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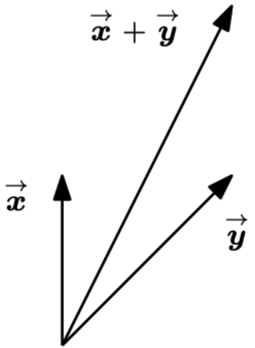
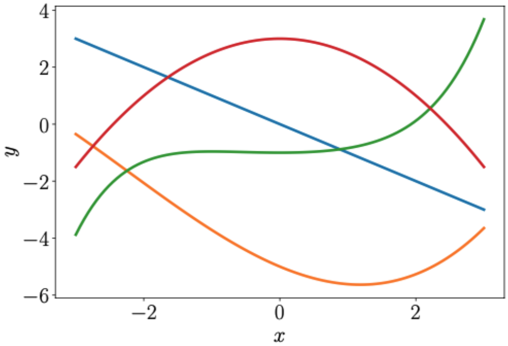
Mathematics for Machine Learning

Roadmap

Roadmap



(a) Geometric vectors.



(b) Polynomials.

Examples of vector spaces: (a) geometric vectors, (b) polynomials.

System of Linear Equations

- For unknown variables $(x_1, \cdots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

- Three cases of solutions
  - No solution: $x_1 + x_2 + x_3 = 3, \quad x_1 - x_2 + 2x_3 = 2, \quad 2x_1 + 3x_3 = 1$
  - Unique solution: $x_1 + x_2 + x_3 = 3, \quad x_1 - x_2 + 2x_3 = 2, \quad x_2 + 3x_3 = 1$
  - Infinitely many solutions: $x_1 + x_2 + x_3 = 3, \quad x_1 - x_2 + 2x_3 = 2, \quad 2x_1 + 3x_3 = 5$
- Question. Under what conditions, one of the above three cases occur?

Matrix Representation

- A collection of linear equations

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
&\vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- Matrix representations:

$$
\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \cdots + \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \Leftrightarrow \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}
$$

  where A is the coefficient matrix, $\mathbf{x}$ is the variable vector, and $\mathbf{b}$ is the constant vector.

- Understanding A is the key to answering various questions about this linear system $A\mathbf{x} = \mathbf{b}$.

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(1) Systems of Linear Equations
(2) Matrices
(3) Solving Systems of Linear Equations
(4) Vector Spaces
(5) Linear Independence
(6) Basis and Rank
(7) Linear Mappings
(8) Affine Spaces

Matrix: Addition and Multiplication

- For two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$,

$$A + B := \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

- For two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times k}$, the elements c_{ij} of the product $C = AB \in \mathbb{R}^{m\times k}$ is:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, k$$

- Example. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$, compute AB and BA.

Identity Matrix and Matrix Properties

- A square matrix I_n with $I_{ii} = 1$ and $I_{ij} = 0$ for $i \neq j$, where n is the number of rows and columns. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Associativity: For $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$, $(AB)C = A(BC)$
- Distributivity: For $A, B \in \mathbb{R}^{m \times n}$, and $C, D \in \mathbb{R}^{n \times p}$,
  - (i) $(A + B)C = AC + BC$
  - (ii) $A(C + D) = AC + AD$
- Multiplication with the identity matrix: For $A \in \mathbb{R}^{m \times n}$, $I_m A = A I_n = A$

Inverse and Transpose

- For a square matrix $A \in \mathbb{R}^{n\times n}$, B is the inverse of A, denoted by A^{-1}, if

$$AB = I_n = BA$$

- Called regular/invertible/nonsingular, if it exists.
- If it exists, it is unique.
- How to compute? For 2×2 matrix,

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

- For a matrix $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$ with $b_{ij} = a_{ji}$ is the transpose of A, which we denote by A^T.
- Example. For $A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$,

$$A^T = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

- If $A = A^T$, A is called symmetric.

Inverse and Transpose: More Properties

- $AA^{-1} = I = A^{-1}A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A + B)^{-1} \neq A^{-1} + B^{-1}$
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- If A is invertible, so is A^T.

