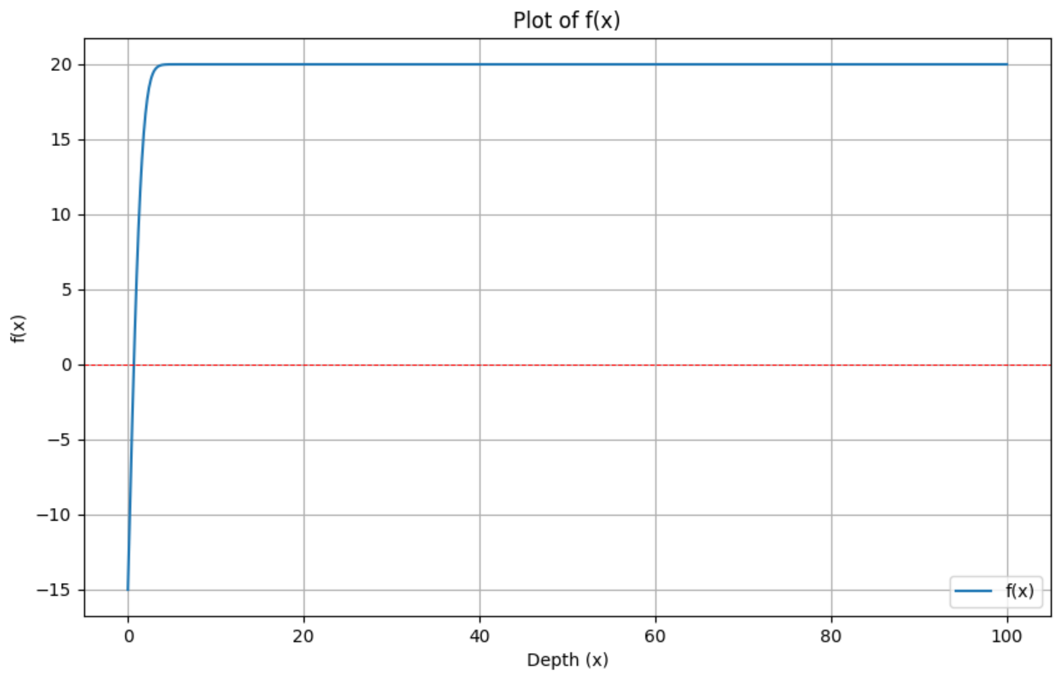
1. f(x) = Ts + (Ti - Ts) * erf(x / (2 * np.sqrt(alpha * exposure_time_seconds)))

f'(x) found using finite differences

Code pushed to Git



b. Depth = 0.6769618544819389 meters

c. Depth = 0.6769618544819407 meters

If you use x0 = x_bar, Newton's method diverges because the initial guess isn't close enough to the root.

2 and 3:

2. a. A root α of multiplicity m can be defined as
$f(x) = (x-\alpha)^m \cdot g(x)$

b. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha - \frac{f(x_n)}{f'(x_n)}$
$e_{n+1} = \alpha - x_{n+1} = \frac{f(x_n)}{f'(x_n)}$
$e_{n+1} = \frac{f''(\alpha)}{2f'(\alpha)} e_n^2$
Thus, if $f''(\alpha)$ is finite and $f'(\alpha)$ is nonzero, $|e_{n+1}| \leq \lambda |e_n|$ so it converges linearly.

c. $e_{n+1} = (x_n - mf(x_n)) - \alpha$
$e_{n+1} = (x_n - m(f(\alpha) + (x_n-\alpha)(f'(\alpha)) + \ldots)) - \alpha$
$e_{n+1} = (1 - mf'(\alpha)) e_n^2$
Thus, if $|1 - mf'(a)|$ is finite and nonzero, fixed point converges quadratically because the equation is of the form $e_{n+1} = K e_n^2$

d. Near roots with $m > 1$, Newton's method may converge linearly, unlike fixed point

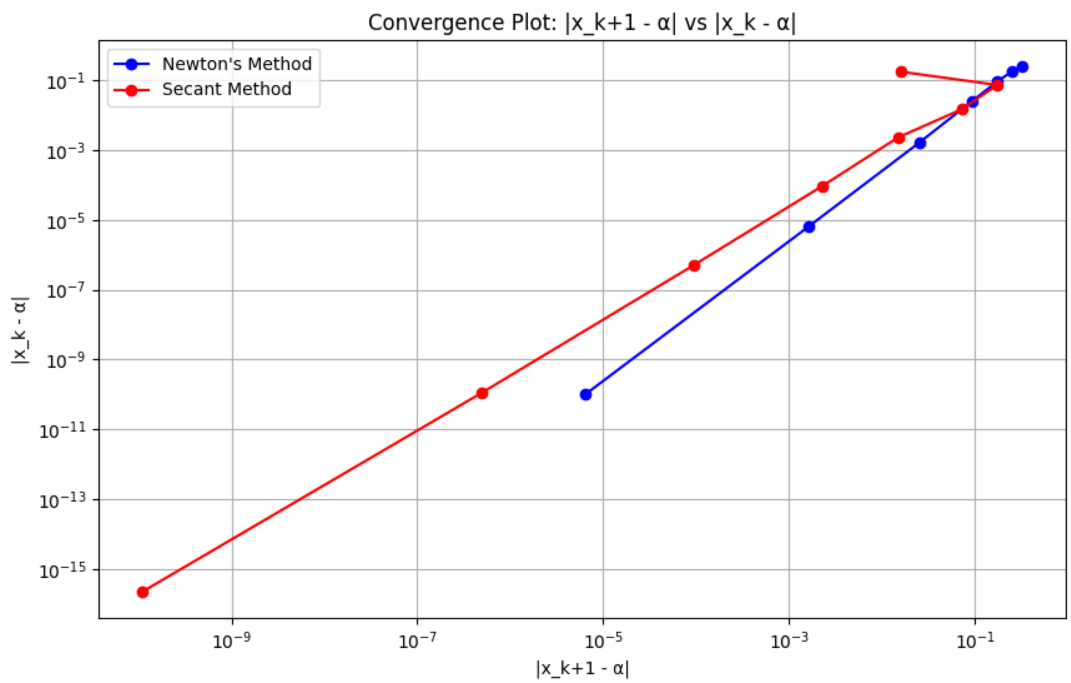
3. A sequence converges when $|x_{k+1} - \alpha| \leq C(|x_k - \alpha|)^p$
$\log(|x_{k+1} - \alpha|) \leq \log(C) + p\log(|x_k - \alpha|)$
$\log(C)$ is constant, so the order p represents the slope of $\log(|x_{k+1} - \alpha|)$ relative to $\log(|x_k - \alpha|)$

4. Newton’s method encounters a divide by 0 error. Both of the modified methods find the root but I prefer the modified method from problem 2 because it does so in the fewest iterations.

5. a.

Iteration	Newton's Error	Secant Error
1	0.31937172774869116	0.016129032258064502
2	0.24988928401224642	0.17444873641857295
3	0.175768032129626	0.07292193773508582
4	0.09343252333612329	0.014875719274581423
5	0.02518515860280779	0.0022852577887197967
6	0.0016227458268762707	9.316205614795514e-05
7	6.389843407950124e-06	4.92342510982624e-07
8	9.870171346904044e-11	1.1030376612097825e-10

b.



Since the slope of the Newton’s method line on a log-log scale is approximately 2, it has quadratic convergence. The slope of the secant method on a log-log scale is closer to 1 thus it has superlinear convergence.