Proposition 1.1.3: (Second Order Sufficient Optimality Conditions) Let $f : \Re^n \mapsto \Re$ be twice continuously differentiable in an open set S. Suppose that a vector $x^* \in S$ satisfies the conditions

$$\nabla f(x^*) = 0, \qquad \nabla^2 f(x^*) : \text{positive definite.}$$

Then, x^* is a strict unconstrained local minimum of f. In particular, there exist scalars $\gamma > 0$ and $\epsilon > 0$ such that

$$f(x) \geq f(x^*) + \frac{\gamma}{2}\|x - x^*\|^2, \qquad \forall\ x \text{ with } \|x - x^*\| < \epsilon. \tag{1.4}$$

Proof: Let λ be the smallest eigenvalue of $\nabla^2 f(x^*)$. By Prop. A.20(b) of Appendix A, λ is positive since $\nabla^2 f(x^*)$ is positive definite. Furthermore, by Prop. A.18(b) of Appendix A, $d'\nabla^2 f(x^*)d \geq \lambda\|d\|^2$ for all $d \in \Re^n$. Using this relation, the hypothesis $\nabla f(x^*) = 0$, and the second order Taylor series expansion, we have for all d

$$\begin{aligned} f(x^* + d) - f(x^*) &= \nabla f(x^*)'d + \tfrac{1}{2}d'\nabla^2 f(x^*)d + o(\|d\|^2) \\ &\geq \frac{\lambda}{2}\|d\|^2 + o(\|d\|^2) \\ &= \left(\frac{\lambda}{2} + \frac{o(\|d\|^2)}{\|d\|^2}\right)\|d\|^2. \end{aligned}$$

It is seen that Eq. (1.4) is satisfied for any $\epsilon > 0$ and $\gamma > 0$ such that

$$\frac{\lambda}{2} + \frac{o(\|d\|^2)}{\|d\|^2} \geq \frac{\gamma}{2}, \qquad \forall\ d \text{ with } \|d\| < \epsilon.$$

Q.E.D.

E X E R C I S E S

1.1.1

For each value of the scalar β, find the set of all stationary points of the following function of the two variables x and y

$$f(x, y) = x^2 + y^2 + \beta xy + x + 2y.$$

Which of these stationary points are global minima?

1.1.2

In each of the following problems fully justify your answer using optimality conditions.

(a) Show that the 2-dimensional function $f(x,y) = (x^2 - 4)^2 + y^2$ has two global minima and one stationary point, which is neither a local maximum nor a local minimum.

(b) Find all local minima of the 2-dimensional function $f(x,y) = \frac{1}{2}x^2 + x\cos y$.

(c) Find all local minima and all local maxima of the 2-dimensional function $f(x,y) = \sin x + \sin y + \sin(x+y)$ within the set

$$\{(x,y) \mid 0 < x < 2\pi,\ 0 < y < 2\pi\}.$$

(d) Show that the 2-dimensional function $f(x,y) = (y - x^2)^2 - x^2$ has only one stationary point, which is neither a local maximum nor a local minimum.

(e) Consider the minimization of the function f in part (d) subject to no constraint on x and the constraint $-1 \le y \le 1$ on y. Show that there exists at least one global minimum and find all global minima.

1.1.3 [Hes75]

Let $f : \Re^n \mapsto \Re$ be a differentiable function. Suppose that a point x^* is a local minimum of f along every line that passes through x^*; that is, the function

$$g(\alpha) = f(x^* + \alpha d)$$

is minimized at $\alpha = 0$ for all $d \in \Re^n$.

(a) Show that $\nabla f(x^*) = 0$.

(b) Show by example that x^* need not be a local minimum of f. *Hint*: Consider the function of two variables

$$f(y,z) = (z - py^2)(z - qy^2),$$

where $0 < p < q$; see Fig. 1.1.5. Show that $(0,0)$ is a local minimum of f along every line that passes through $(0,0)$. Furthermore, if $p < m < q$, then $f(y, my^2) < 0$ if $y \ne 0$ while $f(0,0) = 0$.

