APPM 4600 — HOMEWORK # 11

For all homeworks, you should use Python. **Do not use** symbolic software such as Maple or Mathematica.

1. Assume the error in an integration formula has the asymptotic expansion

$$I - I_n = \frac{C_1}{n\sqrt{n}} + \frac{C_2}{n^2} + \frac{C_3}{n^2\sqrt{n}} + \frac{C_4}{n^3} + \cdots$$

Generalize the Richardson extrapolation process to obtain an estimate of I with an error of order $\frac{1}{n^2\sqrt{n}}$. Assume that three values I_n, $I_{n/2}$ and $I_{n/4}$ have been computed.

Soln:

From the problem statement, we find

$$I - I_{n/2} = \frac{2\sqrt{2}C_1}{n\sqrt{n}} + \frac{4C_2}{n^2} + \frac{4\sqrt{2}C_3}{n^2\sqrt{n}} + \frac{8C_4}{n^3} + \cdots$$

Eliminating the c_1 term, we find

$$\begin{aligned}\Psi(n) &= \frac{2\sqrt{2}(I - I_n) - (I - I_{n/2})}{2\sqrt{2}-1} = \frac{(2\sqrt{2}-1)I - (2\sqrt{2}I_n - I_{n/2})}{2\sqrt{2}-1}\\ &= \frac{1}{2\sqrt{2}-1}\left(\frac{2\sqrt{2}-4}{n^2}c_2 - \frac{2\sqrt{2}}{n^2\sqrt{n}}c_3 + \frac{2\sqrt{2}-8}{n^3}c_4 + \ldots\right)\end{aligned}$$

Then

$$\begin{aligned}\Psi(n/2) &= \frac{2\sqrt{2}(I - I_n/2) - (I - I_{n/4})}{2\sqrt{2}-1} = \frac{(2\sqrt{2}-1)I - (2\sqrt{2}I_n/2 - I_{n/4})}{2\sqrt{2}-1}\\ &= \frac{1}{2\sqrt{2}-1}\left(\frac{2\sqrt{2}-4}{n^2}4c_2 - \frac{2\sqrt{2}}{n^2\sqrt{n}}2\sqrt{2}c_3 + \frac{2\sqrt{2}-8}{n^3}8c_4 + \ldots\right)\end{aligned}$$

Now we need to eliminate c_2. Define

$$\begin{aligned}\Theta(n) &= \frac{4\Psi(n) - \Psi(n/2)}{3} = I - \frac{4(2\sqrt{2}I_n - I_{n/2}) - (2\sqrt{2}I_{n/2} - I_{n/4})}{3(2\sqrt{2}-1)}\\ &= O(\frac{1}{n^2\sqrt{n}})\end{aligned}$$

Thus the estimate of I is

$$I \sim \frac{4(2\sqrt{2}I_n - I_{n/2}) - (2\sqrt{2}I_{n/2} - I_{n/4})}{3(2\sqrt{2}-1)}.$$

2. Use the transformation $t = x^{-1}$ and Composite Simpson's rule with 5 nodes to approximate

$$\int_1^\infty \frac{\cos(x)}{x^3} dx.$$

Soln:

We will use the technique of adding zero. Here the zero is going to be the $P_4(x)$ Taylor polynomial for $\sin(x)$ centered at $a = 0$.

$$P_4(x) = 0 + x + 0(x^2) - x^3 = x - x^3$$

So

$$\int_0^1 x^{-1/4} \sin(x) dx = \int_0^1 \frac{\sin(x) - P_4(x)}{x^{1/4}} dx + \int_0^1 \frac{P_4(x)}{x^{1/4}} dx$$

Let

$$G(x) = \begin{cases} \sin(x) - P_4(x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

We apply Composite Simpson's rule to $\int_0^1 G(x)dx$ and get $2.235868e-01$. The other integral can be evaluated by hand. It is $4/7 + 4/15$.

So the approximation of the integral is $2.235868e - 01 + 4/7 + 4/15$.

3. The gamma function is defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x > 0.$$

Write a program to compute the value of this function from the definition using each of the following approaches:

(a) Truncate the infinite interval of integration and write a composite trapezoidal rule code to perform the numerical integration. You will need to do some experimentation or analysis to determine where to truncate the interval based upon the usual trade-offs between accuracy and efficiency. Please describe your reasoning for your choice of interval and step size for the Trapezoidal Rule. Compare the relative accuracy of this solution with the value given by the Python gamma function (`scipy.special.gamma`) at $x = 2, 4, 6, 8, 10$ (Recall $\Gamma(k) = (k-1)!$ for positive integer k) .

(b) Use the Matlab adaptive quadrature routine `quad` to solve the above integral on the same interval you used for part (a). Compare the accuracy of this solution with the one you obtained in part (a) at the same values of x. Also, compare the number of function evaluations required by the two methods.

