{\wedge}=\bar{X}^{\wedge}/\alpha=7/2=3.5,$$

## Q9:

Using my R code:

```
set.seed(2025)

# known alpha
alpha <- 2
# observed sample
obs <- c(9,5,7)
n <- length(obs)

# observed mle
beta_hat_obs <- mean(obs) / alpha # 3.5

# simulate nsim samples of size n from Gamma(alpha, beta_hat_obs)
nsim <- 1000
beta_hats <- numeric(nsim)
for (i in 1:nsim) {
  samp <- rgamma(n, shape = alpha, scale = beta_hat_obs)
  beta_hats[i] <- mean(samp) / alpha
}

# estimate probability P(beta_hat >= observed)
p_ge <- mean(beta_hats >= beta_hat_obs)

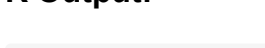
# summary
cat("Observed beta_hat:", beta_hat_obs, "\n")
cat("Mean simulated beta_hats:", mean(beta_hats), "\n")
cat("SD simulated beta_hats:", sd(beta_hats), "\n")
cat("P(beta_hat >= observed) approx:", p_ge, "\n")

# Optional: quick histogram of simulated beta_hats
hist(beta_hats, breaks = 30, col = "lightblue", main = "Simulated beta_hats",
      xlab = expression(hat(beta)))
abline(v = beta_hat_obs, col = "red", lwd = 2)
```

R Output:

```
Observed beta_hat: 3.5
Mean simulated beta_hats: 3.467059
SD simulated beta_hats: 1.421013
P(beta_hat >= observed) approx: 0.447

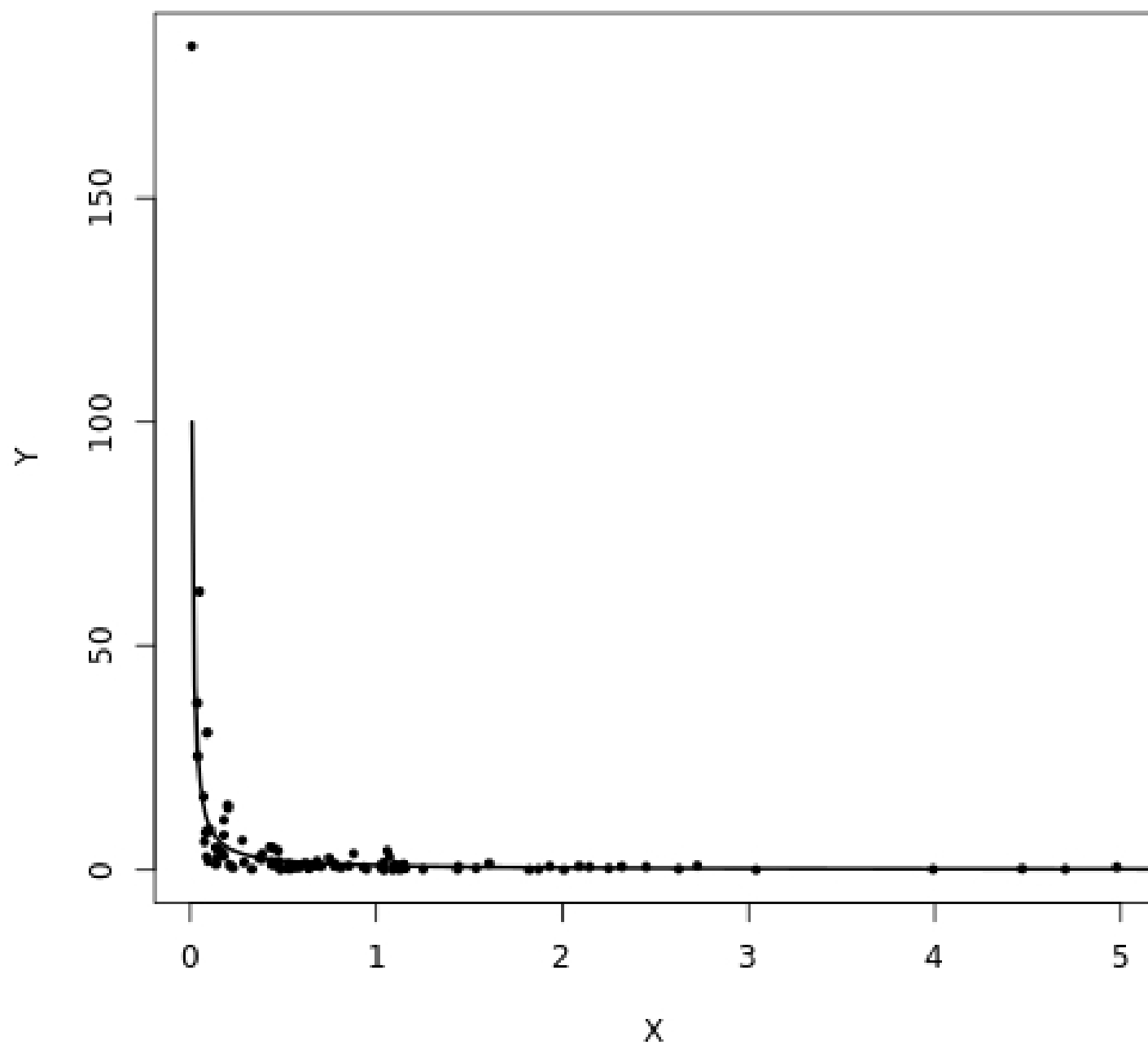
[Execution complete with exit code 0]
```



The estimated probability ≈ 0.47 - which indicates the observed MLE (3.5) is not unusually large under (α=2, β=3.5)

This was built with A MathJax extension for VS Code in a Markdown document, Math was done specifically in MathJax format. This document was exported to HTML & then to PDF prior to submission.

**Scatterplot of Y vs X with conditional mean  $y = 1/x$**



**Simulated beta\_hats**

