Problem Set 1

Alex Ojemann
APPM 4360
1/31/24

1\. a. $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1-2i-1}{1+i} = \frac{2i}{2} = \boxed{i}$

b. $(2+i)(2+i)(2+i) = (4+4i-1)(2+i) = 8+8i-2+4i-4-i = \boxed{2+11i}$

c. $|3-4i| = \sqrt{3^2+(-4)^2} = 5$

2\. a. $(-i)^4 = (-1)^2 = 1 \quad r=1 \quad \theta=0 \quad e^{0i} = \boxed{1}$

b. $r = \sqrt{(-\sqrt{2})^2+(\sqrt{2})^2} = 2 \quad \theta = \arctan\left(\frac{-\sqrt{2}}{\sqrt{2}}\right) = 3\pi/4$ (in Q2)
$\boxed{2e^{3i\pi/4}}$

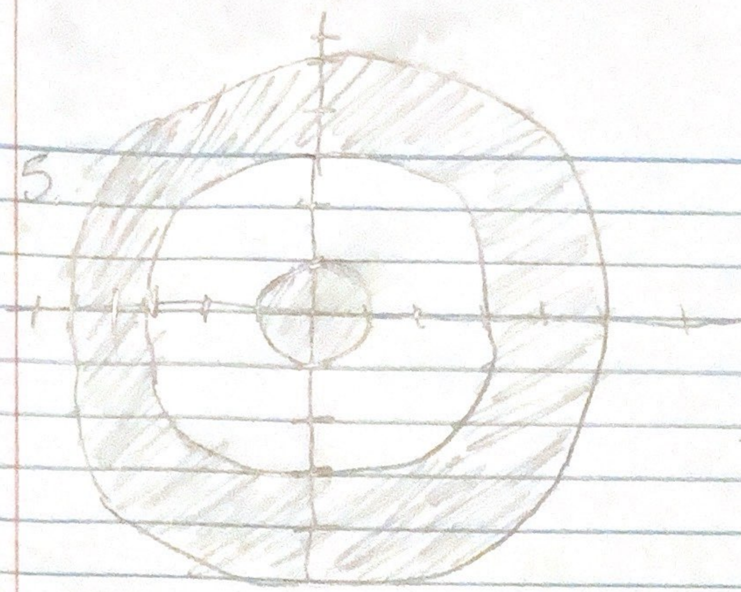
c. $r = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2} = 1 \quad \theta = \arctan\left(\frac{\sqrt{3}}{3}\right) = 7\pi/6$ (in Q3)
$\boxed{e^{7i\pi/6}}$

3\. $z^5+1 = (x+iy)^5+1 = \cancel{x^5+5ix^4y-10x^3y^2-10ix^2y^3+5xy^4+iy^5} = 0$
$(re^{i\theta})^5 = r^5e^{5i\theta} = -1e^{0i} = -1$
$r^5=-1 \quad 5\theta = 0 \rightarrow r=-1 \quad \theta = 2\pi/5,\ 4\pi/5,\ 6\pi/5\ 8\pi/5,\ 0$
$\boxed{z = -e^{2\pi i/5},\ -e^{4\pi i/5},\ -e^{6\pi i/5},\ -e^{8\pi i/5},\ -1}$

4\. a. $|3z_1-2z_2| \leq 3|z_1|+2|z_2|$ by the triangle inequality
$3|z_1|+2|z_2| < 3|z_1|+3|z_2|$ because $|z_2|>0$

b. $|z_1\bar{z}_2+\bar{z}_1\bar{z}_2+z_1z_2| = 3|z_1z_2|$ because conjugates have the same magnitude
$3|z_1z_2| = 3|z_1||z_2|\sin\theta$ where θ is the angle between z_1 and z_2.
$3|z_1||z_2|\sin(\theta) \leq 3|z_1||z_2|$ because $\sin(\theta)\leq 1$ for all θ.

5.



= in region

This region is open and bound
It is not closed, compact
or connected.

