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APPM 4360
Due 3/20/24

# Problem Set 5

1. a. $\frac{z}{1+z^2} = z \cdot \sum_{n=0}^{\infty} (-z^2)^n$

b. $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$

$\frac{\cos(z) - 1}{z^2} = \frac{-1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \dots$

2. $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n$

$\int \frac{1}{1+z} = \log(1+z) = \int \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} \int (-z)^n$ because $\sum (-z)^n$ converges uniformly for $|z| < 1$

$= \boxed{\sum_{n=0}^{\infty} \frac{(-z)^{n+1}}{n+1}}$

3. $\frac{1}{z+1} = \frac{1}{z} \cdot \left( \frac{1}{1 + 1/z} \right) = \boxed{\frac{1}{z} \sum_{n=0}^{\infty} (-1/z)^n}$

4. $|f_n(z) g_n(z)| \le C |g_n(z)|$
Since $g_n(z)$ converges to 0, there exists an $N$
s.t. $g_n(z) < \epsilon$ for all $n \ge N$
Let $w = \epsilon/C$
There also must be an $N$ s.t. $g_n(z) < w$ for all $n \ge N$
Thus, $|f_n(z) g_n(z)| \le C \cdot \epsilon/C = \epsilon$ for all $n \le N$
so it converges to 0.

5. a. $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$
If $x = kz^2$, $\sinh(kz^2) = \sum_{n=0}^{\infty} (kz^2)^{2n+1}/(2n+1)! = \boxed{kz^2 + \frac{(kz^2)^3}{3!} + \dots}$

b. $e^{2z} = 1 + 2z + (2z)^2/2! + (2z)^3/3! + \dots$
$e^{2z} - 1 - 2z = (2z)^2/2! + (2z)^3/3! + \dots$
$\frac{e^{2z} - 1 - 2z}{z^2} = \boxed{\frac{2^2}{2!} + \frac{2^3 z}{3!} + \dots} = \sum_{n=0}^{\infty} \frac{2^{n+2} z^n}{(n+2)!}$

6. $F(z) = \int_{-\infty}^{0} e^{at+izt}\,dt + \int_{0}^{\infty} e^{-bt+izt}\,dt$

For these to converge, $|(a+iz)| < 0$ and $|-b+iz| < 0$

Since only the real part of an exponential affects its magnitude and $z$ is multiplied by $i$, $Im(z)$ is what affects the magnitude of this exponential

So $\boxed{a + im(z) < 0 \text{ and } -b + im(z) < 0}$

b. Since $t^2$ is positive, $e^{-kt^2}$ converges everywhere, So the convergence of $F(z) = \int_{-\infty}^{\infty} e^{-kt^2+izt}\,dt$ must also converge everywhere in $\mathbb{C}$.

7. Given $|f(t)| < Ke^{ct}$, $|\int_0^\infty f(t) e^{-zt}\,dt| \leq \int_0^\infty Ke^{ct}|e^{-zt}|\,dt$

For this to converge, $c - Re(z) < 0$, or $Re(z) > c$.

$F(z)$ is analytic where $Re(z) > c$ because that's where it converges.

8. a. $f(z) = z/(a+z)(a-z) = \frac{1}{a} \cdot \frac{1}{1-z/a} \cdot \frac{z}{z+a}$

$$\frac{1}{1-z/a} = \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n$$

$$f(z) = \frac{z}{a(z+a)} \sum_{n=0}^{\infty} \left(\frac{z}{a}\right)^n = \frac{z}{a(z+a)} + \frac{z^2}{a^2(z+a)} + \frac{z^3}{a^3(z+a)} + \dots$$

b. $f(z) = z/(a+z)(a-z) = \frac{1}{a} \cdot \frac{1}{1+a/z} \cdot \frac{z}{z-a}$

$$\frac{1}{1+a/z} = \sum_{n=0}^{\infty} \left(a/z\right)^n$$

$$f(z) = \frac{z}{a(z-a)} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{z}{a(z-a)} + \frac{1}{z-a} + \frac{a}{z(z-a)} + \dots$$

9. a. z is analytic everywhere in $|z|<1$, so the Laurent series is the ~~Taylor~~ Taylor series for this function, which is
$f(z) = iz - (\tfrac{1}{2} - i)z^2 + (-\tfrac{1}{2} + \tfrac{3i}{4})z^3 + \ldots$

b. $f(z) = \dfrac{8/5 + 9i/5}{z+2i} + \dfrac{2/5 - 4i/5}{z-1}$
1 is in this region and 2i is not, so we don't need to ~~expand~~ expand this

c. both singularities exist in this region, so we need to expand both terms from b.
$f(z) = 2/z + 2-4i/z^2 (-1+2i)\cancel{\tfrac{3+4i}{2}} + \ldots$

10. a. There is a simple pole at $z=2$ and a double pole at $z=-1$ because of the factors in the denominator.

b. sech(z) is entire because it's $2/(e^z + e^{-z})$ and both the numerator and denominator are entire, so there are no singularities.

c. cosh(z) is entire, but $\sqrt{z}$ has a branch point at $z=0$, so there is an essential singularity at $z=0$

d. log(z) has a branch point at $z=0$, $1/z^2$ also has a double pole at $z=0$, but that is not significant because the presence of a branch point already makes $z=0$ an essential singularity.

11. a. The residue of the laurent series is $\frac{a-1}{3a^2}$

so $I = \frac{1}{2\pi i} \cdot 2\pi i \cdot \frac{a-1}{3a^2} = \boxed{a-1/3a^2}$

b. $I = 1/2\pi i \oint_C \tan(2z)\,dz$

$\tan(2z) = \sin(2z)/\cos(2z)$ s.p.s at $\pi/2 + \pi n$

$\tan(2z) = \frac{\sin(\pi/2 + w)}{\cos(\pi/2 + w)} = -\cot(w) = \frac{-1}{2w} + \frac{2w}{3} + \dots$

$I_1 = 1/2\pi i \cdot Res_1 = -1/2$

$I_2 = 1/2\pi i + Res_2 = -1/2$

$I = -1$

c. This is an essential singularity, so the integral does not exist