Equilibrium Solutions to Two-Player Zero Sum Games via Game Theory

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Abstract

Strategic games are a fun way to pass time, but finding the action that maximizes your chances of winning can be challenging. When it depends on the actions of another player it can become even more difficult. Our goal is to find said strategy for two-player zero-sum games. In particular, we will analyze Rock-Paper-Scissors and an arbitrary example involving rough payoffs. Our methods allow us to determine whether a Nash equilibrium solution exists and, if so, determine whether an r-trust maximin equilibrium solution exists.

1 Introduction

Game theory, originally developed by John Von Neumann, is a branch of mathematics that players use to make decisions in their respective game. Usually, it involves taking into account their opponent's possible strategies in order to produce the most optimal decision for the player. In this way, the outcome of the game does not rely on pure chance but rather strategy. Game theory can be applied to a wide variety of games, each with a different desirable outcome.

In addition, there are many applications of game theory in economic and management problems, social policy, and international and national politics. For this project, we chose to focus on two-player zero-sum games. We will be using linear algebra to analyze pay-off matrices and form equilibrium solutions to a variety of two-player zero-sum games.

2 Introduction to Two-Player Zero-Sum Games

In game theory, a two-player zero-sum game is a two person game in which the amount won by one person is equivalent to the combined losses of their opponent. Poker is an example of a zero-sum game, and chess and tennis are examples of two-player zero-sum games. Chess is a two-player zero-sum game because a win for player 1 is of equal magnitude to a loss for player 2, even though there are no numerical payoffs. Singles tennis is a two-player zero-sum game for the same reason.

Figure 1 is a representation of the payoffs of a zero-sum game. The red player can use either strategy 1 or 2 and the blue player can use either strategy A or B. Each of the four potential combinations of strategies used results in an amount won by one player, which is the same amount lost by the other.

Red \ Blue	A	B
1	-30 / 30	10 / -10
2	10 / -10	-20 / 20

Figure 1: Zero Sum Game [3]

3 Introduction to Payoff Matrices

For this project we will be utilizing payoff matrices to construct equilibrium solutions. Payoff matrices are vital to analyzing games and are especially useful when considering two-player zero-sum games. It can be used to determine a dominant strategy and if an equilibrium solution exists.

A payoff matrix is a table where the strategies of one player are listed in the rows and that of the other player in the columns. Each cell represents the payoff of a specific combination of a player one and player two move. This means that the payoff matrix can be constructed by the unique rules of the two-person zero-sum game. Cell a_{11} in the matrix below, for example, contains one value which represents the payoff for player A when they use the strategy denoted by row 1 and player B uses the strategy denoted by column 1. In a two-player zero-sum game, if P denotes the payoff matrix of player A, the payoff matrix of player B would equal to -P.

$$\text{P=}\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Figure 2 is an example of a payoff matrix that involves two different Firms and their strategy for pricing. The blue arrows represent potential changes in strategy that each player could make to increase their payoff when at a certain cell.

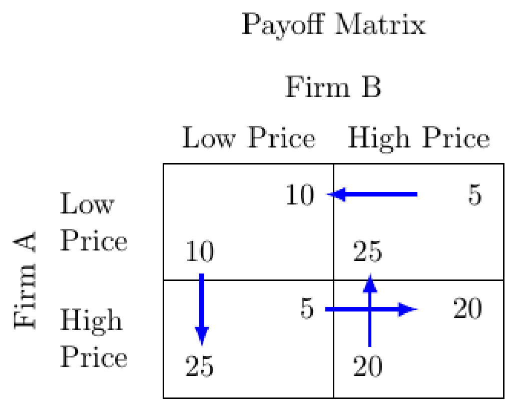

Figure 2: Payoff Matrix [4]

A particular strategy of player A is called a mixed strategy if it satisfies the following:

$$x^T * e_m = 1,$$

where x is a vector in R^m with entries greater than or equal to zero and e is a $m \times 1$. In a mixed strategy, a player does not chose one definite action (like choosing rock every time in Rock-Paper-Scissors), but rather chooses random actions in order to not be predictable. In a mixed strategy, a player picks from a number of pure strategies, which consist of every strategy in the set of strategies available. A pure strategy can be represented as $s_k = (0, 0..., 1, ..., 0)$.

