COSC 414/519I: Computer Graphics

2023W2

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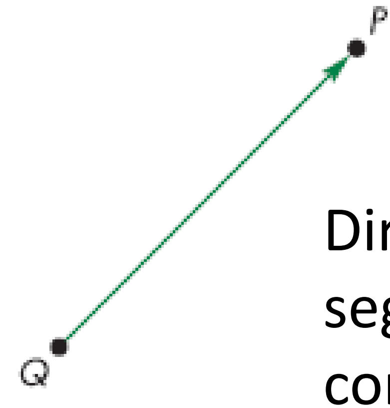
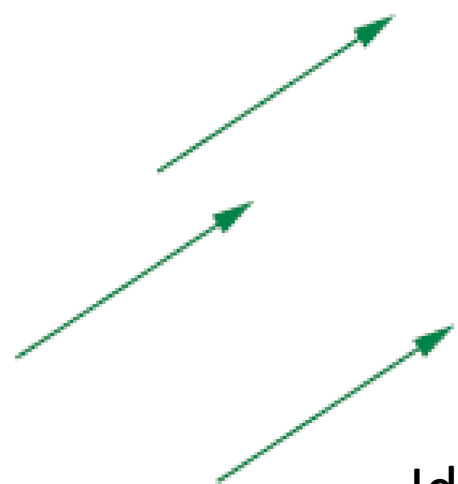
Geometric Objects

- How to represent 3D objects?
  - Using three fundamental types:
    - Scalar – a number, can be used to represent distance.
    - Point – a location in space, no size or shape.
    - Vector – can be used for any quantity with direction and magnitude. Physical quantities, e.g., velocity and force, are vectors. A vector, however, does not have a fixed location in space.

- We often connect points with directed line segments.
- A directed line segment has both magnitude – its length and direction – its orientation, and thus is a vector.



Directed line segment that connects points



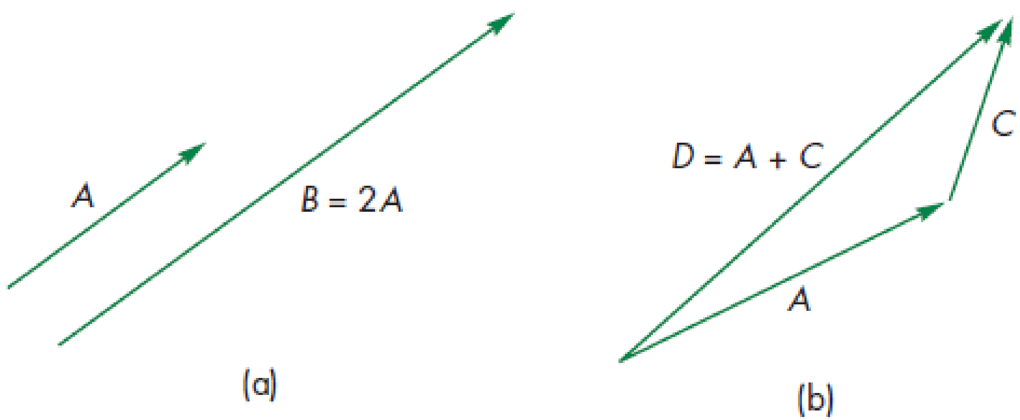
Identical vectors

Vectors

- Vectors can have their lengths increased or decreased without changing their directions.

  (a) $B = 2A$ multiplication of a vector by a scalar
- Two vectors can be added by the ***head-to-tail*** rule.

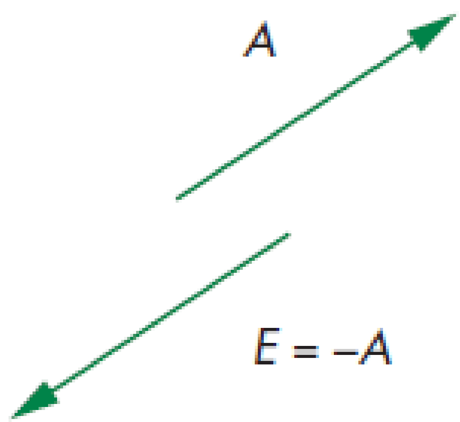
  (b) $D = A + C$ the addition of two vectors



(a) Parallel line segments. (b) Addition of line segments.

