Homework 8

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3.5: 1. a. $\begin{bmatrix} 4 \end{bmatrix} = 4 > 0$ ✓

$\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} = 12 > 0$ ✓

yes.

b. $\begin{bmatrix} -1 \end{bmatrix} = -1$ ✗

No

2. b. $\begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix} + \frac{1}{5}R1$

$\begin{bmatrix} 5 & -1 \\ 0 & 14/5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -1/5 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 0 & 14/5 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -1/5 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 14/5 \end{bmatrix}\begin{bmatrix} 1 & -1/5 \\ 0 & 1 \end{bmatrix}$: positive definite

19. b. $\begin{bmatrix} 4 & -12 \\ -12 & 45 \end{bmatrix} + 3R1$

$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -12 \\ 0 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$\boxed{\begin{bmatrix} 2 & 0 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} 2 & -6 \\ 0 & 3 \end{bmatrix}}$

d. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \frac{1}{2}R1$

$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 1 & 1 & 2 \end{bmatrix} - \frac{1}{2}R1$

$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{bmatrix} - R2/3$

$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \\ 0 & 0 & 4/3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4/3 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & \sqrt{3/2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{6} & \sqrt{4/3} \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{4/3} \end{bmatrix}}$

4.1.1.a. $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} + 2R1$

$\begin{bmatrix} -1 & 2 \\ 0 & 5 \end{bmatrix} /5$

$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} -2R2$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} /-1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓ Basis

$\langle -1, 2\rangle \cdot \langle 2, 1\rangle = -2+2 = 0$ ✓ orthogonal

$\|v_1\| = \|v_2\| = 5$ X orthonormal

$\boxed{\text{Orthogonal basis}}$ $\boxed{\text{i and ii}}$

b. $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} -R1$

$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix} + R2/2$

$\begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{matrix} \cdot\sqrt{2} \\ /\sqrt{2} \end{matrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ Basis ✓

$\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle \cdot \langle -1/\sqrt{2}, 1/\sqrt{2}\rangle = -\sqrt{2}+\sqrt{2} = 0$ Orthogonal ✓

$\|v_1\| = \|v_2\| = 1$ ✓ orthonormal

$\boxed{\text{Orthonormal basis}}$ $\boxed{\text{i, ii, and iii}}$

c. $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are linearly dependent, so $\boxed{\text{not basis}}$

4.1.2.a. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ -R1-R3 \\ \\ \end{matrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} \\ /-3 \\ \\ \end{matrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{matrix} -R2 \\ \\ -R2 \end{matrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓ basis

$\langle 1,1,0\rangle \cdot \langle 1,-1,1\rangle = 1+(-1)+0 = 0$

$\langle 1,1,0\rangle \cdot \langle 0,1,1\rangle = 0+1+0 = 1 \neq 0$ Not orthogonal

(Basis) $\boxed{i}$

b. $\begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{matrix} +R3 \\ \\ \\ \end{matrix}$

$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{matrix} +R2 \\ \\ \\ \end{matrix}$

$\begin{bmatrix} 0 & 0 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ Not linearly independent, Not basis

4.1! 5, a. $\langle a, 1\rangle \cdot \langle -a, 1\rangle = -a^2+1 = 0$

$\boxed{a = \pm 1}$

8. $\langle\langle -1, 2\rangle, \langle 1, 2\rangle\rangle = 0$

$\|\langle -1, 2\rangle\| = 1 \quad \|\langle 1, 2\rangle\| = 0$

$\langle v, w\rangle = \left(\frac{v_1 + w_1}{2}\right)$

4.2.1. a. $y_1 = x_1 = \langle 1, 0, 1\rangle$

$y_2 = x_2 - proj_{y_1} x_2 = \langle 1, 1, 1\rangle - \frac{\langle 1,1,1\rangle \cdot \langle 1,0,1\rangle}{(\sqrt{2})^2} \langle 1, 0, 1\rangle$

$= \langle 0, 1, 0\rangle$

$y_3 = x_3 - proj_{y_1} x_3 - proj_{y_2} x_3 = \langle -1, 2, 1\rangle - \frac{\langle -1,2,1\rangle \langle 1,0,1\rangle}{(\sqrt{2})^2} \langle 1, 0, 1\rangle$

$- \frac{\langle -1,2,1\rangle \langle 0,1,0\rangle}{1^2} \langle 0, 1, 0\rangle = \langle -1, 0, 1\rangle$

basis: $\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$

2. b. $y_1 = x_1 = \langle 1, 0, 0, 1\rangle$

$y_2 = x_2 - proj_{y_1} x_2 = \langle 4, 1, 0, 0\rangle - \frac{\langle 4,1,0,0\rangle \langle 1,0,0,1\rangle}{(\sqrt{2})^2} \langle 1, 0, 0, 1\rangle$

