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- ...in such case I recommended to use `brms` + `projpred`
- `projpred` avoids the overfit in model selection

Use of reference models in model selection

- Background
- First example
- Bayesian and decision theoretical justification
- More examples

Not a novel idea

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  - one key part for practical computation
- Related approaches
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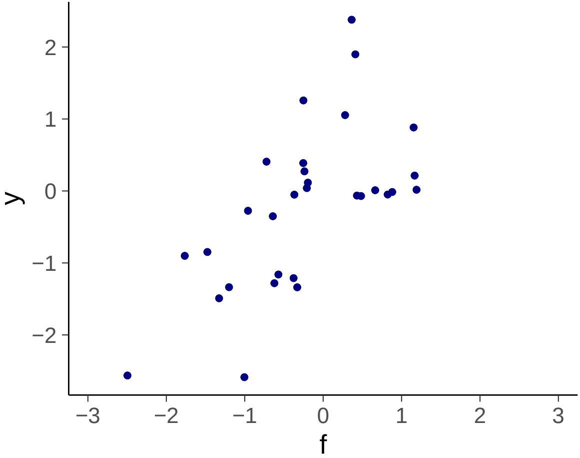
- Motivation in these
  - measurement cost in covariates
  - running cost of predictive model
  - easier explanation / learn from the model

Example: Simulated regression

$$
\begin{aligned}
f &\sim \mathrm{N}(0, 1), \\
y \mid f &\sim \mathrm{N}(f, 1)
\end{aligned}
$$



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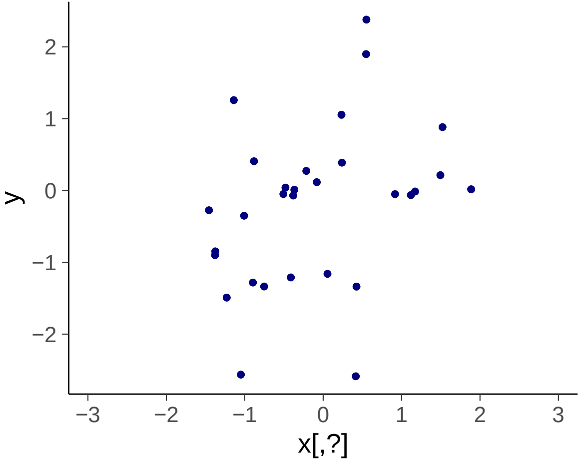

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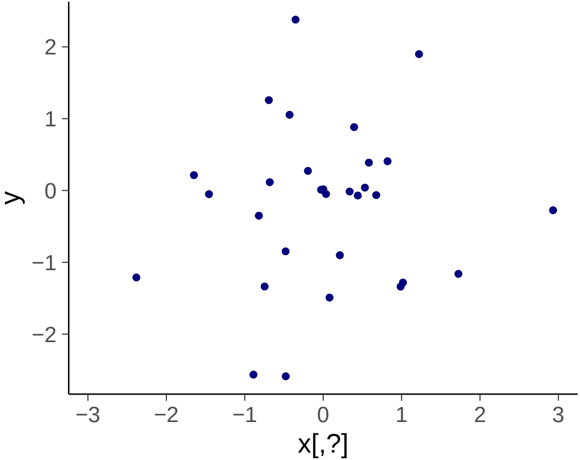

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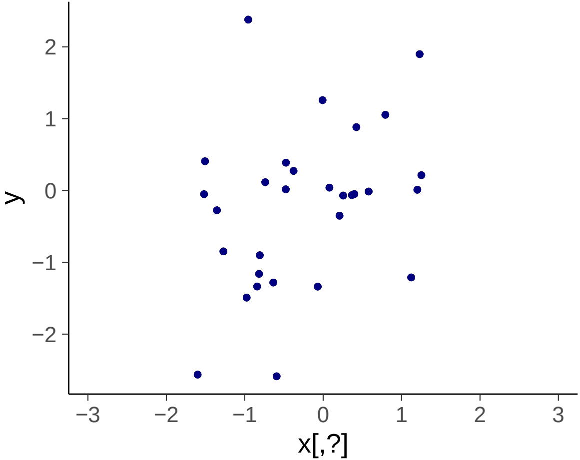

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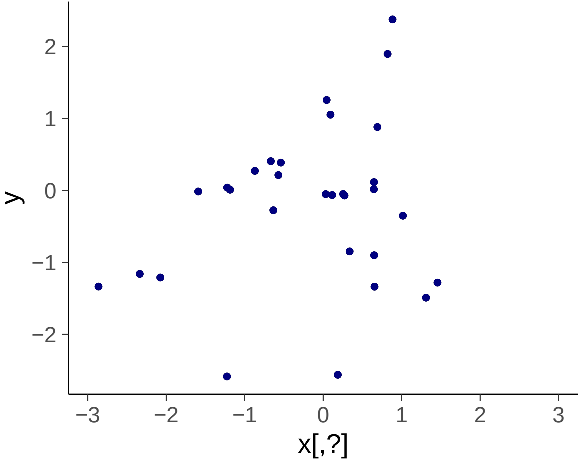