Vectors

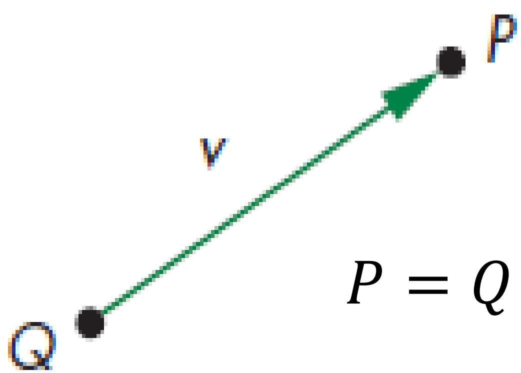
- If two vectors have the same length but opposite directions, their sum will be ***zero vector***, **0** whose magnitude is zero and orientation is undefined.



Inverse vectors.

Vectors

- We can use a directed line segment to move one point to another. We call this operation ***point-vector addition***, and it produces a new point.

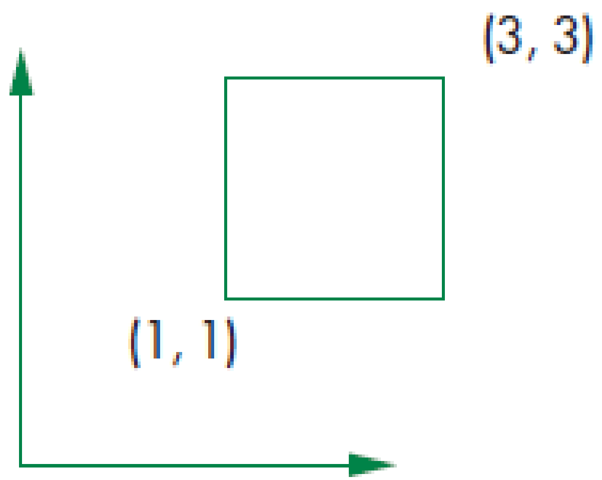


$P = Q + v$

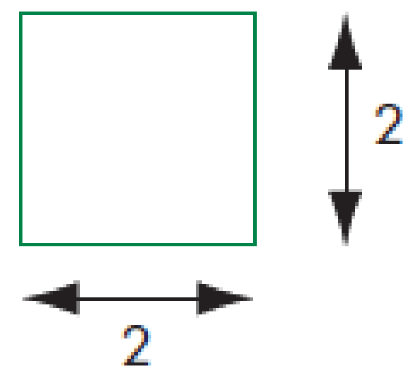
Point-vector addition

Any two points define a directed line segment or vector from one point to the second. We can call this operation ***point-to-point subtraction***, $v = P - Q$.

Coordinate-Free Geometry



Object and coordinate system.



Object without coordinate system.

Vector and Affine Spaces

- Vector space:
  - contains two distinct types of entities, vectors and scalars.
  - We can combine scalars and vectors to form new vector through ***scalar-vector multiplication*** and vectors to vectors through ***vector-vector addition***.
- Euclidean space:
  - an extension of the vector space that adds a measure of size or distance and allows us to define such things as the magnitude of a vector.

Vector and Affine Spaces

- Affine space:
  - An extension of the vector space that includes an additional type of objects: the point.
  - No operations between two points or between a point and a scalar that yields points.
  - Has operation of ***point-vector addition*** that produces a new point, or alternatively, an operation of ***point-point subtraction*** that produces a vector from two points.

Abstract Data Types (ADT)

- In computer graphics, our scalars are real numbers using ordinary addition and multiplication. Our geometric points are locations in space, and our vectors are directed line segments.
- These objects obey the rules of an affine space. We can create the corresponding ADTs in a program.

