1. Simple perspective projection:

Project a 3D point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ onto the projection plane at $z_p = d$.

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \xrightarrow{\quad M \quad} \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{x}{z/d} \\ \frac{y}{z/d} \\ d \\ 1 \end{bmatrix} \quad (p.\ 13\text{-}13)$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix} \quad \text{(simple perspective matrix)} \quad (p.\ 13-15)$$

2. In WebGL, we map any viewing volume onto a 2X2X2 clipping cube (canonical viewing cube). WebGL will clip out the objects outside the clipping cube & show the objects inside on an image by orthogonal projection.

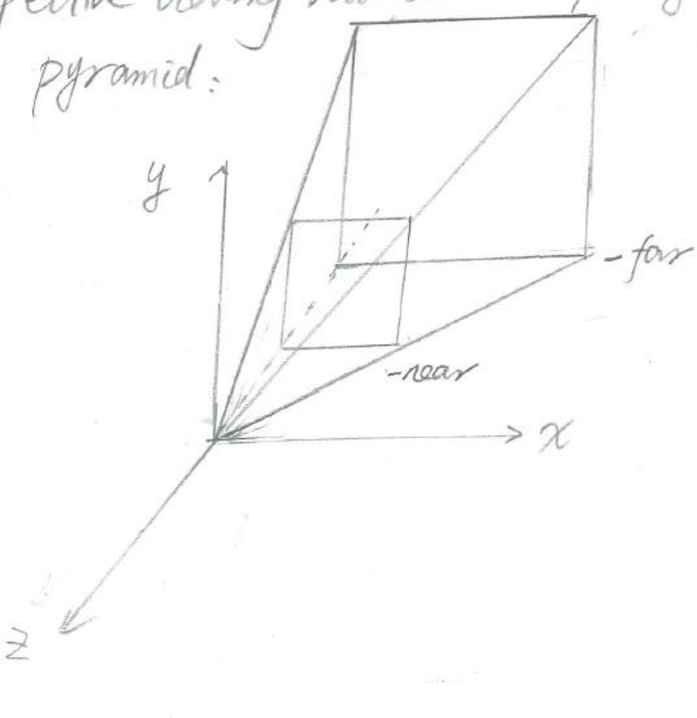
3. We need to find a distortion (mapping) matrix N

which can map a perspective viewing volume to the $2\times 2\times 2$ clipping cube. Followed by the orthogonal projection M_{orth}, we can achieve the simple perspective projection result.

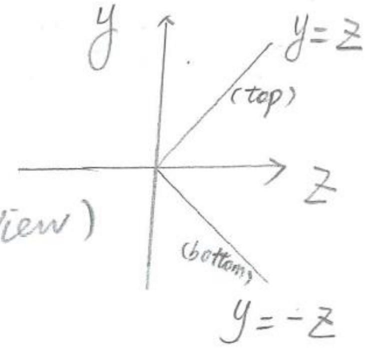
4. Suppose the projection plane $z_p = -1$, $(d=-1)$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad (p.13-24)$$

The perspective viewing volume is a 90° symmetric truncated pyramid:



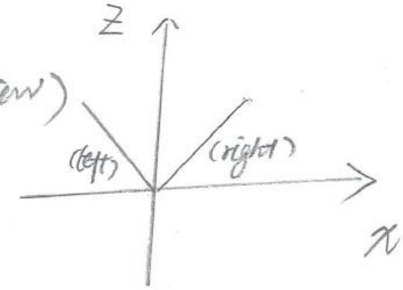
The top & bottom planes are $y = \pm z$.



(side view)

The left & right planes are $x = \pm z$

(top view)



5. N is similar to M but non-singular.

We need $M_{orth} N$ can achieve the same result as M.

Assume $N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix}$ $\quad \alpha, \beta \neq 0$

$$\text{Therefore } M_{orth} N = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{M_{orth}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{N}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = M'$$ (we name it M')

6. Apply M' to point $P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

$$P' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \\ -z \end{bmatrix} = \begin{bmatrix} -\frac{x}{z} \\ -\frac{y}{z} \\ 0 \\ 1 \end{bmatrix}$$

7. Apply M to point $P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

$$P'' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -z \end{bmatrix} = \begin{bmatrix} -\frac{x}{z} \\ -\frac{y}{z} \\ -1 \\ 1 \end{bmatrix}$$

8. Therefore M' & M can get the same projection where

$$\begin{cases} x_p = -\frac{x}{z} & \text{since } d = -1 \\ y_p = -\frac{y}{z} & \text{since } d = -1 \end{cases}$$

9. We then calculate the matrix N which transforms the truncated pyramid to the canonical clipping volume. $\rightarrow$ choose α and β

10. Apply N to a point $\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} \quad \text{(suppose)}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ \alpha z + \beta \\ -z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix}$$

$$\Rightarrow \begin{cases} x' = x \\ y' = y \\ z' = \alpha z + \beta \\ w' = -z \end{cases}$$

divide by w'

$$\Rightarrow \begin{cases} x'' = -\frac{x}{z} \\ y'' = -\frac{y}{z} \\ z'' = -(\alpha + \frac{\beta}{z}) \end{cases} \qquad (p. 13-27)$$

11. N transforms the original left & right planes $x = \pm z$ by $x'' = -\frac{x}{z}$ to the planes $x'' = \pm 1$.

N transforms the original top & bottom planes $y = \pm z$ by $y'' = -\frac{y}{z}$ to the planes $y'' = \pm 1$.

The front plane $z = -near$ is transformed by $z'' = -(\alpha + \frac{\beta}{z})$ to $z'' = 1$

The far plane $z = -far$ is transformed by $z'' = -(\alpha + \frac{\beta}{z})$ to $z'' = -1$

$$\Rightarrow \begin{cases} -(\alpha - \frac{\beta}{near}) = 1 \\ -(\alpha - \frac{\beta}{far}) = -1 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = -\frac{near + far}{near - far} \\ \beta = -\frac{2 * near * far}{near - far} \end{cases}$$

(p. 13-28)