6. $z = x + iy$ $Re(z) = x$ $1/z = 1/(x+iy) = \frac{x-iy}{(x+iy)(x-iy)}$

$= \frac{x-iy}{x^2+y^2}$ $Re(1/z) = \frac{x}{x^2+y^2}$

x^2+y^2 is always positive, so $\left(\frac{x}{x^2+y^2}\right)$ always has the same sign as x.

7. $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0)^n$ where $|z-z_0| < R$

$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$

$\cos(2z) = 1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} + \dots$

$\cos(2z) - 1 = -\frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} + \dots$

$\boxed{\frac{\cos(2z) - 1}{z^2} = -2 + 2z^2/3 - 4z^4/45 + \dots}$

8. a $\lim_{z\to 0} \frac{\sin(\alpha z)}{z} = \lim_{z\to 0} \frac{\alpha \cos(\alpha z)}{1} = 1$ by L'Hospital's rule

Thus, $\lim_{z\to 0} \left(\frac{\sin(\alpha z)}{z}\right)^2 = \lim_{z\to 0} \frac{2\alpha \sin(\alpha z)\cos(\alpha z)}{2z}$

$= \lim_{z\to 0} \left(\frac{2\alpha(\sin(\alpha z)\cdot\alpha \cdot -\sin(\alpha z) + \alpha\cos\alpha z\cdot\cos\alpha z)}{2}\right) = \alpha^2$

b. $\lim_{z\to 0} \frac{\sinh(\alpha x)}{\sinh(x)} = \lim_{z\to 0} \frac{\alpha\cosh(\alpha z)}{\cosh(\alpha z)} = \frac{\alpha}{1} = \boxed{\alpha}$ by L'Hospital's rule

c. $\lim_{z\to\infty} \frac{Mz+1}{(Nz+3)} = \lim_{z\to\infty} \frac{M}{N} = \frac{M}{N}$ by L'Hospital's rule

d. $\sinh(2z) = 2\sinh(z)\cdot\cosh(z)$

$\lim_{z\to\infty} \frac{\sinh(2z)}{\cosh^2(z)} = \lim_{z\to\infty} \frac{2\sinh(z)\cancel{\cosh(z)}}{\cosh(z)\cosh(z)}$

$= \lim_{z\to\infty} \frac{2\sinh(z)}{\cosh(z)} = \lim_{z\to\infty} \frac{2\cosh(z)}{\sinh(z)} = 2$

9. (2): We apply L'Hospital's rule iteratively $k+1$ times

If $f^{(k+1)}(a)$ and $g^{(k+1)}(a)$ are both nonzero, then the limit is the ratio of the two

If $g^{(k+1)}$ is 0, the limit doesn't exist, unless both are zero, in which case you need to look to higher derivatives

(4a) Given $\lim_{z\to z_0} f(z) = 0$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all z within a δ neighborhood of z_0

continued on next page

since $|g(z)| \le M$, we can multiply both sides of the inequality $|f(z)-0| < \epsilon$ by M to get $|f(z)g(z)| \le M|f(z)|$ $M\epsilon$

Now if we choose $\epsilon' = \frac{\epsilon}{M}$ then for $|f(z)|$ we have

$|f(z)g(z)| < M\epsilon' = \epsilon$

(5) $h(z)$ is differentiable if $\lim_{\Delta z \to 0} \frac{h(z_0+\Delta z)-h(z_0)}{\Delta z}$ exists

If $H(z) = Re(z)$, then $h(z) = x$ where $z = x+iy$

As Δz approaches 0 from the real axis, the limit is 1. However, if it approaches 0 from the imaginary axis, the limit does not exist. Similarly, for $Im(z)$, $h(z) = y$, because the limit is 1 if you approach 0 on the ~~imaginary~~ real axis.

10. $w(z) = C_1 + (C_2 \sin \frac{\sqrt{3}}{2} z + C_3 \cos(\frac{\sqrt{3}}{2} z))e^{-z/2}$

If C_1, C_2 and C_3 are complex, their real parts contribute to the real part of the solution, which is:

$Re(w(z)) = C_1 + C_2 \sin(\frac{\sqrt{3}}{2} z)e^{-z/2} + C_3 \cos(\frac{\sqrt{3}}{2} z) e^{-z/2}$