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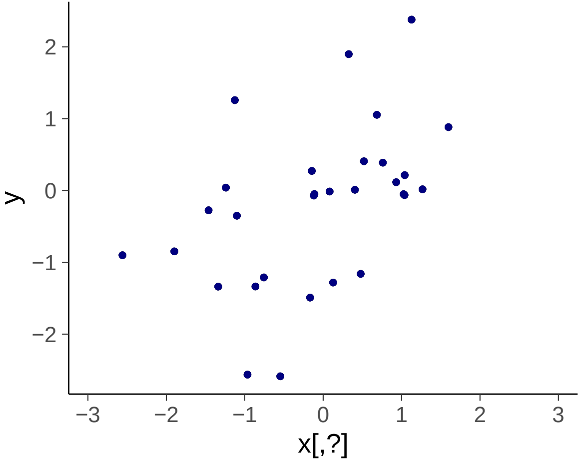

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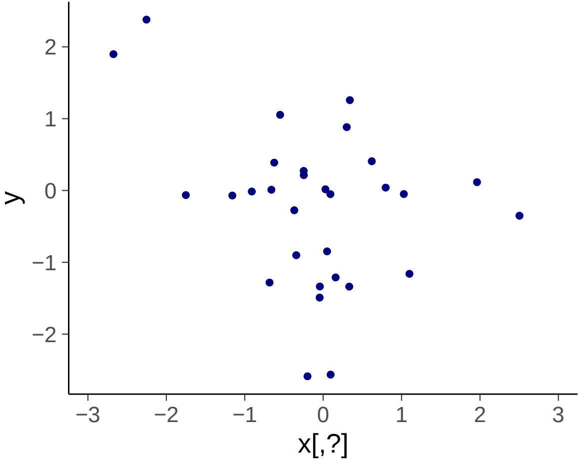

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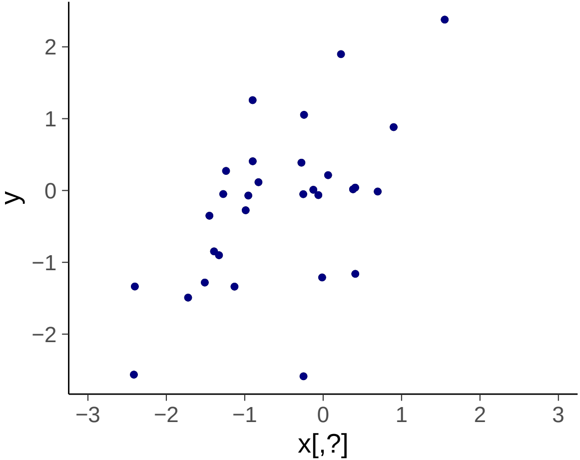

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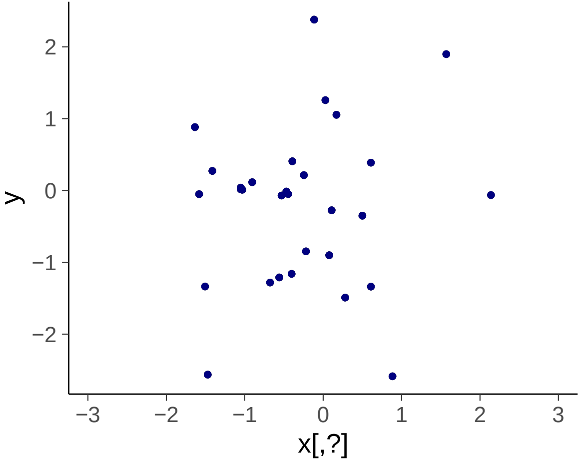

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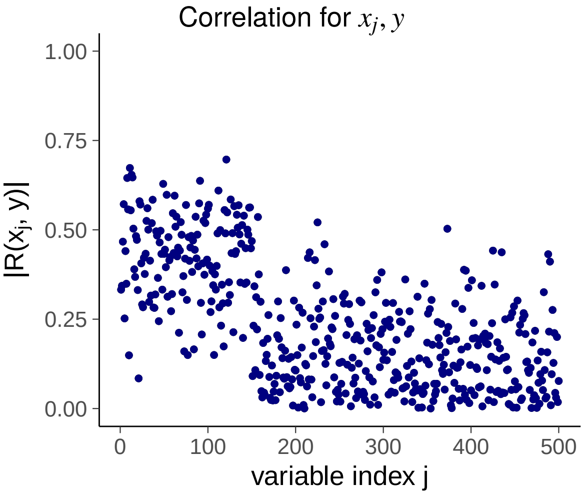

