APPM 4600 — HOMEWORK # 10

For all homeworks, you should use Python. **Do not use** symbolic software such as Maple or Mathematica.

1. For the function $f(x) = \sin(x)$. Determine the Padé approximations of degree 6 with

   (a) Both the numerator and denominator are cubic

   (b) The numerator is quadratic and the denominator is a fourth degree polynomial.

   (c) The numerator is a fourth degree polynomial and the denominator is quadratic.

   Compare the accuracy of these approximations with the sixth order Maclaurin polynomial by ploting the error over the interval $[0, 5]$.

   Soln:

   (a) We need to match the rational function with the truncated Taylor series.

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3} = x - \frac{x^3}{6} + \frac{x^5}{120}$$

   Collecting the terms and matching we get the following equations

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= b_1 \\ a_3 &= b_2 - b_1/6 \\ 0 &= -\frac{b_1}{6} + b_3 \\ 0 &= \frac{1}{120} - \frac{b_2}{6} \\ 0 &= -\frac{b_3}{6} + \frac{b_1}{120} \end{aligned}$$

   The solutions are $a_0 = 0$, $a_1 = 1$, $a_2 = b_1 = 0$, $b_2 = 1/20$, $a_3 = -7/60$ and $b_3 = 0$. So the Padé approximation is

$$P_3^3(x) = \frac{x - 7/60x^3}{1 + 1/20x^2}$$

   (b) $P_4^2(x)$ has the form

$$\frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}{1 + b_1 x + b_2 x^2}$$

   Equating terms in the Padé approximation with the Taylor expansion, we the following equations

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= b_1 \\
a_3 &= b_2 - b_1/6 \\
a_4 &= -\frac{b_1}{6} \\
0 &= \frac{1}{120} - \frac{b_2}{6} \\
0 &= \frac{b_1}{120}
\end{aligned}
$$

The solution this results in the same approximation as in part (a). i.e. $P_2^4(x) = P_3^3(x)$

(c) $P_4^2(x)$ has the form

$$
\frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4}
$$

Equating terms in the Padé approximation with the Taylor expansion, we the following equations

$$
\begin{aligned}
a_0 &= 0 \\
a_1 &= 1 \\
a_2 &= b_1 \\
0 &= b_2 - 1/6 \\
0 &= \frac{-b_1}{6} + b_3 \\
0 &= -\frac{b_2}{6} + \frac{1}{120} + b_4 \\
0 &= \frac{b_1}{120} - \frac{b_3}{6}
\end{aligned}
$$

The resulting rational approximation is

$$
P_4^2(x) = \frac{x}{1 + 1/6x^2 + 7/360x^4}
$$

Figure 1 illustrates the error in the approximations over the interval $[0, 5]$. As expected the error in the Taylor approximation grows the further you get from zero. Close to zero, the Padé approximation P_3^3 is good. While further away the Padé approximation P_4^2 is more accurate.

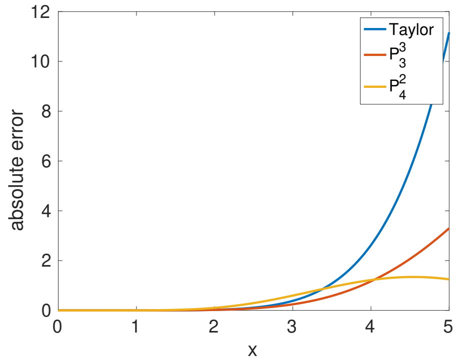

Figure 1: Plot of the error in the different degree 6 approximations of $\sin(x)$

2. Find the constants x_0, x_1 and c_1 so that the quadrature formula

$$\int_0^1 f(x)dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has the highest possible degree of precision.

