Transient, global-in-time, convergent iterative coupling of acoustic BEM and elastic FEM.

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Industrial motivations.

Complete, realistic and efficient simulation of the structure's behaviour when subjected to remote underwater explosions.

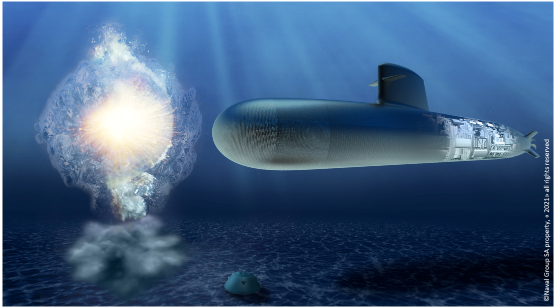
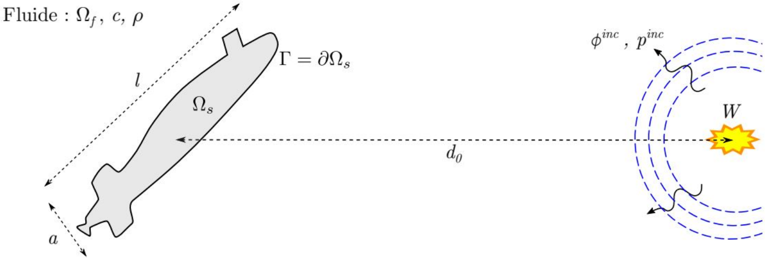


FIGURE : ©Naval Group (2011), on `https://www.inria.fr/fr/modeliser-logiciel-equipe-poems-naval-group`

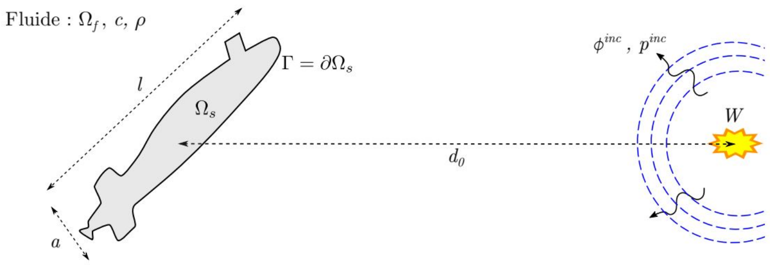
Far-field underwater explosion



1. Acoustic shock wave
2. Bubbles: slow movement of potential fluid

FIGURE : D. Mavaleix-Marchessoux, "Modelling the fluid-structure coupling caused by a far-field underwater explosion". PhD thesis, Institut Polytechnique de Paris, 2020

Far-field underwater explosion



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Overview of the project

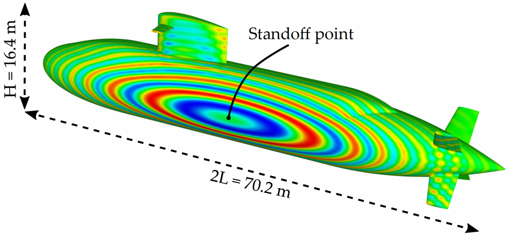
Before 2017: FEM-BEM coupling with approximated equations

↓

Damien Mavaleix-Marchessoux's PhD (2020):
1. FEM-FEM coupling (shock wave)
2. FEM-BEM coupling (bubbles)

↓

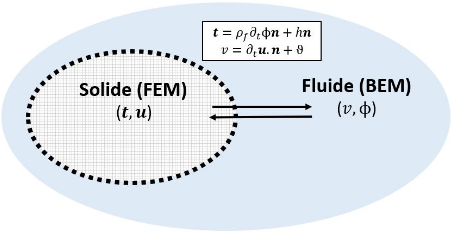
Objective: FEM-BEM coupling for the shock wave



Reflected fluid pressure computed on the ship's surface with a fast FEM/Z-BEM coupling (3.10^6 DOFs).

D. Mavaleix-Marchessoux, "Modelling the fluid-structure coupling caused by a far-field underwater explosion". PhD thesis, Institut Polytechnique de Paris, 2020

Iterative FEM/Z-BEM coupling in time



- Industrial context : commercial solver (structure).
- Possible non-linear solid deformations
  $\Rightarrow$ FEM for structure
- Large fluid domain $\Rightarrow$ BEM for fluid
- High frequencies $\Rightarrow$ Z-BEM with CQM

Time-domain Z-BEM with CQM

Boundary integral equation in time domain (retarded potentials method) :

$$\boxed{\frac{1}{2}\phi(\mathbf{x},t) = -\int_\Gamma \frac{\partial \mathbf{G}}{\partial n}(r,t) \star \phi(\mathbf{y},t) d\Gamma + \int_\Gamma \mathbf{G}(r,t) \star \frac{\partial \phi}{\partial n}(\mathbf{y},t) d\Gamma} \quad t \in [0,T], \mathbf{x} \in \Omega_f, \mathbf{y} \in \Gamma$$

CQM [2] : Z transform of q = Laplace transform $\overline{\mathbf{G}}$ (s) $\times$ Z transform $\Phi(\xi)$

$$q(t) = (\mathbf{G} \star \phi)(t) \quad \Rightarrow \quad Q(\xi) = \overline{\mathbf{G}}(s)\,\Phi(\xi) \qquad \xi \in \mathbb{C}$$

$$\boxed{\frac{1}{2}\Phi(x,\xi) - \int_\Gamma \frac{\partial \overline{G}}{\partial n}(r, \frac{p(\xi)}{\Delta t})\Phi(y,\xi) d\Gamma + \int_\Gamma \overline{G}(r, \frac{p(\xi)}{\Delta t}) \frac{\partial \Phi}{\partial n}(y,\xi) d\Gamma = 0}$$

1. D. Mavaleix-Marchessoux et al., “A fast boundary element method using the Z-transform and high-frequency approximations for large-scale 3D transient wave problems”, 2020
2. C. Lubich. “Convolution quadrature and discretized operational calculus. I.”, 1988
3. T. BETCKE, N. Salles, W. SMIGAJ. “Overresolving in the Laplace Domain for Convolution Quadrature Methods”, 2017

Z-BEM algorithm steps

Time-domain problem on the interval $[0, T]$

$\downarrow$

$t_0, t_1, \ldots t_{N_t}$

$\downarrow$

$$\xi_k = \rho e^{\frac{2\pi i k}{2N_t}} \quad k \in [0, 2N_t]$$

$\downarrow$

$$s_k = \frac{p(\xi_k)}{\Delta t} \quad k \in [0, 2N_t]$$

$\downarrow$

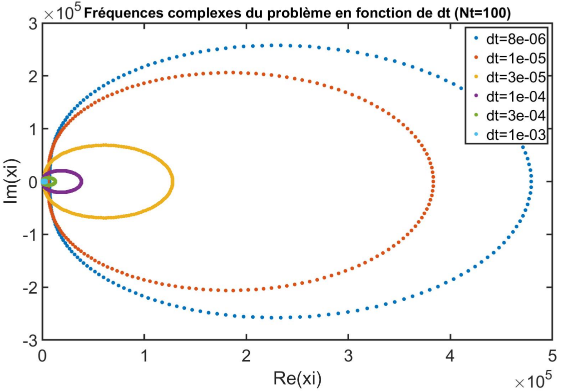
Resolution of (2 Nt-1) frequency BEM problems