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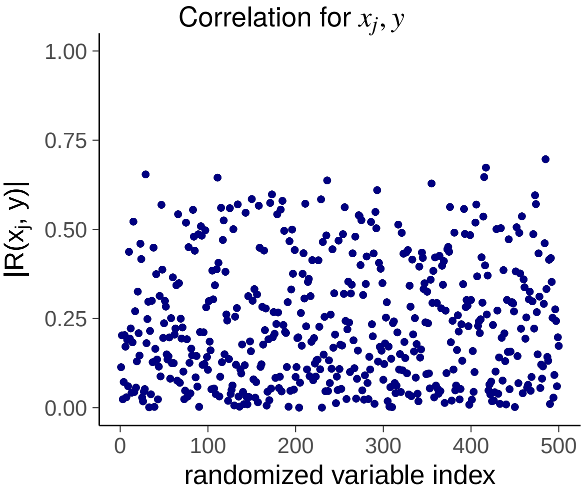

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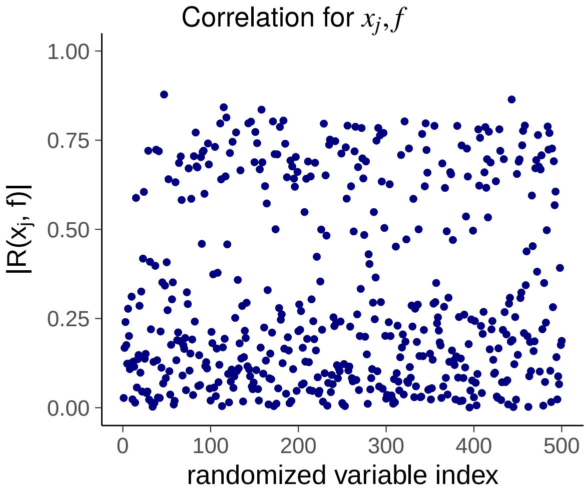

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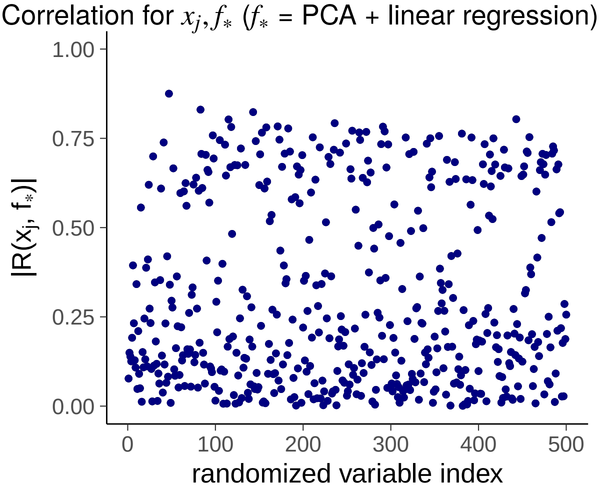
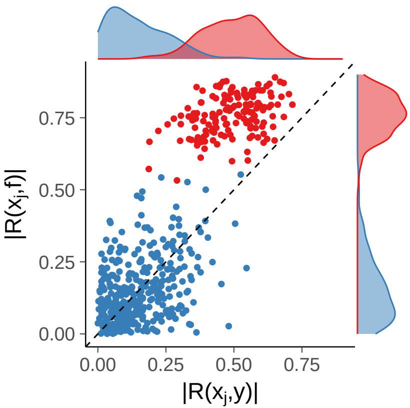


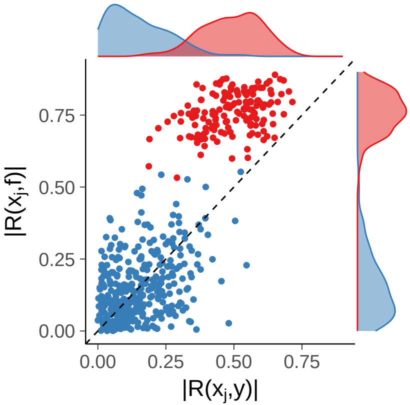
Knowing the latent values would help



irrelevant x_j, relevant x_j

A) Sample correlation with y vs. sample correlation with f

Estimating the latent values with a reference model helps



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A) Sample correlation with y vs. sample correlation with f

