Friday, December 8, 2023 1:56 PM

Power Method.

- a technique for approximating 1 eigenvalue.

Let $A \in \mathbb{R}^{n \times n}$ be full rank.
let $\{(\lambda_i, v_i)\}_{i=1}^{n}$ denote eigenpairs.
& eigenvalues are ordered

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0$$

(Note: we are also assuming the eigenvalues are distinct)

We know that eigenvector form a basis for $\mathbb{R}^n$.
$\Rightarrow$ for any $x \in \mathbb{R}^n$, $\exists \{\alpha_j\}_{j=1}^{n}$ st

$$x = \sum_{j=1}^{n} \alpha_j v_j$$

$$Ax = \sum_{j=1}^{n} \alpha_j \underbrace{Av_j}_{\lambda_j v_j} = \sum_{j=1}^{n} \alpha_j \lambda_j v_j$$

$$A^2 x = \sum_{j=1}^{n} \alpha_j \lambda_j A v_j = \sum_{j=1}^{n} \alpha_j \lambda_j^2 v_j$$

$$A^m x = \sum_{j=1}^{n} \alpha_j \lambda_j^m v_j$$

$$= \lambda_1^m \left(\alpha_1 v_1 + \sum_{j=2}^{n} \alpha_j \left(\frac{\lambda_j}{\lambda_1} \right)^m v_j \right)$$

$$\left|\frac{\lambda_j}{\lambda_1}\right| < 1 \quad \text{for } j = 2, \cdots, n$$

as $m \to \infty$ $\left(\frac{\lambda_j}{\lambda_1}\right)^m \to 0.$

For m large

$$A^m x \approx \lambda_1^m \alpha_1 v_1$$

How do we get λ_1 & v_1?

let $w_1 = \dfrac{\alpha_1 \lambda_1^m v_1}{\|\alpha_1 \lambda_1^m v_1\|}$ $\Rightarrow$ unit length

We can get λ_1 now. since $Aw_1 = \lambda_1 w_1$

So compute Aw_1 & evaluate

$$\lambda_1 \underbrace{w_1^* w_1}_{\|w_1\|^2} = \lambda_1$$

Question: What is the convergence Rate?

$$O(\lambda_2/\lambda_1)$$

Question: What if I want λ_2 not λ_1?

$A - \lambda_1 I$

Apply the power method to this matrix.

The method is called shifted power method.