Abstract Data Types (ADT)

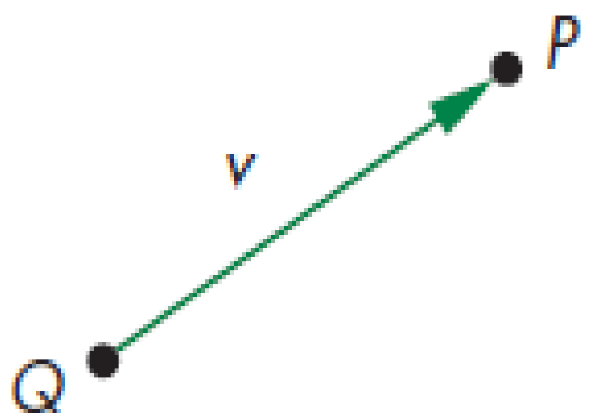
- Greek letters $\alpha, \beta, \gamma, \ldots$ denote scalars.
- Uppercase letters $P, Q, R, \ldots$ denote points.
- Lowercase letters $u, v, w, \ldots$ denote vectors.

Scalars, Points and Vectors

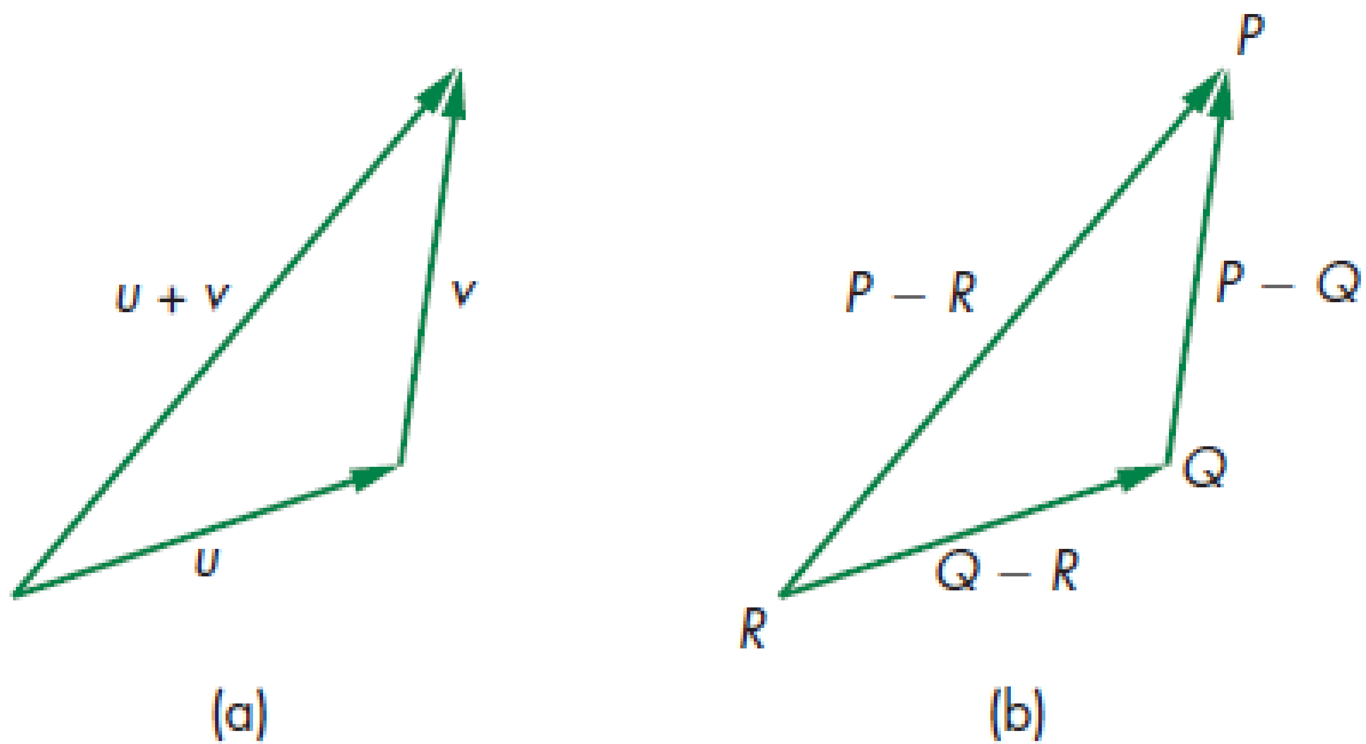
- The magnitude of a vector v is a real number denoted by $|v|$. The operation of vector-scalar multiplication has the property that $|\alpha v| = |\alpha||v|$, and the direction of αv is the same as the direction of v if α is positive and the opposite direction if α is negative.