B) Sample correlation with y vs. sample correlation with f_*

$f_* =$ linear regression fit with 3 principal components

Bayesian justification

- Theory says to integrate over all the uncertainties
  - build a rich model
  - make model checking etc.
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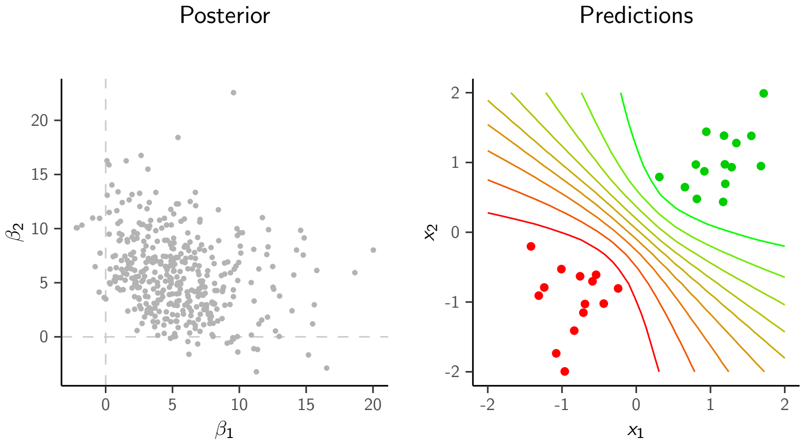
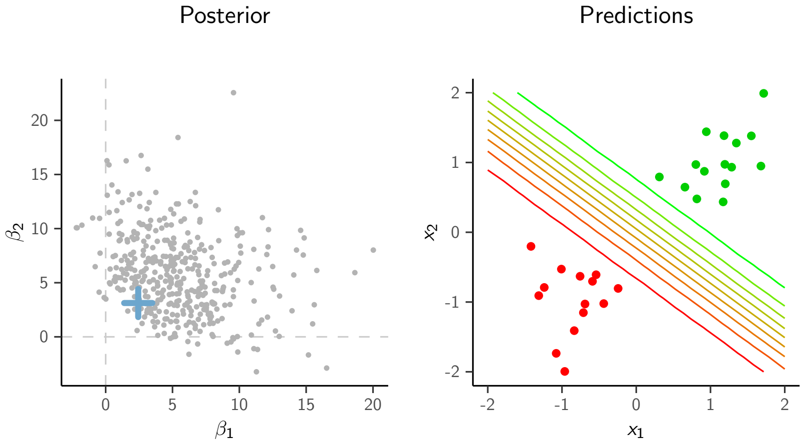
  - Much simpler model
    $\Rightarrow$ "Easier explanation"

Logistic regression with two covariates



Full posterior for β_1 and β_2 and contours of predicted class probability

Logistic regression with two covariates



Projected point estimates for β_1 and β_2

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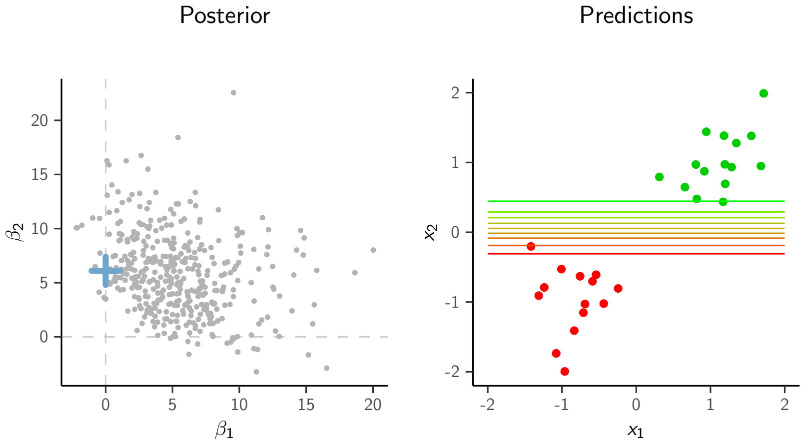

Projected point estimates, constraint $\beta_1 = 0$

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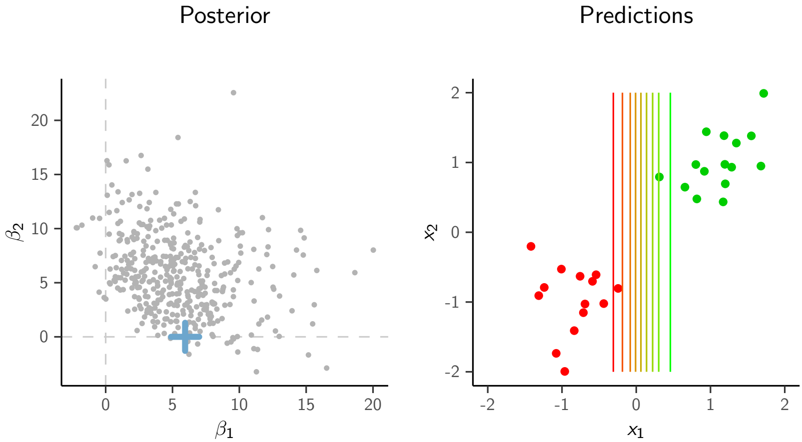