Scalar Multiplication

- Multiplication by a scalar $\lambda \in \mathbb{R}$ to $A \in \mathbb{R}^{m\times n}$
- Example. For $A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$, $3 \times A = \begin{bmatrix} 0 & 6 \\ 3 & -3 \\ 0 & 3 \end{bmatrix}$
- Associativity
  - $(\lambda\psi)C = \lambda(\psi C)$
  - $\lambda(BC) = (\lambda B)C = B(\lambda C) = (BC)\lambda$
  - $(\lambda C)^T = C^T\lambda^T = C^T\lambda = \lambda C^T$
- Distributivity
  - $(\lambda + \psi)C = \lambda C + \psi C$
  - $\lambda(B + C) = \lambda B + \lambda C$

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Example

- $-3x + 2z = -1$
- $x - 2y + 2z = -5/3$
- $-x - 4y + 6z = -13/3$

Express the equation as its augmented matrix and apply Gaussian elimination:

$$\begin{bmatrix} -3 & 0 & 2 & | & -1 \\ 1 & -2 & 2 & | & -5/3 \\ -1 & -4 & 6 & | & -13/3 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} -3 & 0 & 2 & | & -1 \\ 0 & -2 & 8/3 & | & -2 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The two nonzero rows give $-3x + 2z = -1$ and $-2y + (8/3)z = -2$.

Parametrizing Solutions

- From $-3x + 2z = -1$ and $-2y + (8/3)z = -2$:

$$
\begin{aligned}
x &= (1/3) + (2/3)z \\
y &= 1 + (4/3)z \\
z &= z
\end{aligned}
$$

- Solution set:

$$
\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2/3 \\ 4/3 \\ 1 \end{bmatrix} \;\middle|\; z \in \mathbb{R} \right\}
$$

- This helps us understand the set of solutions, e.g., each value of z gives a different solution.

Form of Solution Sets

- General form: $\mathbf{x} = \mathbf{x}_p + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$
- $\mathbf{x}_p$: particular solution
- $\mathbf{v}_1, \ldots, \mathbf{v}_k$: basis of the solution space of $A\mathbf{x} = \mathbf{0}$
- $t_1, \ldots, t_k$: free parameters

Gaussian Elimination Algorithm

1. Write the augmented matrix $[A|\mathbf{b}]$
2. Use row operations to transform to row echelon form
3. Back-substitute to find solutions

Roadmap

Groups

- Definition. A set G with a binary operation $\circ : G \times G \to G$ is called a group $(G, \circ)$ if:
  1. Closure: For all $a, b \in G$, $a \circ b \in G$
  2. Associativity: For all $a, b, c \in G$, $(a \circ b) \circ c = a \circ (b \circ c)$
  3. Identity element: There exists $e \in G$ such that $a \circ e = e \circ a = a$ for all $a \in G$
  4. Inverse element: For each $a \in G$, there exists $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$

- Definition. A set V with two operations: addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$ is called a vector space $(V, +, \cdot)$ over $\mathbb{R}$ if:
    1. $(V, +)$ is an abelian group
    2. Distributivity: $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$
    3. Distributivity: $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$
    4. Associativity: $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$
    5. Identity: $1 \cdot \mathbf{v} = \mathbf{v}$

Examples of Vector Spaces

- $\mathbb{R}^n$ with standard addition and scalar multiplication
- Polynomials of degree at most n
- Continuous functions $C([a, b])$
- Matrices $\mathbb{R}^{m \times n}$
- Elements of $\mathbb{R}^n$

Vector Subspaces

- Definition. Consider a vector space $V = (V, +, \cdot)$ and $U \subset V$. Then, $U = (U, +, \cdot)$ is called a vector subspace (or linear subspace) of V if U is a vector space with operations $+$ and $\cdot$ restricted to $U \times U$ and $\mathbb{R} \times U$.
- Examples
  - For every vector space V, V and $\{\mathbf{0}\}$ are trivial subspaces.
  - The solution set of $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.
  - The solution of $A\mathbf{x} = \mathbf{b}$ $(\mathbf{b} \neq \mathbf{0})$ is not a subspace of $\mathbb{R}^n$.
  - The intersection of arbitrarily many subspaces is a subspace itself.

