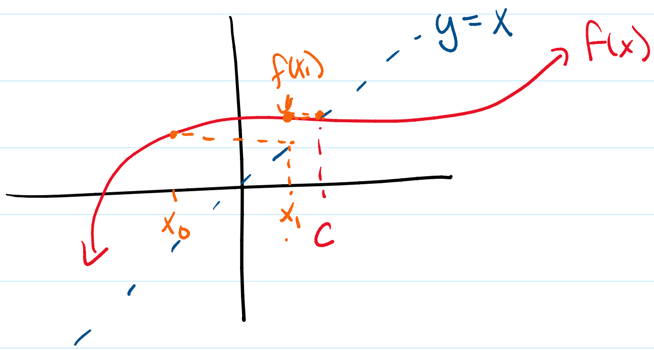
Fixed point

Wednesday, September 13, 2023 2:05 PM

What is a fixed pt? A point c st $f(c) = c$



Graphically, the fixed pt is the pt where the two functions x & $f(x)$ intersect.

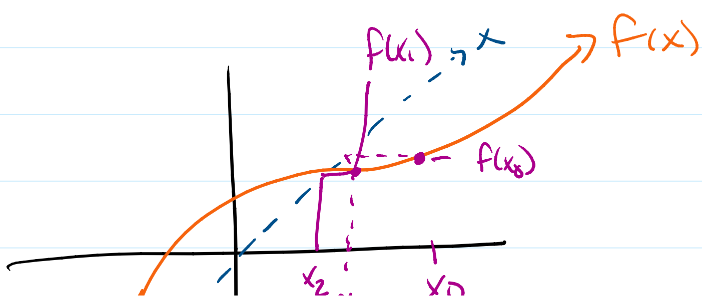
IDEA: march along the line $y = x$

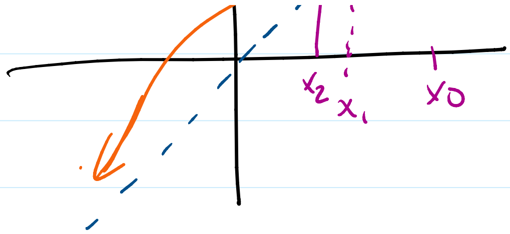
The iteration is $x_{n+1} = f(x_n)$ Given an x_0

Def: The pt x st $f(x) = x$ is called a fixed pt of $f(x)$.

Graphically: This is the pt where $y = x$ & $f(x)$ intersect.

How does iteration work? idea





How is this related to root finding?

the fixed pt α of $f(x)$ is a root of $g(x) = x - f(x)$

In practice, if my goal is to find the root of $g(x)$, I need to rewrite it ($g(x)$) so that it's a fixed pt problem.

So we need to find f.

We want to pick f so the fixed pt iteration converges.

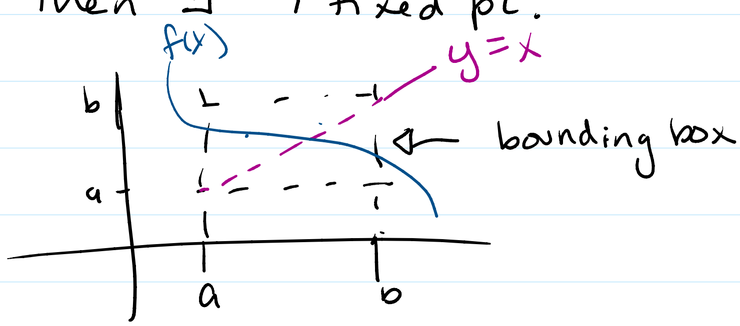
What do we need?

<u>Thm 2.3</u>

(i) If $f \in C[a,b]$ & $f(x) \in [a,b]$ $\forall x \in [a,b]$ then f has a fixed pt in $[a,b]$

(ii) If, in addition, $g'(x)$ exists on $[a,b]$ & $\exists$ a positive constant k $0<k<1$ w/ $|g'(x)| < k$ $\forall x \in [a,b]$, then $\exists$ 1 fixed pt.

<u>Picture</u>: for (i)



Proof:

(i) Given $f(x) \in [a,b]$ $\forall x \in [a,b]$.

Goal: show that $\exists$ at least 1 fixed pt.

Case I: If $f(a)=a$, then a is the fixed pt

If $f(b)=b$, then b is the fixed pt

Case II: $f(a) \neq a$ $\quad f(a) > a$

$f(b) \neq b$ $\quad f(b) < b$

let $g(x) = f(x) - x$

$g(a) > 0$

$g(b) < 0$

By the IVT $\exists$ a pt c st $g(c)=0$

ie $c = f(c)$ this is a fixed pt.

(ii) Assume (i) is true & $|f'(x)| < k < 1$ $\forall x \in (a,b)$

Goal: Show that the fixed pt is unique.

Proof by contradiction. Assume 2 unique fixed pts. α & β i.e. $f(\alpha) = \alpha$ $f(\beta) = \beta$

$\alpha \neq \beta$

$|\alpha - \beta| = |f(\alpha) - f(\beta)|$

for some $c \in [\alpha, \beta]$

$= |f'(c)(\alpha - \beta)|$ by MVT

$\leq |f'(c)|\,|\alpha - \beta|$

MVT if $f \in C^1[a,b]$

let $x, y \in [a,b]$

$\exists$ $c \in [a,b]$

st

$|f(x) - f(y)| = f'(c)(x-y)$

$\leq |f'(c)|\ |\alpha - \beta|$

$< k\ |\alpha - \beta|$

$(1-k)|\alpha-\beta| < 0$

we know

$|\alpha - \beta| > 0$ & $(1-k) > 0$

so $(1-k)|\alpha-\beta| < 0$ is impossible.

$\therefore$ contradiction.

$\Rightarrow \alpha = \beta$ //

st

$|f(x) - f(y)| = f'(c)|x-y|$

$\frac{f(x)-f(y)}{x-y} = f'(c)$

Slope of secant line

Derivative

How do we know if the fixed pt iteration will converge?

Thm 2.4 Let $g \in C[a,b]$ st $g(x) \in [a,b]$ $\forall x \in [a,b]$. Suppose, in addition, that $g'(x)$ exists on $[a,b]$ & $\exists$ a constant k $0<k<1$ w/ $|g'(x)| < k$ $\forall x \in [a,b]$. Then for any initial guess $p_0 \in [a,b]$ the sequence $p_{n+1} = g(p_n)$ converges to a unique fixedpt.

$g \in C[a,b]$
$\Rightarrow$ g is continuous on the interval $[a,b]$

$g \in C'[a,b]$
$\Rightarrow$ g & $g' \in C[a,b]$

Proof: let p denote the fixed pt. i.e. $p = g(p)$

absolute error $|p_{n+1} - p| = |g(p_n) - g(p)|$

$= |g'(c_n)(p_n - p)|$ for some c_n in the interval containing p_n & p by MVT

$\leq |g'(c_n)|\,|p_n - p|$ by Cauchy-Schwarz

$< k\,|p_n - p|$ by assumptions in thm statement.

$< k^2\,|p_{n-1} - p|$ by previous steps.

$\vdots$

$< k^{n+1}\,|p_0 - p|$

To check convergence we look at

$$\lim_{n\to\infty} |p_{n+1} - p|.$$

$$\lim_{n\to\infty} |p_n - p| < \lim_{n\to\infty} k^{n+1}|p_0 - p| = 0$$

since $k < 1$

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