APPM4360: Project Paper

Alex Ojemann
Emanuel Quintana
Felipe Tamayo-Tapia

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Abstract

This report is an exploration of the Riemann Zeta Function. It will start with Euler's Product and consider Riemann's extension into the Complex plane via complex arguments. It will be shown that when the real component of the argument is greater than one, the infinite sum can be represented as an integral. Furthermore the Analytic Continuation of the function beyond its singular points as well as the implications that the function's zeros have on the distribution of prime numbers will be shown as well as discussed. Visualizations not limited to the functions singular points and zeros will be included.

1 Introduction

The Riemann Zeta Function, denoted as $\zeta(s)$, is one of the most significant functions in mathematics. It holds a lot of importance in number theory. In this project, we explore understanding the basic properties of the Riemann Zeta Function using analytical techniques. These properties include its definition, the domain in which it exists and is well-defined, as well as its converging and diverging properties.

Also discussed is Euler's Product which helps us understand how prime numbers work with the Riemann Zeta Function. We discuss and prove how the Riemann Zeta Function is equal to an infinite sequence of prime numbers, this infinite product being Euler's product.

The Riemann Hypothesis also referred to when talking about the zeros of the zeta function, it hypothesizes that all non-trivial zeros lie on the critical line Re(s) $= 1/2$, this hypothesis has not been proven but has been tested to be true for millions of points that lie on the critical line.

The integral representation of the Riemann Zeta Function is shown along with its graphical representation. Using the integral representation, the analytic continuation of the Riemann Zeta Function can be defined.

2 Riemann Zeta Function properties

Initially, we studied the basic properties of the Riemann Zeta Function using analytical techniques. These involved understanding its definition, domain, and basic properties such as convergence and divergence.

The Riemann zeta function $\zeta(s)$ is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad , \ \mathrm{Re}(s) > 1$$

where s is a complex variable and the real part of s is bigger than zero.

$\mathrm{Re}(s)$ being bigger than 1 is important because if we let s be a complex variable and write

$$s = (u + it),$$

where both u and t are real. If we use $n^s = e^{s \log n}$ and substitute back in the real and imaginary form it becomes

$$e^{(u+it) \log n} = n^u e^{it \log n}$$

we can then show that

$$|n^s| = n^u \ , \quad \text{since} \quad |e^{it \log n}| = 1$$

now if we let $u > 1$

$$\frac{1}{|n^s|} < \frac{1}{n}$$

For ϵ approaching zero and $u > 1$,

$$\frac{1}{n^u} < \frac{1}{n^{1+\epsilon}} < \frac{1}{n}$$

To prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^u}$ is absolutely convergent for $u > 1$, consider that since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series, and $\lim_{\epsilon \to \infty} \sum_{n=1}^{\infty} \frac{1}{n^{u+\epsilon}}$ is convergent, since $\frac{1}{n^u}$ is bounded above by $\frac{1}{n+\epsilon}$ for $\epsilon \to 0$, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^u}$ converges absolutely. Therefore the series $\sum_{n=1}^{\infty} \frac{1}{n^u}$ is absolutely convergent for $u > 1$.

On the other hand, when $u < 1$,

$$\frac{1}{n^u} > \frac{1}{n}$$

the series $\sum_{n=1}^{\infty} \frac{1}{n^u}$ diverges by the Comparison Test.

The Riemann zeta function $\zeta(s)$ is an analytic function for s greater than 1, with the derivative being

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s} \qquad , \ \mathrm{Re}(s) > 1$$

3 Euler's Product

The Reimann Zeta function was first connected to prime number theory by Leonhard Euler in 1737, before Bernhard Reimann himself first researched the properties of the function. He showed that the function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is equal to $\prod_p \frac{1}{1-p^{-s}}$ where p is a prime number for the infinite sequence of prime numbers. This infinite product is known as Euler's product.

