Chapter 4

- 4.1 Normal approximation (Laplace's method)
- 4.2 Large-sample theory
- 4.3 Counter examples
  - includes examples of difficult posteriors for MCMC, too
- 4.4 Frequency evaluation*
- 4.5 Other statistical methods*

Normal approximation (Laplace approximation)

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  - bounded, non-singular, the number of parameters don't grow with n
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  - Laplace used this (before Gauss) to approximate the posterior of binomial model to infer ratio of girls and boys born

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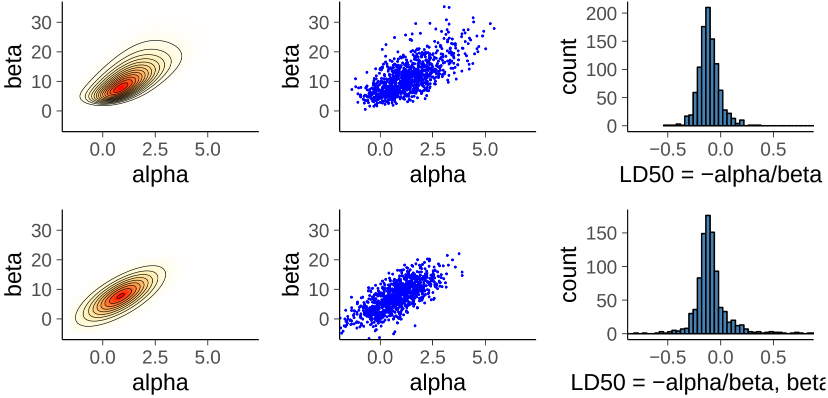

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  - often when $n \to \infty$, $\frac{f^{(3)}(\hat{\theta})}{3!}(\theta - \hat{\theta})^3 + \ldots$ is small

Multivariate Taylor series

- Multivariate series expansion

$$f(\theta) = f(\hat{\theta}) + \frac{df(\theta')}{d\theta'}\bigg|_{\theta'=\hat{\theta}} (\theta - \hat{\theta}) + \frac{1}{2!}(\theta - \hat{\theta})^T \frac{d^2 f(\theta')}{d\theta'^2}\bigg|_{\theta'=\hat{\theta}} (\theta - \hat{\theta}) + \ldots$$

Normal approximation

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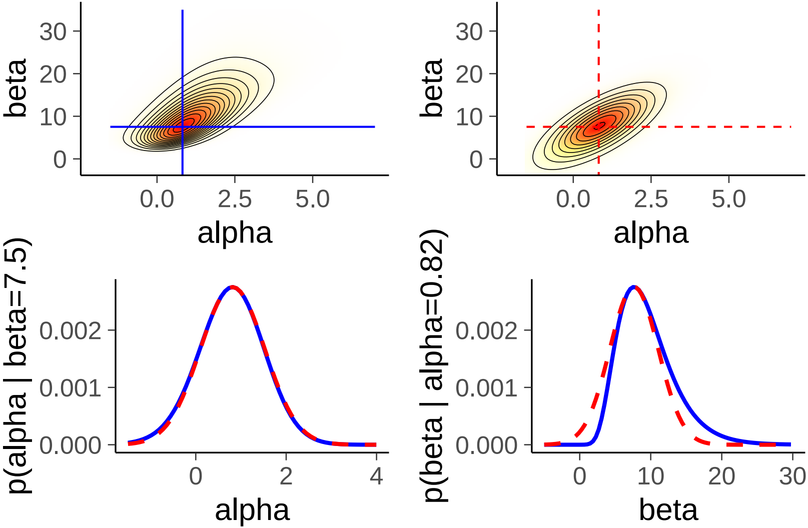

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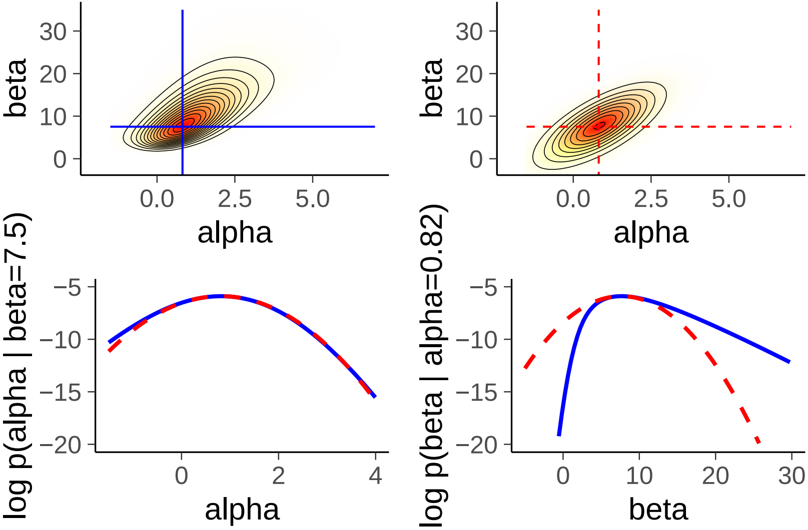