If both player A and player B have mixed strategies, then using the payoff matrix, the specific expected payoff of player A is defined by

$$x^T P y = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j.$$

Now that the payoff can be calculated for each specific player based on their mixed strategies, both player's optimal strategies can be determined. If a mixed strategy denoted as x^* is considered an optimal strategy, then it must result in a larger expected payoff than any other strategy, x, that player A can pick. To ensure that the strategies of x do not vary based on the strategy of y, x^* and x will be compared to the optimal strategy of player B, y^*. Using the formula to determine the specific payoff of player A, this concept can be illustrated as

$$x^T P y^* \leq x^{*T} P y^* \text{ and } x^T P y^* \leq x^{*T} P y.$$

4 Introduction to Rough Variables

In game theory a rough variable is a variable that belongs to a rough set. A rough set is a set that gives a lower and upper approximation of the original set, as shown in the Figure 3.

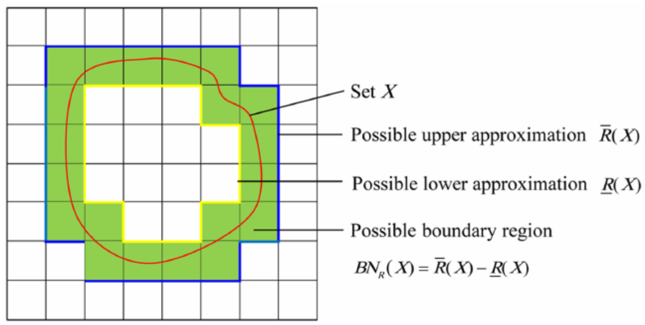

Figure 3: Rough Set Theory [5]

Here, R is the similarity relationship on U, $\underline{R}(X) = \{x \in U | R^{-1}(x) \subset X\}$, and $\overline{R}(X) = \cup_{x \in X} R(x)$.

Let ξ_{ij} be the payoff that player 1 receives and player 2 loses. A rough payoff matrix would then be defined as

$$\text{P}= \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn} \end{bmatrix}.$$

Example: Chinese Poker [1] Each team is constructed of two people. For this example, let Team A be the dealer. The rules are then as follows:

1. If the score Team B gets is less than 40, Team A goes on being a dealer and rises of one grade, denoted as +1.
2. If the score Team B gets is between 40 and 80, Team B becomes the dealer, denoted as 0.
3. If the score Team B gets is more than 80, Team B becomes the dealer and rises of one grade, denoted as 1.

The universe is defined to be $[0, 100]$. The rough variable is then calculated to be $\xi = ([40, 80], [0, 100])$. We can then define ξ_{ij} to be the payoff that Team A receives and Team B looses to construct the payoff matrix above.

If Team A chooses strategy x and Team B chooses strategy y, then the rough payoffs for Team A are

$$x^T Py = \sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_j.$$

5 Introduction to Equilibrium Solutions

In game theory, a Nash equilibrium is a strategy that is both players best response to the strategy of their opponent. A pure Nash equilibrium strategy is where the player makes the same choice every time. A mixed Nash equilibrium is where a probability is assigned to each action. In some cases, there is no pure Nash equilibrium or more than one, and so a mixed Nash equilibrium is used as a solution.

If it is assumed that both players have concrete strategies, a rough payoff of the game can be determined. If player's payoffs can be determined, a mixed Nash equilibrium point can be found and applied to a two-person zero-sum game.

A Nash equilibrium point satisfies 4 conditions:

1. Outcome is independent of what utility function is used
2. Both players can not do better at the same time
3. The outcome is independent of alternatives
4. The outcome is symmetrical

Figure 4 is an example of an Nash equilibrium solution where $(5, 5)$ is the best outcome.

Payoff Matrix		Samsung			
		User Needs First		Carrier Needs First	
Motorola	User Needs First	Nash Equilibrium 5	5	8	2
	Carrier Needs First	2	8	3	3

Figure 4: Nash Equilibrium [6]

6 Introduction to r-trust Maximin Equilibrium Strategy

Using the uncertainties of rough variables, measured by trust, an r-trust maximin equilibrium strategy can be determined. The confidence level, r, is measured by said trust level which functions similar to probability in a probability space. This value is typically pre-selected. We will not directly address this in this paper, but it can be noted that a Genetic Algorithm [1] can be used to solve the r-trust equilibrium strategy which can be used to ultimately find the global optimal solutions.

