Convergence

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<u>Def</u>: Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p w/ $p_n \neq p\ \forall n$. If $\exists$ positive constants α & λ st

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

Then the sequence converges w/ order α & λ is the asymptotic error constant.

If $\alpha = 1$ & $\lambda < 1$, the convergence is linear & λ is the convergence rate

if $\alpha = 2$, the convergence is quadratic

<u>Ex</u>: Show fixed pt iteration is linear if $f'(p) \neq 0$ & p is the fixedpt.

<u>Soln</u>: Taylor expand $f(x)$ about the pt p.

$$f(x) = f(p) + f'(p)(x-p) + \frac{f''(\eta_k)}{2}(x-p)^2$$

for some η_k between x & p

Plug in x_k

$$f(x_k) = f(p) + f'(p)(x_k - p) + \frac{f''(\eta_k)}{2}(x_k - p)^2$$

$$\| \qquad \|$$

$$x_{k+1} = p + f'(p)(x_k - p) + \frac{f''(\eta_k)}{2}(x_k - p)^2$$

$$\underbrace{x_{k+1} - p}_{\text{Numerator}} = f'(p)\underbrace{(x_k - p)}_{\text{denominator}} + \frac{f''(\eta_k)}{2}(x_k - p)^2$$

$$\underbrace{x_{k+1} - p}_{\text{Numerator}} = f'(p)\underbrace{(x_k - p)}_{\text{denominator}} + \frac{f''(c)}{2}(x_k - p)^2$$

need to correct since we are going to toss the quad. term.

$$\frac{x_{k+1} - p}{x_k - p} = f'(\beta_k) \qquad \text{for some } \beta_k \text{ between } x_k \text{ \& } p.$$

$$\lim_{k\to\infty} \left| \frac{x_{k+1} - p}{x_k - p} \right| = \lim_{k\to\infty} |f'(\beta_k)| = |f'(p)|$$

$\therefore$ 1st order as long as $|f'(p)| < 1$
& the convergence rate is $|f'(p)|$

See Textbook Thm 2.8 proof
Thm 2.9 say if $f'(p) = 0$ & $f''(p) \neq 0$
Then the fixed pt iteration has quadratic

<u>Thm</u> Assume that f is twice continuously differentiable on an open interval (a,b) & $p \in (a,b)$ st $f'(p) \neq 0$ but $f(p) = 0$.
Assume the sequence created by Newton's Method $\{p_n\}$ converges to p.
Then for n sufficiently large

$$|p_{n+1} - p| \leq M |p_n - p|^2$$

Where $M > \left| \frac{f''(x)}{2f'(x)} \right| \quad \forall x$ near p.

<u>Proof:</u> let $e_n = p_n - p$ denotes the error at step n

$$p = p_n - e_n$$

Step n

$p = p_n - e_n$

$$f(p) = f(p_n - e_n) = f(p_n) - e_n f'(p_n) + \frac{e_n^2}{2!} f''(\eta_n)$$

for some η_n between p & p_n

We know

$f(p) = 0 \qquad f'(p_n) \neq 0$

$$0 = \frac{f(p)}{f'(p_n)} = \underbrace{\frac{f(p_n)}{f'(p_n)} - e_n}_{\underbrace{\frac{f(p_n)}{f'(p_n)} - p_n}_{-p_{n+1}} + p} + \frac{e_n^2 f''(\eta_n)}{2 f'(p_n)}$$

by def of Newton.

$$0 = -(p_{n+1} - p) + \frac{e_n^2 f''(\eta_n)}{2 f'(p_n)}$$

$$0 = -e_{n+1} + \frac{e_n^2 f''(\eta_n)}{2 f'(p_n)}$$ by definition of e_{n+1}

$$e_{n+1} = \frac{e_n^2 f''(\eta_n)}{2 f'(p_n)}$$

$$|p_{n+1} - p| = |p_n - p|^2 \left| \frac{f''(\eta_n)}{2 f'(p_n)} \right|$$

We know $\left| \frac{f''(\eta_n)}{2 f'(p_n)} \right| < M$ by theorem statement.

$$\Rightarrow \quad |p_{n+1} - p| < |p_n - p|^2 M$$

$\therefore$ We have 2^{nd} order convergence.

Other option

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} = \lim_{n \to \infty} \left| \frac{f''(\eta_n)}{2 f'(p_n)} \right| = \left| \frac{f''(p)}{2 f'(p)} \right| = K$$

What happens if the root has multiplicity greater than 1?

We get 1^{st} order convergence.

Def: A solution p of $f(x) = 0$ is a zero of multiplicity m if for $x \neq p$, we can write

$$f(x) = (x-p)^m q(x) \quad \text{where}$$

$$\lim_{x \to p} q(x) \neq 0.$$

Thm 2.11 The function $f \in C^1[a,b]$ has a simple zero at $p \in [a,b]$ if and only if

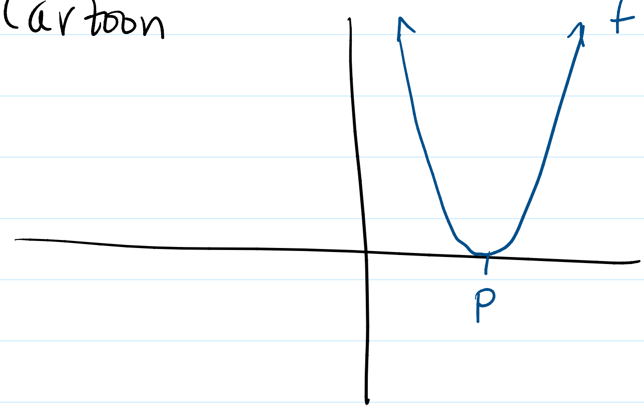
$$f(p) = 0 \;\; \& \;\; f'(p) \neq 0.$$

Thm 2.12 The function $f \in C^m[a,b]$ has a root a $p \in [a,b]$ of multiplicity m if & only if

$$f(p) = f'(p) = \cdots = f^{(m-1)}(p) = 0$$

$$\& \lim_{x \to p} f^{(m)}(x) \neq 0.$$

Ex: Cartoon



it looks like p is a root of f w/ multiplicity 2.

- It turns out you can use a modified Newton iteration

$$x_{n+1} = x_n - m\frac{f(x)}{f'(x)} \quad \text{where } m = \text{multiplicity}$$

If you don't know the multiplicity you can apply Newton to $\mu(x) = \dfrac{f(x)}{f'(x)}$

- p is a root of μ w/ multiplicity 1

- p is a root of μ w/ multiplicity 1

But you need $f''(x)$