CSCI 3104 Fall 2021
Instructors: Profs. Chen and Layer

Problem Set 10

Due Date .... April 26
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1 Instructions

- The solutions **must be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this LaTeX template.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).
- You are welcome and encouraged to collaborate with your classmates, as well as consult outside resources. You must **cite your sources in this document. Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.
- Posting to **any** service including, but not limited to Chegg, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.

2 Standard 26 - Showing problems belong to P

Problem 1. Consider the Shortest Path problem that takes as input a graph $G = (V, E)$ and two vertices $v, t \in V$ and returns the shortest path from v to t. The shortest path decision problem takes as input a graph $G = (V, E)$, two a vertices $v, t \in V$, and a value k, and returns True if there is a path from v to t that is at most k edges and False otherwise. Show that the shortest path decision problem is in P. You are welcome and encouraged to cite algorithms we have previously covered in class, including known facts about their runtime. [**Note:** To gauge the level of detail, we expect your solutions to this problem will be 2-4 sentences. We are not asking you to analyze an algorithm in great detail.]

Answer. For this problem we could use a breadth first traversal. This will find the path with the fewest edges between two vertices v and t by first searching the nodes connected to v by one edge, then those connected by two edges, and so on until t is found or all nodes have been explored thus there is no path to t from v at all. To make the solution into that of a decision problem, we can compare the number of edges of the path with the fewest edges found between v and t to k to return whether there exists a path between v and t with at most k edges. Since these methods solve this problem and have a time complexity of $O(n^2)$ where n is the number of vertices in the graph and the graph is stored as a adjacency matrix, this problem is in P. □

3 Standard 27 - Showing problems belong to NP

Problem 2. Consider the Simple Shortest Path decision problem that takes as input a directed graph $G = (V, E)$, a cost function $c(e) \in \mathbb{Z}$ for $e \in E$, and two vertices $v, t \in V$. The problem returns True if there is a simple path from v to t with edge weights that sum to at most k, and False otherwise. Show this problem is in NP.

Answer. We have shown in the last problem that this problem is in P. If we're given a witness of a solution in the form of the shortest path itself, we can count the number of edges in the path in $O(n)$ time then compare that number to k in $O(1)$ time. Thus, if our original answer was true and we get that the number of edges in the shortest path was less than or equal to k in our comparison, then we're verifying our witness in $O(n)$ time so the problem is in NP. Similarly, if our original answer was false and we get that the number of edges in the shortest path was greater than k in our comparison, then we're verifying our witness in $O(n)$ time so the problem is in co-NP. $\square$

Problem 3. Indiana Jones is gathering n artifacts from a tomb, which is about to crumble and needs to fit them into 5 cases. Each case can carry up to W kilograms, where W is fixed. Suppose the weight of artifact i is the positive integer w_i. Indiana Jones needs to decide if he is able to pack all the artifacts. We formalize the Indiana Jones decision problem as follows.

- Instance: The weights of our n items, $w_1, \ldots, w_n > 0$.
- Decision: Is there a way to place the n items into different cases, such that each case is carrying weight at most W?

Show that Indiana Jones $\in$ NP.

Answer. If we're given the witness of Indiana Jones being able to pack up all of his artifacts in the form of each bag as a list with the items it contains, we can verify that the sum of the weights of the artifacts in each bag is less than or equal to W and that all of the artifacts are in a bag in $O(n)$ time because we would be checking the weights of each of the n total artifacts and adding them together based on what bad they were in in the witness. Thus, if our original answer was true and we get that the combined weight of the artifacts in each bag was less than or equal to W in our comparison, then we're verifying our witness in $O(n)$ time so the problem is in NP. □

4 Standard 27 - NP-compelteness: Reduction

Problem 4. A student has a decision problem L which they know is in the class NP. This student wishes to show that L is NP-complete. They attempt to do so by constructing a polynomial time reduction from L to SAT, a known NP-complete problem. That is, the student attempts to show that $L \leq_p \mathsf{SAT}$. Determine if this student's approach is correct and justify your answer.

Answer. The student's approach is wrong because he's attempting to show that $L \leq_p \mathsf{SAT}$ when he should be attempting to show that $L \geq_p \mathsf{SAT}$ because he needs to show that it's harder or the same hardness to be in NP-hard as well as NP which we are already given. $\square$

Problem 5. Consider the Simple Shortest Path decision problem that takes as input a directed graph $G = (V, E)$, a cost function $c(e) \in \mathbb{Z}$ for $e \in E$, and two vertices $v, t \in V$. The problem returns True if there is a simple path from v to t with edge weights that sum to at most k, and False otherwise. Show this problem is NP-compelte.

Answer. We showed that this problem is in NP in Problem 2. To show that it's NP-hard I will reduce the Max Independent Set Problem to it which is known to be NP-hard. A true instance of the Max Independent Set Problem is a true Instance of the Shortest Path Problem because if the largest independent set in the graph has k vertices, you can't find more than k vertices that don't share an edge so every set of size more than k vertices has at least one edge connecting them so the shortest path between every two nodes must have no more than k edges. A false instance of the Max Independent Set Problem is a false Instance of the Shortest Path Problem because if the largest independent set in the graph has more than k vertices, you can find more than k vertices that don't share an edge so at least one set of k vertices doesn't have an edge connecting them connecting them so the shortest path between two vertices in that set has more than k edges. Thus the Shortest Path Problem must be in NP-hard and we already knew it was in NP so it's in NP-complete. $\square$