$= \langle 2, 1, 0, -2\rangle$

$y_3 = x_3 - proj_{y_1} x_3 - proj_{y_2} x_3 = \langle 1, 0, 2, 1\rangle - \frac{\langle 1,0,0,1\rangle \langle 1,0,2,1\rangle}{(\sqrt{2})^2} \langle 1, 0, 0, 1\rangle$

$- \frac{\langle 1,0,2,1\rangle \langle 2,1,0,-2\rangle}{3^2} \langle 2, 1, 0, -2\rangle = \langle -1, 0, 2, -1\rangle$

$y_4 = x_4 - proj_{y_1} x_4 - proj_{y_2} x_4 - proj_{y_3} x_4 = \langle 0, 2, 0, 1\rangle - \frac{\langle 0,2,0,1\rangle \langle 1,0,0,1\rangle}{(\sqrt{2})^2} \langle 1, 0, 0, 1\rangle$

$- \frac{\langle 0,2,0,1\rangle \langle 2,1,0,-2\rangle}{3^2} \underset{\langle 2,1,0,-2\rangle}{\wedge} - \frac{\langle 0,2,0,1\rangle \langle -1,0,2,-1\rangle}{(\sqrt{6})^2} \langle -1, 0, 2, -1\rangle = \langle -2/3, 2, 1/3, 1/3\rangle$

basis: $\left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} -2 \\ 6 \\ 1 \\ 1 \end{bmatrix} \right)$

4.2: 3. $y_1 = x_1 = \langle 1,-1,0,1\rangle$

$y_2 = \langle 0,-1,1,2\rangle - \left(\frac{\langle 1,-1,0,1\rangle\langle 0,-1,1,2\rangle}{(\sqrt{3})^2}\right)\langle 1,-1,0,1\rangle = \langle -1,0,1,1\rangle$

$y_3 = \langle 2,-1,-1,0\rangle - \left(\frac{\langle 1,-1,0,1\rangle\langle 2,-1,-1,0\rangle}{(\sqrt{3})^2}\right)\langle 1,-1,0,1\rangle -$
$\left(\frac{\langle -1,0,1,1\rangle\langle 2,-1,-1,0\rangle}{(\sqrt{3})^2}\right)\langle -1,0,1,1\rangle = \langle 1/3, 0, -1/3, -1/3\rangle$

This will not result in a basis because $y_2 = -3y_3$ so they're not linearly independent. This happens because the original vectors aren't linearly independent.

8. a. $\begin{bmatrix} 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{3} \end{bmatrix}$

17. a. $y_1 = x_1 = \langle -1,1,2\rangle$

$y_2 = \langle -1,-1,1\rangle - \frac{\langle -1,1,2\rangle\langle -1,-1,1\rangle}{(\sqrt{6})^2}\langle -1,1,2\rangle = \langle -2/3, -4/3, 1/3\rangle$

$y_3 = \langle 0,1,3\rangle - \frac{\langle -1,1,2\rangle\langle 0,1,3\rangle}{(\sqrt{6})^2}\langle -1,1,2\rangle - \frac{\langle -1,-1,1\rangle\langle 0,1,3\rangle}{(\ )}\langle -1,-1,1\rangle$
$= \langle 5/6, -1/2, 1\rangle$

basis: $\left(1/\sqrt{6}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, 1/\sqrt{21}\begin{bmatrix} -2 \\ -4 \\ 1 \end{bmatrix}, 1/\sqrt{70}\begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} \right)$

4.3: 1. a. $Q^TQ = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$\boxed{\text{Not orthogonal}}$

c. $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det = -(-1)(-1) = -1$ improper

$\boxed{\text{Orthogonal, i}}$

d. $\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 10/3 & 0 & 0 \\ 0 & 10/3 & 0 \\ 0 & 0 & 10/3 \end{bmatrix}$

$\boxed{\text{Not orthogonal}}$

4.3: 10. a. False b. True c. True

12.
$$\begin{array}{cc} U^T & U \\ \begin{bmatrix} 1 & 0 & 0 \\ u_{12} & 1 & 0 \\ u_{13} & u_{23} & 1 \end{bmatrix} & \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & u_{12} & u_{13} \\ u_{12} & 1 & u_{12}u_{13}+u_{23} \\ u_{13} & u_{12}u_{13}+u_{23} & 1 \end{bmatrix}$$

This must be the identity so u_{12}, u_{23}, and u_{13} must be 0.

22. a. $m \geq n$ because there can't be fewer vectors than the dimension to form a basis.

b. Yes, A^TA is a diagonal matrix, the diagonal entries are 1s

c. No, AA^T isn't diagonal