7 Mathematical Formulation

When dealing with rough payoffs, optimal strategies are more difficult to attain than when working with regular payoffs. For example, expressing a win as a 1, a loss as -1, and a tie as 0 will directly result in payoff matrix and an equilibrium solution that guarantees the optimal strategy for each player. Using rough variables as defined above to correspond to the amount player A wins or player B looses, a rough payoff matrix and a rough expected maximin equilibrium strategy can be calculated following the definition:

Definition 3.1. [1] *Let rough variable $\xi_{ij}(i = 1, 2, ..., m, j = 1, 2, ...n)$ represent the payoffs that the player receives or player II loses when player I gives the strategy and player II gives the strategy . Then (x^*, y^*) is called a rough expected maximin equilibrium strategy if* $E[x^T Py^*] \leq E[x^{*T} Py^*] \leq E[x^{*T} Py]$ *where P is defined as P=*

$$\begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{m1} & \xi_{m2} & \cdots & \xi_{mn}. \end{bmatrix}.$$

Using matrix multiplication, the inequality defined above is comparing the maximin equilibrium strategies, (x^*, y^*), to any other strategies, (x, y). This ensures that the value of the expected payoff using the equilibrium strategy remains the largest for both players.

It should be noted that the term maximin corresponds to the best action a player could take to assure maximum payoff after assessing the worst payoffs associated with each action.

Since the rough variables ξ_{ij} are independent (meaning they are not influenced by the other variables), and x and y are mixed strategies, then by the properties of linearity, we can simplify the expected payoffs of strategies x and y to be a crisp number using the remark:

Remark 3.2 [1] $E[x^T Py] = E[\sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_i] = \sum_{j=1}^{n} \sum_{i=1}^{m} E[\xi_{ij}] x_i y_j$.

Next, an r-trust maximin equilibrium strategy is introduced in the context of the rough payoff matrix. The r value can be equated to the trust level of payoffs if player A and player B both give pure strategies. In order to determine if the equilibrium solution is a r-trust maximin equilibrium solution, we must consider the trust measure of the product of the strategies and the payoff matrix, and then compare this value to r,

Definition 3.3. [1] *Let rough variable $\xi_{ij}(i = 1, 2, ..., m, j = 1, 2, ...n)$ represent the payoffs that the player receives or player II loses when player I gives the pure strategy i and player II gives the pure strategy j. r is the predetermined level of the payoffs, $r \in R$. Then (x^*, y^*) is called the r-trust*

maximin equilibrium strategy if
$Tr\{x^T Py^*\} \leq Tr\{x^{*T} Py^* \geq r\} \leq Tr\{x^{*T} Py \geq r\}$ *where* P *is defined above.*

From Definition 3.1 it is known that a rough payoff matrix can be calculated from rough variables using strategies from player A and player B. To make the calculation of expected Nash equilibrium easier, we compare pure strategies, labeled s_k for player A, and s_t for player B to the rough variables. This will help determine if a Nash equilibrium point exists. The following inequality can be used:

$$E[s_k^T Py^*] \leq E[x^{*T} Py^*] \leq E[x^{*T} Ps_t].$$

To consider the sufficiency of the above inequality, the following proof can be preformed:

Let $x = (x_1, x_2, ..., x_m)$ be a mixed strategy for player A. For every k where k=1,2,...,m, x_k is multiplied to both sides of the inequality. Then

$$\begin{aligned} & E[s_k^T Py^*]x_k \leq E[x^{*T} Py^*]x_k \\ \Rightarrow & \textstyle\sum_{k=1}^{m} E[s_k^T Py^*]x_k \leq \sum_{k=1}^{m} E[x^{*T} Py^*]x_k \\ \Rightarrow & E[x^T Py^*] \leq E[x^{*T} Py^*] \textstyle\sum_{k=1}^{m} x_k \\ \Rightarrow & E[x^T Py^*] \leq E[x^{*T} Py^*]. \end{aligned}$$

A similar proof can be preformed to prove $E[x^{*T} Py^*] \leq E[x^{*T} Py]$. Thus, (x^*, y^*) are maximin equilibrium strategies to the game.