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$$p(\theta|y) \approx \mathrm{N}(\hat{\theta}, [I(\hat{\theta})]^{-1})$$

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Hessian $H(\theta) = -I(\theta)$

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  - $I(\hat{\theta})$ is the second derivatives at the mode and thus describes the curvature at the mode
  - if the mode is inside the parameter space, $I(\hat{\theta})$ is positive
  - if θ is a vector, then $I(\theta)$ is a matrix

Normal approximation

- BDA3 Ch 4 has an example where it is easy to compute first and second derivatives and there is easy analytic solution to find where the first derivatives are zero

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  - e.g. in R, demo4_1.R:

```
bioassayfun <- function(w, df) {
  z <- w[1] + w[2]*df$x
  -sum(df$y*(z) - df$n*log1p(exp(z)))
}

theta0 <- c(0,0)
optimres <- optim(w0, bioassayfun, gr=NULL, df1, hessian=T)
thetahat <- optimres$par
Sigma <- solve(optimres$hessian)
```

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    - second order autodiff in progress

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- In some cases accuracy for a conditional distribution is sufficient (Ch 13)
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  - Rasmussen & Williams: Gaussian Processes for Machine Learning
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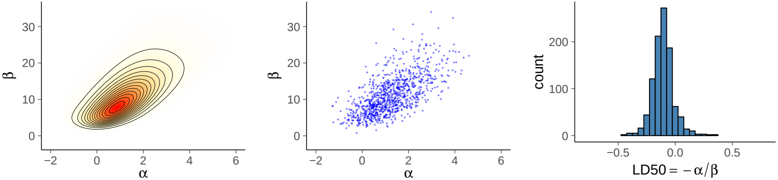
- Accuracy can be improved by importance sampling (Ch 10)

Example: Importance sampling in Bioassay

Grid



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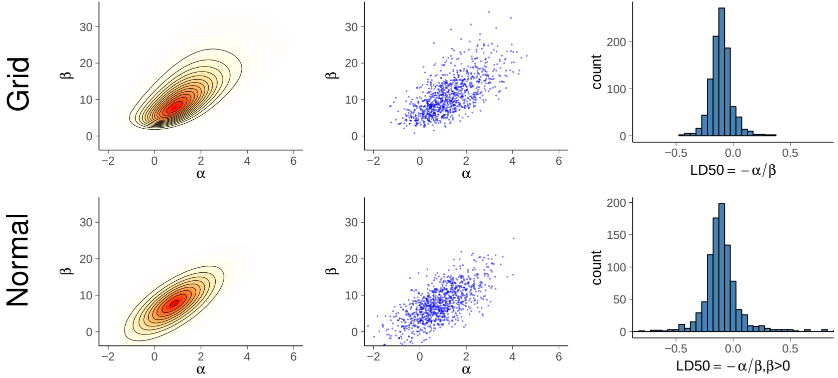

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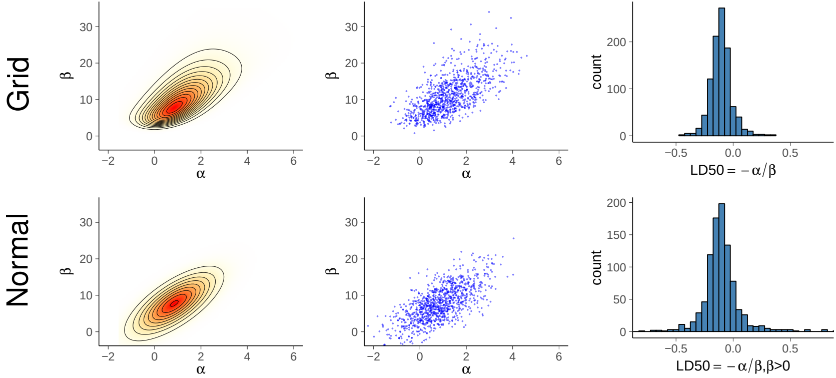
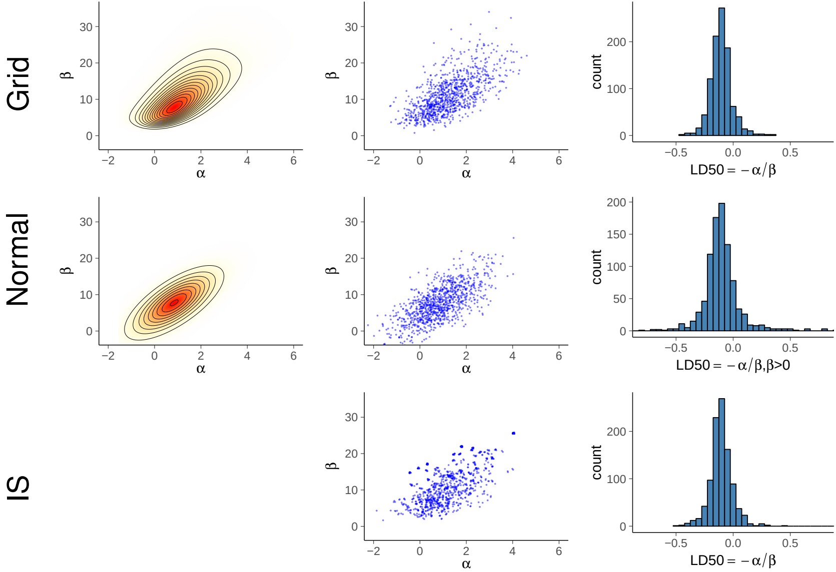