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Linear Independence

- Definition. For a vector space V and vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$, every $\mathbf{v} \in V$ of the form $\mathbf{v} = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_k \mathbf{x}_k$ with $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ is a linear combination of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$.
- Definition. If there is a non-trivial linear combination such that $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent. If only the trivial solution exists, i.e., $\lambda_1 = \cdots = \lambda_k = 0$, $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent.
- Meaning. A set of linearly independent vectors consists of vectors that have no redundancy.
- Useful fact. The vectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ are linearly dependent, iff at least one of them is a linear combination of the others.

Checking Linear Independence

- Use Gaussian elimination to get the row echelon form
- All column vectors are linearly independent iff all columns are pivot columns
- Example: Check if $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ are linearly independent.
- Form matrix and reduce to row echelon form
- Every column is a pivot column. Thus, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent.

Linear Combinations of Linearly Independent Vectors

- Vector space V with k linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$
- m linear combinations $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$. Question: Are they linearly independent?
- If $\mathbf{x}_j = \mathbf{b}_1, \ldots, \mathbf{b}_k \begin{bmatrix} \lambda_{1j} \\ \vdots \\ \lambda_{kj} \end{bmatrix} = B\boldsymbol{\lambda}_j$
- Then $\sum_{j=1}^{m} \psi_j \mathbf{x}_j = \sum_{j=1}^{m} \psi_j B \boldsymbol{\lambda}_j = B \sum_{j=1}^{m} \psi_j \boldsymbol{\lambda}_j$
- $\{\mathbf{x}\}$ linearly independent $\Leftrightarrow$ $\{\boldsymbol{\lambda}\}$ linearly independent

Example

- $\mathbf{x}_1 = \mathbf{b}_1 - 2\mathbf{b}_2 + \mathbf{b}_3 - \mathbf{b}_4$
- $\mathbf{x}_2 = -4\mathbf{b}_1 - 2\mathbf{b}_2 + 4\mathbf{b}_4$
- $\mathbf{x}_3 = 2\mathbf{b}_1 + 3\mathbf{b}_2 - \mathbf{b}_3 - 3\mathbf{b}_4$
- $\mathbf{x}_4 = 17\mathbf{b}_1 - 10\mathbf{b}_2 + 11\mathbf{b}_3 + \mathbf{b}_4$
- Matrix form:

$$A = \begin{bmatrix} 1 & -4 & 2 & 17 \\ -2 & -2 & 3 & -10 \\ 1 & 0 & -1 & 11 \\ -1 & -4 & -3 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The last column is not a pivot column. Thus, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$ are linearly dependent.

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Generating Set and Basis

- Definition. A vector space $V = (V, +, \cdot)$ and a set of vectors $A = \{\mathbf{x}_1, \dots, \mathbf{x}_k\} \subset V$.
  - If every $\mathbf{v} \in V$ can be expressed as a linear combination of $\mathbf{x}_1, \dots, \mathbf{x}_k$, A is called a generating set of V.
  - The set of all linear combinations of A is called the span of A.
  - If A spans the vector space V, we use $V = \text{span}[A]$ or $V = \text{span}[\mathbf{x}_1, \dots, \mathbf{x}_k]$
- Definition. The minimal generating set B of V is called basis of V. We call each element of B a basis vector. The number of basis vectors is called dimension of V.
- Properties
  - B is a maximally linearly independent set of vectors in V.
  - Every vector $\mathbf{x} \in V$ is a linear combination of B, which is unique.

Examples of Bases

- Different bases for $\mathbb{R}^3$:

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$B_3 = \left\{ \begin{bmatrix} 0.5 \\ 0.8 \\ 0.4 \end{bmatrix}, \begin{bmatrix} 1.8 \\ 0.3 \\ 0.3 \end{bmatrix}, \begin{bmatrix} -2.2 \\ -1.3 \\ 3.5 \end{bmatrix} \right\}$$

- Linearly independent, but not maximal. Thus, not a basis:

$$A = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

Determining a Basis

- Want to find a basis of a subspace $U = \text{span}[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]$
  1. Construct a matrix $A = [\mathbf{x}_1 \quad \mathbf{x}_2 \quad \cdots \quad \mathbf{x}_m]$
  2. Find the row-echelon form of A.
  3. Collect the pivot columns.
- Logic: Collect $\mathbf{x}_i$ so that we have only trivial solution. Pivot columns tell us which set of vectors is linearly independent.