To ensure that the found equilibrium strategy is an r-trust maximin equilibrium solution, we need to consider two cases:

1. r is less than a a min payoff denoted c_{ij}
2. r is greater than the max denoted d_{ij}

for certain strategies x and y. To test these two cases a rough variable $\xi_{ij} (i = 1, 2, ..., m, j = 1, 2, ..., n)$ where i represents strategies given by player A and j represents strategies from player B can be written as $\xi_{ij} = [a_{ij}, b_{ij}], [c_{ij}, d_{ij}]$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ ... & ... & ... & ... \\ b_{m1} & b_{m2} & ... & b_{mn} \end{bmatrix}$$
$$C = \begin{bmatrix} c_{11} & c_{12} & ... & c_{1n} \\ c_{21} & c_{22} & ... & c_{2n} \\ ... & ... & ... & ... \\ c_{m1} & c_{m2} & ... & c_{mn} \end{bmatrix} \quad D = \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ ... & ... & ... & ... \\ d_{m1} & d_{m2} & ... & d_{mn}. \end{bmatrix}$$

First, lets consider $r < c_{ij}$:

Theorem 3.7. [1] *If $r < \min[c_{ij}]$ for all $i = 1, 2..., n$, then all strategies (x, y) are r-trust maximin equilibrium strategies.*

No expected payoff resulting from the mixed strategies x and y will lead to a value less than r. Now, we will consider the next case, where r is greater than a max payoff denoted d_{ij} using the theorem:

Theorem 3.8. [1] *If $r < \max[d_{ij}]$ for all $i = 1, 2..., n$, then all strategies (x, y) are r-trust maximin equilibrium strategies.*

Theorem 3.7 and 3.8 deal with two specific cases. Usually, Theorem 3.7 or 3.8 can be used when r is clearly smaller than the lowest value in matrix C, or when r is clearly larger than the maximum value in matrix D. A more general theorem can be used to determine the existence of an r-trust maximin equilibrium strategy when r doesn't immediately satisfy one of those conditions:

Theorem 3.9. [1] *In a two-person zero sum game, rough variables ξ_{ij} $(i = 1, 2, ..., m, j = 1, 2, ..., n)$ represent the payoffs player I receives or player II loses, and the payoff matrix P is defined by (3.1). For a predetermined number r, if for all (x, y), they cannot satisfy anyone of the following conditions: (1) $x^T Dy \leq r$, (2) $x^T By \leq r \leq x^T Dy$, (3) $x^T Ay \leq r \leq x^T By$, (4) $x^T Cy \leq r \leq x^T Ay$, (5) $r \leq x^T Cy$, the there does not exist one r-trust maximum equilibrium strategy.*

The validity of Theorem 3.9 can be proved by comparing complicated trust measure values, with the end result:

$$Tr\{x^{*T} Py \geq r\} \geq Tr\{x^T Cy > r\},$$

which is in conflict with the the definition of r-trust maximin equilibrium strategies. This case involves the C from above, but other cases can be proved using a similar inequality.

8 Examples and Numerical Results

For the Examples and Numerical Results section of this paper, we will be providing two examples of calculating equilibrium solutions to two-player zero-sum games.

Example 1: Rock-Paper-Scissors, Without Rough Variables

Rock-Paper-Scissors is a game involving two players. Each player will choose either rock, paper, or scissors. If the players choose differently, there will be a dominant choice defined by the rules below:

1. Rock dominates Scissors
2. Scissors dominates Paper

3. Paper dominates Rock

The player that chooses the dominating choice is declared the winner and the other player is declared the loser. The winner of the round has a payoff of +1 and the loser has a payoff of -1. If both players choose the same option, both players get a payoff of 0. From these rules, we can generate a payoff matrix without the use of rough variables.

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	$0, 0$	$-1, 1$	$1, -1$
	Paper	$1, -1$	$0, 0$	$-1, 1$
	Scissors	$-1, 1$	$1, -1$	$0, 0$

[2]

Figure 5: Rock-Paper-Scissors Payoff Matrix

Now, using the payoff matrix in Figure 5, we will attempt to calculate a Nash equilibrium. Rock-Paper-Scissors has no pure Nash equilibrium, so we will be calculating the mixed Nash equilibrium.

Let $p1_{rock}$ be the probability that player 1 chooses rock, $p1_{scissors}$ the probability that player 1 chooses scissors, and $p1_{paper}$ the probability that player 1 chooses paper. Let $p2_{rock}$, $p2_{scissors}$, $p2_{paper}$ represents the probabilities for player 2. Finally, let $E_2[...]$ represent the expected payoff of player 2 choosing a specific option.