But the normal approximation is not that good here:
Grid sd(LD50) $\approx$ 0.1, Normal sd(LD50) $\approx$ .75!

Example: Importance sampling in Bioassay



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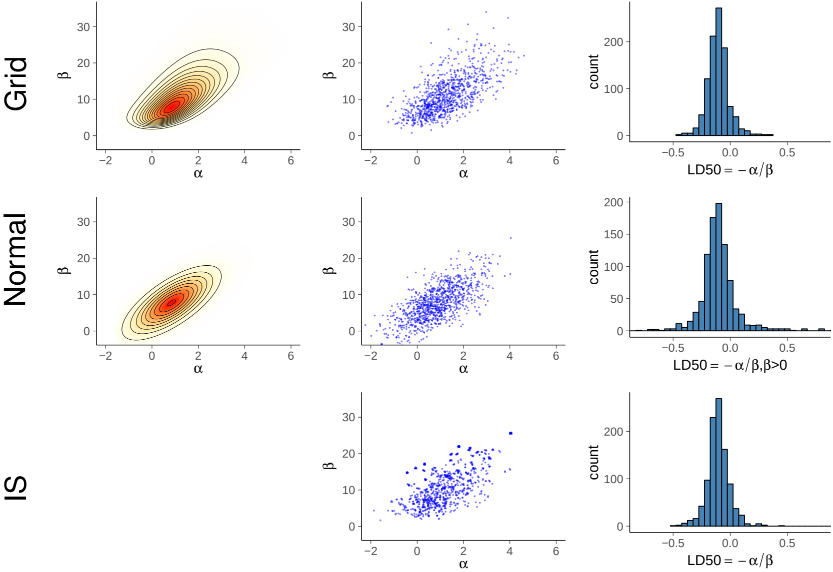

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  - in Bioassay example $k = 0.57$, which is ok
- CmdStan(R) has Laplace algorithm
  - since version 2.33 (2023)
    - + Pareto-k diagnostic via posterior package
    - + importance resampling (IR) via posterior package

Normal approximation and parameter transformations

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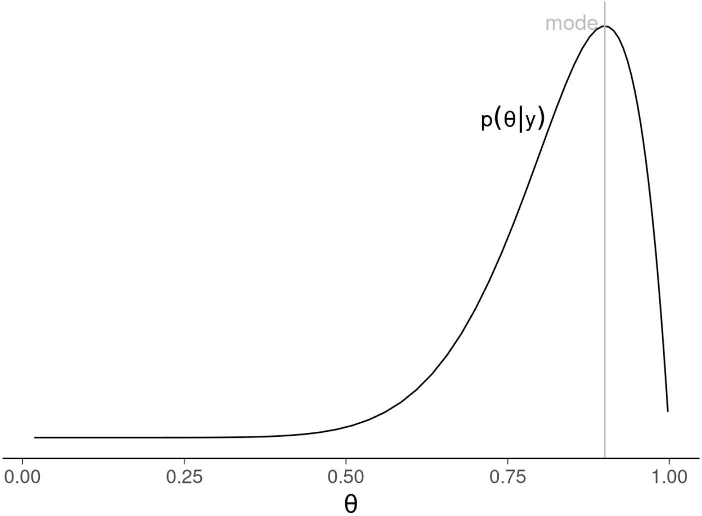
  - density of the transformed parameter needs to include Jacobian of the transformation (BDA3 p. 21)

