Problem Set 4

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1. a. Let $z = t + ti$, $dz = (1+i)\,dt$

$\bar{z} = \overline{(t+ti)} = t - ti$

$\bar{z}^2 = (t-ti)^2 = t^2 - 2ti - t^2 = -2ti$

$\int_0^{1+i} \bar{z}^2\,dz = \int_0^1 -2ti(1+i)\,dt = \int_0^1 -2ti + 2t\,dt =$

$-it^2 + t^2 \Big|_0^1 = \boxed{1-i}$

b. Let $z = t + ti$, $dz = (1+i)\,dt$

$e^z = e^{t+it}$

$\int_0^{1+i} e^z\,dz = \int_0^1 e^{t+it}(1+i)\,dt = (1+i)e^{t+it} + (i-1)e^{t+it}\Big|_0^1$

$= 2ie^{t+it}\Big|_0^1 = 2ie^{1+i}$

2. a. This has a singularity at $z=0$ of order 2

The residue is $\frac{1}{1!}\lim_{z\to 0} \frac{d}{dz}\left(z^2 \frac{e^{iz}}{z^2}\right) = \lim_{z\to 0} ie^{iz} = i$

$\oint_C e^{iz}/z^2 = 2\pi i \cdot \text{residue} = \boxed{-2\pi}$

b. This has a singularity at $z=0$ of order 4

Residue $= \frac{1}{3!}\lim_{z\to 0} \frac{d^3}{dz^3}\left(z^4 \frac{\sin(z)}{z^4}\right) = \frac{1}{6}\lim_{z\to 0} -\cos(z) = -1/6$

$\oint_C \sin(z)/z^4 = 2\pi i \cdot \text{residue} = \boxed{-\pi i/3}$

3. a. There are no singularities in C because $|z_0| > 1$. However, the function has a branch point at $z = z_0$. If a branch cut is chosen that starts from z_0 and runs parallel to the real axis to either ∞ or $-\infty$, whichever direction goes away from the origin from z_0, then the integral is 0 by Cauchy's theorem

b. The singularity at $1/2$ is within the unit circle, while the singularity at $2i$ is outside of it.

The residue of $z = 1/2$ is $\lim_{z\to 1/2} (z - 1/2)\frac{1}{(z-1/2)(z-2i)}$

$= \lim_{z\to 1/2} \frac{1}{z-2i} = \frac{1}{1/2 - 2i}$

$\oint_C \frac{1}{(z-2i)(z-1/2)}\,dz = 2\pi i \cdot \text{residue} = 2\pi i / (1/2 - 2i) = \boxed{\frac{4\pi i}{1-4i}}$

4. The denominator of $f(z)$, $|z^2+a^2|$, is $\geq |z^2-a^2|$ for all real a. $z^2 = (Re^{i\theta})^2 = R^2e^{2i\theta}$ and $|e^{2i\theta}| = 1$, so $|z^2+a^2| \geq |R^2-a^2|$, thus

$$\left|\frac{1}{z^2+a^2}\right| \leq \left|\frac{1}{R^2-a^2}\right|$$

The ML inequality states that $|\int_C f(z)\,dz| \leq \max|f(z)| \cdot$ arclength
The arc length of a semicircle is πR, so

$$\left|\int_C f(z)\,dz\right| \leq \frac{\pi R}{R^2-a^2}$$

5. a. $\dfrac{1-z}{(z^2+1)} = \dfrac{1-z}{(z+i)(z-i)} = \dfrac{A}{z+i} + \dfrac{B}{z-i}$

$A(z-i) + B(z+i) = 1-z$

$A+B=-1 \quad iB-iA=1$

$A+A+1/i=-1 \quad 2A = -1/i - 1 \quad A = \frac{-1-i}{2i} \quad B = \frac{1-i}{2i}$

$$\oint_C \frac{1-z}{z^2+1} = \frac{1}{2i}\left(\oint_C \frac{-1-i}{(z+i)} + \frac{1-i}{z-i}\right) = \frac{(-1-i)(2\pi i)+(1-i)(2\pi i)}{2i}$$

$= \boxed{-2\pi i}$

b. $$\oint_C \sum_{j=1}^{n} \frac{a_j}{z-z_j} = \sum_{j=1}^{n} \oint_C \frac{a_j}{z-z_j} = \boxed{\sum_{j=1}^{n} a_j \cdot 2\pi i}$$

