Newton for nonlinear systems

Wednesday, September 27, 2023 1:49 PM

Vector for of Taylor Series.

$f(\bar{x})$ $x \in \mathbb{R}^n$

Goal of Newton: find $\bar{x}$ st $f(\bar{x}) = \bar{0}$

$$f(a+\Delta x, b+\Delta y) = f(a,b) + \Delta x f_x(a,b) + \Delta y f_y(a,b)$$

$$+ \frac{(\Delta x)^2}{2} f_{xx}(a,b) + \frac{(\Delta y)^2}{2} f_{yy}(a,b)$$

$$+ \frac{(\Delta x \Delta y)}{2} f_{xy}(a,b) + \frac{(\Delta x \Delta y)}{2} f_{yx}(a,b)$$

$$+ \text{HOT}$$

$$= f(a,b) + \nabla f\big|_{(a,b)} \cdot (\Delta x, \Delta y) + \frac{1}{2}\left((\Delta x, \Delta y) \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(a,b)}}_{\text{Hessian}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \right)$$

$$+ \text{HOT}$$

What is the idea behind Newton's method in 1D?

follow the roots of the tangent lines

Goal: Given $f \in C^2[a,b]$, find x st $f(x) = 0$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ → sequence of roots of the tangent lines.

$$x_{n+1} - x_n = - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

Derivation: Via Taylor series (slightly different variant than last time)

let α denote root of $f(x)$ i.e. $f(\alpha) = 0$.

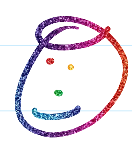
let $\Delta x_n = \alpha - x_n$ = distance between root α & approximation

$$\alpha = x_n + \Delta x_n$$

$$0 = f(\alpha) = f(x_n + \Delta x_n) = \underbrace{f(x_n) + \Delta x_n f'(x_n)}_{\text{Tangent line}} + \text{HOT}$$

Solve for Δx_n

$$\Delta x_n = \frac{-f(x_n)}{f'(x_n)}$$ in practice $\Delta x_n = x_{n+1} - x_n$

$$\rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ Same as before 

$$= g(x_n)$$ use fixed pt theory on $g(x)$

so Newton converges if $|g'(x_n)| < 1$

Now for Systems of equations.

Given functions $f(x,y)$ & $g(x,y)$. Our goal is to find (α,β) st $f(\alpha,\beta)=0$ & $g(\alpha,\beta)=0$

Vector form: Find $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ st

$$\bar{F}(\alpha,\beta) = \begin{bmatrix} f(\alpha,\beta) \\ g(\alpha,\beta) \end{bmatrix} = \bar{0}$$

let's Taylor expand

let $\alpha = x_n + \Delta x_n \quad \beta = y_n + \Delta y_n)$

$$0 = f(\alpha,\beta) = f(x_n+\Delta x_n, y_n+\Delta y_n)$$

$$= f(x_n,y_n) + \Delta x_n f_x(x_n,y_n) + \Delta y_n f_y(x_n,y_n) + HOT$$

$$0 = g(\alpha,\beta) = g(x_n+\Delta x_n, y_n+\Delta y_n)$$

$$= g(x_n,y_n) + \Delta x_n g_x(x_n,y_n) + \Delta y_n g_y(x_n,y_n) + HOT$$

1st order in vector form

$$\begin{pmatrix}0\\0\end{pmatrix} = \begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix} + \underbrace{\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{(x_n,y_n)}}_{\text{Jacobian} = J(x_n,y_n)} \begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix}$$

Solve for $\begin{pmatrix}\Delta x_n\\ \Delta y_n\end{pmatrix}$ to get

$$\begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix} = -J(x_n,y_n)^{-1} \begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix}$$

$$\begin{bmatrix} \cdots \\ \Delta y_n \end{bmatrix} = \underbrace{\cdots(x_n, y_n)}_{J_n^{-1}} \begin{bmatrix} \cdots \\ g(x_n, y_n) \end{bmatrix}$$

We know the distance between the iterates

$$\Delta x_n = x_{n+1} - x_n \qquad \Delta y_n = y_{n+1} - y_n$$

$$\Rightarrow \begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} \Delta x_n \\ \Delta y_n \end{bmatrix}$$

So the full step in the iteration is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - (J_n)^{-1} \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix}$$

1D

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

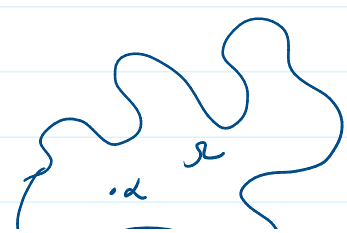
<u>Note:</u> - Quadratic convergence unless the root is not simple (i.e. multiplicity > than 1)

- It is very important to be close to the root.

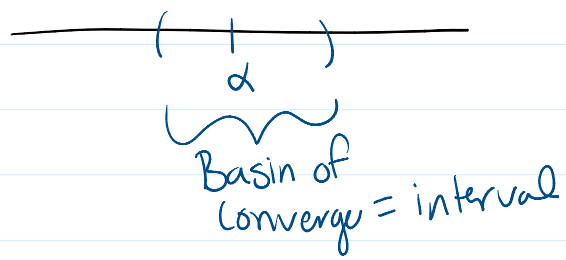
<u>Basin of convergence</u>

1D

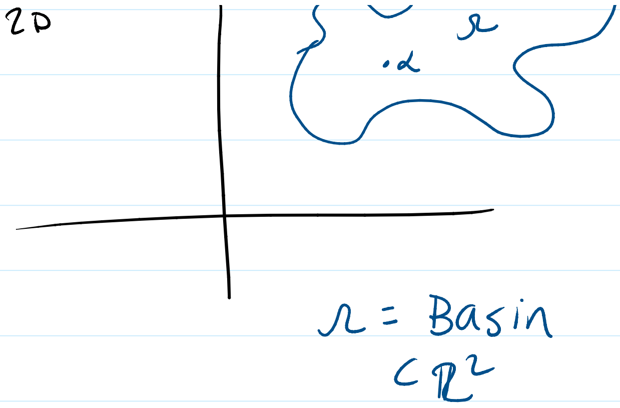
2D



1D



2D



In practice if you wanted to find the basin of convergence you have to explore $\mathbb{R}^n$ with the Jacobian.

<u>Numerical Example</u> - Basin of convergence.

let $z = x + iy$ where $x, y \in \mathbb{R}$

Use Newton to approximate the roots of $z^3 - 1$

We know that there are 3 distinct roots.

$$z_1 = 1 \quad , \quad z_{2,3} = -\frac{1}{2} \pm i\, \sqrt{3}/2$$

In terms of x & y, this is equivalent to apply Newton to the 2 equation Non-linear system

$$\begin{cases} x^3 - 3xy - 1 = 0 \\ 3x^2 y - y^3 = 0 \end{cases}$$

Code: plots in $[-1,1]^2$ in complex plane.

Green = convergence to $z_2 = -1/2 - i\sqrt{3}/2$

with that pt as initial guess.