Interpolation

Monday, October 9, 2023 1:44 PM

What is interpolation?

Given a set of pts $\{x_j\}_{j=0}^{n}$ & function evaluations $\{f(x_j)\}_{j=0}^{n}$ construct an approximation of the function that goes through the pts

In APPM 4600, we do interpolation w/polynomials & trig polynomials.

Naive approach: Given $\{(x_j, f(x_j))\}_{j=0}^{n}$ we can construct a degree n polynomial.

$$p_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

Plug in each of that a pts.

$$p_n(x_j) = a_0 + a_1 x_j + \cdots + a_n x_j^n = f(x_j)$$

This is one row eqn. but we have n rows.

$$\underbrace{\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & & & \\ 1 & x_n & & x_n^n \end{bmatrix}}\begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{bmatrix}$$

This is called the <u>Vandermonde</u> matrix.
This is ill-conditioned for large n.

<u>Def</u>: We call the pts $\{x_j\}_{j=0}^n$ the <u>interpolation nodes</u>.

How do we know that this idea of interpolation is sane?

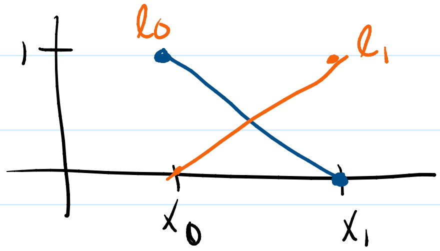
<u>Thm</u> (Weierstrass approximation Thm)
Suppose f is continuous on $[a,b]$. For $\varepsilon > 0$, There exists a polynomial $p(x)$ st

$$|p(x) - f(x)| < \varepsilon \qquad \forall x \in [a,b]$$

How we build this polynomial?

let's do something other than monomials.
let's start w/ a linear approximation.

$x_0 < x_1$ Data: $(x_0, f(x_0))$, $(x_1, f(x_1))$



our polynomial is going to be the sum of 2 lines: $l_0(x)$ & $l_1(x)$ that satisfy

$$l_0(x_0) = 1 \qquad l_1(x_0) = 0$$
$$l_0(x_1) = 0 \qquad l_1(x_1) = 1$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \qquad l_1(x) = \frac{x - x_0}{x_1 - x_0}$$

We can easily write our polynomial

$$p_1(x) = f(x_0)\, l_0(x) + f(x_1)\, l_1(x)$$

How do we extend this to build higher order polynomials?

Data: $\{(x_j, f(x_j))\}_{j=0}^{n}$ How do build my basis of $l_j(x)$ functions?

Start w/ l_0: We want

$$\ell_0(x_0) = 1 \qquad \ell_0(x_j) = 0 \qquad j \neq 0$$

need $\ell_0(x)$ must be degree n.

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}$$

← makes roots

← scales for 1 at x_0

$$\ell_j(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{j-1})(x-x_{j+1})\cdots(x-x_n)}{(x_j-x_0)(x_j-x_1)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_n)}$$

for $j = 0, \dots, n$

$$\ell_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{n} \frac{(x-x_i)}{(x_j-x_i)}$$

<u>Def</u>: $\ell_j(x)$ is called a <u>Lagrange Polynomial</u> of degree n.

our interpolating polynomial

$$P_n(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x) + \cdots + f(x_n)\ell_n(x)$$

$$= \sum_{j=0}^{n} f(x_j)\ell_j(x)$$

Ex. Consider the function $f(x) = e^x$

(a) construct $p_1(x)$ w/ $x_0 = 0$ & $x_1 = 1$

Soln: $p_1(x) = mx + b$

$= m(x - x_0) + f(x_0)$

solve for m.

$$p_1(x) = f(x_0) \frac{(x - x_1)}{x_0 - x_1} + f(x_1) \frac{(x - x_0)}{(x_1 - x_0)}$$

$$= \underbrace{\frac{x-1}{-1}}_{\ell_0} + e \underbrace{\frac{x}{1}}_{\ell_1}$$

(b) Construct $P_2(x)$ w/ $x_0 = 0$, $x_1 = 1/2$, $x_2 = 1$

Soln: $P_2(x) = f(x_0)\,\ell_0(x) + f(x_1)\,\ell_1(x) + f(x_2)\,\ell_2(x)$

$$= 1 \frac{(x - 1/2)(x-1)}{(0 - 1/2)(0-1)} + e^{1/2} \frac{(x-0)(x-1)}{(1/2 - 0)(1/2 - 1)}$$

$$+ e \frac{(x-0)(x-1/2)}{}$$

$$+\, e\,\frac{(x-0)(x-1/2)}{(1-0)(1-1/2)}$$

How accurate is this?

Thm 3.3 Suppose $x_0, \cdots x_n$ are distinct numbers in the interval $[a,b]$ & $f \in C^{(n+1)}[a,b]$

Then, for each $x \in [a,b]$, $\exists\, \eta$ (generally not known) betwee $x_0, x_1, \cdots, x_n$ in $[a,b]$ st

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\eta)}{(n+1)!}(x-x_0)\cdots(x-x_n)$$

where $P_n(x)$ is the Lagrange approximation of degree n through the pts $\{(x_j, f(x_j))\}_{j=0}^{n}$

Compare w/ Taylor: for $\alpha \in [a-\varepsilon, a+\varepsilon]$

$$f(x) = T_n(x) + \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-a)^{n+1}$$