$\downarrow$

Inverse Z-transform

$$\boxed{\psi(., t_n) = \simeq \frac{1}{L} \sum_{k=0}^{L-1} \Psi(., \xi_k) \xi_k^{-n} \quad \forall n \in [0, M]}$$

= Solution on the whole time interval



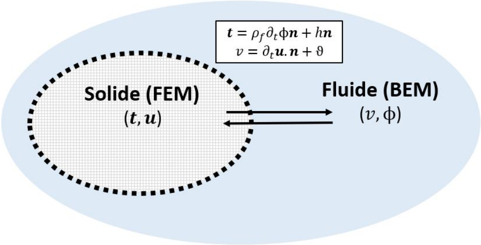
Goal : global-in-time FEM/Z-BEM iterative algorithm.

$$\phi^{\mathrm{tot}} = \phi^{\mathrm{inc}} + \phi^{\mathrm{sc}}$$

Fluid variables : $v = \nabla\phi.\mathbf{n} = \partial_n\phi, \quad p = -\rho_f\partial_t\phi$

Solid variables : $\mathbf{u}, \quad \mathbf{t} = \sigma[\mathbf{u}].\mathbf{n}$

Pressure and velocity jumps : $\nu := -\partial_n\phi^{\mathrm{inc}} \qquad h := -p^{\mathrm{inc}}$



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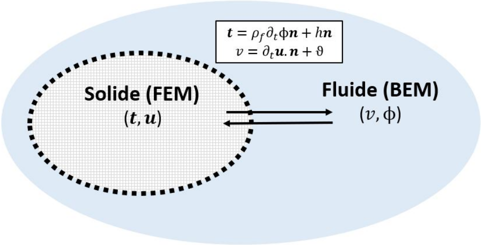

$$\begin{cases} \Delta\phi^{\text{sc}}(t,\mathbf{x}) - \frac{1}{c^2}\partial_{tt}\phi^{\text{sc}}(t,\mathbf{x}) = 0 & \text{on } \Omega_f \times [0,T] \\ -div\sigma[\mathbf{u}] + \rho_s\partial_{tt}\mathbf{u} = \mathbf{0} & \text{on } \Omega_s \times [0,T] \\ \\ \mathbf{t} = h\mathbf{n} + \rho_f\partial_t\phi^{\text{sc}}\mathbf{n} & \text{on } \Gamma \times [0,T] \\ \partial_n\phi^{\text{sc}} = \partial_t\mathbf{u}.\mathbf{n} + \nu & \text{on } \Gamma \times [0,T] \\ \\ \phi^{\text{sc}}(0,x) = \partial_t\phi^{\text{sc}}(0,x) = 0 & \text{on } \Omega_f \\ \mathbf{u}(0,x) = \partial_t\mathbf{u}(0,x) = 0 & \text{on } \Omega_s \end{cases}$$

Solvability mapping results for transient FSI problem

Standard energy identities

For a data $(h, \nu) \in H^1([0,T]; H^{-1/2}(\Gamma))$, the transmission problem admits a unique solution $\rightarrow (\phi, \mathbf{u}) \in C^0([0,T]; H^1(\Omega))$.

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Additional result (less standard)[1]

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Sobolev interpolation [1]

For a data $(h,\nu) \in H^1([0,T]; L^2(\Gamma))$, the transmission problem admits a unique solution $\to (\phi, \mathbf{u}) \in C^0([0,T]; H^{3/2}(\Omega))$
and the velocities have boundary traces (Rellich- Nečas identity) :

$\phi'|_\Gamma, \mathbf{u}'|_\Gamma \in L^2([0,T]; L^2(\Gamma))$ and we have $\partial_n \phi \in L^2([0,T]; L^2(\Gamma))$ and $\mathbf{t}[u] \in L^2([0,T]; L^2(\Gamma))$.

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1. M. BONNET, S. CHAILLAT, A. NASSOR, *in preparation.*
2. **In frequency domain :** H. BARUCQ, R. DJELLOULI, E. ESTECAHANDY. "On the existence and the uniqueness of the solution of a fluid-structure interaction scattering problem" 2014.
3. **In time domain :** G. BAO, GAO et LI. "Time-Domain Analysis of an Acoustic–Elastic Interaction Problem" 2018.

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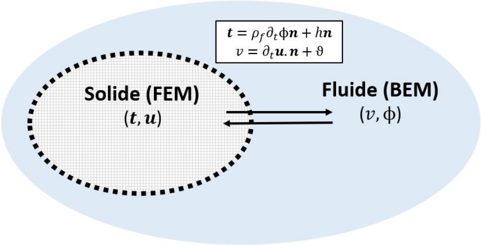

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"Naïve" iterative coupling strategy : two Neumann IBVPs

Global-in-time iteration : $\mathrm{i} \geq 1$, on $[0, T]$:

$$\boxed{\begin{cases} \Delta\phi^i(t) - \frac{1}{c_f^2}\partial_{tt}\phi^i(t) = 0 & \text{on } \Omega_f \\ \partial_n\phi^{\mathbf{i}}(t) = \partial_t\mathbf{u}^{\mathbf{i-1}}(t).\mathbf{n} + \nu & \text{on } \Gamma \\ \phi^i(0) = \partial_t\phi^i(0) = 0 \end{cases}}$$

Acoustic Neumann problem

$$\boxed{\begin{cases} -div(\sigma[\mathbf{u}^{\mathbf{i}}]) + \rho_s\partial_{tt}\mathbf{u}^{\mathbf{i}}(t) = 0 & \text{on } \Omega_s \\ \mathbf{t}[\mathbf{u}^{\mathbf{i}}](t) = \rho_f\partial_t\phi^{\mathbf{i-1}}\mathbf{n} - h\mathbf{n} & \text{on } \Gamma \\ \mathbf{u}^{\mathbf{i}}(0) = \partial_t\mathbf{u}^{\mathbf{i}}(0) = 0 \end{cases}}$$

Elastodynamic Neumann problem

Global-in-time iteration : i ≥ 1, on $[0,T]$:

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A Neumann data in $H^1([0,T]; L^2(\Gamma))$ gives a velocity solution $\partial_t\phi \in L^2([0,T]; L^2(\Gamma))$ and the definition of a trace in velocity is not possible.

