Hermite

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What if we have more info about $f(x)$ at the nodes? What if we have $f'(x)$?

Specifically, suppose at $\{x_j\}_{j=0}^{n}$, we have

$$\{f(x_j)\}_{j=0}^{n} \quad \& \quad \{f'(x_j)\}_{j=0}^{n}$$

$2n+2$ data $\Rightarrow$ we should be able approx with a polynomial of degree $2n+1$. through the pts.

So we want to build $p(x)$ st

$$p(x_j) = f(x_j) \quad \& \quad p'(x_j) = f'(x_j) \quad \forall j = 0, \dots, n$$

We want to write our polynomial in terms of basis function that behave "like" Lagrange polynomials.

$$p(x) = \sum_{j=0}^{n} f(x_j)\, Q_j(x) + \sum_{j=0}^{n} R_j(x)\, f'(x_j)$$

What properties do we want of $Q_j(x)$?

$$Q_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$Q_j'(x_i) = 0 \quad \forall i$$

What properties do we want of $R_j(x)$?

$$R_j(x_i) = 0 \quad i = 0, \cdots, n$$

$$R_j'(x_i) = \delta_{ij}$$

$$R_j(x) = (x - x_j)\, \ell_j^2(x)$$

Let's verify the properties

$$R_j(x_i) = (x_i - x_j)\, \ell_j^2(x_i) = 0$$

$$R_j'(x) = (x - x_j)\, 2\, \ell_j(x)\, \ell_j'(x) + \ell_j^2(x)$$

$$R_j'(x_i) = 2(x_i - x_j)\, \ell_j(x_i)\, \ell_j'(x_i) + \ell_j^2(x_i)$$

if $i \neq j \Rightarrow \ell_j(x_i) = 0$ both terms have this
we get $= 0$.

if $i = j$ the first term is 0 since it has $(x_j - x_j)$
the second term is 1^2

$$\therefore R_j'(x_i) = \delta_{ij}$$

Q_j is more "fun"

$$Q_j(x) = [1 - 2(x - x_j)\, \ell_j'(x_j)]\, \ell_j^2(x)$$

Check properties:

$$Q_j(x_i) = (1 - 2(x_i - x_j)\,\ell_j'(x_j))\,\ell_j^2(x_i)$$

if $i \neq j$ $\quad \ell_j(x_i) = 0$.

if $i = j$ $\quad 2(x_i - x_j)\,\ell_j'(x_j) = 0$

$$Q_j'(x) = 2(1 - 2(x - x_j)\,\ell_j'(x_j))\,\ell_j(x)\,\ell_j'(x) + \ell_j^2(x)(-2)\,\ell_j'(x_j)$$

$$Q_j'(x_i) = 2(1 - 2(x_i - x_j)\,\ell_j'(x_j))\,\ell_j(x_i)\,\ell_j'(x_i) + \ell_j^2(x_i)(-2)\,\ell_j'(x_j)$$

if $i \neq j$ $\quad \ell_j(x_i) = 0 \Rightarrow Q_j(x_i) = 0$.

if $i = j$

$$Q_j'(x_j) = 2\,\underbrace{\ell_j(x_j)}_{=1}\,\ell_j'(x_j) + -2\,\underbrace{\ell_j^2(x_j)}_{=1}\,\ell_j'(x_j)$$

$$= 0$$

How accurate is this approximation?

Thm 3.9 If $f \in C^1[a,b]$ $\ni$ $\{x_j\}_{j=0}^n \in (a,b)$ are distinct. The unique polynomial of degree distinct. The unique polynomial of degree at most $2n+1$ agreeing w/ f & f' at $\{x_j\}_{j=0}^{n}$

is the Hermite polynomial

$$P_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)\, Q_j(x) + \sum_{j=0}^{n} f'(x_j)\, R_j(x)$$

where R_j & Q_j are as defined above.

If $f \in C^{2n+2}[a,b]$ then

$$f(x) = P_{2n+1}(x) + \frac{\left((x-x_0)(x-x_1)\cdots(x-x_n)\right)^2}{(2n+2)!} f^{(2n+2)}(\eta)$$

for some $\eta \in [a,b]$

Ex: Suppose $f(x) = ax^2 + bx + c$. How much data do we need to build $f(x)$ exactly?

order of Hermite $\rightarrow 2n+1 = 2 \leftarrow$ goal

Solve for n: $2n = 1 \Rightarrow n = 1/2$

This is impossible so take $n = 1 \Rightarrow$ 2 nodes.
you will get a exact approx.

Ex: If you use 3 pts for Hermite interpolation what order polynomial do you get?

If you use 3 pts for Hermite interpolation What order polynomial do you get?

$n+1 = 3$ pts $\Rightarrow$ $n = 2$ $\quad$ x_0, x_1, x_2 $\quad$ $f(x_0)\, f(x_1)\, f(x_2)$ $\quad$ $f'(x_0)\, f'(x_1)\, f'(x_2)$

degree order $2n+1 \Rightarrow 2(2)+1 = 5^{th}$ degree polynomial

You can extend divided difference to Hermite. This is easiest w/a table.

x_j	$f(x_j)$	$f'(x_j) / f[x_j, x_{j+1}]$
x_0	$f(x_0)$	$f'(x_0)$
x_0	$f(x_0)$	$f[x_0, x_1]$
x_1	$f(x_1)$	$f'(x_1)$
x_1	$f(x_1)$	$f[x_1, x_2]$
x_2	$f(x_2)$	$f'(x_2)$
x_2	$f(x_2)$	

Use divide difference formulas

$$P_5(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^2(x - x_1) + \cdots$$