Rank (1)

- Definition. The rank of $A \in \mathbb{R}^{m \times n}$ denoted by $\text{rk}(A)$ is the number of linearly independent columns.
  - Same as the number of linearly independent rows
- Example: $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$
- Thus, $\text{rk}(A) = 2$.
- $\text{rk}(A) = \text{rk}(A^T)$

Rank (2)

- The columns (resp. rows) of A span a subspace U (resp. W) with $\dim(U) = \text{rk}(A)$ (resp. $\dim(W) = \text{rk}(A)$), and a basis of U (resp. W) can be found by Gaussian elimination of A (resp. A^T).
- For all $A \in \mathbb{R}^{n \times n}$, $\text{rk}(A) = n$ iff A is regular (invertible).
- The linear system $A\mathbf{x} = \mathbf{b}$ is solvable iff $\text{rk}(A) = \text{rk}(A|\mathbf{b})$.
- For $A \in \mathbb{R}^{m \times n}$, the subspace of solutions for $A\mathbf{x} = \mathbf{0}$ possesses dimension $n - \text{rk}(A)$.
- $A \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions. The rank of the full-rank matrix A is $\min(\# \text{ of cols}, \# \text{ of rows})$.

Roadmap

Linear Mapping (1)

- Interest: A mapping that preserves the structure of the vector space
- Definition. For vector spaces V, W, a mapping $\Phi : V \to W$ is called a linear mapping (or homomorphism/linear transformation), if, for all $\mathbf{x}, \mathbf{y} \in V$ and all $\lambda \in \mathbb{R}$,
  - $\Phi(\mathbf{x} + \mathbf{y}) = \Phi(\mathbf{x}) + \Phi(\mathbf{y})$
  - $\Phi(\lambda \mathbf{x}) = \lambda \Phi(\mathbf{x})$
- Definition. A mapping $\Phi : V \to W$ is called
  - Injective, if $\forall \mathbf{x}, \mathbf{y} \in V, \Phi(\mathbf{x}) = \Phi(\mathbf{y}) \Rightarrow \mathbf{x} = \mathbf{y}$
  - Surjective, if $\Phi(V) = W$
  - Bijective, if it is both injective and surjective

Linear Mapping (2)

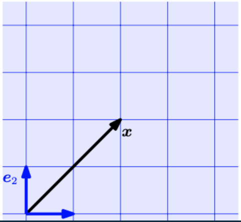
- Properties
  - $\Phi(\mathbf{0}_V) = \mathbf{0}_W$
  - Φ is injective iff $\ker(\Phi) = \{\mathbf{0}_V\}$
  - Φ is surjective iff $\mathrm{Im}(\Phi) = W$
- Isomorphism: If Φ is bijective, then V and W are isomorphic, denoted $V \cong W$.
- Theorem: Two finite-dimensional vector spaces are isomorphic iff they have the same dimension.

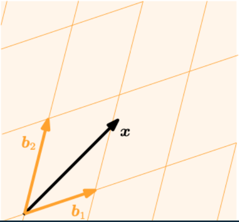
Coordinates

- Let $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be a basis of vector space V.
- Every $\mathbf{x} \in V$ can be uniquely written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 + \cdots + \alpha_n \mathbf{b}_n$$

- We call $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ the coordinate of $\mathbf{x}$ with respect to $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$.
- Basis change $\Rightarrow$ Coordinate change





Transformation Matrix

- Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ be a basis of V and $C = (\mathbf{c}_1, \ldots, \mathbf{c}_m)$ be a basis of W.
- For a linear mapping $\Phi : V \to W$, the transformation matrix A_Φ is defined such that:

$$\Phi(\mathbf{x})_C = A_\Phi \mathbf{x}_B$$

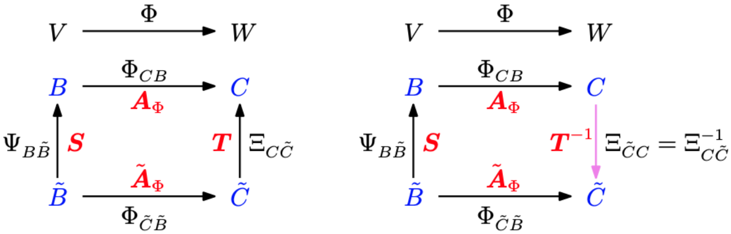
  where $\mathbf{x}_B$ and $\Phi(\mathbf{x})_C$ are coordinates with respect to B and C.
- The columns of A_Φ are the coordinates of $\Phi(\mathbf{b}_1), \ldots, \Phi(\mathbf{b}_n)$ with respect to C.