Projected point estimates, constraint $\beta_2 = 0$

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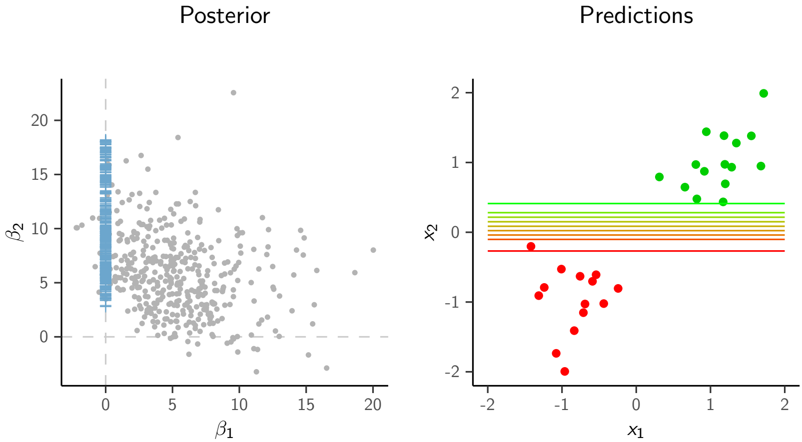

Draw-by-draw projection, constraint $\beta_1 = 0$

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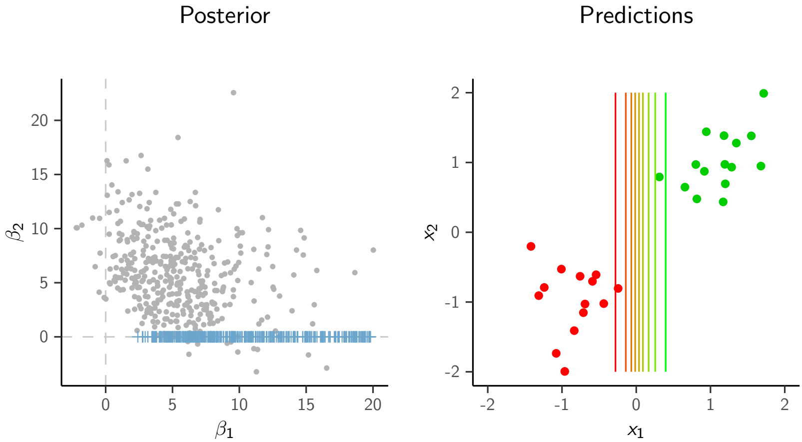

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    - solves the problem of how to do the inference after the model selection

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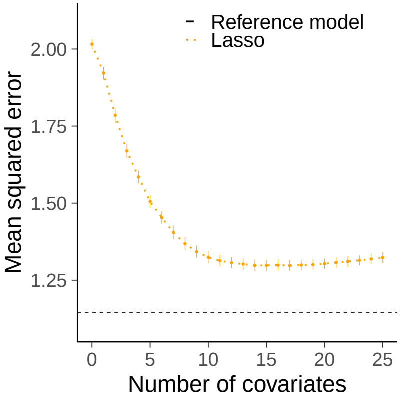
- Search heuristics, e.g.
    - Monte Carlo search
    - Forward search
    - L_1-penalization (as in Lasso)
- Use cross-validation to select the appropriate model size
    - need to cross-validate over the search paths

Projective selection vs. Lasso

Same simulated regression data as before,
$n = 50$, $p = 500$, $p_{\text{rel}} = 150$, $\rho = 0.5$



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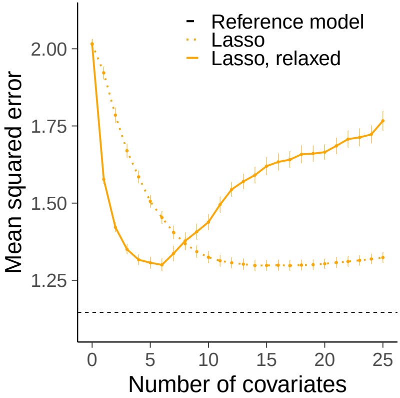

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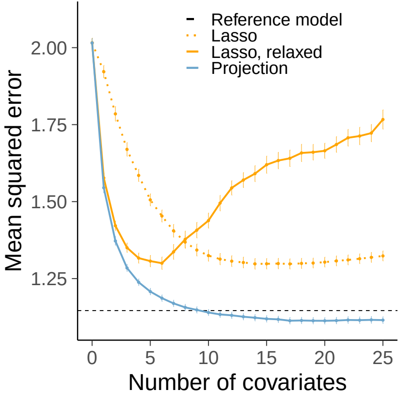

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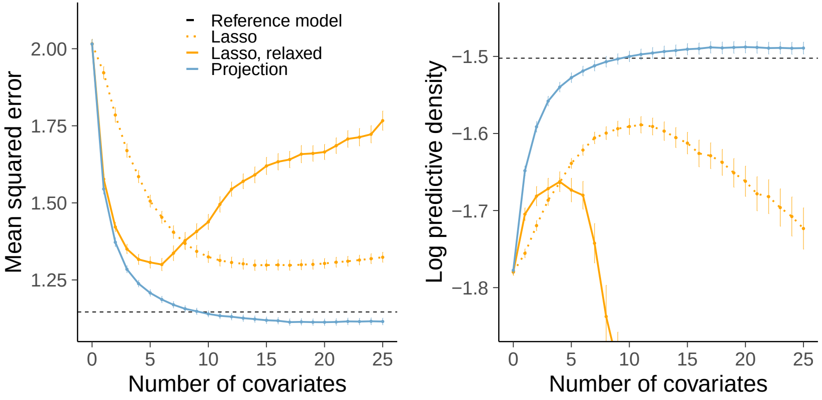

Bodyfat: small p example of projection predictive

Predict bodyfat percentage. The reference value is obtained by immersing person in water. $n = 251$.

