QR iteration

Monday, December 4, 2023 2:10 PM

Ex: Show that for $A \in \mathbb{C}^{n \times n}$ Hermitian all the eigenvalues are real.

Soln: A Hermitian $A^* = A$

let (λ, v) be an eigenpair of A

$\Rightarrow Av = \lambda v$

Goal: Show λ is real.

$$v^*(Av) = v^*\lambda v = \lambda v^* v$$

$$(Av)^* = (\lambda v)^* = \bar{\lambda} v^*$$

$$\begin{array}{c} (Av)^* \\ \parallel \\ v^* A^* \\ v^* A \end{array}$$

$$\lambda v^* v = v^*(Av) = (v^* A) v = (Av)^* v = \bar{\lambda} v^* v$$

$$(\lambda - \bar{\lambda}) v^* v = 0$$

$$v^* v \neq 0 \Rightarrow \lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda}$$

only way this is true is if $\lambda \in \mathbb{R}$

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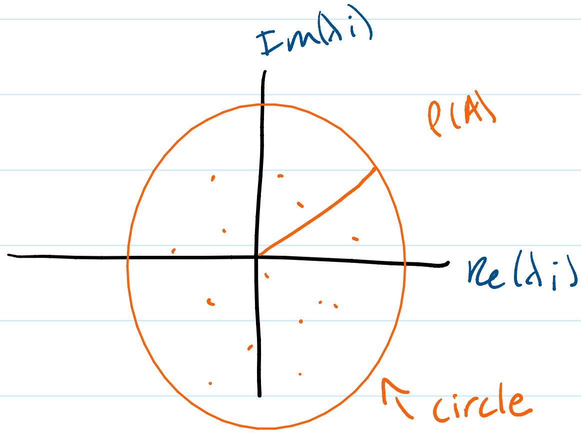
Def: The <u>spectral radius</u> of a matrix is the magnitude of the largest eigenvalue of A.

$$\rho(A) = \max_i |\lambda_i| \quad \text{where } \lambda_i \text{ is an eigenvalue of } A.$$



Thm 9.1 (Gersgorhin Thm)

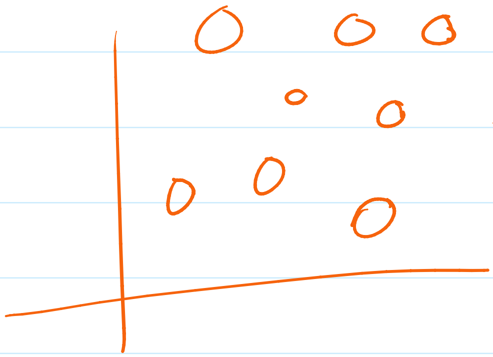
let A be an $n \times n$ matrix. Consider the circles in the complex plane

$$R_i = \Big\{ z \in \mathbb{C} : \underbrace{|z - a_{ii}|}_{\text{center}} \le \underbrace{\sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}|}_{\text{radius}} \Big\}$$

Then (i) all eigenvalues λ of A are contained in the union of the circles; $\bigcup_{i=1}^{n} R_i$

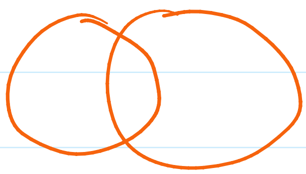
(ii) If an k circles intersect, there ar k eigenvalues in the union of the circles.

Ex: (i)



1 eigenvalue each

(ii)



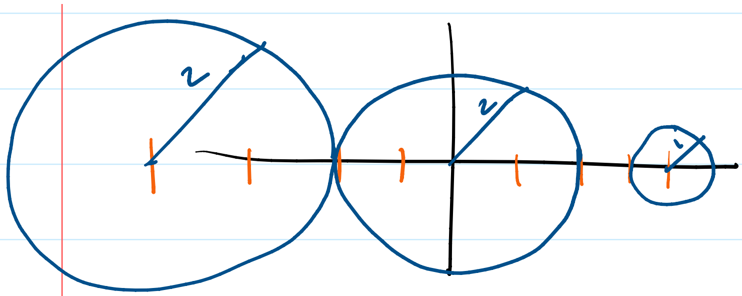
2 eigenvalues in the union

Ex: find the circles containing the eigenvalues of A.

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & -4 \end{bmatrix}$$

Soln:

$$R_1 = |z-4| \leq 1$$
$$R_2 = |z-0| \leq 2$$
$$R_3 = |z+4| \leq 2$$



We can use info to find specific eigenvalue via the Power Method.

<u>QR iteration</u> - Computes approximation of all eigenvalues

let $A \in \mathbb{C}^{n \times n}$

Recall: $A = QR$ where Q is unitary (orthogonal) & R is upper triangular

Goal: Make A similar to an upper triangular matrix. Then read off the eigenvalues.

<u>Def</u>: A is similar to B if $\exists$ a matrix P st $A = PBP^{-1}$

Goal ↑ upper triangular

<u>Cartoon</u>:

Set $A_0 = A$

$(Q_1, R_1) = qr(A_0)$ $\quad\quad\quad$ $A_0 = Q_1 R_1$

$\Rightarrow R_1 = Q_1^* A_0$

Set $A_1 = R_1 Q_1$

What is the relationship between A_0 & A_1?

$A_1 = R_1 Q_1 = Q_1^* A_0 Q_1$

$\Rightarrow$ They are similar.

Why is this better? A_1 is closer to upper Hessenberg.

Continue.

$(Q_2, R_2) = qr(A_1)$

$A_2 = R_2 Q_2$

$\vdots$

Continue until A_k is upper triangular or "nearly" upper triangular.

Read off eigenvalues.

<u>Computational Cost:</u>

Cost 1 QR: $O(n^3)$

If you need all eigenvalues $O(n^4)$