LU factorization

Monday, November 27, 2023 1:35 PM

Ex: Solve the following linear system in 4 digit arithmetic via Gaussian Elimination

$$\begin{cases} 10^{-4} x_1 + 20 x_2 = 30 \\ 10 x_1 + 10 x_2 = 40 \end{cases}$$

Soln:

$$\begin{array}{c} ① \\ ② \end{array} \left[\begin{array}{cc|c} 10^{-4} & 20 & 30 \\ 10 & 10 & 40 \end{array}\right]$$

$$\underbrace{② \to}_{\text{replace}} ② - \frac{10}{10^{-4}} ①, \quad \frac{10}{10^{-4}} = 10^5$$

$$\left[\begin{array}{cc|c} 10^{-4} & 20 & 30 \\ 0 & -2\times10^{6} & -3\times10^{6} \end{array}\right]$$

4 digit Arith

$$10 - \underbrace{10^5(20)}_{2\times10^6} = -2\times10^6$$

$$x_2 = 3/2 \quad x_1 = 0$$

Exact soln: $x_2 = 1.49999$

$x_1 = 2.50001$

While x_2 is close, x_1 is completel incorrect.

How can fix this?

let's swap the rows

$$\left[\begin{array}{cc|c} 10 & 10 & 40 \\ 10^{-4} & 20 & 30 \end{array}\right] \qquad ② \to ② - \frac{10^{-4}}{10} ① \quad " 10^{-5}$$

$$\left[\begin{array}{cc|c} 10 & 10 & 40 \\ 0 & 20 & 30 \end{array}\right] \qquad 20 - 10^{-5} \cdot 10 = 20 \text{ in 4 digit arithmetic}$$

$$x_2 = 3/2 \qquad x_1 = \frac{1}{10}\left(40 - 10(3/2)\right) = 2.5$$

This is much better!

$$\frac{|x_2 - 1.49999|}{|1.49999|} \sim O(10^{-4})$$

$$\frac{|x_1 - 2.50001|}{|2.50001|} \sim O(10^{-4})$$

This swapping of rows is called <u>row pivoting</u>.

How do we choose which row swap?

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

nitude

$\left[\, a_{n1} \ \cdots \ a_{nn} \,\right]$

Pick the biggest entry in magnitude in the 1st column & swap the correspond row to the 1st row.

let row k have the largest a_{k1} entry
Then swap row k & row 1

What is the matrix that swaps row k & row 1?

PA where P is the permutation matrix

$$P = \begin{array}{r c} & k\text{th column} \\ & \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ 1 & 0 & \cdots & 0 & & \cdots & 0 \\ & & & & 1 & & \\ & 0 & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} \\ k\text{th row} \rightarrow & \end{array}$$

What if I need to row pivot for the second column?

row j has largest $|a_{j2}|$ entry.

Then swap row 2 & row j
Permutation matrix

$$P_2 = \begin{array}{r c} & \text{column } j \\ & \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & \\ 0 & 1 & 0 & \cdots & & & 0 \end{bmatrix} \\ j \rightarrow & \end{array}$$

$$
\begin{array}{c}
2 \\
j \rightarrow \\
\\
\\
\end{array}
\begin{bmatrix}
0 & \cdots & 0\,1\,0 & \cdots & \cdots & 0 \\
\vdots & & & & & \\
0 & 1\,0 & \cdots & \cdots & \cdots & 0 \\
 & & 1 & & & \\
 & & & \ddots & & \\
 & & & & & 1
\end{bmatrix}
$$

$P_2 P A$ ← Swaps row k & 1
& row j & 2

After your done w/ your pivoting you get

$$PA = LU$$

↑ pivot or permutation matrix

$$P = P_{n-1} \cdots P_2 P_1$$

How do you solve a linear system w/ pivoted LU?

Goal: Solve $Ax = b$

We know $PA = LU$

multiply by P

$PAx = Pb$ ← Move the rows so they are better ordered for LU stability

Now Solve $L\underbrace{Ux}_{u} = Pb$

y

1^{st} Solve $Ly = Pb$

2^{nd} Solve $Ux = y$

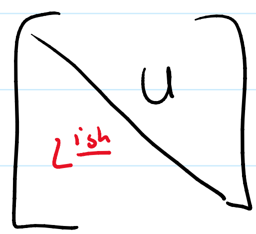
The Software package that does LU & other basic linear algebra is called LAPACK.

Input : $A \in \mathbb{R}^{n \times n}$

output : P - permutation matrix

LU - overwriten into the A matrix that was inputted.

$$LU = \begin{bmatrix} u_{11} & - & - & - & u_{1n} \\ l_{21} & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ l_{n1} & - & - & l_{n,n-1} & u_{nn} \end{bmatrix} =$$

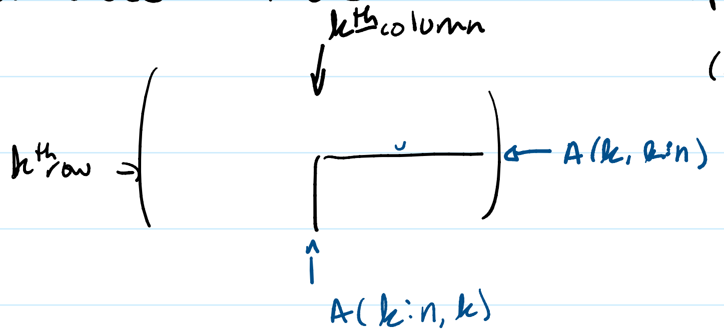


$$L = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ L^{ish} & & 1 \end{bmatrix}$$

This is more stable than LU w/out pivots but this is not the most stable version of LU.

To create the most stable LU, you need to do <u>complete pivoting</u>. This involves both row & column pivots.

How does this ork? Suppose we are processing column k.



1^{st} check $A(k, k:n)$ ① & $A(k:n, k)$ ② for the largest entry (in magnitude)

If the largest entry is in ①, then it is a column swap. Then check if row swap is needed.

If the largest entry is in ②, the do a row swap.

The result of the factorization is

$$\underset{\substack{\uparrow \\ \text{row permutations}}}{P} A \underset{\substack{\uparrow \\ \text{column permutations}}}{Q} = LU$$

If I want to solve $Ax=b$, what do we need to do to be able to use the LU?

$$\underbrace{PAQ}_{LU}\underbrace{Q^{-1}x}_{w} = Pb$$

$$LU\, w = Pb$$

Solve $Ly = Pb$

Solve $Uw = y$

Solve $Q^{-1}x = w$ $\Rightarrow$ $x = Qw$ & re order the answer from LU solve

Note you never need to make Q^{-1}