Normal approximation and parameter transformations

Binomial model $y \sim \mathrm{Bin}(\theta, N)$, with data $y = 9, N = 10$

With $\mathrm{Beta}(1, 1)$ prior, the posterior is $\mathrm{Beta}(9 + 1, 1 + 1)$



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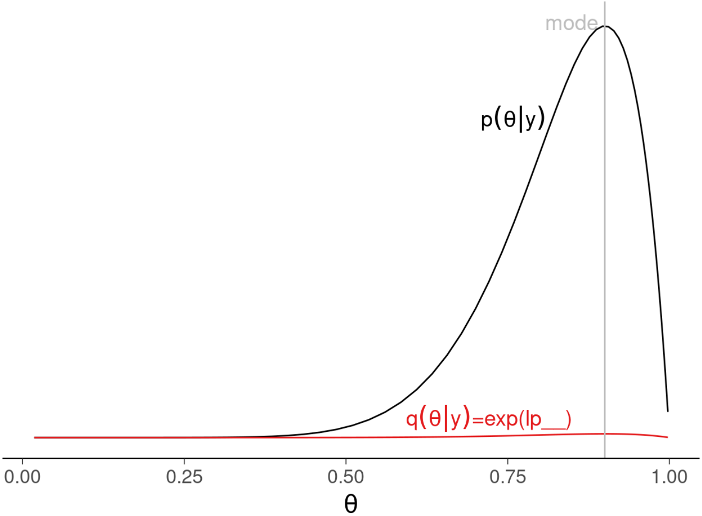
Stan computes only the unnormalized posterior $q(\theta|y)$



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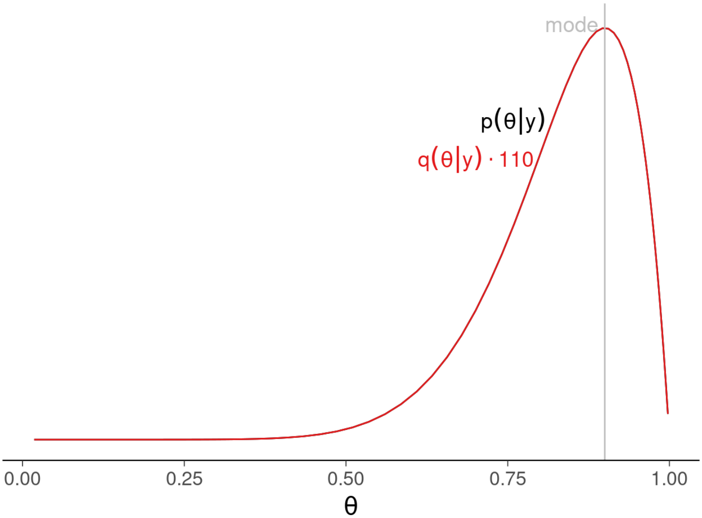
For illustration purposes we normalize Stan result $q(\theta|y)$



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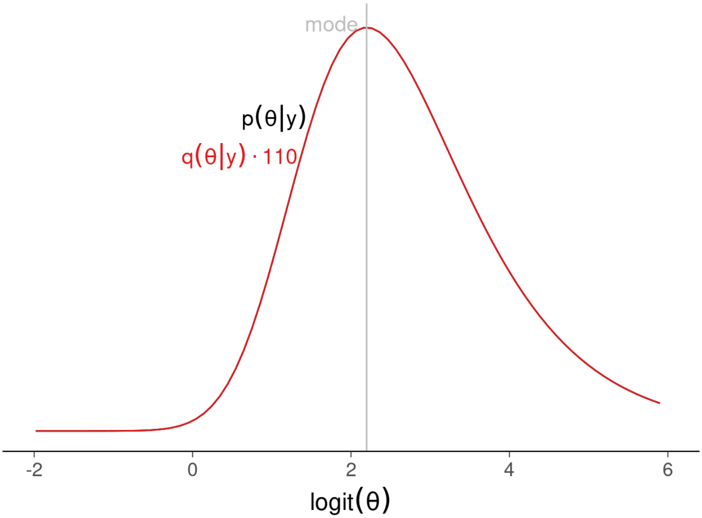
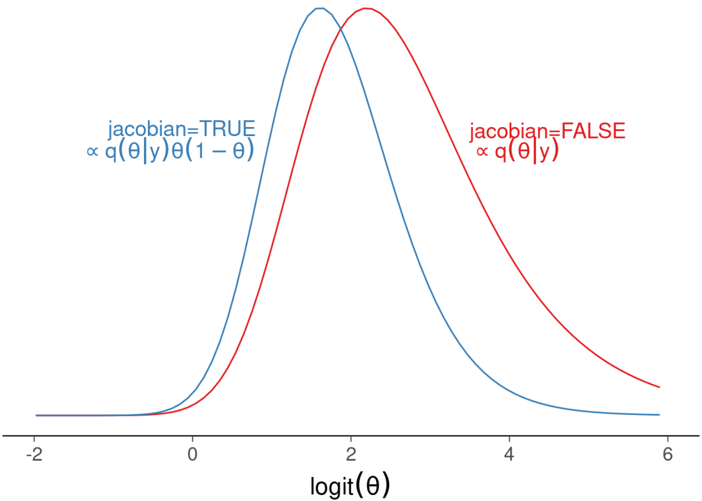
$\mathrm{Beta}(9 + 1, 1 + 1)$, but x-axis shows the unconstrained $\mathrm{logit}(\theta)$