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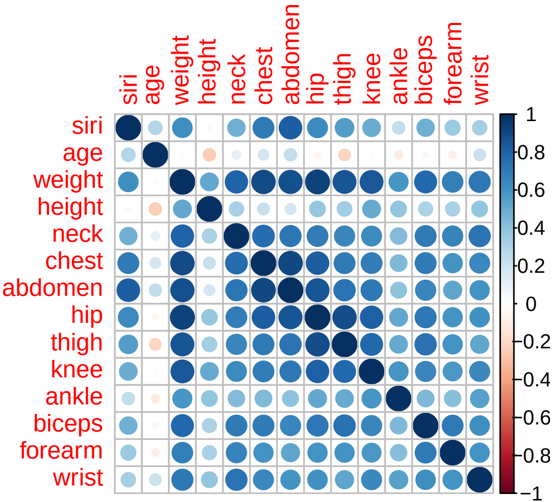
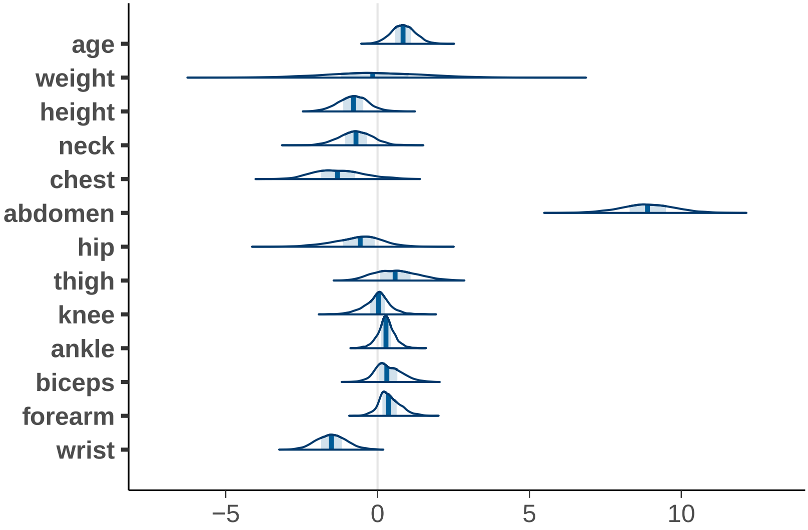