Neumann-Neumann iterates can not remain in a fixed space.

D. Lasiecka et Triggiani, "Sharp regularity theory for second order hyperbolic equations of Neumann type - Part I. -L2 nonhomogeneous data", 1990

Iterative coupling strategy : two Robin IBVPs

Global-in-time iteration : $\mathrm{i} \geq 1$, on $[0, T]$:

$$\boxed{\begin{cases} -\Delta\phi^i(t) + \frac{1}{c_f^2}\partial_{tt}\phi^i(t) = 0 & \text{on } \Omega_f \\ \mathrm{k}\partial_n\phi^i(t) - \rho_f\partial_t\phi^i(t) = f^{i-1}(t) & \text{on } \Gamma \\ \phi^i(0) = \partial_t\phi^i(0) = 0 \end{cases}}$$

Acoustic Robin problem

$$\boxed{\begin{cases} -div(\sigma[\mathbf{u^i}]) + \rho_s\partial_{tt}\mathbf{u^i}(t) = 0 & \text{sur } \Omega_s \\ \mathbf{t}[\mathbf{u^i}](t) + \mathrm{k}\partial_t\mathbf{u^i} = g^{i-1}(t) & \text{sur } \Gamma \\ \mathbf{u^i}(0) = \partial_t\mathbf{u^i}(0) = 0 \end{cases}}$$

Elastodynamic Robin problem

with $\mathrm{k} > 0$

$f^{i-1}(t) = F(\partial_n\phi^{i-1}, \partial_t\phi^{i-1}, \mathbf{t}[\mathbf{u}]^{i-1}, \partial_t\mathbf{u}^{i-1})$ and $g^{i-1}(t) = G(\partial_n\phi^{i-1}, \partial_t\phi^{i-1}, \mathbf{t}[\mathbf{u}]^{i-1}, \partial_t\mathbf{u}^{i-1})$

Robin IBVPs

$$\left\{\begin{array}{l} \Delta\phi(t) + \frac{1}{c_f^2}\partial_{tt}\phi(t) = 0 \quad \text{on } \Omega_f \\ k\partial_n\phi^i(t) - \rho_f\partial_t\phi^i(t) = f^{i-1}(t) \quad \text{on } \Gamma \\ \phi(0) = \partial_t\phi(0) = 0 \end{array}\right.$$

or

$$\left\{\begin{array}{l} -div(\sigma[\mathbf{u}]) + \rho_s\partial_{tt}\mathbf{u}(t) = 0 \quad \text{on } \Omega_s \\ \mathbf{t}[\mathbf{u}^{\mathbf{i}}](t) + k\partial_t\mathbf{u}^{\mathbf{i}} = g^{i-1}(t) \quad \text{on } \Gamma \\ \mathbf{u}(0) = \partial_t\mathbf{u}(0) = 0 \end{array}\right.$$

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For example, variational formulation for the acoustic Robin IBVP :

$$\int_\Omega \nabla\phi\nabla\widetilde{\phi}d\Omega + \frac{1}{c_f^2}\int_\Omega \phi''\widetilde{\phi}d\Omega + \frac{\rho_f}{k}\int_\Gamma \phi'\widetilde{\phi}d\Gamma = \left(\frac{1}{k}f, \widetilde{\phi}\right)_\Gamma$$

$$\begin{cases} a\left(\phi(t), \widetilde{\phi}\right) + b\left(\phi''(t), \widetilde{\phi}\right) + c\left(\phi'(t), \widetilde{\phi}\right) = \frac{1}{k}\left(f(t), \widetilde{\phi}\right)_\Gamma \\ \forall\widetilde{\phi} \in H^1(\Omega), \quad t \in [0, T] \\ \phi(0) = \partial_t\phi(0) = 0 \end{cases}$$

and similarly for the elastodynamic Robin IBVP.

Robin IBVPs

Find $\phi(t) \in H^1(\Omega)$ such as

$$\begin{cases} a\left(\phi(t), \widetilde{\phi}\right) + b\left(\phi''(t), \widetilde{\phi}\right) + c\left(\phi'(t), \widetilde{\phi}\right) = \frac{1}{k}\left(f(t), \widetilde{\phi}\right)_\Gamma \\ \forall \widetilde{\phi} \in H^1(\Omega), \quad t \in [0,T] \\ \phi(0) = \partial_t \phi(0) = 0 \end{cases}$$

Energy identity : $\quad \|\phi\|^2_{1,\Omega,T} + \|\partial_t \phi\|^2_{0,\Omega,T} + \|\partial_t \phi\|^2_{0,\Gamma,T} \leq C \, \|f\|^2_{0,\Gamma,T}$

Solvability result for the Robin evolution problem :

$f \in L^2([0,T]; L^2(\Gamma))$ gives a velocity trace solution $(\partial_t \phi \,|_\Gamma, \mathbf{t}[u]) \in L^2([0,T]; L^2(\Gamma))$

Velocity trace solution and Robin boundary data have same regularity.
$\rightarrow$ Preservation of $L^2_T(\Gamma)$ regularity for all Robin iterates.

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Yes !

Convergence proof of Robin-Robin iterations using energy estimates

1. The error fields ($\phi^i_{err} = \phi^i - \phi$) at iteration i in Ω_f verify

$$a(\phi^i(\tau), \partial_t \phi^i(\tau)) + b(\partial_{tt}\phi^i(\tau), \partial_t \phi^i(\tau)) = -(v^i(\tau), \partial_t \phi^i(\tau))_\Gamma \quad \tau \in [0, T] \qquad \text{(a.)}$$
$$A(\mathbf{u}^i, \partial_t \mathbf{u}^i(\tau)) + B(\partial_{tt}\mathbf{u}^i(\tau), \partial_t \mathbf{u}^i(\tau))_\Gamma = (\mathbf{t}^i(\tau), \partial_t \mathbf{u}^i(\tau))_\Gamma \quad \tau \in [0, T] \qquad \text{(b.)}$$

2. Integrating on $\tau \in$ [0,t] and adding a). and b). we obtain the energy identity

$$\boxed{\mathbf{E^i(t) = -(v^i, \partial_t \phi^i)_{\Gamma,t} + (t^i, \partial_t u^i)_{\Gamma,t}}}$$

3. We introduce the incoming and outcoming traces [4] :

$$\mathbb{B}^i = \begin{Bmatrix} f^i \\ g^i \end{Bmatrix} = \begin{Bmatrix} -\rho_f \partial_t \phi^i + k v^i \\ \mathbf{t}^i + k\mathbf{u}^i \end{Bmatrix} \quad \text{et} \quad \overline{\mathbb{B}}^i = \begin{Bmatrix} \rho_f \partial_t \phi^i + k v^i \\ \mathbf{t}^i - k\mathbf{u}^i \end{Bmatrix}$$

