Problem Set 2

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1. A function $f(z_0)$ is continuous at z_0 if for every $\epsilon>0$ there exists a $\delta>0$ such that for all z within a δ-neighborhood of z_0, $|f(z)-f(z_0)|<\epsilon$

Since ϵ can be arbitrarily small and f is continuous for all z, there must be a neighborhood in which $f(z)\neq 0$ for all z.

2. $\partial u/\partial x = \cos(x)\cosh(y) = \partial v/\partial y$

$\partial u/\partial y = \sin(x)\sinh(y) = -\partial v/\partial x$

$f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

$= \sin(x)\cos(iy) + i\cos(x)\sin(iy)$

$= \sin(x+iy)$

$= \boxed{\sin(z)}$

3. a. $\partial v/\partial y = -x = \partial u/\partial x$

$\partial v/\partial x = k-y = -\partial u/\partial y$

$\partial u/\partial y = y-k$

$u(x,y) = y^2/2 - x^2/2 - ky$

$f(z) = u(x,y) + i(v(x,y)) = y^2/2 - x^2/2 - ky + i(x(k-y))$

b. $v(r,\theta) = -\sin\theta/r$

$\partial u/\partial r = 1/r\,\partial v/\partial\theta = -\cos\theta/r^2$

$\partial u/\partial\theta = -r\,\partial v/\partial r = -\sin\theta/r$

$u(r,\theta) = \cos\theta/r + \cos\theta/r$

$f(z) = u(r,\theta) + iv(r,\theta) = \frac{2\cos(\theta)}{r}$ ~~+~~ $+ i(\sin\theta/r)$

$= \frac{2x}{x^2+y^2} + \frac{iy}{x^2+y^2} = \frac{2x+iy}{x^2+y^2}$

4. a. This function is undefined at $\pm i$

$$f(x,y) = \frac{1}{(x+iy)^2+1} = \frac{1}{x^2+2xiy-y^2+1}$$

$$u(x,y) = \frac{x^2-y^2+1}{(x^2-y^2+1)^2+(2xy)^2} \qquad v(x,y) = \frac{-2xy}{(x^2+y^2+1)^2+(2xy)^2}$$

$\partial u/\partial x \neq \partial v/\partial y$, so C-R does not hold and the function is not analytic

b. $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$ and $\cosh(z)$ is never 0, so the function is defined everywhere

$$u(x,y) = \frac{\sinh(2x)}{2\cos^2(y)+\cosh(x)} \qquad v(x,y) = \frac{\sin(2y)}{\cos(2y)+\cosh(2x)}$$

$\partial u/\partial x = \partial v/\partial y$ and $\partial v/\partial x = -\partial u/\partial y$, so the function is analytic everywhere

c. The functions e^z and $\sin(z)$ are both well defined everywhere, so $e^{\sin(z)}$ is as well

$$u(x,y) = e^{\sin(x)\cosh(y)}\cos(\cos(x)\sinh(y))$$
$$v(x,y) = e^{\sin(x)\cosh(y)}\sin(\cos(x)\sinh(y))$$

$\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$, so the function is analytic everywhere

5. We propose the solution $w = e^{rt}$

$$\frac{d^3}{dt^3}(e^{rt}) - k^3 e^{rt} = 0$$

$$r^3 e^{rt} - k^3 e^{rt} = 0$$

$$r^3 - k^3 = 0$$

$$r^3 = k^3$$

Solutions to this are k, $ke^{2\pi i/3}$ and $ke^{4\pi i/3}$

Thus, $w(t) = C_1 e^{kt} + C_2 e^{(-k/2 + i\sqrt{3}k/2)t} + C_3 e^{(-k/2 - i\sqrt{3}k/2)t}$

In terms of real functions,

$$w(t) = A_1 e^{kt} + e^{-kt/2}\left(A_2 \cos\left(\frac{\sqrt{3}kt}{2}\right) + A_3 \sin\left(\frac{\sqrt{3}kt}{2}\right)\right)$$

6. ~~If $f(z) = x + iy$, $f(\bar{z})$~~

Given that $u(x,y)$ and $v(x,y)$ of $f(\bar{z})$ are twice differentiable, Laplace equation is satisfied if we operate under the assumption of harmonic functions

Cauchy Riemann equations can't be satisfied for any function of $\bar{z}$ unless it's constant, thus it can't be analytic unless it's constant.

7. a. $e^z = i\pi(z-1)$
$z = \ln(i\pi(z-1))$

6. a. i. This branch point occurs at $z=i$ because the function changes values when approaching $z=i$ from different sides

ii. This has a branch point when $z=1$ because $\log(1/0)$ is undefined and at $z \to \infty$ because $\log(1/-\infty)$ is undefined, so there is a branch cut along the positive real axis from 1 to ∞.

8. a. $z = \tanh(w)$

$z = e^w - e^{-w} / e^w + e^{-w}$

$z(e^w + e^{-w}) = e^w - e^{-w}$

$e^w(1-z) = e^{-w}(1+z)$

$\ln(e^w(1-z)) = \ln(e^{-w}(1+z))$

$w + \ln(1-z) = -w + \ln(1+z)$

$2w = \ln(1+z) - \ln(1-z)$

$w = 1/2 \ln\left(\frac{1+z}{1-z}\right)$

Thus, $\tanh^{-1}(z) = 1/2\left(\frac{1+z}{1-z}\right)$

b. $d/dz(\tanh^{-1}(z)) = d/dz\left(1/2 \ln\left(\frac{1+z}{1-z}\right)\right)$

$= 1/2 \cdot \frac{1-z}{1+z} \cdot \frac{d}{dz}\left(\frac{1+z}{1-z}\right)$

$= 1/2 \cdot \frac{1-z}{1+z} \cdot \left(\frac{(1-z)-(-1-z)}{(1-z)^2}\right)$

$= 1/2 \frac{1-z}{1+z} \cdot \frac{2}{(1-z)^2}$

$= 1/(1-z)(1+z)$

$= \boxed{1/(1-z^2)}$

9. $\Omega(z) = k \log(z - z_0)$

~~In complex~~,

This can be written as $\log(z-z_0) = \log|z-z_0| \cdot i\,\text{sign}(z-z_0)$

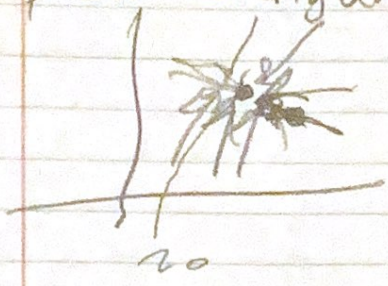
Thus, $\Omega(z) = k\log|z-z_0| + ik\,\text{sign}(z-z_0)$

$\bar{V} = d\Omega/dz = d/dz(k\log(z-z_0)) = k/(z-z_0)$

$M = \oint V_r\, ds$

$M = \int_0^{2\pi} k/r \; r\, d\theta = \int_0^{2\pi} k\, d\theta = 2\pi k$

Flow configuration



10. a. $z^2 - 1 = 0$

$z = \pm 1$

Thus, the branch cut is between -1 and 1 on the positive real axis because values on either side of that cut are different for principal branch $0 \leq \theta < 2\pi$

b. $z^2 + 1 = 0$

$z = \pm i$

Thus, the branch cut is between $-i$ and $+i$ along the imaginary axis because values on either side of that cut are different for principal branch $0 \leq \theta \leq 2\pi$