Normal approximation and parameter transformations

...but we need to take into account the absolute value of the determinant of the Jacobian of the transformation $\theta(1-\theta)$



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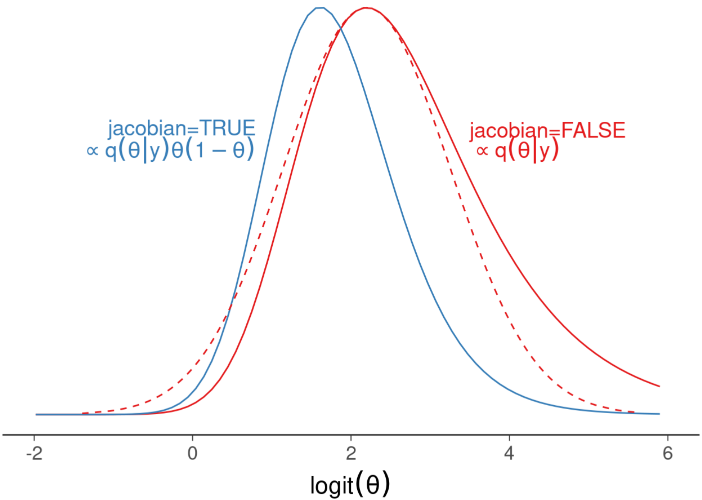

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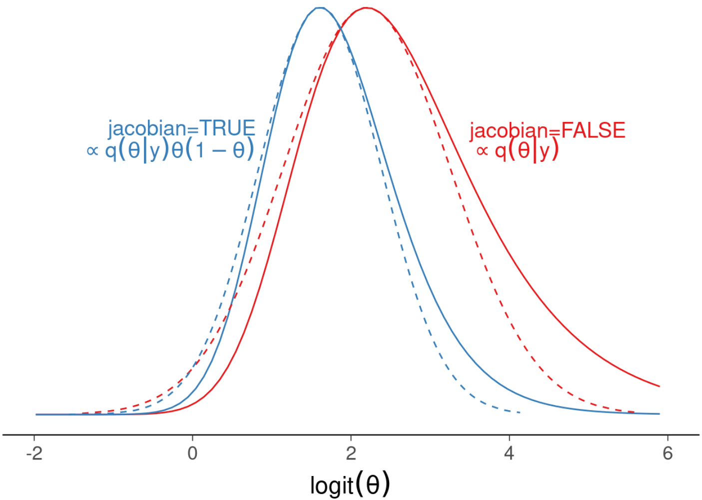

Normal approximation and parameter transformations

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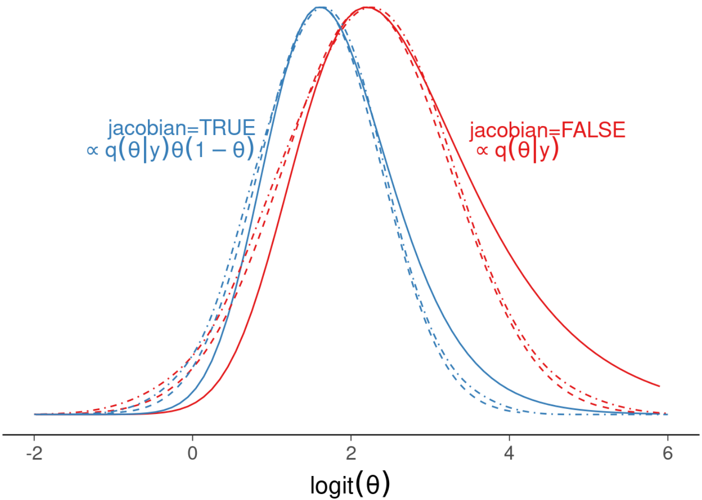
Sample from both approximations and show KDEs for draws