To prove this, one can start with the Reimann Zeta function and expand the sum,

$$\zeta(s) = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + ...$$

Then multiply both sides by $1/2^s$,

$$1/2^s * \zeta(s) = 1/2^s + 1/4^s + 1/6^s + 1/8^s + 1/10^s + ...$$

Then, subtracting that from the original function,

$$(1 - 1/2^s) * \zeta(s) = 1 + 1/3^s + 1/5^s + 1/7^s + 1/9^s + ...$$

Now this is repeated by multiplying by $1/3^s$ and subtracting from the the result of the previous step,

$$(1 - 1/3^s) * (1 - 1/2^s) * \zeta(s) = 1 + 1/5^s + 1/7^s + 1/11^s + 1/13^s + ...$$

Both sides can be multiplied an infinite number of times by $1 - 1/p^s$ where p is a prime number for all primes because there are infinitely many primes. This will ultimately result in 1 on the right side of the equation because all of the other terms will be eliminated when multiplying by $1 - 1/p^s$ because all numbers can be decomposed into a product of prime numbers.

Thus,

$$\prod_p (1 - 1/p^s) * \zeta(s) = 1$$

Solving for $\zeta(s)$,

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

4 Integral Representation of $\zeta(s)$

The Zeta Function's Analytic Continuation is obtained through its integral representation in terms of the Gamma Function. Let $s = \sigma + it$, then

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx$$

making the substitution that $x = (n + a)t$ so $dx = (n + a)dt$ for $n \geq 0$ then

$$\Gamma(s) = (n + a)^s \int_0^{\infty} e^{-(n+a)t} t^{s-1}\, dt$$

.
dividing both sides by $(n+a)^s$

$$(n+a)^{-s}\Gamma(s) = \int_0^\infty e^{-(n+a)t}t^{s-1}\,dx$$

If we sum over n, then

$$\zeta(s)\Gamma(s) = \sum_{n=0}^{\infty}\int_0^\infty e^{-(n+a)t}t^{s-1}$$

and because the integral is a Lebesgue Integral, the sum and integral interchange. Notice that if $t > 0$ then $0 \leq e^{-t} \leq 1$ and

$$\sum_{n=0}^{\infty} e^{-nt} = \frac{1}{1-e^{-t}}$$

so finally

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}}\,dx$$

This integral representation converges uniformly and represents an analytic function for all s with $\sigma > 1$.

5 Analytic Continuation

For a complex number $s = \sigma + it$, $\zeta(s)$ can be represented for values of s where $\sigma \leq 1$. This is done through the contour integral shown. It encloses the negative real axis. The function defined by this contour integral is entire.

$$I(s,a) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^{az}}{1-e^z}\,dz$$

which cleans up the Integral Representation of $\zeta(s)$

$$\zeta(s,a) = \Gamma(1-s)I(s,a)$$

This last representation is analytic for all s except for a pole at s=1 with residue 1. $I(s,a)$ is entire so the singular values of $\zeta(s)$ are the pole of $\Gamma(1-s)\quad s = 1, 2, 3, \ldots$. But we know that $\zeta(s)$ is analytic for $s = 2, 3, \ldots$ so $s = 1$ is the only pole for the Analytic Continuation of $\zeta(s)$ in the entire complex plane.

6 Zeros Distribution:

It has been well established that there are zeros of the Riemann zeta function at $s = -2, -4, -6, \ldots$ for all even integers on the negative real axis, which are known as the trivial zeros. However, there are more zeros which do not follow

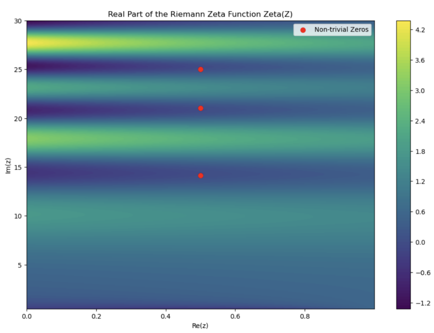

Figure 1: Real part of $\zeta(Z)$.

