Newton for nonlinear systems

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Vector for of Taylor Series.

$f(\bar{x})$ $x \in \mathbb{R}^n$

Goal of Newton: find $\bar{x}$ st $f(\bar{x}) = \bar{0}$

$$f(a+\Delta x, b+\Delta y) = f(a,b) + \Delta x f_x(a,b) + \Delta y f_y(a,b) + \frac{(\Delta x)^2}{2} f_{xx}(a,b) + \frac{(\Delta y)^2}{2} f_{yy}(a,b) + \frac{(\Delta x \Delta y)}{2} f_{xy}(a,b) + \frac{(\Delta x \Delta y)}{2} f_{yx}(a,b) + \text{HOT}$$

$$= f(a,b) + \nabla f\Big|_{(a,b)} \cdot (\Delta x, \Delta y) + \frac{1}{2}\Bigg((\Delta x, \Delta y) \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}_{(a,b)}}_{\text{Hessian}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \text{HOT}$$

What is the idea behind Newton's method in 1D?

follow the roots of the tangent lines

Goal: Given $f \in C^2[a,b]$, find x st $f(x) = 0$.

$$x_{n+1} = x_n - \frac{f'(x_n)}{f'(x_n)}$$ $\rightarrow$ sequence of roots of the tangent lines.

$$x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

Derivation: Via Taylor series (slightly different variant than last time)

let α denote root of $f(x)$ i.e. $f(\alpha) = 0$.

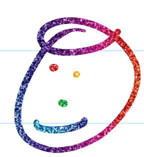
let $\Delta x_n = \alpha - x_n$ = distance between root α & approximation

$$\alpha = x_n + \Delta x_n$$

$$0 = f(\alpha) = f(x_n + \Delta x_n) = \underbrace{f(x_n) + \Delta x_n f'(x_n)}_{\text{Tangent line}} + HOT$$

Solve for Δx_n

$$\Delta x_n = \frac{-f(x_n)}{f'(x_n)}$$ in practice $\Delta x_n = x_{n+1} - x_n$

$$\rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ Same as before 

$$= g(x_n)$$ use fixed pt theory on $g(x)$

so Newton converges if $|g'(x_n)| < 1$

Now for Systems of equations.

Given functions $f(x,y)$ & $g(x,y)$. Our goal is to find (α, β) st $f(\alpha,\beta) = 0$ & $g(\alpha,\beta) = 0$

Vector form: Find $\binom{\alpha}{\beta}$ st

$$\bar{F}(\alpha,\beta) = \begin{bmatrix} f(\alpha,\beta) \\ g(\alpha,\beta) \end{bmatrix} = \bar{0}$$

let's Taylor expand

let $\alpha = x_n + \Delta x_n \quad \beta = y_n + \Delta y_n)$

$$0 = f(\alpha,\beta) = f(x_n + \Delta x_n, y_n + \Delta y_n)$$

$$= f(x_n, y_n) + \Delta x_n f_x(x_n, y_n) + \Delta y_n f_y(x_n, y_n) + HOT$$

$$0 = g(\alpha,\beta) = g(x_n + \Delta x_n, y_n + \Delta y_n)$$

$$= g(x_n, y_n) + \Delta x_n g_x(x_n, y_n) + \Delta y_n g_y(x_n, y_n) + HOT$$

1st order in vector form

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{bmatrix} + \left.\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\right|_{(x_n, y_n)} \begin{bmatrix} \delta x_n \\ \delta y_n \end{bmatrix}$$

Jacobian $= J(x_n, y_n)$

Solve for $\begin{pmatrix} \Delta x_n \\ \Delta y_n \end{pmatrix}$