Splines

Wednesday, October 18, 2023 1:26 PM

Goal for interpolation:

make a function that goes through a set of Data pts $\{(x_j, f(x_j))\}_{j=0}^{n}$

So far:

- 4 ways to create a "global polynomial" approximate
  - monomials
  - Lagrange
  - Barycentric
  - Newton DD

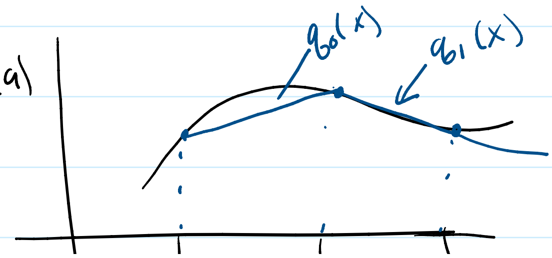
If we have $\{(x_j, f(x_j)), (x_j, f'(x_j))\}_{j=0}^{n}$

we only have one way = Hermite

§3.5 Splines

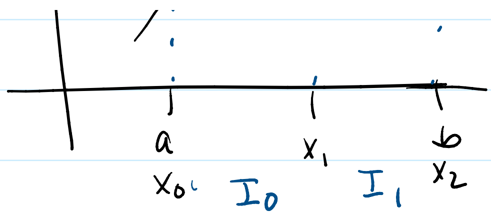
Instead of "global" approximations on an interval $[a,b]$ w/ one polynomial, we will create a bunch of polynomial approximations defined on subintervals.

Ex: (a)

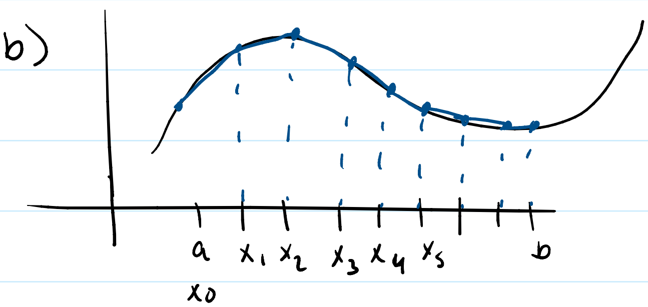


Piecewise linear

linear spline



(b)

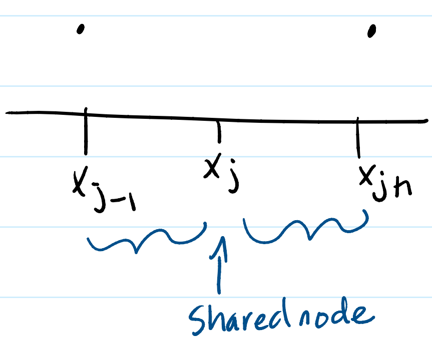


Piecewise linear approximation is easy to understand but it provides a poor approximation. We need more information/data to get better approximations.

let's try for higher order & enforce continuity of the 1^{st} & 2^{nd} derivatives.

let S denote the piecewise defined polynomial.

← enforce S, S' & S'' are continuous

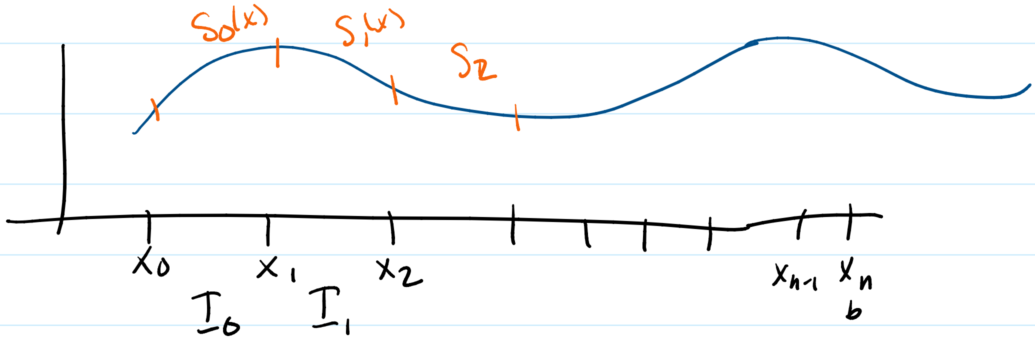


Our given data is the same as before $\{(x_j, f(x_j)\}_{j=0}^{n}$

and we want to build our approximation to be in C^2

Let's build the approximation in a piecewise manner.

Introduce Notation



let $S_i(x)$ denote the polynomial approx. on interval $I_i = [x_i, x_{i+1}]$

Wish list for $S_i(x)$

$$S_i(x_i) = f(x_i) \qquad S_i(x_{i+1}) = f(x_{i+1})$$

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1})$$

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1})$$

4 pieces of data. to create 1 Polynomial

4 pieces of data $\Rightarrow$ degree 3 approximation.

Note: At the endpts, there is not enough

Note: At the endpts, there is not enough info to make a cubic.
There are 2 standard approaches for dealing w/ this.

1- Natural spline (free boundary)

$$S_0''(x_0) = S_{n-1}''(x_n) = 0$$

2- Clamped boundary

$$S_0'(x_0) = f'(x_0) \quad S_{n-1}'(x_n) = f'(x_n)$$

let's build $S_i(x)$

let's write $S_i''(x)$ in a Lagrange form

$$S_i''(x) = M_i \frac{(x_{i+1} - x)}{h_i} + \frac{M_{i+1}(x - x_i)}{h_i}$$

where $h_i = x_{i+1} - x_i$ & M_i are unknowns

- The enforcement of C^2 is done.

