Problem Set 6

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1.a. Singularities when $1-e^z=0$ or $e^z=1$, at $z=2\pi in$

Let $e^z = e^{2\pi in + w} = e^w$ because $e^{2\pi in}=1$

$e^w = 1+w+\frac{w^2}{2}+\ldots$

$1-e^z = 1-e^w = -w - w^2/2 + \ldots$

$e^z = e^w = 1+w+w^2/2+w^3/3+\ldots$

$$\frac{e^z}{1-e^z} = \frac{1+w+w^2/2+\ldots}{-w(1+w/2+w^2/6+\ldots)} = \frac{-1}{w}\cdot\left(\frac{1+w+w^2/2+\ldots}{1+w/2+w^2/6+\ldots}\right)$$

This behaves like $-1/w$ as $w\to 0$, so the singularity is a simple pole

$$\frac{1}{1+w/2+w^2/6+\ldots} = 1-(w/2+w^2/6+\ldots)+(w/2+w^2/6+\ldots)^2 = 1-w/2-w^2/6+w^2/4+\ldots$$

$$\frac{e^z}{1-e^z} = \frac{-1}{w}\left(1+w+\frac{w^2}{2}+\ldots\right)\left(1-\frac{w}{2}-\frac{w^2}{12}\right)$$

$= -1/w - 1/2 + \ldots$

The first two terms of the Laurent series are $-1/w$ and $-1/2$

b. Singularities at $z=0$ and $z=2$

There is no Laurent series around $z=0$, it's a branch point

Around $z=2$, $w=z-2$ so $z(z-2)=(w+2)(w)=2w(1+w/2)$

$\log(z)=\log(2+w)=\log(2(1+w/2)) = \log(2)+w/2-w^2/8+\ldots$

Since w is very small, $\frac{1}{1+w/2}\approx 1-w/2$, so $\frac{1}{2w(1+w/2)} = \frac{1}{2w}-1/4$

$$\frac{\log(2+w)}{2w(1+w/2)} = (\log(2)+w/2-w^2/8+\ldots)(1/2w-1/4)$$

$$= \frac{\log(2)}{2w}+\frac{(1-\log(2))}{4}+\ldots$$

This is a simple pole with the first two terms $\frac{\log(2)}{2w}$ and $\frac{(1-\log(2))}{4}$

1\. c. There is a singularity at $z=0$

$\frac{\sinh(\sqrt{z})}{z^{3/2}} = \frac{\sqrt{z} - (\sqrt{z})^3/6 + \dots}{z^{3/2}} = \frac{1}{z} - \frac{1}{6} + \dots$

This is a simple pole with the first two terms being $1/z$ and $-1/6$

d. Singularities exist when $\sin(1/z)=0$, at $z = 1/n\pi$ $n>0$

$\cot(1/z) = z - 1/3z + \dots$ around $z=0$

This is an essential singularity because the negative powers of z continue to get larger, the first two Laurent series terms are z and $-1/3z$

At $z = 1/n\pi$, $w = z - 1/n\pi$, $\cot(w) = \frac{\sin(w)}{\cos(w)} = 1/w - w/3 + \dots$

So $z = 1/n\pi$ are all simple poles with the first two terms $1/w$ and $-w/3$

2\. a. There's a 3rd order pole at $z=0$

$f(z) = \frac{z^2+1}{z^3} = \frac{1}{z} + \frac{1}{z^3}$

The residue is 1, so $I = \frac{1}{2\pi i} \cdot 2\pi i \cdot 1 = \boxed{1}$

b. There's an essential singularity at $z=0$ because of $e^{-1/z}$

$e^{-1/z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{-n}$, so $f(z) = z^2 e^{-1/z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{2-n}$

The residue term occurs when $n=3$, which is $-1/6z$

Thus $I = 1/2\pi i \cdot 2\pi i \cdot -1/6 = \boxed{-1/6}$

3.a. $z = 1/w$, $\frac{z^{2n}+a^2}{z^{2m}+b^2} = \frac{w^{-2n}+a^2}{w^{-2m}+b^2} = \frac{a^2w^{2n}+1}{b^2w^{2n}+w^{2(n-m)}}$