Normal approximation and parameter transformations

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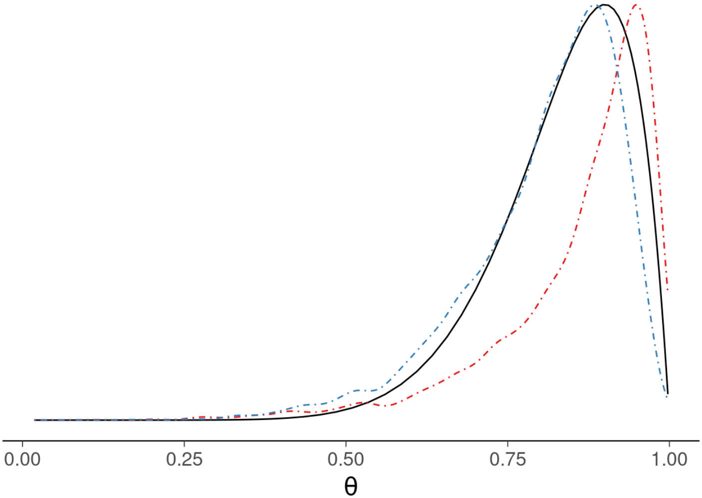
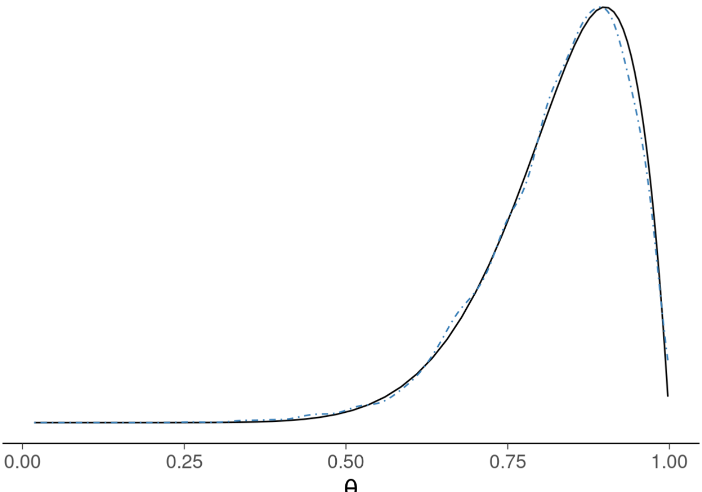
Inverse transform draws and show KDEs



Normal approximation and parameter transformations

Laplace approximation can be further improved with importance resampling



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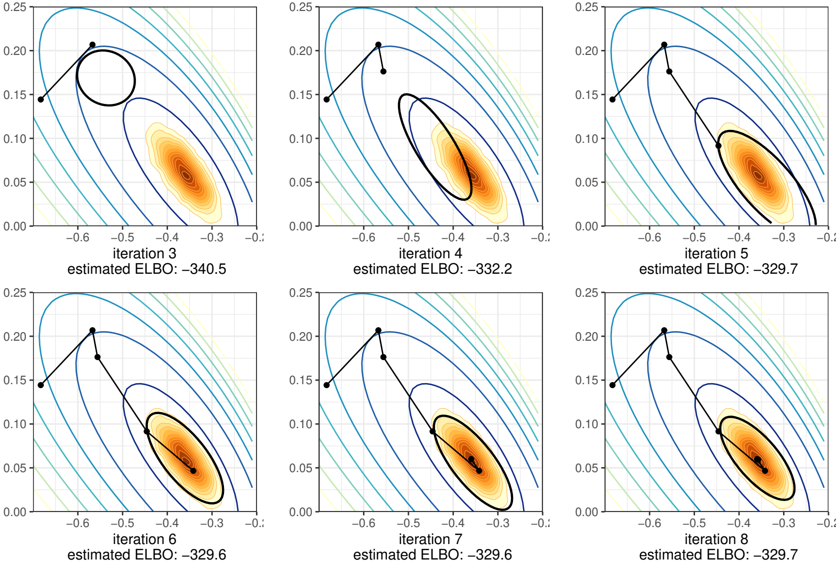
- Instead of mode and Hessian at mode, e.g.
  - variational inference (Ch 13)
    - CS-E4820 - Machine Learning: Advanced Probabilistic Methods
    - CS-E4895 - Gaussian Processes
    - Stan has the ADVI algorithm (not very good implementation)
    - Stan has Pathfinder algorithm (CmdStanR, brms)
    - instead of normal, methods with flexible flow transformations
  - expectation propagation (Ch 13)
  - speed of these is usually between optimization and MCMC
    - stochastic variational inference can be even slower than MCMC

Pathfinder: Parallel quasi-Newton variational inference.



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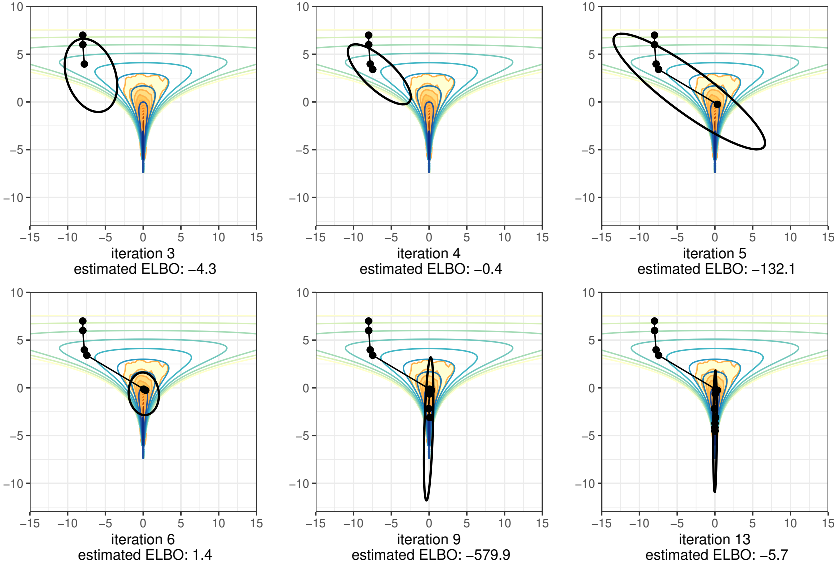