Basis Change

- Consider two bases $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ and $\tilde{B} = (\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_n)$ of V.
- Change of basis matrix S: $\tilde{\mathbf{b}}_j = \sum_{i=1}^{n} s_{ij} \mathbf{b}_i$
- For a vector $\mathbf{x}$: $\mathbf{x}_B = S\mathbf{x}_{\tilde{B}}$
- For a linear mapping $\Phi : V \to V$:

$$A'_\Phi = S^{-1} A_\Phi S$$

where A_Φ is the transformation matrix with respect to B and A'_Φ is with respect to $\tilde{B}$.



Basis Change: General Case

- For $\Phi : V \to W$ with bases $B, \tilde{B}$ of V and $C, \tilde{C}$ of W:
- (inter) transformation matrices A_Φ from B to C and A'_Φ from $\tilde{B}$ to $\tilde{C}$
- (intra) transformation matrices S from $\tilde{B}$ to B and T from $\tilde{C}$ to C
- Theorem. $A'_\Phi = T^{-1} A_\Phi S$

Image and Kernel

- Kernel (null space): $\ker(\Phi) = \{\mathbf{v} \in V \mid \Phi(\mathbf{v}) = \mathbf{0}_W\}$
- Image (range): set of vectors $\mathbf{w} \in W$ that can be reached by Φ from any vector in V
- V: domain, W: codomain

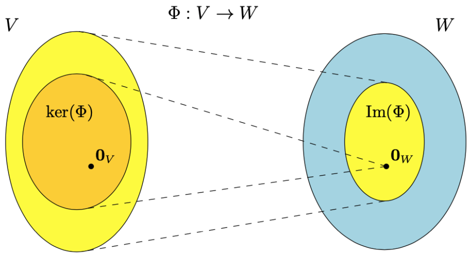
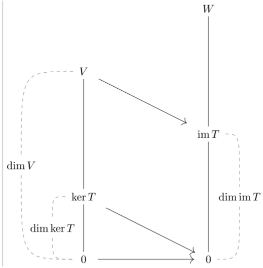


Image and Kernel: Properties

- $\ker(\Phi)$ is a subspace of V
- $\mathrm{Im}(\Phi)$ is a subspace of W
- Φ is injective iff $\ker(\Phi) = \{\mathbf{0}_V\}$
- Φ is surjective iff $\mathrm{Im}(\Phi) = W$
- Rank-Nullity Theorem: $\dim(\ker(\Phi)) + \dim(\mathrm{Im}(\Phi)) = \dim(V)$
- Simplified version. For $A \in \mathbb{R}^{m\times n}$,

$$\mathrm{rk}(A) + \mathrm{nullity}(A) = n$$



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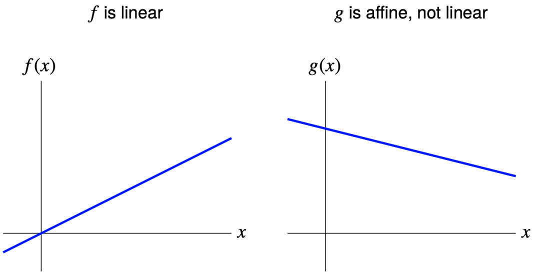
Linear vs. Affine Function

- Distinction between linear and affine functions:
  - linear function: $f(\mathbf{x}) = a\mathbf{x}$
  - affine function: $f(\mathbf{x}) = a\mathbf{x} + b$
  - sometimes (ignorant) people refer to affine functions as linear



Affine Subspace

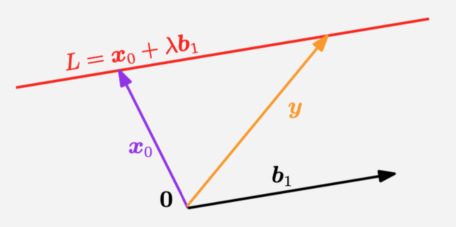
- Definition. Let U be a subspace of V and $\mathbf{x}_0 \in V$. The set

$$L = \mathbf{x}_0 + U = \{\mathbf{x}_0 + \mathbf{u} \mid \mathbf{u} \in U\}$$

  is called an affine subspace (or affine set) with support point $\mathbf{x}_0$ and direction space U.
- Parametric form:

$$\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{b}_1 + \cdots + \lambda_k \mathbf{b}_k, \quad \lambda_1, \ldots, \lambda_k \in \mathbb{R}$$

  where $\mathbf{b}_1, \ldots, \mathbf{b}_k$ is a basis of U.