Scalars, Points and Vectors

- Subtraction of two points: $v = P - Q$
- Point-vector addition: $P = Q + v$



Scalars, Points and Vectors

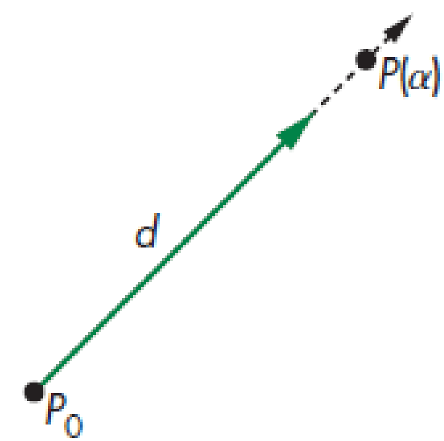


Use of the head-to-tail rule. (a) For vectors. (b) For points.

$$(P - Q) + (Q - R) = P - R$$

Lines

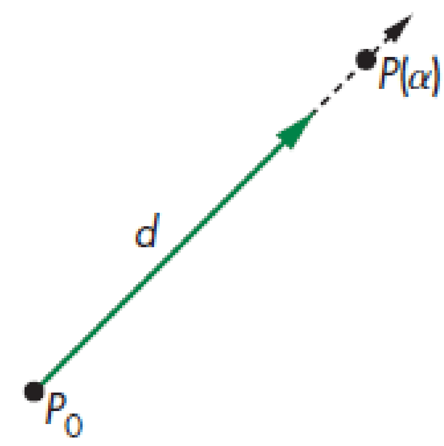
- The sum of a point and a vector (or the subtraction of two points) leads to the notion of a line in an affine space.
- Consider all points of the form $P(\alpha) = P_0 + \alpha d$, where P_0 is an arbitrary point, d is an arbitrary vector, and α is a scalar that can vary over some range of values.



Line in an affine space.

Lines

- For any value α, $P(\alpha)$ yields a point.
- These points lie on a line. This form is known as the parametric form of the line.
- If α is nonnegative, we get a ray emanating from P_0 and going in the direction of d.



Line in an affine space.

Affine Sums

- For any point Q, vector v, and positive scalar α,

$$P = Q + \alpha v$$

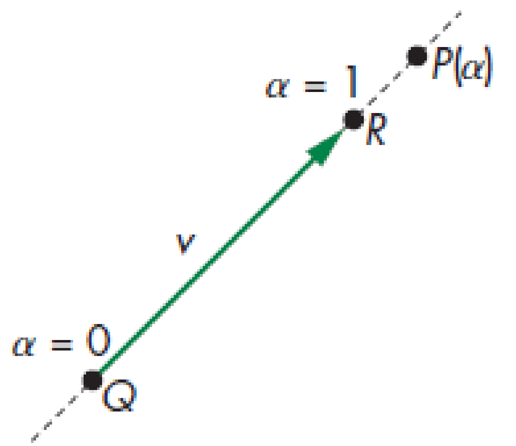
describes all points on the line from Q in the direction of v.

- We can always find a point R such that

$$v = R - Q$$

- Thus

$$P = Q + \alpha(R - Q)$$
$$= \alpha R + (1 - \alpha)Q$$



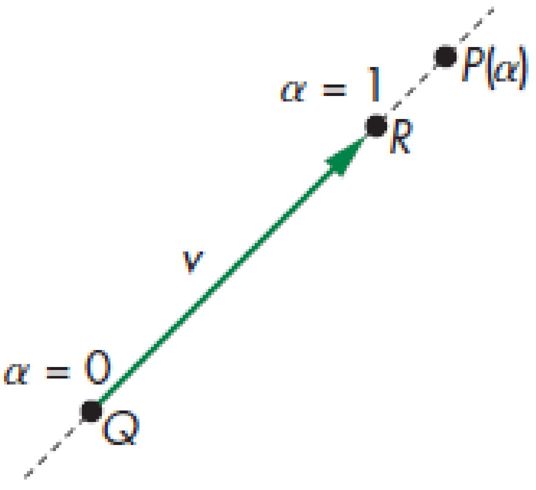
Affine addition.