1.1.4

Use optimality conditions to show that for all $x > 0$ we have

$$\frac{1}{x} + x \ge 2.$$

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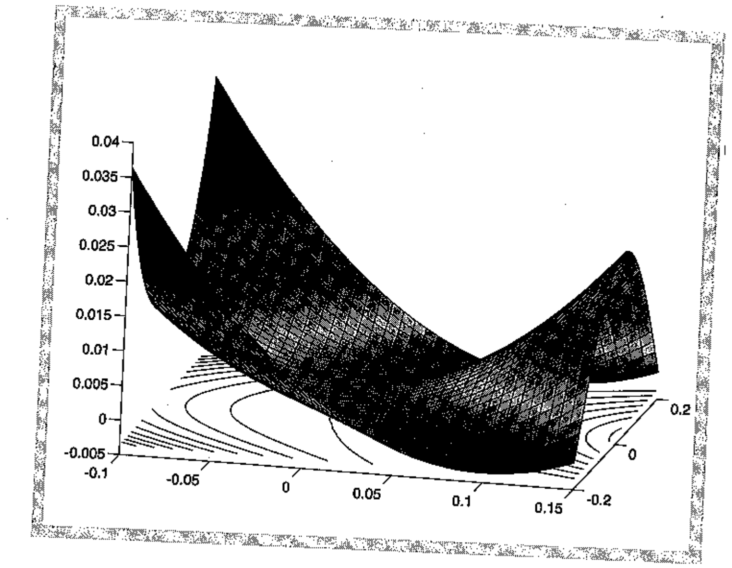

Figure 1.1.5. Three-dimensional graph of the function $f(y, z) = (z - py^2)(z - qy^2)$ for $p = 1$ and $q = 4$ (cf. Exercise 1.1.3). The origin is a local minimum with respect to every line that passes through it, but is not a local minimum of f.

1.1.5

Find the rectangular parallelepiped of unit volume that has the minimum surface area. *Hint*: By eliminating one of the dimensions, show that the problem is equivalent to the minimization over $x > 0$ and $y > 0$ of

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}.$$

Show that the sets $\{(x, y) \mid f(x, y) \le \gamma,\ x > 0,\ y > 0\}$ are compact for all scalars γ.

1.1.6 (The Weber Point of a Set of Points)

We want to find a point x in the plane whose sum of weighted distances from a given set of points $y_1, \ldots, y_m$ is minimized. Mathematically, the problem is

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^{m} w_i \|x - y_i\| \\ &\text{subject to} && x \in \Re^n, \end{aligned}$$

where $w_1, \ldots, w_m$ are given positive scalars.

(a) Show that there exists a global minimum for this problem and that it can be realized by means of the mechanical model shown in Fig. 1.1.6.

(b) Is the optimal solution always unique?

(c) Show that an optimal solution minimizes the potential energy of the mechanical model of Fig. 1.1.6, defined as $\sum_{i=1}^{m} w_i h_i$, where h_i is the height of the ith weight, measured from some reference level.

Note: This problem stems from Weber's work [Web29], which is generally viewed as the starting point of location theory.

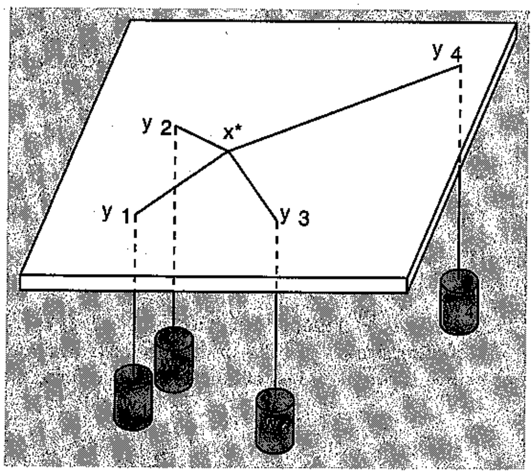

Figure 1.1.6. Mechanical model (known as the Varignon frame) associated with the Weber problem (Exercise 1.1.6). It consists of a board with a hole drilled at each of the given points y_i. Through each hole, a string is passed with the corresponding weight w_i attached. The other ends of the strings are tied with a knot as shown. In the absence of friction or tangled strings, the forces at the knot reach equilibrium when the knot is located at an optimal solution x^*.

1.1.7 (Fermat-Torricelli-Viviani Problem)

Given a triangle in the plane, consider the problem of finding a point whose sum of distances from the vertices of the triangle is minimal. Show that such a point is either a vertex, or else it is such that each side of the triangle is seen from that point at a 120 degree angle (this is known as the Torricelli point). *Note*: This problem, whose detailed history is traced in [BMS99], was suggested by Fermat to Torricelli who solved it. Viviani also solved the problem a little later and proved the following generalization: Suppose that x_i, $i = 1, \ldots, m$, are points in the plane, and x is a point in their convex hull such that $x \neq x_i$ for all i, and the angles $x_i x \overline{x}_{i+1}$, $i < m$, and $x_m \overline{x} x_1$ are all equal to $2\pi/m$. Then x minimizes $\sum_{i=1}^{m} \|z - x_i\|$ over all z in the plane (show this as an exercise by using sufficient optimality conditions; compare with the preceding exercise). Fermat is credited with being the first to study systematically optimization problems in geometry.