APPM 4600 — HOMEWORK # 8

1. Consider the task of interpolating the function $f(x) = \frac{1}{1+x^2}$ on the interval $[-5, 5]$. Using equispaced nodes with $n = 5, 10, 15$ and 20, interpolate the function using the methods below:

   (a) Lagrange interpolation.

   (b) Hermite interpolation.

   (c) Natural Cubic spline.

   (d) Clamped Cubic spline.

   Which method performs best? Do you have an intuition why? **Soln:**The code for this problem is provided below. Figure 2 reports on a semilogy plot the errors using the different methods. Changing the nodes improves the performance of the Lagrange and Hermite interpolation. In fact the Hermite interpolation is better than the other interpolation techniques. It is also the highest order.

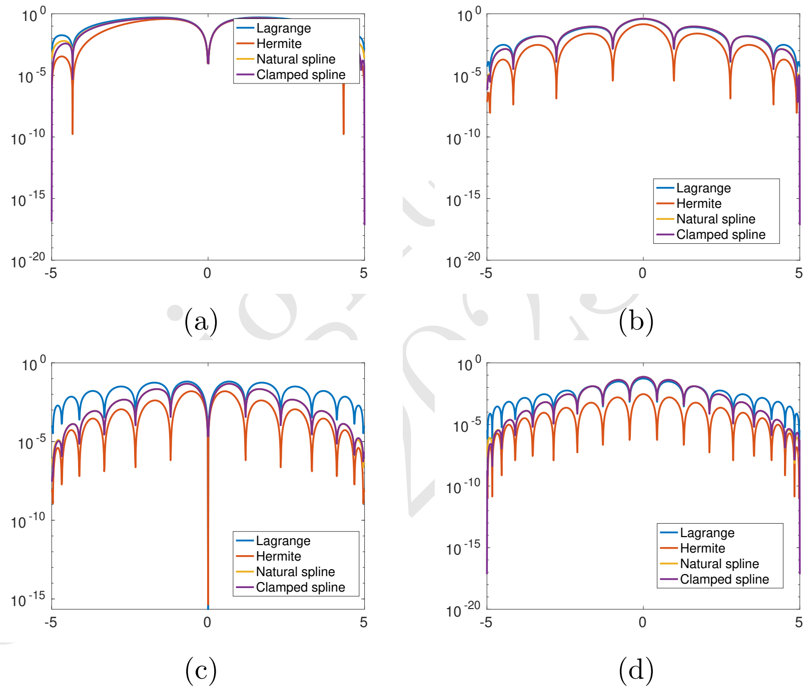

Figure 1: Illustration of errors when $n = 5, 10, 15$ and 20 respectively (a,b,c,d). Here the interpolation nodes are Chebychev instead of uniformly spaced on the interval.

2. Repeat the experiment from the previous problem but with Chebychev nodes. How does this impact the performance of the different interpolation techniques?

   Soln: The code for this problem is provided below. Figure 2 reports on a semilogy plot the errors using the different methods. Changing the nodes improves the performance of the Lagrange and Hermite interpolation. In fact the Hermite interpolation is better than the other interpolation techniques. It is also the highest order.

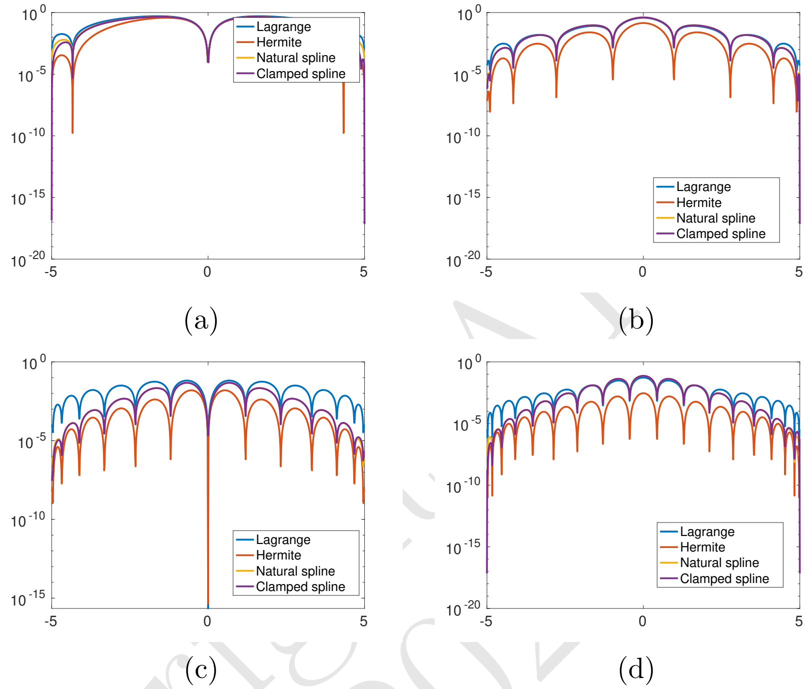

Figure 2: Illustration of errors when $n = 5, 10, 15$ and 20 respectively (a,b,c,d). Here the interpolation nodes are Chebychev instead of uniformly spaced on the interval.

3. Consider the task of approximating a periodic function such as $f(x) = \sin(10x)$ on the interval $[0, 2\pi]$ using the cubic spline. How do you modify the end point conditions on the coefficients so that the spline is naturally periodic?

   Soln: For periodic problems we have that $f(x_0) = f(x_n)$. We also have that the first and second derivatives should be equal at the end points; i.e. $S'_{n-1}(x_n) = S'_0(x_0)$ and $S''_{n-1}(x_n) = S''_0(x_0)$. This means that $M_n = M_0$. Now we need to create an equation for the finding M_0 and the periodicity conditions.

   For M_0 we end up with the same equation as for the interior nodes. Specifically, we have

$$\frac{h_{n-1}}{6}M_{n-1} + \frac{h_0 + h_{n-1}}{3}M_0 + \frac{h_1}{6}M_1 = \frac{f(x_1) - f(x_0)}{h_0} - \frac{f(x_0) - f(x_{n-1})}{h_{n-1}}$$

   Also, the corresponding to M_{n-1} on the diagonal gets updated with M_0 replacing M_n.

$$\frac{h_{n-2}}{6}M_{n-2} + \frac{h_{n-2} + h_{n-1}}{3}M_{n-1} + \frac{h_{n-1}}{6}M_0 = \frac{f(x_0) - f(x_{n-1})}{h_{n-1}} - \frac{f(x_{n-1}) - f(x_{n-2})}{h_{n-2}}$$

   The resulting linear system is still tridiagonal but there are entries at the end of the first row and first column that enforce the periodicity.