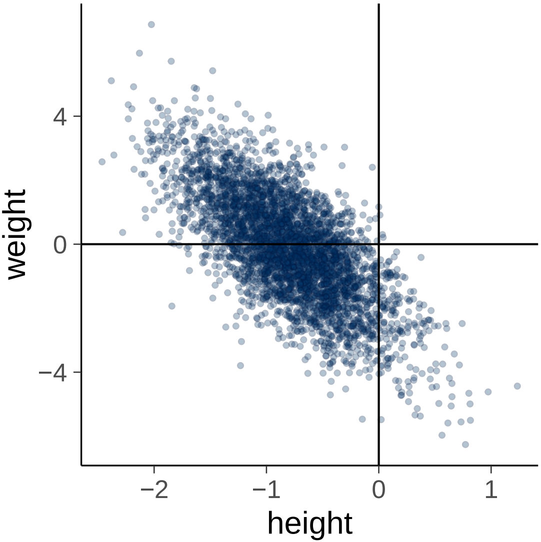
Bodyfat

Marginal posteriors of coefficients



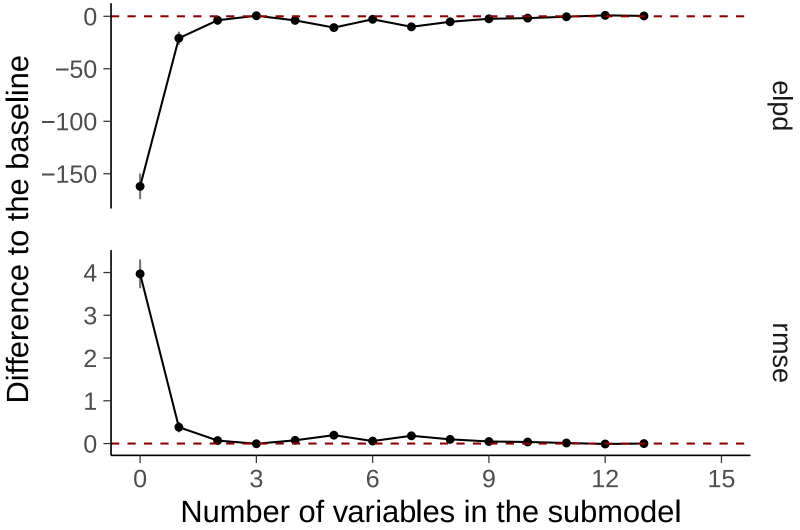
Bodyfat

Bivariate marginal of weight and height



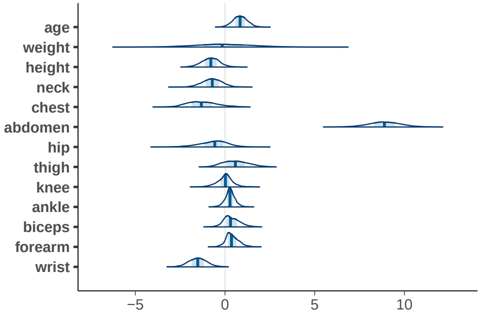
Bodyfat

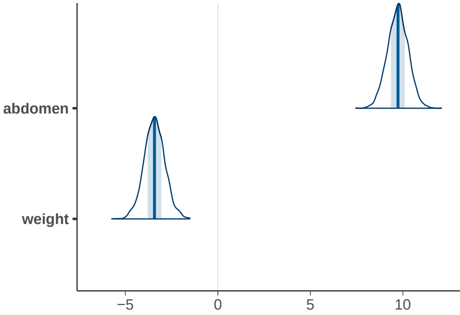
The predictive performance of the full and submodels



Bodyfat

Marginals of the reference and projected posterior





Predictive performance vs. selected variables

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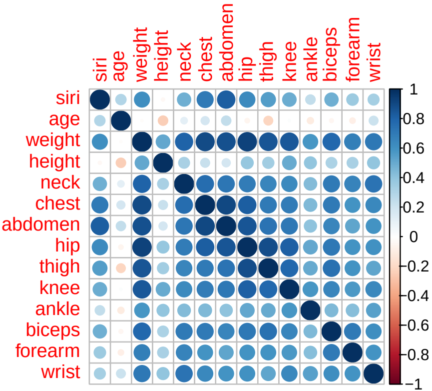
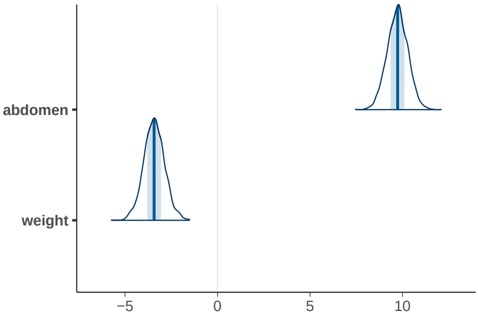
Predictive performance vs. selected variables

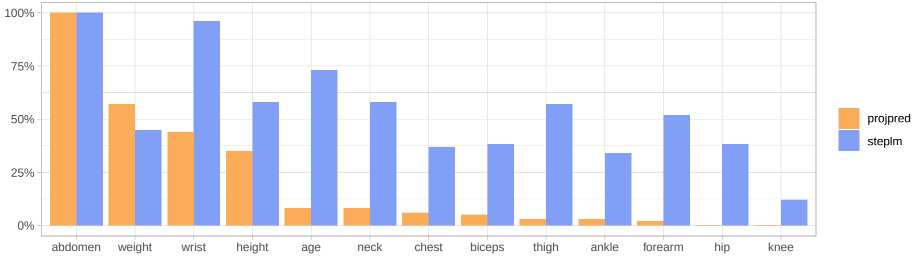
- The initial aim: find the minimal set of variables providing similar predictive performance as the reference model
- Some keep asking can it find the true variables
  - What do you mean by true variables?





Variability under data perturbation

Comparing projection predictive variable selection (projpred) and stepwise maximum likelihood over bootstrapped datasets



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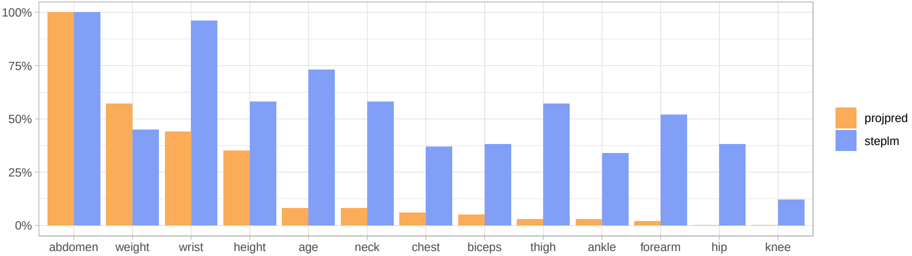
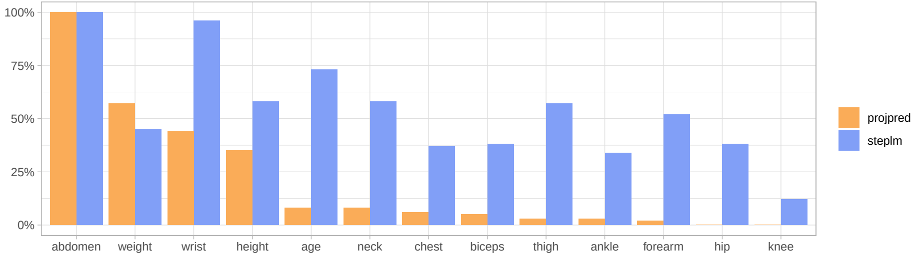