Using the above payoff matrix we can calculate

$$E_2[rock] = (0) * p1_{rock} + (-1) * p1_{paper} + (1) * p1_{scissors}$$
$$E_2[paper] = (1) * p1_{rock} + (0) * p1_{paper} + (-1) * p1_{scissors}$$
$$E_2[scissors] = (-1) * p1_{rock} + (1) * p1_{paper} + (0) * p1_{scissors}.$$

We know that

$$E_2[rock] = E_2[paper]$$
$$E_2[paper] = E_2[scissors]$$

and therefore,

$$E_2[rock] = E_2[scissors].$$

Now, all the probabilities must add to 1 and so the three equations to solve the three unknowns are

$$(0) * p1_{rock} + (-1) * p1_{paper} + (1) * p1_{scissors} =$$
$$(1) * p1_{rock} + (0) * p1_{paper} + (-1) * p1_{scissors}$$
$$(1) * p1_{rock} + (0) * p1_{paper} + (-1) * p1_{scissors} =$$
$$(-1) * p1_{rock} + (1) * p1_{paper} + (0) * p1_{scissors}$$
$$p1_{rock} + p1_{paper} + p1_{scissors} = 1.$$

We then solve for each variable to find the mixed Nash equilibrium,

$$p1_{scissors} = \frac{1}{3}\ p1_{paper} = \frac{1}{3}\ p1_{rock} = \frac{1}{3}.$$

This means that if player 1 chooses rock $\frac{1}{3}$ of the time, paper $\frac{1}{3}$ of the time, and scissors $\frac{1}{3}$ of the time, it will make player 2 indifferent over his pure strategies.

Since in Rock-Paper-Scissors the payoffs are symmetric, the Mixed Nash equilibrium for player 2 is the same as player 1,

$$p2_{scissors} = \frac{1}{3}\ p2_{paper} = \frac{1}{3}\ p2_{rock} = \frac{1}{3}.$$

Example 2: Game with Rough Variables [1]

For this second example, we will be considering a made up game which involves two players, player A and player B.

If player A chooses strategy i and player B chooses strategy j, then player B will pay player A a specific value based off their strategies. This value will be between a_{ij} and c_{ij} at the least and between b_{ij} and d_{ij} at the most. We can then represent each ξ_{ij} as $([a_{ij}, c_{ij}], [b_{ij}, d_{ij}])$ in the payoff matrix below.
For the sake of the example, these values will be made up,

$$\begin{array}{ccc} \xi_{11} = ([15, 25], [10, 28]) & \xi_{12} = ([13.5, 22], [8, 25]) & \xi_{13} = ([15, 20], [11.2, 21]) \\ \xi_{21} = ([17, 30], [9, 35]) & \xi_{22} = ([16.2, 26], [12, 28]) & \xi_{23} = ([13, 27], [10, 30]) \\ \xi_{31} = ([18, 20], [11, 24]) & \xi_{32} = ([18, 24], [12, 29]) & \xi_{33} = ([13, 20], [12, 25]). \end{array}$$

Using the mathematical definition of a rough payoff from Section 4, Introduction to Rough Variables and Remark 3.2 from Section 6, Mathematical Formulation, we get the following equation,

$$E[x^T Py] = E[\sum_{j=1}^{n} \sum_{i=1}^{m} \xi_{ij} x_i y_j] = \sum_{j=1}^{n} \sum_{i=1}^{m} E[\xi_{ij}] x_i y_j = x^T P' y.$$

We calculate P' by averaging the values $a_{ij}, c_{ij}, b_{ij}, d_{ij}$,

$$\begin{bmatrix} \mathrm{E}[\xi_{11}] & \mathrm{E}[\xi_{12}] & \mathrm{E}[\xi_{13}] \\ \mathrm{E}[\xi_{21}] & \mathrm{E}[\xi_{22}] & \mathrm{E}[\xi_{23}] \\ \mathrm{E}[\xi_{31}] & \mathrm{E}[\xi_{32}] & \mathrm{E}[\xi_{33}] \end{bmatrix} = \begin{bmatrix} 19.5 & 17.125 & 16.8 \\ 22.75 & 20.55 & 20 \\ 18.25 & 20.75 & 17.5 \end{bmatrix}.$$

From this matrix we get the maximin equilibrium solution to be 20. Here, player A gains the most while minimizing the loss for player B, therefore it is a maximin solution. Then, we use the equation from above to solve for x and y. The equilibrium solution is calculated to be when player A chooses $x = (0, 0, 1)$ and player B chooses $y = (0, 1, 0)$.

However, we will continue to test for different $r-$trust maximin equilibrium.