4. The traces verify : $-(v^i, \partial_t \phi^i)_{\Gamma,t} + (\mathbf{t}^i, \partial_t \mathbf{u}^i)_{\Gamma,t} = \frac{1}{4} \left\| \mathbb{B}^i \right\|^2_{\Gamma,t,k} - \frac{1}{4} \left\| \overline{\mathbb{B}}^i \right\|^2_{\Gamma,t,k}$

$$\rightarrow \quad \boxed{\mathbf{E^i(t) = \frac{1}{4} \left\| \mathbb{B}^i \right\|^2_{\Gamma,t,k} - \frac{1}{4} \left\| \overline{\mathbb{B}}^i \right\|^2_{\Gamma,t,k}}}$$

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$$\boxed{\mathbf{E^i(t) = -(v^i, \partial_t \phi^i)_{\Gamma,t} + (t^i, \partial_t u^i)_{\Gamma,t}}}$$

3. We introduce the incoming and outcoming traces [4] :

$$\mathbb{B}^i = \begin{Bmatrix} f^i \\ g^i \end{Bmatrix} = \begin{Bmatrix} -\rho_f \partial_t \phi^i + k v^i \\ \mathbf{t}^i + k\mathbf{u}^i \end{Bmatrix} \quad \text{et} \quad \overline{\mathbb{B}}^i = \begin{Bmatrix} \rho_f \partial_t \phi^i + k v^i \\ \mathbf{t}^i - k\mathbf{u}^i \end{Bmatrix}$$

4. The traces verify : $-(v^i, \partial_t \phi^i)_{\Gamma,t} + (\mathbf{t}^i, \partial_t \mathbf{u}^i)_{\Gamma,t} = \frac{1}{4}\left\|\mathbb{B}^i\right\|^2_{\Gamma,t,k} - \frac{1}{4}\left\|\overline{\mathbb{B}}^i\right\|^2_{\Gamma,t,k}$

$$\rightarrow \quad \boxed{\mathbf{E^i(t) = \frac{1}{4}\left\|\mathbb{B}^i\right\|^2_{\Gamma,t,k} - \frac{1}{4}\left\|\overline{\mathbb{B}}^i\right\|^2_{\Gamma,t,k}}}$$

4. COLLINO, JOLY et LECOUVEZ, "Exponentially convergent non overlapping domain decomposition methods for the Helmholtz equation", *ESAIM : Mathematical Modelling and Numerical Analysis*, 2020

5. Relation between iterations i and i+1 : $\mathbb{B}_{i+1} = \mathbb{B}_i + \mathbb{X}\overline{\mathbb{B}}_{i+1} + \mathbb{L}\mathbb{T}$ with $\mathbb{X}$: isometry for $\|\cdot\|_{\Gamma,t,k}$

6. We sum the $E^i(t) \geq 0$ for N successive RR iterations

$$\sum_{i=0}^{N-1} \underbrace{E^i(t)}_{\geq 0} + \underbrace{\|\mathbb{B}_N\|^2_{\Gamma,t,k}}_{\text{independent of i}} = \underbrace{\|\mathbb{B}_0\|^2_{\Gamma,t,k}}_{\text{independent of i}} \quad \Rightarrow \lim_{i\to\infty} E^i(t) = 0 \qquad \forall t \in [0,T]$$

The error field tend toward 0 when $i \to \infty$

$$\lim_{i\to\infty} \left\|\phi^i(t) - \phi(t)\right\|^2_{H^1(\Omega)} = 0 \quad \text{and} \quad \lim_{i\to\infty} \left\|\partial_t\phi^i(t) - \partial_t\phi(t)\right\|^2_{L^2(\Omega)} = 0 \qquad \forall t \in [0,T]$$

7. The **solvability result for the transient fluid-structure interaction problem** allows to conclude :

For a given data $(h,\nu) \in H^1([0,T]; L^2(\Gamma))$, the Robin iterations converge to the coupled problem's solution in $L^2([0,T]; L^2(\Gamma))$.

1. M. BONNET, S. CHAILLAT, A. NASSOR, *in preparation.*

Non uniform convergence property

Relaxed Robin iterations : $\mathbb{B}_{i+1} = (1-r)\mathbb{B}_i + r\mathbb{X}\overline{\mathbb{B}}_{i+1} + r\mathbb{L}\mathbb{T}$ with $r \in [0,1[$

In frequency domain : relaxed Robin iterations have a uniform convergence [4] :

$$\|\mathbb{B}_{p+1} - \mathbb{B}^*\| \leq q\,\|\mathbb{B}_p - \mathbb{B}^*\| \quad \text{with } 0 < q < 1$$

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In time domain : For a given Robin transmission data $\mathbb{T} \in L^2_T(\Gamma)$, **relaxed iterations geometrically convergent with data-dependent rate q($\mathbb{T}$) <1.**

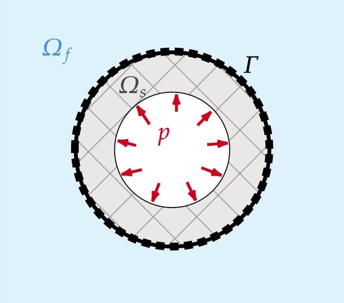
$$\|\mathbb{B}_{p+1} - \mathbb{B}^*\|_{\Gamma,T,k} \leq q(\mathbb{T})\,\|\mathbb{B}_p - \mathbb{B}^*\|_{\Gamma,T,k}$$

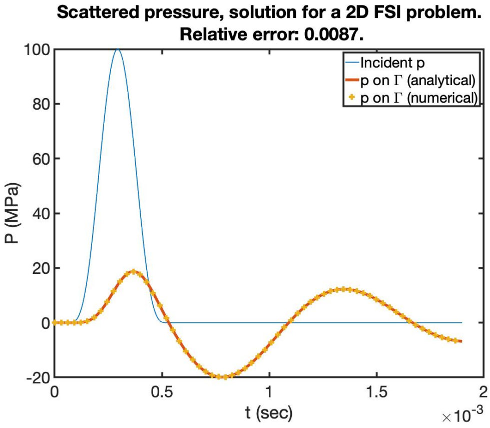
q($\mathbb{T}$) depends on $\mathbb{T}$ through $\boxed{\eta(\mathbb{T}) := \dfrac{\|\mathbb{T}\|_{H^1_T(\Gamma)}}{\|\mathbb{T}\|_{L^2_T(\Gamma)}}}$

- geometrical convergence faster for low η ("less high-frequency content").
- $1 - q(\mathbb{T})$ may be arbitrarily small.