6. a. By Green's theorem, $\oint_C F_1\,dx + F_2\,dy = \iint_R (\partial F_2/\partial x - \partial F_1/\partial y)\,dA$
The mixed partials of a continuously differentiable function are equal, thus $\partial F_2/\partial x - \partial F_1/\partial y = 0$
Thus, $\iint_R (\partial F_2/\partial x - \partial F_1/\partial y)\,dA = 0$

b. $\Omega(z)$ must satisfy $\phi_x = \psi_y$ and $\phi_y = -\psi_x$

$\Omega'(z) = \phi_x + i\psi_x = \phi_x - i\phi_y$

$\oint \Omega'(z)\,dz = \oint_C (\phi_x - i\phi_y)\,dz = \oint_C \phi_x\,dx - \phi_x\, i\,dy - i\phi_y\,dx - \phi_y\,dy$

This integral must be 0 when ϕ is continuously differentiable

7. Given that $f(z)$ is analytic in the domain except for isolated singularities, $\oint f(z) = 2\pi i \cdot \Sigma$ residues

Residue $= \frac{1}{2!} \lim_{z \to 2} \frac{d^2}{dz^2} \left((z-2)^3 (z^3+z+2) / (z-2)^3 \right)$

$= 6z/2 = 3z$

$\oint_C \frac{z^3+z+2}{(z-2)^3} = 2\pi i \cdot 3z = \boxed{6\pi i z}$

8. By the extended Cauchy integral formula,

$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \right|$

$|f^{(n)}(0)| \leq \frac{n!}{2\pi i} \oint_C \frac{C|z|}{|z|^{n+1}} |dz| = \frac{n! C}{2\pi i} \oint_C \frac{1}{|z|^n} |dz|$

Since $|z| = R$ when C is a circle with radius R,

$|f^{(n)}(0)| \leq \frac{n! C}{2\pi i} \cdot \frac{2\pi R}{R^n} = |n! C / R^{n-1}$

As $R \to \infty$, this approaches 0, so all derivatives are 0, so $f(z) = Az$

9. The sequence converges $\overset{\text{to } 0}{\vee}$ for any $z \neq 0$.

$|(nz)^{-2} - 0| = 1/n^2|z|^2$

Since $|z| > \alpha > 0$, the series converges when $1/n^2\alpha^2 < \epsilon$. Given that this doesn't depend on z, the convergence is uniform.

If $\alpha = 0$, then the sequence does not converge because $1/n^2|z|^2$ can be arbitrarily large.

10. a. $|z^{4n}| < R^{4n}$ because $|z| < R$

$\sum_{n=1}^{\infty} R^{4n} = \frac{R^4}{1-R^4}$ because this is a geometric series with $R < 1$

Since this is a finite value that is $\geq \sum_{n=1}^{\infty} |z|^{4n}$, this must converge

10. b. Since only the real part of an exponential effect its magnitude, $|e^{-3nz}| = e^{-3nRe(z)}$

Since $R < Re(z)$, $e^{-3nRe(z)} \leq e^{-3nR}$

$\sum_{n=1}^{\infty} e^{-3nR} = \frac{e^{-3R}}{1-e^{-3R}}$ because $R <$

Since this is a constant that is $\geq \sum_{n=1}^{\infty} |e^{-3nz}|$, this must converge

11. We use the limit ratio test to find these radii of convergence

a. $\lim_{n\to\infty} \left| \frac{(-z)^{3(n+1)}}{(-z)^{3n}} \right| = \lim_{n\to\infty} |(-z)^3| = |z|^3$

Thus, $|z| < 1$ must be true for convergence

b. $\lim_{n\to\infty} \left| \frac{(n+1)^{n+1} z^{2(n+1)}}{n^n z^{2n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{n^n} \right| |z^2| = e|z|^2$

Thus, for the series to converge, $|z| < 1/\sqrt{e}$

12. Let $h(z) = f(z)/g(z)$

$\lim_{|z|\to\infty} h(z) = 1$. Thus, $h(z)$ must be bounded in the entire complex plane. It also must be entire because $f(z)$ and $g(z)$ are entire. Since it's entire and bounded, by Liouville's theorem, it must be a constant. Because $\lim_{|z|\to\infty} h(z) = 1$, that constant must be 1, thus $f(z) = g(z)$ for all z.