Rational approximations

What is idea?

We are going to build an approximation that is written as the quotient of polynomials.

i.e. $f(x) \approx \frac{p(x)}{q(x)}$ where $p(x)$ & $q(x)$ are polynomials

Padé approximation

The general idea is that we know how to create the Taylor expansion but we also know computing w/ a Taylor expansion is not a good plan. Instead we will write a rational approximation & match terms.

In other words, we will match the following

$$P_m^n(x) = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_n x^n} = \text{Taylor expansion w/ } m+n+1 \text{ terms (order } m+n)$$

Ex: Create $P_3^2(x)$ approximation of $f(x) = e^{-x}$

This means degree 3 polynomial in the numerator & degree 2 polynomial in the denominator

numerator & degree 2 polynomial
in the denominator.

Soln:

$$P_3^2(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2}$$

We will match with $T_5(x)$

$$T_5(x) = \sum_{n=0}^{5} \frac{f^{(n)}(0)}{n!} x^n$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

Set the two expressions equal & solve for the unknown coefficients

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}\right)\left(1 + b_1 x + b_2 x^2\right)$$

I like to build a Table collect terms 1 at a time.

x^j Term	equation for coefficients
Constant	$a_0 = 1$
x	$a_1 = b_1 - 1$

$x \quad a_1 = b_1 - 1$

$x^2 \quad a_2 = \frac{1}{2} - b_1 + b_2$

$x^3 \quad a_3 = -\frac{1}{6} + \frac{b_1}{2} - b_2$

$x^4 \quad 0 = \frac{1}{24} - \frac{1}{6} b_1 + \frac{1}{2} b_2$

$x^5 \quad 0 = -\frac{1}{120} + \frac{1}{24} b_1 - \frac{1}{6} b_2$

The last two equations are independent $\{a_j\}_{j=0}^{3}$ so we can solve them to find b_1 & b_2

$$\begin{bmatrix} -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{24} & -1/6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -1/24 \\ 1/120 \end{bmatrix}$$

The result is $b_1 = 2/5$ & $b_2 = 1/20$. with these we get $a_1 = -3/5$, $a_2 = 3/20$ & $a_3 = -1/60$

Thus our rational approximation is

$$P_3^2(x) = \frac{1 - 3/5\, x + 3/20\, x^2 - \frac{1}{60} x^3}{1 + 2/5\, x + \frac{1}{20} x^2}$$

Fourier Series

The idea is to approximate functions as the sum of sines & cosines on $[-\pi, \pi]$

i.e. $f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin(kx) \right)$

We know that $\{1, \sin(x), \cdots, \sin(nx), \cos(x), \cdots, \cos nx\}$

- are linearly independent on $[-\pi, \pi]$
- orthogonal i.e. $\int_{-\pi}^{\pi} \cos(kx) \sin(lx)\, dx = 0$

Let's use this to find our constants.

let $l \neq 0$.

$$\int_{-\pi}^{\pi} f(x) \cos(lx)\, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx) \right] \cos(lx)\, dx$$

$$= a_l \int_{-\pi}^{\pi} (\cos(lx))^2\, dx$$

$$= a_l \int_{-\pi}^{\pi} \frac{1}{2}(1 + \cos(2lx))\, dx$$

$$= a_l \pi$$

$$\Rightarrow a_l = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(lx)\, dx.$$

Sameprocess to find

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$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$b_\ell = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(\ell x)\,dx$$

Implementation Comments:

- Only use a finite # of terms
- Use numerical quadrature to approximate the integrals. (our next topic)

Does $S_N(x) = a_0 + \sum_{k=1}^{N}\left[a_k\cos(kx) + b_k\sin(kx)\right]$ converge to $f(x)$ as $N\to\infty$?

Up until 1876 people thought f continuous was enough.

But this is not so.

Need: f to be continuous & piecewise differentiable on $[-\pi,\pi]$.

What happens if the derivative blows up at a pt in $[-\pi,\pi]$?

In this case we are approximating something
that is continuous but not infinitely differentiable
by something that is infinitely differentiable
& smooth.

<u>Ex</u>: look at the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$