4. COLLINO, JOLY et LECOUVEZ, "Exponentially convergent non overlapping domain decomposition methods for the Helmholtz equation", *ESAIM : Mathematical Modelling and Numerical Analysis*, 2020

Numerical validation : radially-symmetric coupled problem



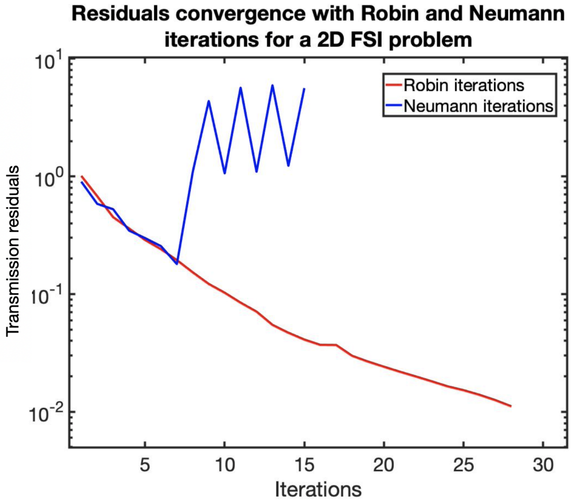


Neumann conditions :

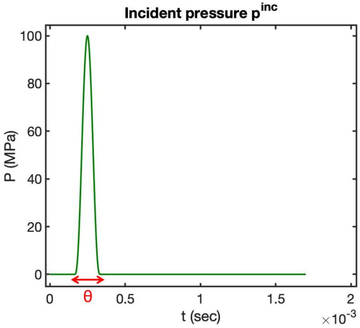
$$\begin{cases} \partial_n \phi^i(t) = & \partial_t \mathbf{u}^{i-1}(t).\mathbf{n} + \nu \\ \mathbf{t}[\mathbf{u}]^i(t) = & \rho_f \partial_t \phi^{i-1} \mathbf{n} - h\mathbf{n} \end{cases}$$

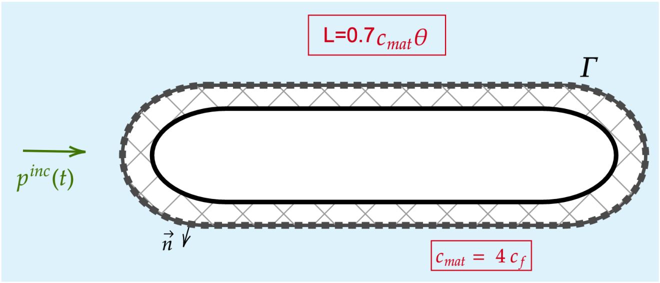
Robin conditions :

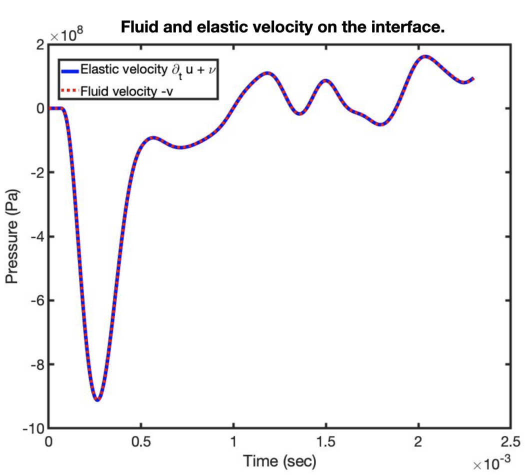
$$\begin{cases} k\partial_n \phi^i(t) - \rho_f \partial_t \phi^i(t) = & f^{i-1}(t) \\ \mathbf{t}[\mathbf{u}]^i(t) + k\partial_t \mathbf{u}^i = & g^{i-1}(t) \end{cases}$$

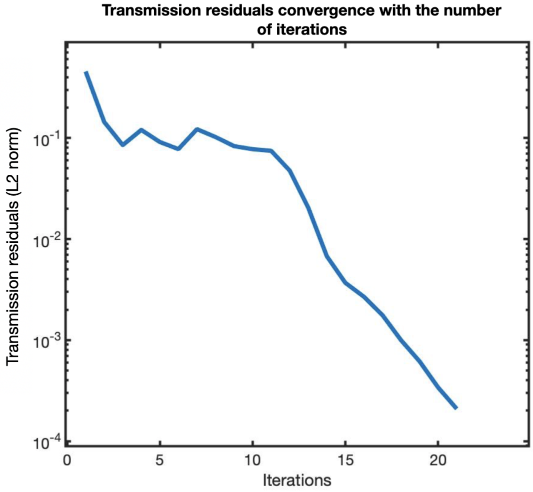


Numerical validation : 2D coupled problem

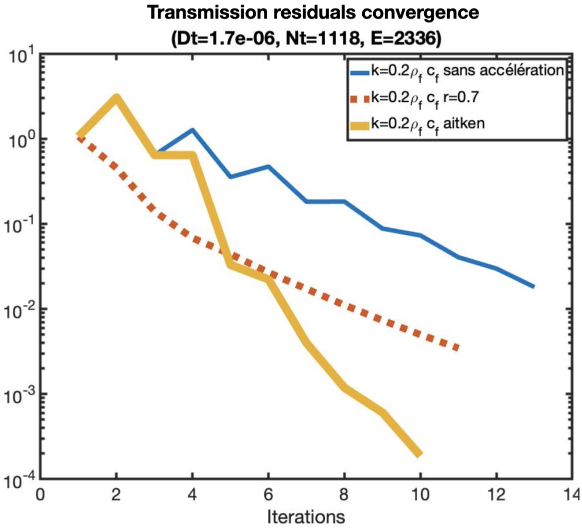








$$\boxed{v_n = \partial_t \mathbf{u}.\mathbf{n} + \nu}$$



Convergence rate and Aitken acceleration

$$\forall i \geq 0$$

$$\mathrm{B}^i = \begin{Bmatrix} f^i(t) \\ g^i(t) \end{Bmatrix} \quad \text{and} \quad \Delta^i = B^i - B^{i+1}$$

$$\mathrm{t}^* = \frac{(\Delta^i, \Delta^i - \Delta^{i-1})}{\left\|\Delta^i - \Delta^{i-1}\right\|_2^2}$$

$$B^{i+1} = B^i + t^* \left(B^{i-1} - B^i\right)$$

$$\Updownarrow$$

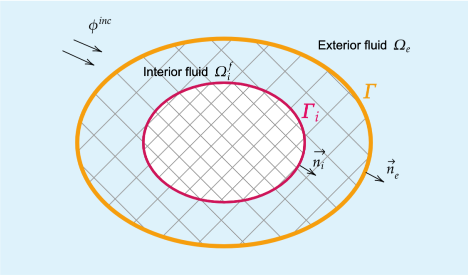
$$\begin{cases} k\partial_n \phi^{i+1}(t) - \rho_f \partial_t \phi^{i+1}(t) = f^i(t) + t^* \left(f^{i-1}(t) - f^i(t)\right) \\ \mathbf{t}[\mathbf{u}^{\mathbf{i+1}}](t) + k\partial_t \mathbf{u}^{\mathbf{i+1}} = g^i(t) + t^* \left(g^{i-1}(t) - g^i(t)\right) \end{cases}$$

- A transient **Robin** problem admits a velocity solution on the boundary with the **same regularity** as the data.
  $\rightarrow$ We define an iterative procedure with **guaranteed** convergence.