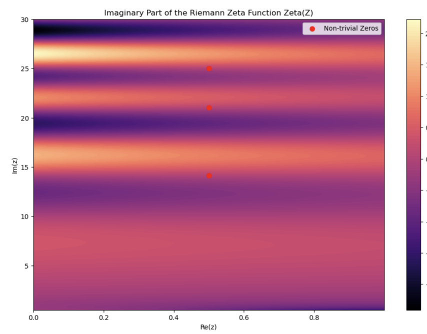

Figure 2: Imaginary part of $\zeta(Z)$.

this pattern called the non-trivial zeros. The Riemann Hypothesis which hypothesises that all non-trivial zeros lie on the critical line $\mathrm{Re}(s) = 1/2$. Jacques Hademard and Charles Jean de la Vallée Poussin proved that all non-trivial zeros of the analytically continued Riemann Zeta function lie in the critical strip, between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$.

Figures 1 and 2 show the real and imaginary parts of the Riemann Zeta function within the critical strip as contour maps for $0.5 < \mathrm{Im}(s) < 30$. Zeros were found using Müller's method with initial guesses at an evenly spaced grid of points within and $0 < \mathrm{Re}(s) < 1$ and $0.5 < \mathrm{Im}(s) < 30$ and plotted on each of the maps as red points. This verifies that all of the zeros of the function within this range of values for the imaginary part of s lie on the line $\mathrm{Re}(s) = 1/2$ as Riemann hypothesized. However, nobody has ever been able to mathematically prove that all zeros are on this line.

Figure 3 shows the real and imaginary parts as blue and red lines respectively when the real part of the input is fixed at 1/2 and the imaginary part is on the x axis where $0.5 < \mathrm{Im}(s) < 30$. There are several points where both the real and imaginary part both equal zero.

7 Prime Number Thoerem

The Prime Number Theorem:

$$\lim_{x\to\infty} \frac{\pi(x)\log(x)}{x} = 1$$

where $\pi(x)$ is the number of primes $\leq$ x. There are an infinite number of primes so $\pi(x) \to \infty$ as $x \to \infty$. So the Prime Number Theorem is saying that $\pi(x)$ In this section we are essentially working with two functions.
The Mangoldt Function:

$$\Lambda(n) = \begin{cases} \log p, & \text{n=p}^m \text{ for some prime p and some m} \geq 1, \\ 0, & otherwise \end{cases}$$

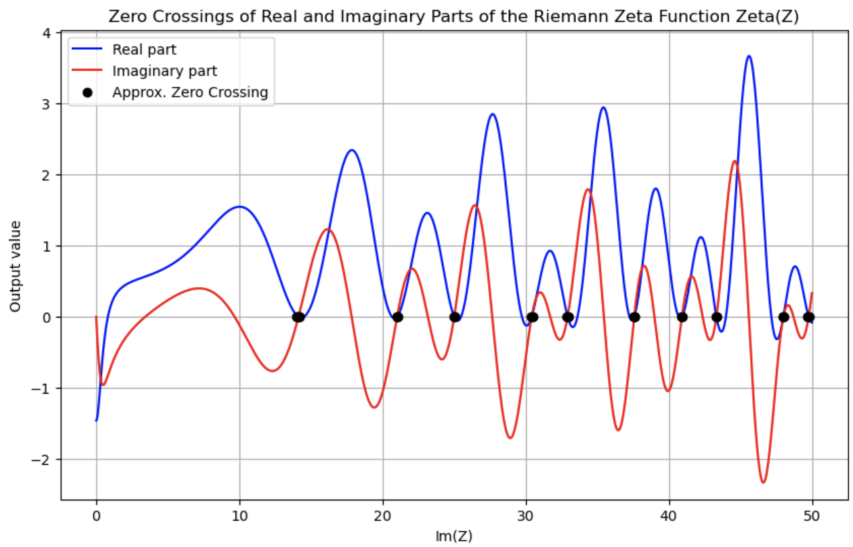

Figure 3: Real and Imaginary Parts of $\zeta(Z)$ when $\mathrm{Re}(s) = 1/2$.