Since $n > m > 0$, $w^{2(n-m)}$ goes to 0 slower than w^{2n}, so the function behaves as $1/w^{2(n-m)}$ as $w \to 0$, so this is a pole of order $2(n-m)$ at $z=\infty$

b. $z = 1/w$, $\log(z^{2}-a^2) = \log(w^{-2}-a^2)$. As $w \to 0$, this behaves like $\log(w^{-2})$. Log has a branch point at ∞ so this is a branch point at $z=\infty$.

c. $z = 1/w$, $\sin(z) = \sin(1/w) = w^{-1} - w^{-3}/3! + w^{-5}/5! - \ldots$

This is an essential singularity because the negative powers of w continue infinitely

4.a. $\text{Res}(\phi(z)/(z-z_0)^m) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz}\phi(z)|_{z=z_0}$

$= d/dz(e^{2z}) = 2e^{2z}$ which is $2e^2$ at pole $z=1$

Thus, $1/2\pi i \oint_C \frac{e^{2z}}{(z-1)^2}dz = \frac{1}{2\pi i}\cdot 2\pi i \cdot 2e^2 = \boxed{2e^2}$

b. Singularity at 0 inside contour and $\pm 2i$ outside contour

Residue $= P(0)/(0^2+4)^2 = P(0)/16$

$\frac{1}{2\pi i}\oint_{C_0} P(z)/z(z^2+4)^2 \, dz = \frac{1}{2\pi i}\cdot 2\pi i \cdot P(0)/16 = \boxed{P(0)/16}$.

5. Since $f(z)=0$ and $f'(z) \neq 0$ at N points, the contribution from these points to the integral is $2\pi i \cdot N(z)/D'(z) \cdot N$

$= 2\pi i f'(z)/f'(z) \cdot N = 2\pi i N$

For poles, $f(z) = g(z)/z-p$ and $f'(z) = \frac{g'(z)(z-p)-g(z)}{g(z)(z-p)}$

As z approaches p, $g'(z)(z-p)$ approaches 0 but $-g(z)$ does not, so the function behaves like $-g(z)/g(z)(z-p)$ or $-1/(z-p)$.

Thus, the contribution from these M poles is $2\pi i \cdot M \cdot \text{Res}$

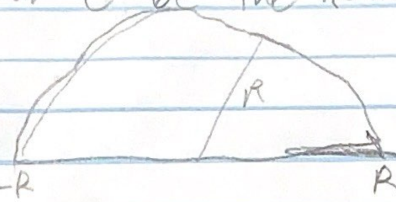
$= -2\pi i \cdot M$

Thus, $\frac{1}{2\pi i}\cdot \oint_C f'(z)/f(z)\, dz = 1/2\pi i \cdot 2\pi i \cdot (N-M) = N-M$

when $f(z)$ has N simple 0s and N simple poles.

6. Consider $f(z) = 1/(z^2+\beta^2)^2$, singularities at $z = \pm i\beta$

Let contour C be the following



Where $R > \beta$

The only pole inside this contour is $z = i\beta$ which is of order 2.

$Res(f, i\beta) = \lim_{z \to i\beta} \frac{d}{dz}[(z-i\beta)^2 f(z)] = \lim_{z \to i\beta} \frac{d}{dz} \frac{1}{(z+i\beta)^2}$

$= -2\left(1/(z+i\beta)^3\right) = -2/(2i\beta)^3 = 1/4i\beta^3$

Thus, $\oint_C f(z)dz = 2\pi i \cdot res(f, i\beta) = 2\pi i / 4i\beta^3 = \boxed{\pi/2\beta^3}$

7. a. $\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2} = e^{ikx}$ for the purpose of this integral because e^{ikx} and e^{-ikx} contribute equally

$\int_{-\infty}^{\infty} \frac{xe^{ikx}}{x^2+4x+5}$

$x^2+4x+5 = (x+2)^2+1$, thus, poles at $-2 \pm i$

Residue at $-2+i = \lim_{z \to -2+i} (z+2-i)f(z) = \lim_{z \to -2+i} \frac{ze^{ikz}}{z+2+i} = \frac{(-2+i)e^{ik(-2+i)}}{2i}$

$\int_{-\infty}^{\infty} \frac{xe^{ikx}}{x^2+4x+5} = 2\pi i \cdot \frac{(-2+i)e^{ik(-2+i)}}{2i} = \pi(-2+i)e^{ik(-2+i)}$

the real part of this corresponds to the value of the real integral

b. $x^4+1 = 0$ at $y = e^{(2n-1)i\pi/4}$ for $n = 1,2,3,4$

For an integral over a semicircle in the upper half plane, only $e^{i\pi/4}$ and $e^{3i\pi/4}$