- The 2D numerical results corroborate the theoretical conclusions on convergence.

Problem : in an industrial context, inhomogeneous Robin conditions may not be available for the solid domain (Abaqus for example).

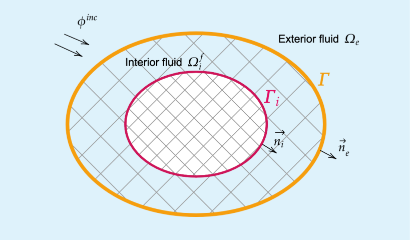
Acoustic-acoustic coupling with Robin boundary conditions



Advantages :

- No Robin condition imposed in the solid part.
- Guaranteed convergence.
- Non-linearities in the FEM-fluid part.

$$
\begin{aligned}
\text{Interior fluid variables :}\quad & v_i = \nabla\phi_i.\mathbf{n}, \quad p_i = -\rho_f\partial_t\phi_i \\
\text{Exterior fluid variables :}\quad & v_e = \nabla\phi_e.\mathbf{n}, \quad p_e = -\rho_f\partial_t\phi_e \\
\text{Solid variables :}\quad & \mathbf{u}, \quad \mathbf{t} = \sigma[\mathbf{u}].\mathbf{n} \\
\text{Pressure and velocity jumps on } \Gamma_e \text{ :}\quad & s_p := \rho_f\partial_t\phi_e^{\text{inc}} \quad \text{and} \quad s_v := \partial_n\phi_e^{\text{inc}}
\end{aligned}
$$



$$
\begin{cases}
\Delta\phi(t,\mathbf{x}) - \frac{1}{c_f^2}\partial_{tt}\phi(t,\mathbf{x}) = 0 \quad \text{on } \Omega_e \bigcup \Omega_i^f \\
-div\sigma[\mathbf{u}] + \rho_s\partial_{tt}\mathbf{u} = \mathbf{0} \quad \text{on } \Omega_i^s \\
\\
\mathbf{t} = \rho_f\partial_t\phi_i\mathbf{n} \quad \text{on } \Gamma_i \\
\partial_n\phi^{\mathrm{i}} = \partial_t\mathbf{u}.\mathbf{n} \quad \text{on } \Gamma_i \\
\rho_f\partial_t\phi_i = \rho_f\partial_t\phi_e + s_p \quad \text{on } \Gamma_e \\
\partial_n\phi_i = s_v + \partial_n\phi_e \quad \text{on } \Gamma_e \\
\\
+ \text{ initial conditions}
\end{cases}
$$

Global-in-time iteration : $\text{i} \geq 1$, on the whole time interval $[0, T]$:

$$\boxed{\begin{cases} \Delta\phi_e - \frac{1}{c_f^2}\partial_{tt}\phi_e = 0 & \text{on } \Omega_e \times [0,T] \\ -\rho_f \partial_t \phi_e^{\text{i}} + k_e v_e^{\text{i}} = g_e^{\text{i-1}} & \text{on } \Gamma_e \times [0,T] \\ \quad + \text{C.I.} & \end{cases}}$$

Acoustic Robin problem

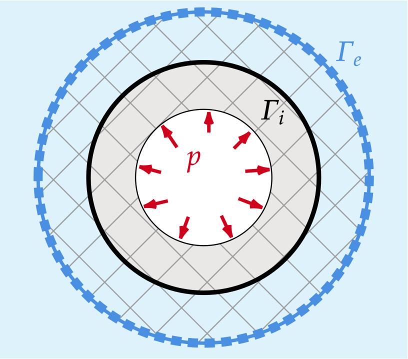
$$\boxed{\begin{cases} \Delta\phi_i - \frac{1}{c_f^2}\partial_{tt}\phi_i = 0 & \text{on } \Omega_i^f \times [0,T] \\ -div\sigma[\mathbf{u}] + \rho_s \partial_{tt}\mathbf{u} = \mathbf{0} & \text{on } \Omega_i^s \times [0,T] \\ v_i = \partial_t \mathbf{u}.\mathbf{n} \quad \text{and } \mathbf{t} = \rho_f \partial_t \phi_i \mathbf{n} & \text{on } \Gamma_i \times [0,T] \\ -\rho_f \partial_t \phi_i^{\text{i}} + k_i v_i^{\text{i}} = g_i^{\text{i-1}} & \text{on } \Gamma_e \times [0,T] \\ \quad + \text{C.I.} & \end{cases}}$$

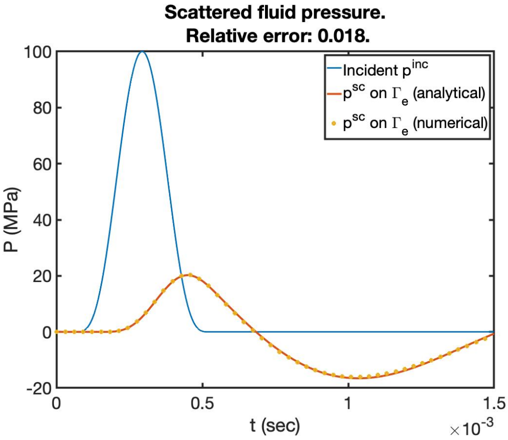
Acoustic-elastodynamic Robin

with $k_e, k_i > 0$

and g_e, g_i two Robin data depending on the previous iteration.

Acoustic-acoustic coupling with Robin boundary conditions





Conclusion and outlook

Conclusion :

1. Solvability mapping results for the transient fluid-structure interaction problem.
2. Robin global-in-time coupling algorithms with guaranteed convergence :
   - coupling at fluid-structure interface
   - coupling at fluid-fluid interface.
3. Preliminary validations with numerical 2D examples.

Prospects :

- 3D numerical implementation : Code Aster (FEM) and Coffee (CQM-BEM).
- Taking free surface or sea bed into account (method of images).
- Non linear fluid phenomena near ship hull (cavitation).

Thank you for you attention !

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Poems (CNRS, ENSTA Paris, INRIA) - Institut Polytechnique de Paris
Waves 2022 - 29/07/2022

Méthode des éléments de frontière (BEM) : Représentation intégrale

Fonction de Green acoustique 2D

$$\Delta G(\mathbf{x},\mathbf{y}) + k^2 G(\mathbf{x},\mathbf{y}) = -\delta(\mathbf{y}), \qquad \left(\text{acoustique 2D : } G(\mathbf{x},\mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{y}-\mathbf{x}|)\right)$$

La formule de Green ne peut pas s'appliquer car $\delta_y \notin L^2$.