Chebyshev's Function:

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

Proving the Prime Number Theorem is equivalent to proving that $\psi(x) \sim x$ as $x \to \infty$. Using an integral representation

$$\psi_1(x) = \int_1^{\infty} \psi(t)dt$$

we can show that

$$\psi_1(x) \sim \frac{1}{2}x^2 \quad \text{as} \quad x \to \infty \Rightarrow \psi(x) \sim x \quad as \quad x \to \infty$$

This is done by representing $\psi(x)_2$ in terms of the Contour Representation of $\zeta(s)$

$$\frac{\psi_1(x)}{x^2} = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds \quad , c > 1$$

Note that $\left(-\frac{\zeta'(s)}{\zeta(s)}\right)$ has a first order pole at $s = 1$ with residue 1. Subtracting this pole gives

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^{s-1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right) ds \quad , c > 1$$

let

$$h(s) = \frac{1}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \right)$$

then

$$\frac{\psi_1(x)}{x^2} - \frac{1}{2}\left(1 - \frac{1}{x}\right)^2 = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{s-1}h(s)ds = \frac{x^{c-1}}{2\pi}\int_{-\infty i}^{+\infty i} h(c+it)e^{it\log x}dt$$

Finally

$$\lim_{x\to\infty}\frac{x^{c-1}}{2\pi}\int_{-\infty}^{+\infty} h(c+it)e^{it\log(x)}dt = 0$$

because the Riemann-Lebesgue lemma in the theory of Fourier series states that

$$\lim_{x\to\infty}\int_{-\infty}^{+\infty} f(t)e^{itx}dt = 0$$

8 Conclusion and Future Directions

This project included demonstrations of the convergence of the series definition of the Reimann zeta function when $\mathrm{Re}(s) > 1$, proof that the function is equivalent to Euler's product over all prime numbers, the steps to represent the function as an integral and extend into the region of the complex plane where $\mathrm{Re}(s) \leq 1$, numerical experiments that verify that all non-trivial zeros over a certain range of imaginary values in the critical strip are on the line $\mathrm{Re}(s) = 1/2$ as Riemann hypothesized, and a proof of the prime number theorem which represents a fascinating application of the Riemann zeta function.

One area in which it would've been fascinating to explore if we had more time is the error bound of the prime number theorem. It has been shown that the error bound is $O(\sqrt(n) * log(n))$ if the Riemann hypothesis is true. It also would've been interesting to try to visualize $\pi(x)$ and its approximation and error bound.

9 References

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2. Alexander, Anita Nicole. A Thesis Submitted to Kent State University in Partial Fulfillment of the Requirements for the Degree of Master of Arts, etd.ohiolink.edu/acprod/odb$_etd/ws/send_file/send?accession = case1212682598disposition = inline.Accessed30Apr.$2024.

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10 Appendix

Python code to generate figures:

```
import numpy as np
import matplotlib.pyplot as plt
from mpmath import zeta, findroot, mp

# Set higher precision
mp.dps = 30

# Grid setup for initial root guesses
x = np.linspace(0, 0.9999, 40)
y = np.linspace(0.5, 30, 60)
X, Y = np.meshgrid(x, y)

# Convert grid to complex numbers
initial_guesses = X + 1j * Y

# Function to find zeros of Zeta using findroot
def find_zeta_zero(z):
    try:
        # Using the Muller method to find roots starting from initial complex nu
        root = findroot(lambda s: zeta(s), z, solver='muller', maxsteps=1000, to
        return root
    except Exception as e:
        return None

# Array to store roots
zeros = []

# Process each initial guess
for z in initial_guesses.flatten():
    root = find_zeta_zero(z)
    if root and all(np.abs(root - zp) > 1e-5 for zp in zeros):
# Check uniqueness within tolerance
        zeros.append(root)

# Print and plot results
zeros = np.array(zeros)

print("Zeros found:")
print(zeros)

from mpmath import mpc, re, im

# Function to check if a new zero is close to any zero in the list
def is_unique(new_zero, existing_zeros, tolerance=1e-4):
```