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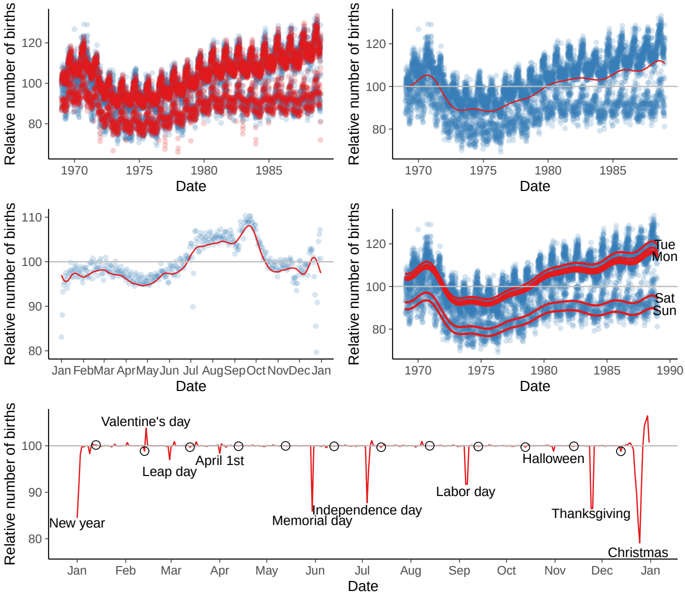
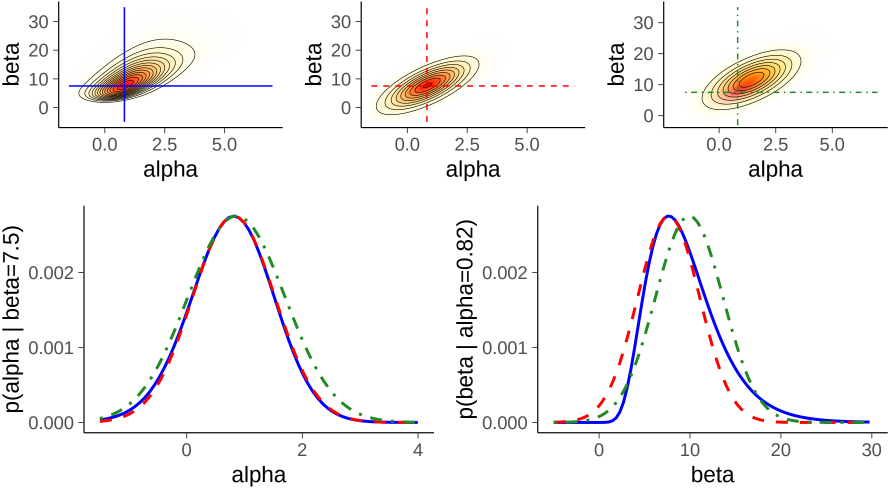
Birthdays case study uses Pathfinder to speed up workflow

https://users.aalto.fi/~ave/casestudies/Birthdays/birthdays.html



Distributional approximations

Exact, Normal at mode, Normal with variational inference



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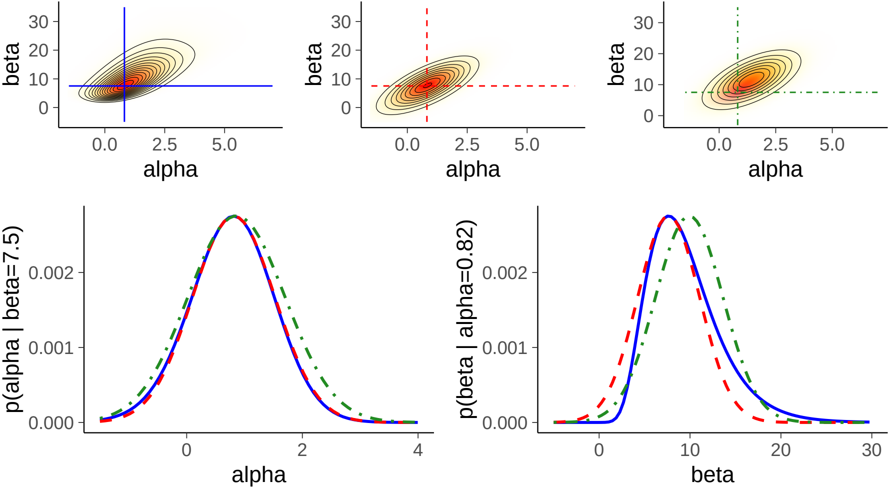

