Newton's method

Monday, September 18, 2023 1:27 PM

Newton requires $f \in C^2[a,b]$ w/ a root $\alpha \in [a,b]$

$\Rightarrow f, f'$ & f'' are continuous on $[a,b]$

Derivation:

let $p_0 \in [a,b]$ be the initial guess & $f'(p_0) \neq 0$

Write down the 2^{nd} order Taylor evaluation of $f(x)$ centered at p_0

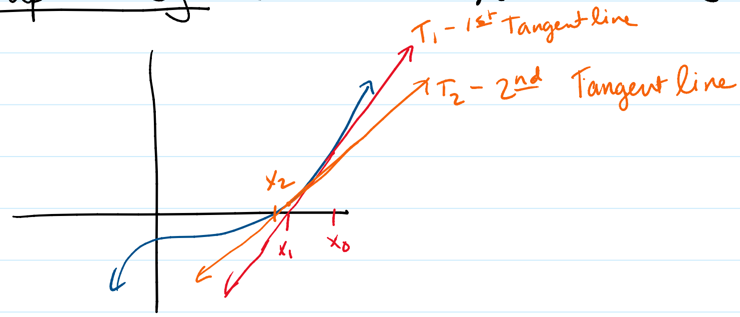
$$f(x) = \underbrace{f(p_0) + f'(p_0)(x - p_0)}_{\text{Tangent line}} + \underbrace{\frac{f''(\eta_0)}{2!}(x - p_0)^2}_{\text{Remainder for some } \eta_0 \text{ betwen } x \text{ \& } p_0}$$

To create our iteration we find the root of the Tangent line

$$0 = f(p_0) + f'(p_0)(x - p_0) \qquad \text{solve for } x.$$

$$x = p_0 - \frac{f(p_0)}{f'(p_0)}$$

Graphically our iteration looks as follows



Cliff notes of our iteration:

Given x_0, $x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{= g(x_k)}$

$$= g(x_k)$$

So this is a fixed pt iteration. You can use your fixed pt theory.

Pseudo code: Newton's method

Input: $f(x)$ - function evaluator
$f'(x)$ - derivative of the function
x_0 - initial guess
tol - tolerance
N_{max} - Max # of iterations

Output: α^* - approximation of the root
ier - error message $\begin{cases} 0 & \text{success} \\ 1 & \text{failure} \end{cases}$:(

Steps:

Step1: if $f'(x_0) == 0$ ier=1; $\alpha^* = x_0$ return :(

Step2: count = 0
Step3: while count < N_{max}

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$ (iterate) Root of the tangent line.

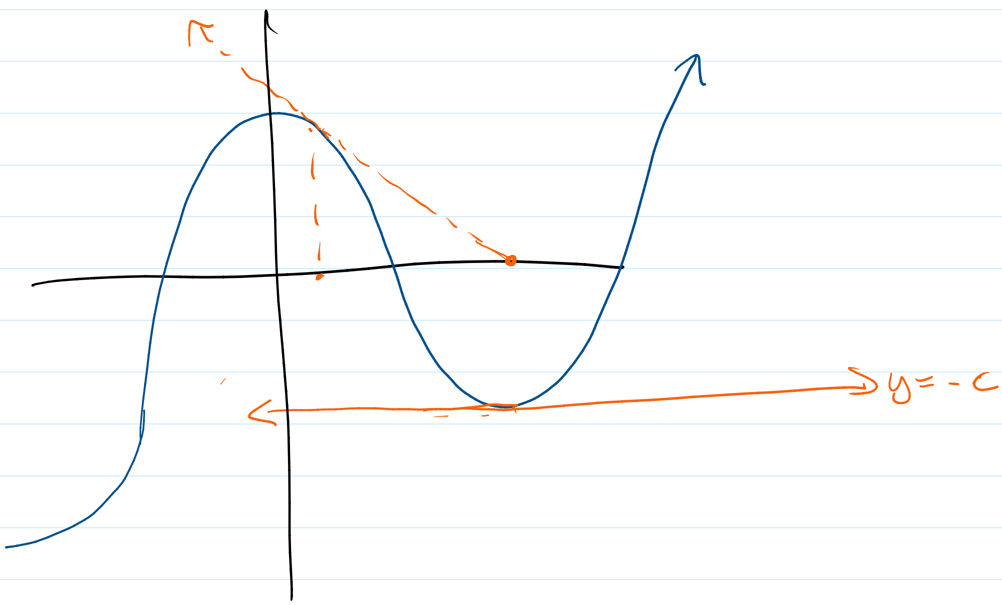
If $|x_1 - x_0| < tol$ (or $\frac{|x_1 - x_0|}{|x_1|} < tol$)
$\alpha^* = x_1$
ier=0 :)
return

count = count + 1
$x_0 = x_1$ - Reset for next iteration.

If $|f'(x_0)| == 0$
ier = 1
$\alpha^* = x_0$
return

Step 4: ier = 1;
$\alpha^* - x_0$
return

Drawing of a situation



Thm 2.6 Let $f \in C^2[a,b]$. If $p \in [a,b]$ st $f(p) = 0$ & $f'(p) \neq 0$ Then $\exists$ a $\delta > 0$ st Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ that converges to p for any $p_0 \in (p-\delta, p+\delta)$

Proof: See textbook

Question: How do we know if our guess is close enough?

Newton $\quad \underbrace{p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} = g(p_n)}$ ie it's a fixed pt iteration if $|g'(x_0)|<1$ we know the iteration converges.

Goal: pick x_0 st $|g'(x_0)|<1$.

Double Check my Calc :/

$$\left\{ g'(x) = 1 + \frac{f(x)}{(f'(x))^2} f''(x) - \frac{1}{f'(x)} f'(x) \right.$$

Def: All x_0 st $|g'(x_0)|<1$ where $g(x)$ is defined by the Newton iteration are said to be in the basin of convergence.