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    return all(abs(im(new_zero) - im(zero)) > tolerance for zero in existing_zer

# Filter to retain only those zeros where the real part is less than 0 and imagi
# and ensure uniqueness within 1e-10 tolerance
nontrivial_zeros = []
for z in zeros:
    if re(z) > 0 and im(z) < 30 and im(z) > 0 and is_unique(z, nontrivial_zeros)
        nontrivial_zeros.append(z)

print("Non-trivial zeros (Real part > 0)")
for zero in nontrivial_zeros:
    print(zero)

#Grid setup for display
x = np.linspace(0, 0.9999, 400)
y = np.linspace(0.5, 30, 600)
X, Y = np.meshgrid(x, y)

# Calculate Zeta function values with appropriate type conversion
Z = np.array([[complex(zeta(complex(x, y))) for x in X[0]] for y in Y[:, 0]])

# Ensure no NaNs or infinities
Z_real = np.real(Z)
Z_imag = np.imag(Z)

np.nan_to_num(Z_real, copy=False, nan=0.0, posinf=0.0, neginf=0.0)
np.nan_to_num(Z_imag, copy=False, nan=0.0, posinf=0.0, neginf=0.0)


# Extract real and imaginary parts for plotting
zero_reals = [re(z) for z in nontrivial_zeros]
zero_imags = [im(z) for z in nontrivial_zeros]

# Plotting the real part of the Zeta function
plt.figure(figsize=(12, 8))
plt.contourf(X, Y, Z_real, levels=100, cmap='viridis')
plt.colorbar()
plt.scatter(zero_reals, zero_imags, color='red', label='Non-trivial Zeros', s=50
# scatter non-trivial zeros
plt.title('Real Part of the Riemann Zeta Function Zeta(Z)')
plt.xlabel('Re(z)')
plt.ylabel('Im(z)')
plt.legend()
plt.show()

# Plotting the imaginary part of the Zeta function
```

```
plt.figure(figsize=(12, 8))
plt.contourf(X, Y, Z_imag, levels=100, cmap='magma')
plt.colorbar()
plt.scatter(zero_reals, zero_imags, color='red', label='Non-trivial Zeros', s=50
# scatter non-trivial zeros
plt.title('Imaginary Part of the Riemann Zeta Function Zeta(Z)')
plt.xlabel('Re(z)')
plt.ylabel('Im(z)')
plt.legend()
plt.show()

import numpy as np
import matplotlib.pyplot as plt
from mpmath import zeta

# Define the range of the imaginary part t
t_values = np.linspace(0, 50, 1000)
real_parts = []
imag_parts = []

# Calculate Zeta function values for sigma = 0.5 + i*t
for t in t_values:
    z = zeta(0.5 + 1j * t)
    real_parts.append(float(z.real))
    imag_parts.append(float(z.imag))

# Threshold to consider close to zero
threshold = 0.1

# Find approximate zero crossings for both real and imaginary parts
approx_zero_crossings = []

# Check where both the real and imaginary parts are close to zero
for i in range(len(real_parts)):
    if abs(real_parts[i]) < threshold and abs(imag_parts[i]) < threshold:
        approx_zero_crossings.append(t_values[i])

# Plotting
plt.figure(figsize=(10, 6))
plt.plot(t_values, real_parts, label='Real part', color='blue')
plt.plot(t_values, imag_parts, label='Imaginary part', color='red')

# Mark approximate zero crossings on the plot
for z in approx_zero_crossings:
    plt.scatter(z, 0, color='black', zorder=5, label="Approx. Zero Crossing" if
```

```
plt.xlabel('Im(Z)')
plt.ylabel('Output value')
plt.title('Zero Crossings of Real and Imaginary Parts of the Riemann Zeta Functi
plt.legend()
plt.grid(True)
plt.ylim(1.1 * min(real_parts + imag_parts), 1.1 * max(real_parts + imag_parts))
plt.show()
```