(c) Gauss-Laguerre quadrature is designed for the interval $[0, \infty)$ and the weight function $w(t) = e^{-t}$. It is therefore ideal to use for this problem. Call the Numpy subroutine `numpy.polynomial.laguerre.laggauss` to to obtain the n weights $\mathbf{w}$ and n abscissae $\mathbf{x}$ for Gauss-Laguerre quadrature and use this to approximate $\Gamma(x)$.

Soln:

(a) *This is one possible solution.*
Assume $x > 1$. (This is the case we are asked about anyway.)
Lets expand the definition of $\Gamma(x)$.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \int_0^b t^{x-1}e^{-t}dt + \int_b^\infty t^{x-1}e^{-t}dt \tag{1}$$

First we need to find b such that the last integral in (1) is bounded by the tolerance ϵ. We will begin by bounding the integrand by $Me^{-\alpha t}$ for $0 < \alpha < 1$ and M some constant to be found.

$$t^{x-1}e^{-t} = t^{x-1}e^{-t+\alpha t-\alpha t} = t^{x-1}e^{-\alpha t}e^{(\alpha-1)t}.$$

Our goal has been changed now to find the maximum of $f(t) = t^{x-1}e^{(\alpha-1)t}$. Note $f'(t) = (x-1)t^{x-2}e^{(\alpha-1)t} + (\alpha-1)t^{x-1}e^{(\alpha-1)t} = [x-1+(\alpha-1)t]t^{x-2}e^{(\alpha-1)t}$. This equals zero when $t = t_m = \frac{1-x}{\alpha-1}$.
Since $f(t) \geq 0$ for all t and $f(0) = \lim_{t\to\infty} f(t) = 0$, $f(t_m)$ is a maximum. Thus we set $M = f(t_m) = \left(\frac{1-x}{\alpha-1}\right)^{x-1} e^{1-x}$.
We can now bound the last integral in (1).

$$|\int_b^\infty t^{x-1}e^{-t}dt| \leq \int_b^\infty Me^{-\alpha t} = \frac{M}{\alpha}e^{-\alpha b}.$$

x	$\max_{t\in(0,b)} \lvert f''(t)\rvert \leq$
2	2
4	1
6	6
8	150
10	6000

Table 1: Bounds for the second derivative of the integrand in problem 2.

So for a given tolerance ϵ, we want b to be large enough so that $\frac{M}{\alpha}e^{-\alpha b} < \epsilon$. This implies $b \geq \frac{1}{\alpha}\ln(\frac{\alpha\epsilon}{M})$. Then

$$|\Gamma(x) - \int_0^b t^{x-1}e^{-t}dt| < \epsilon$$

.

We can determine an approximate number of intervals needed via the same process as the last homework. We know

$$|E_n(f)| = |\int_0^b t^{x-1}e^{-t}dt - I_n(f)| \leq \frac{h^2}{12}\max_{t\in(0,b)}|f''(t)|$$

where

$$f''(t) = e^{-t}t^{x-3}\left(t^2 - 2t(x-1) + (x-2)(x-3)\right)$$

and $h = b/n$.

Table 1 reports upper bounds on $\max_{t\in(0,b)}|f''(t)|$. From here it is easy to approximate n.

(b) The error for evaluating $\Gamma(8)$ are as reported below.

```
 Error from trap = 4.6384230e-11
Error quad = 2.2274435e-08
```

So our approximation using b from part (a) and trapezoidal is better.

(c) The errors are reported in Figure 1. With $n = 4$ nodes, we get an absolute error of approximately 10^{-11}. The code is provided below.

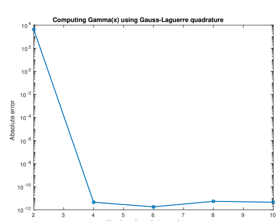

Figure 1: Semilogy plot of the absolute error when using Gauss-Laguerre quadrature

4. Given the linear system

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

(a) Find the value(s) of α for which the system has no solutions.

(b) Find the value(s) for α for which the system has an infinite number of solutions.

(c) Assuming a unique solution exists for a given α, find the solution.

Soln:

(a) & (b) Let $\mathsf{A} = \begin{bmatrix} 2 & -6\alpha \\ 3\alpha & -1 \end{bmatrix}$. The determinant of A is 0 when $\alpha = \pm\frac{1}{3}$.

When $\alpha = 1/3$, the linear system becomes

$$2x_1 - 2x_2 = 3$$
$$x_1 - x_2 = \frac{3}{2}$$

This means that $x_1 = \frac{3}{2} + x_2$ which can always be true for any x_2.

When $\alpha = -1/3$, the linear system becomes

$$2x_1 + 2x_2 = 3$$
$$-x_1 - x_2 = \frac{3}{2}$$

There is no way for both of these equations to be satisfied at the same time so there is no solution for this choice of α.

(c) For any other α there is a unique solution.

If $\alpha = 0$, $x_1 = 3/2$ and $x_2 = -3/2$.

If $\alpha \neq 0$, $x_2 = \frac{3/2 - 1/\alpha}{6\alpha + \frac{2}{3\alpha}}$ and $x_1 = 1/2\,(3 + 6\alpha x_2)$.