Problem Set 3

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APPM4360
2/21/24

1. a. $\log(t) = \log(re^{i(\theta_p+2\pi n)}) = \log(r) + i(\theta_p + 2\pi n)$

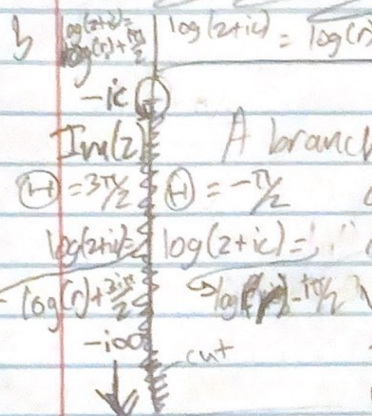
$t = z + ic$

$t = 0$ when $z = -ic$, so branch point at $z = -ic$

There are infinitely many Reimann sheets because $z = \infty$ also a branch point

~~b.~~

b

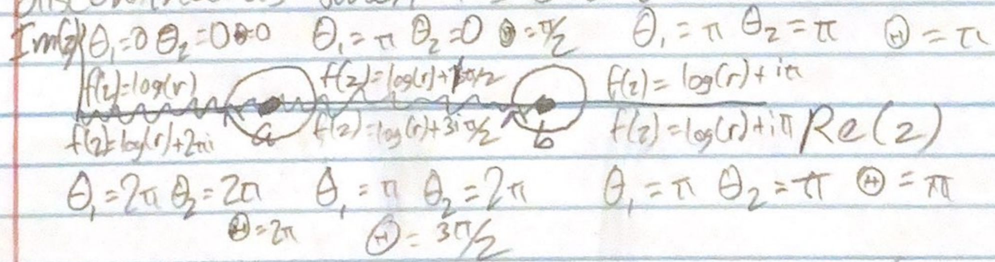


A branch cut between $z = -ic$ and $z = -i\infty$ along the negative imaginary axis as shown is used because the function's values differ there when crossing from $3\pi/2$ to $-\pi/2$ when moving in a circle, whereas above $-ic$ on the imaginary axis, the value doesn't change.

2. $\log(t) = \log(re^{i(\theta_p+2\pi n)}) = \log(r) + i(\theta_p + 2\pi n)$

$t = (z-a)/(z-b)$

$t = 0$ when $z = a$ and $t = \infty$ when $z = b$ so a and b are the branch points because $\log(t)$ is discontinuous when $t = 0$ or ∞.



The branch cut is along the real axis from $-\infty$ to b as shown.

3. a. For this boundary condition, you use a logarithmic function with a discontinuity at $x = 1$. This would be $f(z) = k\log(x-1)$ where k is adjusted to account for α and β.

b. For this boundary condition, you use a log function with a discontinuity at l_1 and l_2. This would be $f(z) = k_1 \log(x - l_1) + k_2 \log(x - l_2)$ where k_1 and k_2 are adjusted to account for α and β.

4. a. $\oint_C f(z)\,dz = \oint 2\bar{z}\,dz = 2\oint \frac{1}{z}\,dz$
because $z = e^{it}$ and $\bar{z} = e^{-it}$ thus $z \cdot \bar{z} = 1$
Substituting in $dz = ie^{it}$, we get $\oint_C ie^{it} \cdot e^{-it} = \oint_C i\,dt$
$\int_0^{2\pi} i\,dt = it \Big|_0^{2\pi} = \boxed{2\pi i}$

b. $\oint (z-1)/z\,dz = \int_0^{2\pi} \frac{e^{it}-1}{\cancel{e^{it}}} \cancel{i}e^{it}\,dt = i\int_0^{2\pi} (1-e^{-it})\,dt$
$= i \cdot t + \frac{e^{-it}}{-i}\Big|_0^{2\pi} = i \cdot t - ie^{-it}\Big|_0^{2\pi} = i(2\pi - i(e^{-2\pi i} - 1))$
$= \boxed{2\pi i}$

5. a. $|a| < 1$
By Cauchy's theorem, $\oint_C f(z)\,dz = 2\pi i \cdot$ # of singular points. There is one singular point at $z = a$ so
$\oint_C f(z)\,dz = \boxed{2\pi i}$

b. $|a| > 1$: By Cauchy's theorem, $\oint f(z)\,dz = 2\pi i \cdot$ # of singular points. in this case, a is no longer in the contour, so $\oint f(z)\,dz = 0$

c. When $|a| = 1$, a lies exactly on the contour. Thus, the integral along the contour can not be well defined.

6 (extra credit).

$$\Omega(z, z_0) = \frac{k}{2\pi|z_0|}\left[\log(z-|z_0|e^{i\theta_0}) - \log(z)\right]$$

$$= \frac{-k}{2\pi}\left(\frac{\log(z) - \log(z-|z_0|e^{i\theta_0})}{|z_0|}\right)$$

$$= \lim_{|z_0|\to 0} \frac{-k}{2\pi}\left(\frac{\log(z) - \log(z-|z_0|e^{i\theta_0})}{|z_0|}\right)$$

This is the limit definition of a derivative, specifically for log (z), which is 1/z, resulting in:

$$\Omega(z) = \frac{-ke^{i\theta_0}}{2\pi}\cdot\frac{1}{z}$$

Split into real and imaginary parts using Euler's formula

$$\Omega(z) = \frac{-ke^{i\theta_0}}{2\pi z} = \frac{-k}{2\pi}\cdot\frac{\cos\theta_0 + i\sin\theta_0}{z}$$

$$\Phi(x,y) = \frac{-k\cos\theta_0}{2\pi}Re\left(\frac{1}{z}\right) \quad \Psi = \frac{-k\sin\theta_0}{2\pi}Im\left(\frac{1}{z}\right)$$

For $z = x+iy$, $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$, so

$$\Phi(x,y) = \frac{-k\cos\theta_0}{2\pi}\cdot\frac{x}{x^2+y^2}, \quad \Psi(x,y) = \frac{-k\sin\theta_0}{2\pi}\cdot\frac{y}{x^2+y^2}$$

$\theta_0 = 0$

$\theta_0 = \pi/2$