M	projpred	Freq %	steplm	Freq %
1	abdom., weight	39	abdom., age, forearm, height, hip, neck, thigh, wrist	4
2	abdom., wrist	10	abdom., age, chest, forearm, height, neck, thigh, wrist	4
3	abdom., height	10	abdom., forearm, height, neck, wrist	2
4	abdom., height, wrist	9	abdom., forearm, neck, weight, wrist	2
5	abdom., weight, wrist	8	abdom., age, height, hip, thigh, wrist	2
6	abdom., chest, height, wrist	2	abdom., age, height, hip, neck, thigh, wrist	2
7	abdom., biceps, weight, wrist	2	abdom., age, ankle, forearm, height, hip, neck, thigh, wrist	2
8	abdom., height, weight, wrist	2	abdom., age, biceps, chest, height, neck, wrist	2
9	abdom., age, wrist	2	abdom., age, biceps, chest, forearm, height, neck, thigh, wrist	2
10	abdom., age, height, neck, thigh, wrist	2	abdom., age, ankle, biceps, weight, wrist	2

Variability under data perturbation

Comparing projection predictive variable selection (projpred) and stepwise maximum likelihood over bootstrapped datasets



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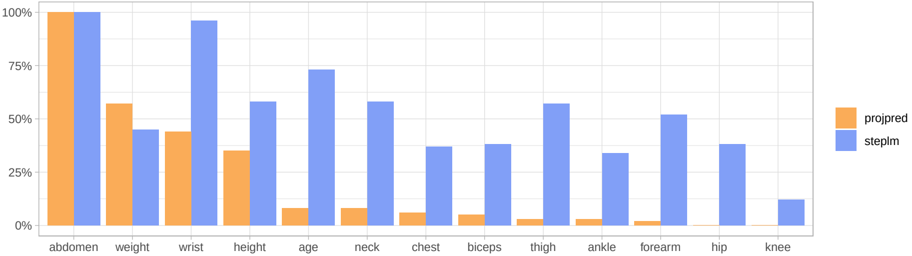

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    - Bayesian inference for the reference
    - The reference model
    - Projection for submodel inference

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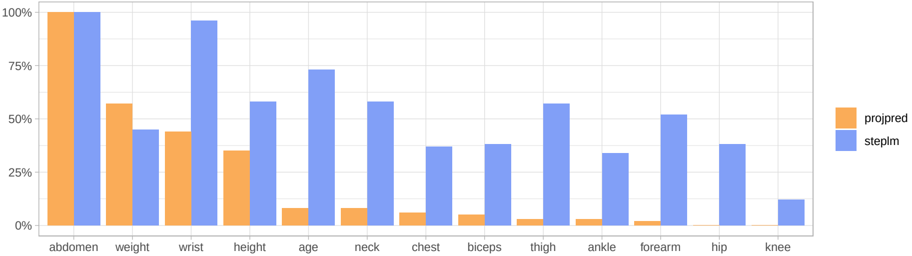

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Multilevel regerssion and GAMMs

- `projpred` supports also hierarchical models in `brms`
  Catalina, Bürkner, and Vehtari (2022). Projection predictive inference for generalized linear and additive multilevel models. *Proceedings of the 24th International Conference on Artificial Intelligence and Statistics (AISTATS), PMLR* 151:4446–4461. https://proceedings.mlr.press/v151/catalina22a.html

Scaling

- So far the biggest number of variables we've tested is 22K
  - 96s for creating a reference model
  - 14s for projection predictive variable selection

Intro paper and brms and rstanarm + projpred examples

- McLatchie, Rögnvaldsson, Weber, and Aki Vehtari (2024). Advances in projection predictive inference. *Statistical Science*. https://arxiv.org/abs/2306.15581
- https://mc-stan.org/projpred/articles/projpred.html
- https://users.aalto.fi/~ave/casestudies.html
- Fast and often sufficient if $n \gg p$

```
varsel <- cv_varsel(fit, method='forward', cv_method='loo',
                    validate_search=FALSE)
```

- Slower but needed if not $n \gg p$

```
varsel <- cv_varsel(fit, method='forward', cv_method='kfold', K=10,
                    validate_search=TRUE)
```

- If p is very big

```
varsel <- cv_varsel(fit, method='L1', cv_method='kfold', K=5,
                    validate_search=TRUE)
```