Integrate twice to get $S_i(x)$

$$S_i(x) = \frac{(x_{i+1}-x)^3}{6h_i} M_i + \frac{(x-x_i)^3}{6h_i} M_{i+1} + C(x_{i+1}-x)$$
$$+ D(x-x_i)$$

Now let's plug in our given data

$$S_i(x_i) = f(x_i) = \frac{h_i^3 M_i}{6h_i} + Ch_i$$

$$\Rightarrow C = \frac{f(x_i)}{h_i} - \frac{h_i}{6} M_i$$

$$S_i(x_{i+1}) = f(x_{i+1}) = \frac{h_i^3 M_{i+1}}{h_i} + Dh_i$$

$$\Rightarrow D = \frac{f(x_{i+1})}{h_i} - \frac{h_i}{6} M_{i+1}$$

let's update $S_i(x)$ w/ our constants

On $I_i = [x_i, x_{i+1}]$

$$S_i'(x) = \frac{-(x_{i+1}-x)^2 M_i + (x-x_i)^2 M_{i+1}}{2h_i} + \frac{f(x_{i+1}) - f(x_i)}{h_i} - \frac{(M_{i+1} - x_i)h}{6}$$

on $I_{i-1} = [x_{i-1}, x_i]$

$$S_{i-1}'(x) = \frac{-(x_i - x)^2 M_{i-1} + (x - x_{i-1})^2 M_i}{2h_{i-1}} + \frac{f(x_i) - f(x_{i-1})}{h_{i-1}} - \left(\frac{M_i - M_{i-1}}{6}\right) h_{i-1}$$

We need these to be continuous (or equal) at x_i

$$S_i'(x_i) = \frac{-h_i^2 M_i}{2h_i} + \frac{f(x_{i+1}) - f(x_i)}{h_i} - \frac{(M_{i+1} - M_i)h_i}{6}$$

$$S_{i-1}'(x_i) = \frac{h_{i-1}^2 M_i}{2h_{i-1}} + \frac{f(x_i) - f(x_{i-1})}{h_{i-1}} - \left(\frac{M_i - M_{i-1}}{6}\right) h_{i-1}$$

Set these equal & collect M_i's on one

Set these equal & collect M_i's on one side (put the f's on the other side)

$$\frac{h_{i-1}}{6} M_{i-1} + \frac{h_i + h_{i+1}}{3} M_i + \frac{h_i}{6} M_{i+1} = \frac{f(x_{i+1}) - f(x_i)}{h_i} - \frac{f(x_i) - f(x_{i-1})}{h_{i-1}}$$

for $i = 1, \dots, n-1$

But what about endpts?

Natural spline: $S_0''(x_0) = 0 \quad S_{n-1}''(x_n) = 0$

$\Rightarrow M_0 = 0 \quad M_n = 0$

The system that we have solve is

Independent of boundary condition

$$\begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ \frac{h_0}{6} & \frac{h_1 + h_0}{3} & \frac{h_1}{6} & \cdots & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{h_{n-2}}{6} & \frac{h_{n-1} + h_{n-2}}{3} & \frac{h_{n-1}}{6} \\ 0 & \cdots & & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ \vdots \\ \vdots \\ M_n \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{f(x_2) - f(x_1)}{h_1} - \frac{f(x_1) - f(x_0)}{h_0} \\ \vdots \\ \vdots \\ \frac{f(x_n) - f(x_{n-1})}{h_{n-1}} - \frac{f(x_{n-1}) - f(x_{n-2})}{h_{n-2}} \\ 0 \end{bmatrix}$$

What about the clamped spline?

Recall $S_0'(x_0) = f'(x_0)$ $\quad S_{n-1}'(x_n) = f'(x_n)$

$$S_0'(x) = \frac{-(x_1-x)^2 M_0 - (x-x_0)^2 M_1}{2h_0} + \frac{f(x_1) - f(x_0)}{h_0} - \left(\frac{M_1 - M_0}{6}\right) h_0$$

Plug in x_0 to get

$$S_0'(x_0) = \frac{-h_0^2 M_0}{2h} + \frac{f(x_1) - f(x_0)}{h_0} - \left(\frac{M_1 - M_0}{6}\right) h_0 = f'(x_0)$$

$$\frac{h_0}{3} M_0 + \frac{h_0}{6} M_1 = -f'(x_0) + \frac{f(x_1) - f(x_0)}{h_0}$$

You get a similar equation for the other end pt. This results in the following linear system

$$\begin{bmatrix} \frac{h_0}{3} & \frac{h_0}{6} & \cdots & & & 0 \\ \frac{h_0}{6} & \frac{h_1+h_0}{3} & \frac{h_1}{6} & \cdots & & \\ 0 & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{h_{n-2}}{6} & \frac{h_{n-1}+h_{n-2}}{3} & \frac{h_{n-1}}{6} \\ 0 & \cdots & & 0 & \frac{h_{n-1}}{6} & \frac{h_{n-1}}{3} \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \vdots \\ \vdots \\ M_n \end{bmatrix} = \begin{bmatrix} -f'(x_0) + \frac{f(x_1) - f(x_0)}{h_0} \\ \frac{f(x_2) - f(x_1)}{h_1} - \frac{f(x_1) - f(x_0)}{h_0} \\ \vdots \\ \frac{f(x_n) - f(x_{n-1})}{h_{n-1}} - \frac{f(x_{n-1}) - f(x_{n-2})}{h_{n-2}} \\ -f'(x_n) + \frac{f(x_n) - f(x_{n-1})}{h_{n-1}} \end{bmatrix}$$