Examples of Affine Subspaces

- Lines in $\mathbb{R}^2$ or $\mathbb{R}^3$
- Planes in $\mathbb{R}^3$
- Solution sets of linear equations $A\mathbf{x} = \mathbf{b}$ (for $\mathbf{b} \neq \mathbf{0}$)
- Hyperplanes: $\mathbf{a}^T\mathbf{x} = c$

Summary

- Linear algebra is the study of vector spaces and linear mappings
- Key concepts: basis, dimension, rank, kernel, image
- Applications: solving systems of equations, computer graphics, machine learning
- Understanding the structure of linear transformations is crucial

Review Questions

1. What is the difference between a vector space and a subspace?
2. How do you determine if vectors are linearly independent?
3. What is the relationship between rank and dimension?
4. How does basis change affect the transformation matrix?
5. What is the difference between linear and affine functions?

Thank You!

Math Tutorial: Introduction

- This tutorial section provides deeper explanations and worked examples
- Each topic from the main lecture is explored in greater detail
- Includes step-by-step solutions and practical applications
- Topics covered:
  1. Systems of Linear Equations - Detailed Examples
  2. Matrix Operations - Computational Methods
  3. Vector Spaces - Theoretical Foundations
  4. Linear Independence - Practical Techniques
  5. Basis and Rank - Advanced Concepts
  6. Linear Mappings - Transformation Matrices
  7. Affine Spaces - Geometric Interpretation

Tutorial: Systems of Linear Equations - Example 1

Problem: Solve the system:

$$\begin{cases} 2x + 3y - z = 8 \\ x - y + 2z = 1 \\ 3x + 2y + z = 7 \end{cases}$$

Solution: Using Gaussian elimination, write the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 1 & -1 & 2 & 1 \\ 3 & 2 & 1 & 7 \end{array}\right]$$

After row reduction:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Answer: $x = 2, y = 1, z = -1$

Tutorial: Systems of Linear Equations - Example 2

Problem: Determine the nature of solutions:

$$\begin{cases} x + 2y - z = 3 \\ 2x + 4y - 2z = 6 \\ 3x + 6y - 3z = 9 \end{cases}$$

Analysis: Notice that equations 2 and 3 are multiples of equation 1.

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 3 & 6 & -3 & 9 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Result: Infinitely many solutions (dependent system)

$$\mathbf{x} = \begin{bmatrix} 3 - 2y + z \\ y \\ z \end{bmatrix}, \quad y, z \in \mathbb{R}$$

Tutorial: Matrix Operations - Detailed Example

Problem: Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$, compute AB, BA, and A^{-1}.

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1(2)+2(1) & 1(0)+2(3) \\ 3(2)+4(1) & 3(0)+4(3) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2(1)+0(3) & 2(2)+0(4) \\ 1(1)+3(3) & 1(2)+3(4) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 10 & 14 \end{bmatrix}$$

Note: $AB \neq BA$ (matrix multiplication is not commutative)

For A^{-1}: $\det(A) = 1(4) - 2(3) = -2$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Tutorial: Vector Spaces - Subspace Verification

Problem: Verify that $U = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}$ is a subspace of $\mathbb{R}^3$.

Solution: Check three conditions:

1. **Zero vector:** $(0, 0, 0)$ satisfies $0 + 0 - 0 = 0$ ✓
2. **Closure under addition:** If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ satisfy the equation:

$$\begin{aligned}(u_1 + v_1) + (u_2 + v_2) - (u_3 + v_3) &= (u_1 + u_2 - u_3) + (v_1 + v_2 - v_3) \\ &= 0 + 0 = 0 \quad \checkmark\end{aligned}$$

3. **Closure under scalar multiplication:** If $\mathbf{u}$ satisfies the equation and $\lambda \in \mathbb{R}$:

$$(\lambda u_1) + (\lambda u_2) - (\lambda u_3) = \lambda(u_1 + u_2 - u_3) = \lambda \cdot 0 = 0 \quad \checkmark$$

Conclusion: U is a subspace of $\mathbb{R}^3$

Tutorial: Linear Independence - Detailed Example

Problem: Determine if $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ are linearly independent.