Théorème de représentation pour le problème extérieur

$$\int_\Gamma \left(-G(\mathbf{x},\mathbf{y})\left[\partial_n \psi\right]^s(\mathbf{y}) + \partial_n G(\mathbf{x},\mathbf{y})\left[\psi(\mathbf{y})\right]^s\right) d\Gamma = \begin{cases} 0, & \mathbf{x} \in \Omega_s \\ \psi(\mathbf{x}), & \mathbf{x} \in \Omega_f \end{cases}$$

Opérateurs intégraux de simple et double couche :

$$Sp(\mathbf{x}) := \int_\Gamma G(\mathbf{x}-\mathbf{y}) p(\mathbf{y}) d\Gamma_y, \quad Dq(\mathbf{x}) := \int_\Gamma \frac{\partial G}{\partial \mathbf{n}(\mathbf{y})}(\mathbf{x}-\mathbf{y}) q(\mathbf{y}) d\Gamma_y, \quad \mathbf{x} \notin \Gamma \qquad \text{(G connue)}$$

Représentation intégrale (= produits matrice-vecteur)

$$\boxed{\psi(\mathbf{x}) = D\left[\psi\right]_\Gamma(\mathbf{x}) - S\left[\partial_n \psi\right]_\Gamma(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_f} \qquad [f]_\Gamma = f^s - f^f, \text{ la différence des traces}$$

Il faut résoudre une équation intégrale pour déterminer $[\psi]_\Gamma$ et $[\partial_n \psi]_\Gamma$

Z-BEM [1] en temps avec CQM

Équation intégrale de frontière dans le domaine temporel (méthode des potentiels retardés) :

$$\boxed{\frac{1}{2}\phi(\mathbf{x},t) = -\int_\Gamma \frac{\partial \mathbf{G}}{\partial n}(r,t)\star\phi(\mathbf{y},t)d\Gamma + \int_\Gamma \mathbf{G}(r,t)\star\frac{\partial\phi}{\partial n}(\mathbf{y},t)d\Gamma} \quad t\in[0,T], \mathbf{x}\in\Omega_f, \mathbf{y}\in\Gamma$$

Convolution quadrature method (CQM) [2]

Transformée Z d'un produit de convolution = transformée de Laplace $\overline{\mathbf{G}}$ $\times$ transformée en Z de ϕ

$$q(t) = (\mathbf{G}\star f)(t) = \int_0^t \mathbf{G}(t-\tau)f(\tau)d\tau \qquad \forall t\geq 0 \text{ avec } f = \frac{\partial\phi}{\partial n}\text{connu}$$

$$Q(\xi) = \overline{\mathbf{G}}\,(s)\,F(\xi) \qquad\qquad \xi\in\mathbb{C}, \quad s = \frac{p(\xi)}{\Delta t}$$

$$\boxed{\frac{1}{2}\Phi(x,\xi) - \int_\Gamma \frac{\partial\overline{G}}{\partial n}(r,\frac{p(\xi)}{\Delta t})\Phi(y,\xi)d\Gamma + \int_\Gamma \overline{G}(r,\frac{p(\xi)}{\Delta t})\frac{\partial\Phi}{\partial n}(y,\xi)d\Gamma = 0}$$

1. D. Mavaleix-Marchessoux, “Modelling the fluid-structure coupling caused by a far-field underwater explosion”. PhD thesis, Institut Polytechnique de Paris, 2020
2. C. Lubich. “Convolution quadrature and discretized operational calculus. I.” Numerische Mathematik 52(1988) : 129-145

Preuve de convergence utilisant des estimations d'énergie

1.Les champs d'erreur ($\phi_{err}^q = \phi_e^q - \phi_e$) à l'itération q dans Ω_e et Ω_i vérifient, $\forall t \in [0, T]$

$$a(\phi_e^q, \partial_t \phi_e^q) + b(\partial_{tt}\phi_e^q, \partial_t \phi_e^q) = -\rho_f (v_e^q, \partial_t \phi_e^q)_{\Gamma_e}$$

$$A(\mathbf{u}^q, \partial_t \mathbf{u}^q) + B(\partial_{tt}\mathbf{u}^q, \partial_t \mathbf{u}^q)_\Gamma + a(\phi_i^q, \partial_t \phi_i^q) + b(\partial_{tt}\phi_i^q, \partial_t \phi_i^q) = -\rho_f (v_i^q, \partial_t \phi_i^q)_{\Gamma_i} + \rho_f (v_i^q, \partial_t \phi_i^q)_{\Gamma_e} + (\mathbf{t}^q, \partial_t \mathbf{u}^q)_{\Gamma_i} \quad ($$

$$(\mathbf{t}^q, \partial_t \mathbf{u}^q)_{\Gamma_i} = (\rho_f \partial_t \phi_i^q(t), \mathbf{n}.\partial_t^q \mathbf{u}(t))_{\Gamma_i} = \rho_f (\partial_t \phi_i^q(t), v_i^q(t))_{\Gamma_i}$$

2. En intégrant sur t dans [0,s] et sommant les deux lignes on obtient l'identité d'énergie

$$\rightarrow \qquad \mathbf{E^q(s)} = -\rho_f (v_e^q, \partial_t \phi_e^q)_{\Gamma_e, s} - \cancel{\rho_f (v_i^q, \partial_t \phi_i^q)_{\Gamma_i, s}} + \rho_f (v_i^q, \partial_t \phi_i^q)_{\Gamma_e, s} + \cancel{\rho_f (\partial_t \phi_i^q, v_i^q)_{\Gamma_i}}$$

3. On introduit les traces entrantes et sortantes :

$$\mathbb{B}^q = \begin{Bmatrix} -\rho_f \partial_t \phi_e^q + k_e v_e^q \\ \rho_f \partial_t \phi_i^q + k_i v_i^q \end{Bmatrix} \quad \text{et} \quad \overline{\mathbb{B}}^q = \begin{Bmatrix} \rho_f \partial_t \phi_e^q + k_e v_e^q \\ -\rho_f \partial_t \phi_i^q + k_i v_i^q \end{Bmatrix}$$

4. On montre que :

$$\boxed{E^q(s) = -\rho_f (v_e^q, \partial_t \phi_e^q)_{\Gamma_e, s} + \rho_f (v_i^q, \partial_t \phi_i^q)_{\Gamma_e, s} = \frac{1}{4} \left\| \mathbb{B}^q \right\|_{\Gamma_e, s, K}^2 - \frac{1}{4} \left\| \overline{\mathbb{B}}^q \right\|_{\Gamma_e, s, K}^2}$$

La somme des $E^q(s) \geq 0$ sur les itérations q, s'écrit

$$\sum_{q=0}^{N-1} \underbrace{E^q(s)}_{\geq 0} + \underbrace{\|\mathbb{B}_N\|^2_{\Gamma,s,K}}_{\text{indépendant de q}} = \underbrace{\|\mathbb{B}_0\|^2_{\Gamma_e,s,K}}_{\text{indépendant de q}} \quad \Rightarrow \lim_{q\to\infty} E^q(s) = 0 \qquad \forall s \in [0,T]$$

5. Les champs d'erreur tendent vers 0 quand $q \to \infty$

$$\lim_{q\to\infty} \|\phi_e^q - \phi\|^2_{H^1(\Omega)} = 0 \quad \text{et} \quad \lim_{q\to\infty} \|\partial_t\phi_e^q - \partial_t\phi\|^2_{L^2(\Omega)} = 0$$

6. Le **résultat de solvabilité du problème d'interaction fluide-structure transitoire** permet de conclure que :

Avec une donnée dans $H^1_T(L^2(\Gamma))$, les itérations convergent vers la solution du problème couplé.