We can separate each ξ_{ij} into four different matrices, where a_{ij} values are placed into matrix A, b_{ij} values into matrix B, and so on,

$$A = \begin{bmatrix} 15 & 13.5 & 15 \\ 17 & 16.2 & 13 \\ 18 & 18 & 13 \end{bmatrix} \quad B = \begin{bmatrix} 25 & 22 & 20 \\ 30 & 26 & 27 \\ 20 & 24 & 20 \end{bmatrix}$$
$$C = \begin{bmatrix} 10 & 8 & 11.2 \\ 9 & 12 & 10 \\ 11 & 12 & 12 \end{bmatrix} \quad D = \begin{bmatrix} 28 & 25 & 21 \\ 35 & 28 & 30 \\ 24 & 29 & 25 \end{bmatrix}.$$

We will consider three different predetermined r values and determine if they are $r-$ trust maximins or not.

- *Case 1 (r=5)* The minimum value of matrix C is 8 and $0 \leq x, y \leq 1$. This means that (x, y) satisfy the inequality $x^T Cy \geq 5$. We can then use Theorem 3.7, mentioned in Section 6, Mathematical Formulation to say that all (x, y) are $5-$trust maximin equilibrium strategy for this example.
- *Case 2 (r=40)* The maximum value of matrix D is 35 and $0 \leq x, y \leq 1$. This means that (x, y) satisfy the inequality $x^T Cy \leq 40$. We can then use Theorem 3.8, mentioned in Section 6, Mathematical Formulation to say that all (x, y) are $40-$trust maximin equilibrium strategy for this example.
- *Case 3 (r=25)* The maximum value of matrix B is 30, the minimum value of matrix D is 21, and $0 \leq x, y \leq 1$. This means that not all (x, y) satisfy $x^T By \leq 25 \leq x^T Dy$. Therefore, according to Theorem 3.9, mentioned in Section 6, Mathematical Formulation this example does not have a $25-$trust maximin equilibrium strategy.

To continue the problem, it is possible to use a Genetic Algorithm mentioned in Section 4 of *A Class of Two-Person Zero-Sum Matrix Games with Rough Payoffs*[1] to solve for optimal solutions.

9 Discussion and Conclusion

We discovered that the equilibrium strategy in Rock-Paper-Scissors is to use each of the three options an equal percentage of the time. In the arbitrary game with rough payoffs, the equilibrium strategy was for player 1 to use strategy (0,0,1) and player 2 to use strategy (0,1,0) which we were able to determine by finding the expected rough payoffs. We also determined that two of our three test values of r resulted in the existence of an r-trust maximin equilibrium strategy. These results are only pertinent to two specific games, but these methods could be applied to other types of games, such as non-zero sum games, to determine Nash equilibrium solutions. In addition, our research could be furthered by using the aforementioned Genetic Algorithm to find the r- trust maximin equilibrium solutions in the cases where they exist.

10 Attribution

Olivia designed research and performed experiments; Alex analyzed the results; Gabreece designed research and performed experiments; all three contributed to preparing the report.

References

[1] Jiuping Xu, Liming Yao. *A Class of Two-Person Zero-Sum Matrix Games with Rough Payoffs.* Attila Gilanyi, *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 404792, 22 pages, 2010.

[2] "Mixed Strategy Nash Equilibrium." *Games, Strategies, and Decision Making, Second Edition* by Joseph E. Harrington, Jr., Worth Publishers, 2009.

[3] *Humanity Is Transcending Out Of A ZeroSum Ego Based Game, Evolving Into The NonZeroSum Heart Consciousness* Dr. Goodhart *A Green Road Journal*, www.agreenroadjournal.com/2017/10/humanity-is-transcending-out-of-zerosum.html.

[4] "Spieltheorie Auszahlungsmatrix mit Pfeilen zur Darstellung von Equilibria." JohannesB *TeXwelt*, https://texwelt.de/fragen/14775/spieltheorie-auszahlungsmatrix-mit-pfeilen-zur-darstellung-von-equilibria.

[5] *Rough Set Theory — An Introduction* AvikDutta *Geeks for Geeks* https://www.geeksforgeeks.org/rough-set-theory-an-introduction/.

[6] *The Basics of Game Theory: Dominant Strategies and Nash Equilibrium* Jake Nielson *Ignition Framework* https://www.ignitionframework.com/basics-game-theory-make-right-decision-every-time/.