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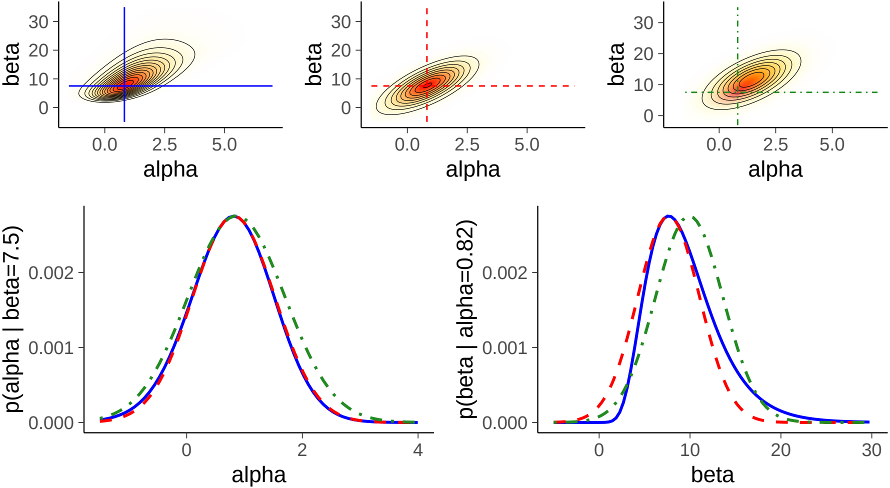

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  - with increasing number of posterior dimensions, the stochastic divergence estimate gets worse and flows have problems, too (Dhaka, Catalina, Andersen, Welandawe, Huggins, and Vehtari, 2021)

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    - eventually likelihood dominates the prior
    - the higher order terms in Taylor series increase slower than the second order term
  - see counter examples

Large sample theory

- Assume "true" underlying data distribution $f(y)$
  - observations $y_1, \ldots, y_n$ are independent samples from the joint distribution $f(y)$
  - "true" data distribution $f(y)$ is not always well defined
  - in the following we proceed as if there were true underlying data distribution
  - for the theory the exact form of $f(y)$ is not important as long at it has certain regularity conditions

Large sample theory

- Consistency
  - if true distribution is included in the parametric family, so that $f(y) = p(y|\theta_0)$ for some θ_0, then posterior converges to a point θ_0, when $n \to \infty$

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- But a finite data from this data generating process may lack the joint height and weight observations, and thus the the finite data likelihood doesn't have information about ρ
- If the likelihood is weakly informative for some parameters, priors and integration are more important

Large sample theory – counter examples

- If the number of parameter increases as the number of observation increases
  - in some models number of parameters depends on the number of observations
  - e.g. time series models $y_t \sim \mathrm{N}(\theta_t, \sigma^2)$ and θ_t has prior in time
  - posterior of θ_t does not converge to a point, if additional observations do not bring enough information

Large sample theory – counter examples

- Aliasing (valetoisto in Finnish)
  - special case of under-identifiability where likelihood repeats in separate points
  - e.g. mixture of normals

$$p(y_i|\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \lambda) = \lambda \, \mathrm{N}(\mu_1, \sigma_1^2) + (1 - \lambda) \, \mathrm{N}(\mu_2, \sigma_2^2)$$

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- For MCMC makes the convergence diagnostics more difficult, as it is difficult to identify aliasing from other multimodality

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  - asymptotic results assume that probability sums to 1
  - e.g. Binomial model, with $\mathrm{Beta}(0, 0)$ prior and observation $y = n$
    - posterior $p(\theta|n, 0) = \theta^{n-1}(1-\theta)^{-1}$
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- Prior distribution does not include the convergence point
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- Should have a positive prior probability/density where needed

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  - if θ_0 is on the edge of the parameter space, Taylor series expansion has to be truncated, and normal approximation does not necessarily hold
  - e.g. $y_i \sim \mathrm{N}(\theta, 1)$ with a restriction $\theta \geq 0$ and assume that $\theta_0 = 0$
    - posterior of θ is left truncated normal distribution with $\mu = \bar{y}$
    - in the limit $n \to \infty$ posterior is half normal distribution
- Can be easy or difficult for MCMC

Frequency evaluations

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  - Calibration
    - α%-posterior interval has the true value in α% cases
    - α%-predictive interval has the true future values in α% cases
    - approximate calibration with shorter intervals for likely true values more important than exact calibration with very bad intervals for all possible values.

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  - unbiased estimate for strictly positive parameter can be negative
- Confidence interval is defined to have true value inside the interval in $\alpha\%$ cases of repeated data generation from the data generating mechanism
  - doesn't need be useful to have perfect calibration

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- Bayesian inference
  - easier for complex, e.g. hierarchical, models
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  - a consistent way to add prior information
- A lot of machine learning is not pure frequentist or Bayesian, but there is often a probabilistic flavor