Affine Sums

- This operation looks like the addition of two points and leads to the equivalent form

$$P = \alpha_1 R + \alpha_2 Q$$

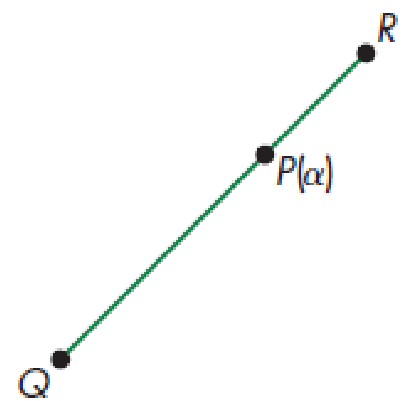
where $\alpha_1 + \alpha_2 = 1$



Affine addition.

Convexity

- A convex object is one for which any point lying on the line segment connecting any two points in the object is also in the object.
- We can use affine sums to help us gain a deeper understanding of convexity. For $0 \leq \alpha \leq 1$, the affine sum defines the line segment connecting R and Q; thus this line segment is a convex object.



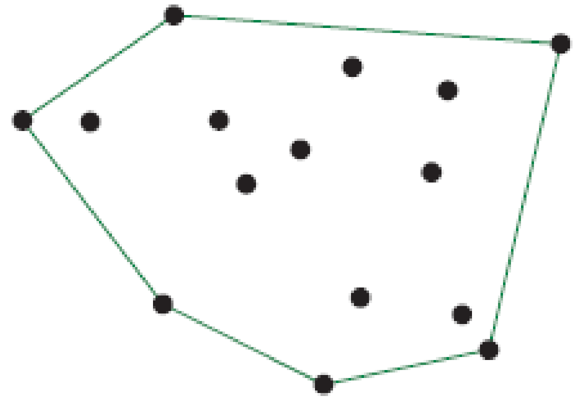
Line segment that connects two points.

Convexity

- We can extend the affine sums to include objects defined by n points $P_1, P_2, \ldots, P_n$. Consider the form $P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n$ which is defined if and only if

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

- The set of points formed by the affine sum of *n* points, under the additional restriction $\alpha_i \geq 0, i = 1,2, \ldots, n$ is called the ***convex hull*** of the set of points.

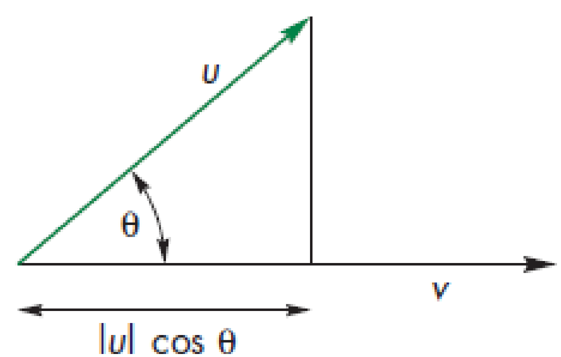


Convex hull.

Dot and Cross Products

- The dot product of u and v is written $u \cdot \mathrm{v}$. If $u \cdot \mathrm{v} = 0$, u and v are orthogonal.
- The magnitude of a vector u can be given by the dot product $|\mathrm{u}| = \sqrt{u \cdot \mathrm{u}}$.
- The cosine of the angle between two vectors is given by

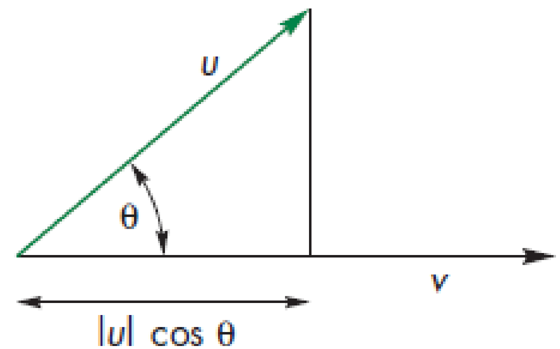
$$cos\theta = \frac{u \cdot \mathrm{v}}{|u||v|}$$



Dot product and projection.