Solution semi-analytique

Problème IFS axisymétrique :

$$\boxed{\begin{cases} \frac{1}{r}\partial_r\left(r\partial_r u\right) - \frac{u}{r^2} - \frac{1}{c^2}\partial_{tt}u = 0 \quad \text{(élastodynamique)} \\ \frac{1}{r}\partial_r\left(r\partial_r \phi\right) - \frac{1}{c^2}\partial_{tt}\phi = 0 \quad \text{(équation des ondes)} \end{cases}}$$

Passage dans le domaine de Laplace

$$\boxed{\begin{cases} \frac{1}{r}\partial_r\left(r\partial_r \overline{u}\right) - \frac{\overline{u}}{r^2} - \frac{1}{c^2}s^2\overline{u} = 0 \quad \text{(élastodynamique)} \\ \frac{1}{r}\partial_r\left(r\partial_r \overline{\phi}\right) - \frac{1}{c^2}s^2\overline{\phi} = 0 \quad \text{(équation des ondes)} \end{cases}}$$

= équations de Bessel modifiées

Solutions analytiques fréquentielles

$$\begin{cases} \overline{\mathbf{u}}(r,s) = A(s)I_1(\frac{sr}{c}) + B(s)K_1(\frac{sr}{c}) \\ \overline{\mathbf{\Phi}}(r,s) = C(s)K_0(\frac{sr}{c}) \end{cases}$$

avec (A,B,C) proportionnels à $\overline{\mathbf{P}}(s)$ donc

$$\begin{cases} \overline{\mathbf{u}}(r,s) = \overline{\mathbf{U_1}}(r,s)\overline{\mathbf{P}}(s) \\ \overline{\mathbf{\Phi}}(r,s) = \overline{\Phi_1}(r,s)\overline{\mathbf{P}}(s) \end{cases}$$

Retour temporel par transformée en Z inverse

$$\Rightarrow \begin{cases} u(r,t_n) = \frac{1}{L}\sum_{k=0}^{L-1}\overline{\mathbf{u}}(r,\xi_k)\xi_k^{-n} \\ \phi(r,t_n) = \frac{1}{L}\sum_{k=0}^{L-1}\overline{\mathbf{\Phi}}(r,\xi_k)\xi_k^{-n} \end{cases} \qquad n \in [0, N$$

Propriété de convergence géométrique.

Pour toute donnée de Robin initiale $\mathbb{T} \in L^2_T(\Gamma)$, les itérations RR convergent géométriquement avec un ratio de convergence q($\mathbb{T}$) <1.

Soient $\mathbb{T} = \begin{Bmatrix} h \\ \nu \end{Bmatrix} \in H^1_T(L^2(\Gamma))$ le vecteur des données de transmission et $\mathbb{X}$ et $\mathbb{L}$ tels que :

$$\mathbb{B}_p = \begin{Bmatrix} -\rho_f \partial_t \phi_p + k v_p \\ \mathbf{t}_p + k\mathbf{u}_p \end{Bmatrix} \quad \text{et} \quad \overline{\mathbb{B}}_p = \begin{Bmatrix} \rho_f \partial_t \phi_p + k v_p \\ \mathbf{t}_p - k\mathbf{u}_p \end{Bmatrix}, \quad \mathbb{X}\mathbb{X} = \mathbb{I} \quad \text{et} \quad \|\mathbb{X}\mathbb{B}\|^2 = \|\mathbb{B}\|^2$$

et $\mathbb{S}$ est un "scattering operator" tel que : $\mathbb{S}\mathbb{B} = \overline{\mathbb{B}}$.

Itérations RR $(0 < r \leq 1)$: $\mathbb{B}_{p+1} = A_r \mathbb{B}_p + r\mathbb{L}\mathbb{T} \quad \forall p \in \mathbb{N}$ avec $A_r := (1-r)\mathbb{I} + r\mathbb{X}\mathbb{S}$

RR converge dans $L^2_T(\Gamma)$ et à convergence les conditions de transmissions sont vérifiées :

$$\mathbb{B}^* = A_r \mathbb{B}^* + \mathbb{L}\mathbb{T} \Leftrightarrow (\mathbb{I} - \mathbb{X}\mathbb{S})\mathbb{B}^* = \mathbb{L}\mathbb{T} \qquad (r = 1)$$

On veut montrer que :

$$\boxed{\exists \quad q = q(\mathbb{T}) \text{ tel que } \forall p \in \mathbb{N}, \quad \|\mathbb{A}_r(\mathbb{B}_p - \mathbb{B}^*)\|_{\Gamma,T,\mathbb{K}} \leq q\, \|\mathbb{B}_p - \mathbb{B}^*\|_{\Gamma,T,\mathbb{K}}}$$

Propriété de convergence géométrique dans le domaine fréquentiel

<u>En fréquentiel :</u> $\mathbb{A}_r$ est strictement contractant et les itérations sont géométriques :

$$\|\mathbb{B}_{p+1} - \mathbb{B}^*\| \leq q \, \|\mathbb{B}_p - \mathbb{B}^*\| \qquad \text{avec } 0 < q < 1$$

<u>En temporel :</u> $\mathbb{A}_r$ est contractant et les itérations ne sont géométriques :

$$\|\mathbb{B}_{p+1} - \mathbb{B}^*\|_{\Gamma,T,\mathbb{K}} \leq q(\mathbb{T}) \, \|\mathbb{B}_p - \mathbb{B}^*\|_{\Gamma,T,\mathbb{K}}$$

Le ratio de convergence géométrique q dépend de la donnée $\mathbb{T}$ selon $\boxed{\eta := \dfrac{\|\mathbb{L}\mathbb{T}\|_{H^1_T(\Gamma)}}{\|\mathbb{L}\mathbb{T}\|_{L^2_T(\Gamma)}}}$

et la convergence géométrique est plus rapide pour les faibles valeurs de η (à fréquences plus faibles).