Soln: Since there are three degrees of freedom, we can expect to integrate polynomials up to degree 2 exactly. Thus, we need to find x_0, c_1 and x_1 such that 1, x and x^2 can be integrated exactly on $(0, 1)$.

$$\int_0^1 1dx = 1 = \frac{1}{2} + c_1$$
$$\int_0^1 xdx = \frac{1}{2} = \frac{1}{2}x_0 + c_1 x_1$$
$$\int_0^1 x^2dx = \frac{1}{3} = \frac{1}{2}x_0^2 + c_1 x_1^2$$

The first equation tells us that $c_1 = \frac{1}{2}$. The second equation says $x_1 = 1 - x_0$. We can plug this into the third equation so that our goal is now to find the roots of a quadratric

$$x_0^2 + (1 - x_0)^2 = \frac{2}{3}.$$

The roots are $x_0 = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$. Both roots are the interval. In fact, one is x_1 since $x_1 = 1 - x_0$. Let $x_0 = \frac{1}{2} + \frac{\sqrt{3}}{6}$ then $x_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$.

3. (a) Write a code to approximate $\int_{-5}^{5} \frac{1}{1+s^2} ds$ using a composite Trapezoidal rule. To do this, partition the interval $[-5, 5]$ into equally spaced points $t_0, t_1, \ldots, t_n$.

Write another code to approximate $\int_{-5}^{5} \frac{1}{1+s^2} ds$ using a composite Simpson's rule. To do this, partition the interval $[-5, 5]$ into equally spaced points $t_0, t_1, \ldots, t_n$ where $n = 2k$ is even. The even indexed points should be the endpoints of your subintervals.

You may combine the two into one code that selects the desired method if you wish.

Turn in a listing of your code(s).

b) Use the error estimates derived in class to choose n so that

$$\left| \int_{-5}^{5} \frac{1}{1+s^2} ds - T_n \right| < 10^{-4} \quad \text{and} \quad \left| \int_{-5}^{5} \frac{1}{1+s^2} ds - S_n \right| < 10^{-4},$$

where T_n is the result of the composite Trapezoidal rule and where S_n is the result of the composite Simpson's rule. Be sure to explain your reasoning for choosing n in both cases (these n values will be different in the two cases).

c) Run your code with the predicted values of n and compare your computed values S_n and T_n with that of `SCIPY`'s `quad` routine on the same problem. Run the built in quadrature twice, once with the default tolerance of 10^{-6} and another time with the set tolerance of 10^{-4}. Report the number of function evaluations required in both cases and compare these to the number of function values your codes (both S_n and T_n) required to meet the tolerance

Turn in your codes and the results of this test.

Soln:

(a)

(b) We know from class that E_{trap} error from the trapezoidal rule is

$$E_{\text{trap}} = -\frac{h^2}{12}(b-a)f''(\eta) \text{ for some } \eta \in (a, b).$$

We are not going to be able to find η so instead we will use the upper bound. We do know that $h = (b-a)/n$.

So

$$|E_{\text{trap}}| \leq \frac{(b-a)^3}{12n^2} \max_{\eta \in (a,b)} |f''(\eta)|$$

Now we can plug in specific information $f''(x) = \frac{8x^2}{(1+x^2)^3} - \frac{2}{(1+x^2)^2}$ has a maximum at $x = 0$ of 2. So want

$$|E_{\text{trap}}| \leq \frac{1000}{6n^2} < 10^{-4}.$$

Solving for n, we find the number of intervals should be larger than 1291.

Applying the same technique to Simpson's rule we find

$$|E_{\text{Simp}}| \leq \frac{10^5}{180n^4}(24).$$

Making $\frac{10^5}{180n^4}(24) < 10^{-4}$ requires $n > 108$.

(c) Below are the results from the code provided in part (a).

```
Number of quad evaluations for 1e-4 =  41
Number of quad evaluations for 1e-6 =  81
Error trap for tol = 1e-4: 6.4415310e-06
Error trap for tol = 1e-6 2.5330259e-07
Error Simp for tol = 1e-4 6.4100873e-06
Error Simp for tol = 1e-6 2.2185888e-07
```

This means we achieved the desired accuracy with the less function evaluations than the theory gives as a maximum number of function evaluations.