Dot and Cross Products

- $|u|cos\theta = \frac{u \cdot \mathrm{v}}{|v|}$ is the length of the orthogonal projection of u on to v.



Dot product and projection.

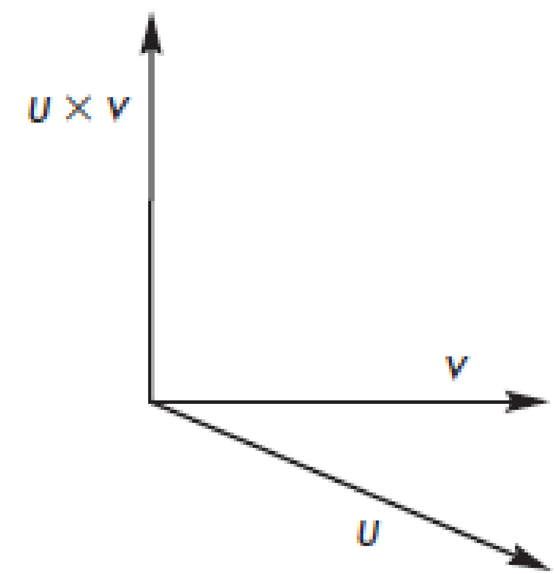
- A set of vectors is linear independent if we cannot write one of the vectors in terms of the others using scalar-vector multiplication and vector addition.

Dot and Cross Products

- A vector space has a dimension, which is the maximum number of linearly independent vectors that we can find.

Dot and Cross Products

- We can use two nonparallel vectors, u and v, to determine a third vector n that is orthogonal to them. This vector is the cross product $\mathrm{n} = u \times v$.
- We can use cross product to derive three mutually orthogonal vectors in 3D space from any two nonparallel vectors.



Cross product.

Dot and Cross Products

$$\mathrm{n} = u \times v$$
$$\mathrm{w} = u \times n$$

and then u, n, and w are mutually orthogonal.

- The magnitude of the cross product gives the magnitude of the sine of the angle θ between u and v,

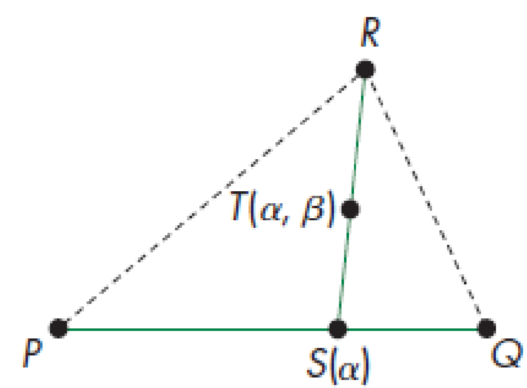
$$|sin\theta| = \frac{|u \times \mathrm{v}|}{|u||v|}$$

Planes

- A plane in an affine space can be defined as a direct extension of the parametric line.
- The line segment that joins *P* and *Q* is the set of points of the form

$$S(\alpha) = \alpha P + (1 - \alpha)Q \qquad 0 \leq \alpha \leq 1$$

- Suppose we take an arbitrary point on this line segment and form the line segment from this point to *R*, we can use a second parameter β to describe points along it.



Formation of a plane.

