Homework 3

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Markov Processes
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1. $\dfrac{f(x)}{g(x)} = \dfrac{\sqrt{\pi}}{2\sqrt{2}} e^{x^2/2 - x}$

$\lim_{x\to\infty} \dfrac{\sqrt{\pi}}{2\sqrt{2}} e^{x^2/2 - x} = \infty$

Since the limit as $x \to \infty$ of $f(x)/g(x)$ is ∞, there is no constant c such that $f(x) \leq cg(x)$ for all $x \in \mathbb{R}$.

2. a. CDF: $F(x) = \int_{-\pi/2} \cos(x)/2 \, dx = \sin(x)/2 + 1/2$

Inverse of cdf: $u = \sin(x)/2 + 1/2 \Rightarrow x = \arcsin(2u-1)$

Pseudocode:

Generate $u \sim U(0,1)$
Calculate $x = \arcsin(2u-1)$
Return x

b. While x not accepted:
  generate $x \sim U(-\pi/2, \pi/2)$
  generate $u \sim U(0,1)$
  Calculate the acceptance probability as $\cos(x)/2$
  If $u \leq$ the acceptance probability, then accept and return x

We know that $\max(\cos(x)/2) = 1/2$, so the condition that there exists a c such that $f(x) \leq cg(x)$ where $g(x)$ is a uniform distribution is met.

3. a. $\dfrac{f(x)}{g(x)} = \dfrac{\cancel{\sqrt{2\pi}}\, e^{-x^2/2}}{\frac{1}{\pi(1+x^2)}} = \dfrac{\pi e^{-x^2/2}(1+x^2)}{\sqrt{2\pi}}$

$\dfrac{d}{dx}\left(\dfrac{\pi e^{-x^2/2}(1+x^2)}{\sqrt{2\pi}}\right) = \dfrac{\pi}{\sqrt{2\pi}}\left(-xe^{-x^2/2} - x^3e^{-x^2/2} + 2xe^{-x^2/2}\right)$

$= \pi/\sqrt{2\pi}\,(xe^{-x^2/2} - x^3e^{-x^2/2}) = \pi/\sqrt{2\pi}\, x(1+x)(1-x)e^{-x^2/2}$

This occurs at $x=-1$, $x=0$, and $x=1$. It goes from positive to negative at $x=0$.

So the minimum c is $\dfrac{f(0)}{g(0)} = \pi/\sqrt{2\pi} = \boxed{\sqrt{\dfrac{\pi}{2}}}$

b. CDF: $G(y) = \int_{-\infty}^{y} \frac{1}{\pi(1+x^2)} dx = \tan^{-1}(y)/\pi + 1/2$
Inverse: $u = \tan^{-1}(y)/\pi + 1/2$ $y = \tan(\pi u + \pi/2)$
Pseudocode:
  Generate $U \sim U(0,1)$
  Calculate $Y = \tan(\pi u + \pi/2)$
  return Y

c. Set $c = \sqrt{\pi/2}$
  While ~~Y~~ not accepted:
    Generate Y using inverse transform from (b)
    Calculate the acceptance probability $p = (f(Y)/(c \cdot g(Y)))$
    Generate $U \sim U(0,1)$
    if $(U < p)$:
      return Y

5. a. $S = \{0, 1, 2, \ldots, 99, 100\}$ representing the number of working light bulbs

b. This is a Markov Chain because the number of working light bulbs depends only on p and the number of working light bulbs in the previous step, satisfying the Markovian property

c. The number of ways to choose $j-i$ failing light bulbs in state $t+1$ is $\binom{i}{j-i}$
The probability that $j-i$ light bulbs fail is $p^{j-i}(1-p)^i$
Thus, $P(X_{t+1} = i \mid X_t = j) = \boxed{\binom{j}{j-i} p^{j-i} (1-p)^i}$

6. a. Possible Paths: $3 \to 2 \to 3 \to 2 \to 1$, $3 \to 4 \to 3 \to 2 \to 1$, $3 \to 2 \to 1 \to 1 \to 1$

$P(3 \to 2 \to 3 \to 2 \to 1 \mid x_0 = 3) = 1/4 \cdot 3/4 \cdot 1/4 \cdot 1/4 = 3/256$

$P(3 \to 4 \to 3 \to 2 \to 1 \mid x_0 = 3) = 3/4 \cdot 1/4 \cdot 1/4 \cdot 1/4 = 3/256$

$P(3 \to 2 \to 1 \to 1 \to 1 \mid x_0 = 3) = 1/4 \cdot 1/4 \cdot 1 \cdot 1 = 1/16$

Thus, $P(x_4 = 1 \mid x_0 = 3) = 3/256 + 3/256 + 1/16 = 22/256 = 0.086$

b. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

c. $A^4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.298 & 0.07 & 0 & 0.211 & 0.422 \\ 0.086 & 0 & 0.141 & 0 & 0.773 \\ 0.016 & 0.023 & 0 & 0.07 & 0.891 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The probability of getting to state 1 at X_4 from state 3 at X_0 is $A^4(3,1)$, which is 0.086, same as in part A

d. $A^{100} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.325 & 0 & 0 & 0 & 0.625 \\ 0.1 & 0 & 0 & 0 & 0.9 \\ 0.025 & 0 & 0 & 0 & 0.975 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$P(X_{100} = 1 \mid X_0 = 3) = A^{100}(3,1) = \boxed{0.01}$

7. a. $S = \{0, 1, 2, 3, 4\}$

b. $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

c. The shortest possible path from 0 to 4 is 4 steps

$A^4 = \begin{bmatrix} 0.222 & 0 & 0.667 & 0 & 0.111 \\ 0 & 0.556 & 0 & 0.444 & 0 \\ 0.167 & 0 & 0.667 & 0 & 0.167 \\ 0 & 0.444 & 0 & 0.556 & 0 \\ 0.111 & 0 & 0.667 & 0 & 0.222 \end{bmatrix}$

$P(X_4 = 4 \mid X_0 = 0) = A^4(0,4) = 0.111$

8. a. $\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.1 & 0.7 & 0.2 \\ 0.2 & 0.8 & 0 \end{bmatrix}$

b. $A^7 = \begin{bmatrix} 0.108 & 0.697 & 0.194 \\ 0.109 & 0.697 & 0.194 \\ 0.109 & 0.698 & 0.194 \end{bmatrix}$ $P(X_7 = 1 \mid X_0 = 1) = A^7(1,1) = \boxed{0.108}$

c. $A^{50} = \begin{bmatrix} 0.109 & 0.698 & 0.194 \\ 0.109 & 0.698 & 0.194 \\ 0.109 & 0.698 & 0.194 \end{bmatrix}$

Dish 2 will be eaten on any day in the far future with probability $\boxed{0.698}$