Final Exam

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Section 002

1. a. True
   b. False
   c. True
   d. True
   e. True
   f. True
   g. False
   h. False
   i. True
   j. True

2. $\begin{bmatrix} 9 & -9 & 9 \\ -9 & 10 & -7 \\ 9 & -7 & 14 \end{bmatrix} \begin{matrix} \\ +R1 \\ \\ \end{matrix}$

$\begin{bmatrix} 9 & -9 & 9 \\ 0 & 1 & 2 \\ 9 & -7 & 14 \end{bmatrix} \begin{matrix} \\ \\ -R1 \end{matrix}$

$\begin{bmatrix} 9 & -9 & 9 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{matrix} \\ \\ -2R2 \end{matrix}$

$\begin{bmatrix} 9 & -9 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 9 & -9 & 9 \\ -9 & 10 & -7 \\ 9 & -7 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 9 & -9 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

3. $u_1 = \langle 1,-1,0 \rangle$ $v_2 = \langle 0,3,0 \rangle$ $u_3 = \langle 2,0,1 \rangle$

let $\langle a,b,c \rangle \in \mathbb{R}^3$ such that $\langle a,b,c \rangle = pu_1 + qu_2 + ru_3$

$\langle a,b,c \rangle = p\langle 1,-1,0 \rangle + q\langle 0,3,0 \rangle + r\langle 2,0,1 \rangle$

$= \langle p+2r, 3q-p, r \rangle$

$a = p+2r$ $b = 3q-p$ $c = r$

$a = p+2r = p+2c$ $p = a-2c$

$b = 3q-p = 3q-a+2c$ $3q = b+a-2c$ $q = \frac{1}{3}(b+a-2c)$

$r = c$

$\langle a,b,c \rangle = (a-2c)\langle 1,-1,0 \rangle + \frac{1}{3}(b+a-2c)\langle 0,3,0 \rangle + c\langle 2,0,1 \rangle$

$$L(x,y,z) = \begin{pmatrix} x+2y-2z \\ 4x+2y \\ -z \end{pmatrix}$$

$L(1,-1,0) = \langle -1,2,0 \rangle$

$L(0,3,0) = \langle 6,6,0 \rangle$

$L(2,0,1) = \langle 0,8,-1 \rangle$

$\langle -1,2,0 \rangle = (-1+2(0))\langle 1,-1,0 \rangle + \frac{1}{3}(2+(-1)-2(0))\langle 0,3,0 \rangle + 0\langle 2,0,1 \rangle$

$= -1\langle 1,-1,0 \rangle + \frac{1}{3}\langle 0,3,0 \rangle + 0\langle 2,0,1 \rangle$

$\langle 6,6,0 \rangle = 6\langle 1,-1,0 \rangle + 4\langle 0,3,0 \rangle + 0\langle 2,0,1 \rangle$

$\langle 0,8,1 \rangle = -2\langle 1,-1,0 \rangle + 2\langle 0,3,0 \rangle + 1\langle 2,0,1 \rangle$

Thus, the matrix representation is $\boxed{\begin{bmatrix} -1 & 1/3 & 0 \\ 6 & 4 & 0 \\ -2 & 2 & 1 \end{bmatrix}}$

4. $p = \beta_0 + \beta_1 \frac{w}{h^2} + \beta_2 t + \beta_3 t^2$

$$\nabla p = \begin{bmatrix} 0 \\ w/h^2 \\ t \\ t^2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ w/h^2 \\ t \\ t^2 \end{bmatrix}$$

5. $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\det(AA^T - \lambda I) = (2-\lambda)(2-\lambda) - 4 = \lambda^2 - 4\lambda = 0$

$\lambda = 0, 4$

$\lambda = 4$: $\begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} /-2$

$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} + 2R1$

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$x_1 = -x_2$

$\begin{bmatrix} -x_2 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\lambda = 0$: $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} + R1$

$\begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} /2$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x_1 = x_2$

$\begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} / \sqrt{2} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\Sigma = \begin{bmatrix} \sqrt{4} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ $U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$V_i = \frac{1}{\sigma_i} A^T \cdot u_i$

$V = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$A = U \Sigma V^T = \boxed{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$