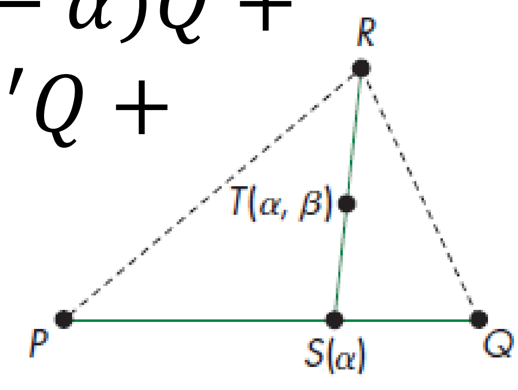
Planes

$$T(\beta) = \beta S + (1 - \beta) R \quad 0 \le \beta \le 1$$

Such points are determined by α and β and form the triangle determined by *P*, *Q*, and *R*.

$$T(\alpha, \beta) = \beta[\alpha P + (1 - \alpha) Q] + (1 - \beta) R$$

If we rewrite $T(\alpha, \beta) = \beta\alpha P + \beta(1 - \alpha) Q + (1 - \beta) R \Rightarrow T(\alpha', \beta', \gamma') = \alpha' P + \beta' Q + \gamma' R$, where $\alpha' + \beta' + \gamma' = 1$



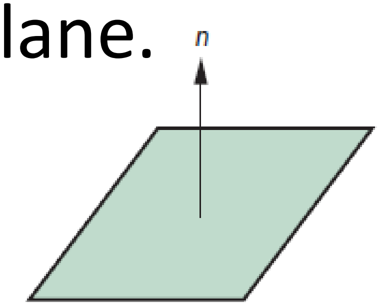
Formation of a plane.

Planes

- The representation of a point by $(\alpha', \beta', \gamma')$ is called its ***barycentric coordinate*** representation of *T* with respect to *P*, *Q*, *R*.
- For $0 \leq \alpha, \beta \leq 1$, all the points $T(\alpha, \beta)$ lie in the triangle formed by *P, Q, R.*

Planes

- If a point P lies in the plane, u and v are two nonparallel arbitrary vectors, then $P - P_0 = \alpha u + \beta v$ where P_0 is the joint point of u and v.
- We can find a vector *n* that is orthogonal to both *u* and *v*. If we use the cross product $n = u \times v$, then the equation of the plane becomes $n \cdot (P - P_0) = 0$.
- The vector *n* is perpendicular, or orthogonal, to the plane; it is called the ***normal*** to the plane.



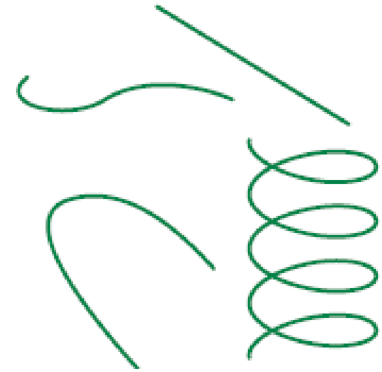
Normal to a plane.

Planes

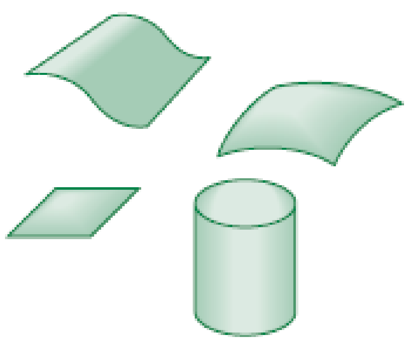
- The forms $P(\alpha)$, for the line, and $T(\alpha, \beta)$, for the plane, are known as ***parametric forms*** because they give the value of a point in space for each value of the parameters α and β.

3D Objects

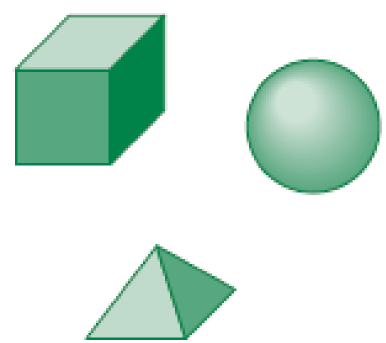
In 3D, objects are no longer restricted to lie in the same plane.



Curves in three dimensions.



Surfaces in three dimensions.



Volumetric objects.

3D Objects

- Three features characterize 3D objects that fit well with existing graphics hardware and software:

1. The objects are described by their surfaces and can be thought of as being hollow.
2. The objects can be specified through a set of vertices in 3D.
3. The objects either are composed of or can be approximate by flat, convex polygons.