$Res(f, e^{i\pi/4}) = \frac{e^{ik/(\sqrt{2}(1+i))}}{4i}$

$Res(f, e^{3i\pi/4}) = \frac{e^{ik/(\sqrt{2}(1+i))}}{-4}$

Thus $\int_{-\infty}^{\infty} \frac{e^{ikx}}{x^4+1} = 2\pi i \cdot Res(f, e^{i\pi/4}) + Res(f, e^{3i\pi/4}) =$

$(1/i - 1) \cdot \frac{\pi i e^{ik/(\sqrt{2}(1+i))}}{2}$

The real part of this is the solution to the real integral

8. Let C be a semicircular contour in the left half plane with Radius R where $R > |b/a|$

Pole at $z = -b/a$ inside contour

$\mathrm{Res}(f, -b/a) = \cancel{\frac{e^{t(-b/a)}}{a}} = e^{-tb/a}/a$

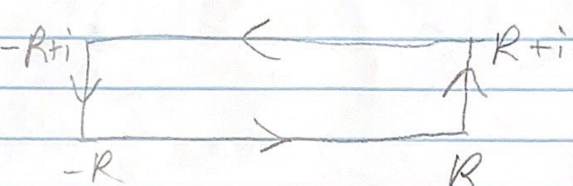
$\oint_C e^{tz}/az-b \, dz = 2\pi i \cdot \mathrm{Res}(e^{tz}/az-b, -b/a) = 2\pi i e^{-tb/a}/a$

The contribution of the semicircle vanishes by Jordan's lemma, so we're left with the line from $-iR$ to iR along the imaginary axis

$$\lim_{R \to \infty} \int_{-iR}^{iR} \frac{e^{tz}}{az+b} dz = \boxed{2\pi i \cdot \frac{e^{-tb/a}}{a}}$$

9. Singularities occur when $\cosh(\pi x) = 0$ at $x = \frac{(2n+1)i}{2}$ where n is an integer

Consider the following rectangular contour



$$\int_0^{\infty} \frac{\cosh(ax)}{\cosh(\pi x)} dx = \int_{-R}^{R} \frac{\cosh(ax)}{\cosh(\pi x)} dx + \int_{R+i}^{-R+i} \frac{\cosh((w+i)a)}{\cosh((w+i)\pi)} dx$$

Residue $= N/D' = \lim_{x \to i/2} \frac{\cosh(ax)}{\pi \sinh(\pi x)} = \frac{\cos(a/2)}{\pi i}$

The vertical segments from R to $R+i$ and $-R+i$ to $-R$ decay to 0 by Jordan's lemma

The integral along the horizontal contours is $2\pi i \cdot \mathrm{Res}$

$= 2\pi i \cdot \frac{\cos(a/2)}{\pi i} = 2\cos(a/2)$

$\int_0^{\infty} f(x)dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x)dx = \cos(a/2)$

due to symmetry

10. The poles of $\frac{1}{x^7+1}$ are at $e^{i\left(\frac{\pi+2n\pi}{7}\right)}$ for $n=1,2,3,4,5,6$

Consider a semicircular contour in the upper half plane with radius R

Only the poles where $n=0,1,2$ lie in the upper half plane

$\text{Res}\left(\frac{1}{x^7+1}, e^{i\pi/7}\right) = N/D' = 1/7x^6 = 1/7e^{6i\pi/7}$

$\text{Res}\left(\frac{1}{x^7+1}, e^{3i\pi/7}\right) = N/D' = 1/7x^6 = 1/7e^{18i\pi/7} = 1/7e^{4\pi i/7}$

$\text{Res}\left(\frac{1}{x^7+1}, e^{5i\pi/7}\right) = N/D' = 1/7x^6 = 1/7e^{30i\pi/7} = 1/7e^{2\pi i/7}$

Thus $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \frac{1}{7e^{6\pi i/7}} + \frac{1}{7e^{4\pi i/7}} + \frac{1}{7e^{2\pi i/7}}$

$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \underset{\text{due to symmetry}}{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \boxed{\frac{1}{2}\left(\frac{1}{7e^{6\pi i/7}} + \frac{1}{7e^{4\pi i/7}} + \frac{1}{7e^{2\pi i/7}}\right)}$