Solution: Form matrix and reduce:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -3 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Analysis: The third column is not a pivot column, so $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Verification: $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \neq \mathbf{v}_3$

Actually, the vectors are linearly dependent.

Tutorial: Basis and Dimension - Finding a Basis

Problem: Find a basis for the subspace $U = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$

Solution: Form matrix and reduce:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Result: Pivot columns are 1 and 2, so a basis is:

$$B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension: $\dim(U) = 2$

Tutorial: Rank Calculation - Detailed Example

Problem: Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Solution: Row reduce to echelon form:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -3 \end{bmatrix} \xrightarrow{\text{swap}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Result: Number of pivot columns $= 2$

Answer: $\text{rk}(A) = 2$

Tutorial: Linear Mappings - Transformation Matrix

Problem: Find the transformation matrix for $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Phi(x, y) = (2x + y, x - y)$ with respect to the standard basis.

Solution: Apply Φ to basis vectors:

$$\Phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \Phi \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The transformation matrix is:

$$A_\Phi = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Verification: $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x - y \end{bmatrix}$ ✓

Problem: For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$, find $\ker(A)$ and $\mathrm{Im}(A)$.

Solution for Kernel: Solve $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row reduce: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

General solution: $x_1 = -2x_2 - 3x_3$

$$\ker(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Solution for Image: Pivot columns of A form a basis:

Tutorial: Basis Change - Detailed Example

Problem: Given bases $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\tilde{B} = \{\tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2\}$ where:

$$\tilde{\mathbf{b}}_1 = \mathbf{b}_1 + \mathbf{b}_2, \quad \tilde{\mathbf{b}}_2 = \mathbf{b}_2$$

Find the change of basis matrix S.

Solution: Express new basis in terms of old:

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

For a vector $\mathbf{x}$: $\mathbf{x}_B = S\mathbf{x}_{\tilde{B}}$

Example: If $\mathbf{x}_{\tilde{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then:

$$\mathbf{x}_B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Tutorial: Affine Spaces - Geometric Example

Problem: Find the parametric equation of the line passing through $\mathbf{p} = (1, 2, 3)$ in the direction of $\mathbf{d} = (2, -1, 1)$.

Solution: The line is an affine subspace:

$$L = \mathbf{p} + \operatorname{span}\{\mathbf{d}\} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Parametric form:

$$\begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 3 + t \end{cases}$$

Cartesian form: Eliminate t:

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-3}{1}$$

Points on the line:

- $t = 0$: $(1, 2, 3)$ (support point)

Definition: For a square matrix A, a non-zero vector $\mathbf{v}$ is an eigenvector with eigenvalue λ if:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Example: Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

Solution: Solve $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (3-\lambda)(2-\lambda) = 0$$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = 2$

For $\lambda_1 = 3$: $(A - 3I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For $\lambda_2 = 2$: $(A - 2I)\mathbf{v} = \mathbf{0}$ gives $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Tutorial: Summary and Key Takeaways

- **Systems of Equations:** Gaussian elimination is the fundamental tool
- **Matrices:** Represent linear transformations and systems
- **Vector Spaces:** Abstract structures with specific properties
- **Linear Independence:** Key concept for basis and dimension
- **Rank:** Determines solvability and solution space dimension
- **Linear Mappings:** Preserve vector space structure
- **Kernel and Image:** Fundamental subspaces of a transformation
- **Basis Change:** Allows computation in different coordinate systems
- **Affine Spaces:** Generalization of linear subspaces
- **Applications:** Computer graphics, machine learning, physics, engineering

Tutorial: Practice Problems

1. Solve the system: $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + y - z = 0 \end{cases}$
2. Find the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$
3. Determine if $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are linearly independent.
4. Find the kernel and image of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$
5. Find the transformation matrix for $\Phi(x, y) = (x + 2y, 3x - y)$