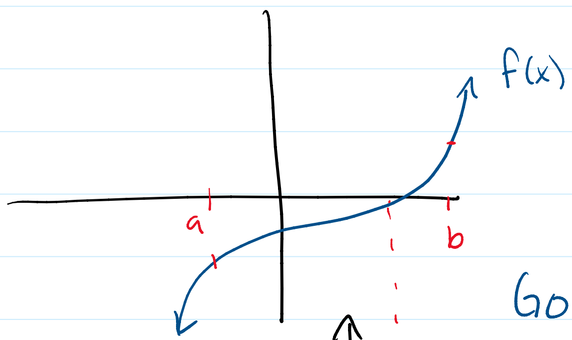
Monday, September 11, 2023 1:58 PM

For general functions root finding is hard so numerical methods are helpful.



f - continuous

$f(a) < 0$, $f(b) > 0$

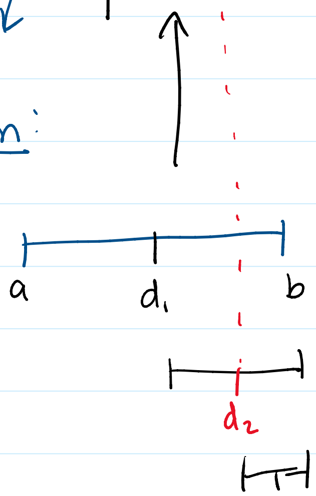
By IVP we know

$\exists$ a pt c st $f(c) = 0$

Goal: Find c st $f(c) = 0$.

Bisection:

Cartoon:



$d_1 = 0.5(a+b)$ $\quad f(d_1) < 0$

$d_2 = 0.5(d_1 + b)$

Question: How we check for a change of sign on the computer?

$f(a)$ $\quad f(b)$

Answer: if $f(a)f(b) < 0$ then sign change.

$\Rightarrow$ root in interval.

Pseudocode: Bisection

Input: a - left end pt

b - right endpt

f(x) - function

tol - tolerance

$f(x)$ - function

tol - tolerance

Output: c^* = approximation of root

ier = $\begin{cases} 0 & \text{success!} \\ 1 & \text{failure! :(} \end{cases}$ error message

Steps:

Step1: if $f(b) \cdot f(a) > 0$

$c^* = b$

ier = 1

return

Step2: $d = 0.5(a+b)$; $fa = f(a)$; $fd = f(d)$

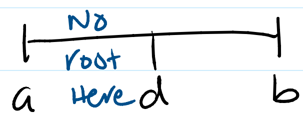
Step3: while $|d-a| > tol$ $\left(\text{or } \frac{|d-a|}{|d|} > tol \right)$

option: If $fd \cdot fa = 0$; $c^* = d$ ier=0.

If $fd \cdot fa > 0$

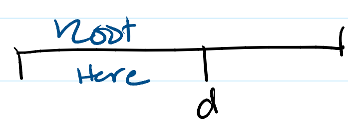
$a = d$

$fa = fd$



else

$b = d$



end

$d = 0.5(a+b)$

$fd = f(d)$

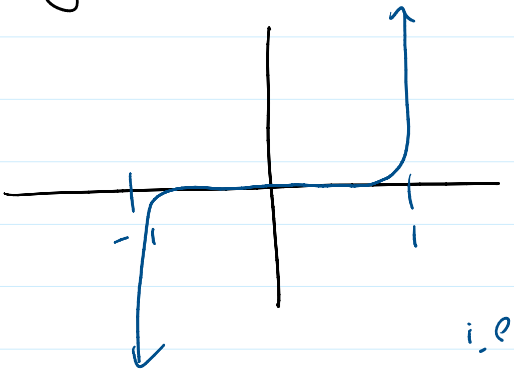
end while

$c^* = d$

ier = 0

Why don't I check $|f(d)| < tol$?

Why don't I check $|f(d)| < tol$?



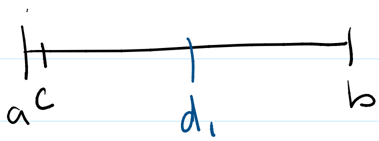
$f(x) = x^{101}$ $\quad c = 0$

with this function you can have many wrong answers w/ bisection

i.e. lots of values of x satisfy $|x^{101}| < tol$

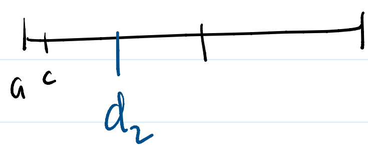
<u>Convergence:</u> How fast will bisection converge?

It. 1



$|d_1 - c| < \frac{1}{2}(b-a)$

It. 2



$|d_2 - c| < \left(\frac{1}{2}\right)^2 (b-a)$

It n $\qquad |d_n - c| < \left(\frac{1}{2}\right)^n (b-a)$

Question: Does it converge?

Answer: $\lim_{n\to\infty} |d_n - c| < \lim_{n\to\infty} (1/2)^n |b-a| = 0$

Can we find a max # of iterations needed to achieve an absolute convergence error of tol?

Answer: $\left(\frac{1}{2}\right)^n (b-a) < tol$

Solve for n = give N_{max} = max # of iterations

Thm 2.1 Suppose that $f \in C[a,b]$ & $f(a)f(b) < 0$

then the bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero (named p) of $f(x)$ with

$$|p_n - p| \leq \left(\tfrac{1}{2}\right)^n (b-a) \quad \text{when } n \geq 1$$