Romberg

Monday, November 6, 2023 1:59 PM

$(x-a)(x-(\frac{a+b}{2}))(x-b)$

Trapezoidal & Simpson's rule are low order quadratures. This mean that convergence is slow but they are efficient to evaluate.

We can do some post processing of low order approximations to get high order accuracy.

## Recall Composite Trapezoidal Rule

$$\underbrace{\int_a^b f(x)\,dx}_{I} = \underbrace{\frac{h}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right)}_{\text{Quadrature.}} + \underbrace{k_1 h^2 + k_2 h^4 + \cdots}_{\text{Series rep. of error}}$$

$$T(h) + k_1 h^2 + k_2 h^4 + k_3 h^6 + \cdots$$

where $k_j$ are constants that we don't need to know.

① $I = T(h) + k_1 h^2 + k_2 h^4 + k_3 h^6 + \cdots$

② $I = T(h/2) + k_1 \frac{h^2}{4} + k_2 \frac{h^4}{2^4} + k_3 \frac{h^6}{2^6} + \cdots$

4② − ① The $h^2$ term will go away

$$\underbrace{4I - I}_{3I} = 4\,T(h/2) - T(h) + \tilde{c}_1 h^4 + \tilde{c}_2 h^6 + \cdots$$

wher $\tilde{c}_i$ are constants

$$③\quad I = \underbrace{\frac{4\,T(h/2) - T(h)}{3}}_{N(h)} + c_1 h^4 + c_2 h^6 + \cdots$$

$$④\quad I = N(h/2) + c_1 \frac{h^4}{2^4} + c_2 \frac{h^6}{2^6} + \cdots$$

$2^4$ ④ − ③

$$(2^4 - 1) I = 2^4 N(h/2) - N(h) + d_1 h^6 + d_2 h^8 + \cdots$$

$$I = \underbrace{\frac{2^4 N(h/2) - N(h)}{2^4 - 1}}_{N_2(h)} + d_1 h^6 + d_2 h^8 + \cdots$$

What evaluations are need to create $N(h/2)$?

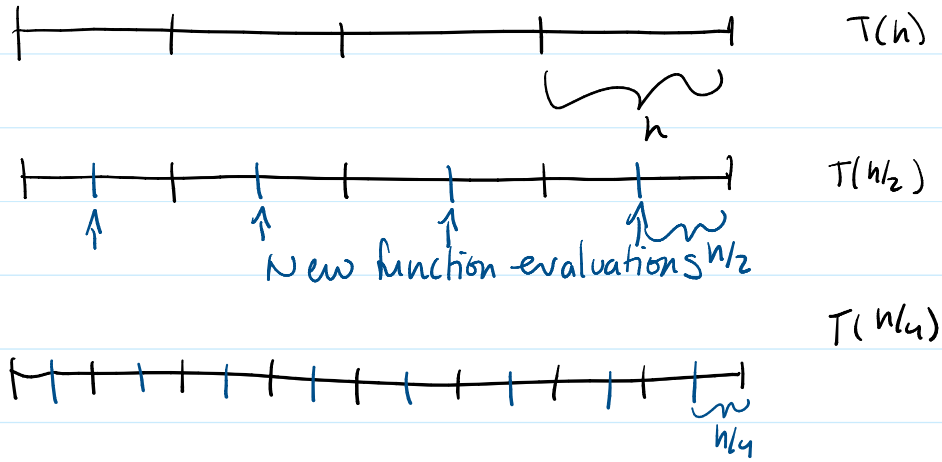
What do we need to create $N(h)$?

$T(h/2)$ & $T(h)$

To get $N(h/2)$ we need $T(h/2)$ & $T(h/4)$

To get $N_2(h)$ we need: $T(h)$, $T(h/2)$ & $T(h/4)$

function evaluations




Many evaluations can be reused to further reduce the cost of creating $T(h/2^j)$

You can repeat the "elimination process" to get an $O(h^{2j})$ approximation of $I$

$$N_j(h) = N_{j-1}(h/2) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}$$

You can use a Newton Divided-Difference type table.

| Evaluations | $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|
| $j=0$ | $j=1$ | $j=2$ | $j=3$ |

| Evaluations | $N_1$ | $N_2$ | $N_3$ |
|---|---|---|---|
| $j=0$ | $j=1$ | $j=2$ | $j=3$ |
| $T(h)$ | $N(h)$ | $N_2(h)$ | $N_3(h)$ |
| $T(h/2)$ | $N(h/2)$ | $N_2(h/2)$ | |
| $T(h/4)$ | $N(h/4)$ | | |
| $T(h/8)$ | | | |
| Order $h^2$ | $h^4$ | $h^6$ | $h^8$ |

Every entry is an approximation of $I$.
The accuracy improves as you move to the right.