Solution to Homework 2

1) The moment generating functions of X and Y are, respectively,

$$\phi_X(s) = (pe^s + (1-p))^n \quad \text{and} \quad \phi_Y(s) = (pe^s + (1-p))^m$$

Therefore, by independence,

$$\phi_{X+Y}(s) = \phi_X(s)\,\phi_Y(s) = (pe^s + (1-p))^n (pe^s + (1-p))^m$$

$$= (pe^s + (1-p))^{n+m},$$

which is the MGF of a binomial with $n+m$ trials and probability p.

This makes sense because, if all the trials are independent, the number of successes in n trials plus the number of successes in m trials is the same as the number of successes in $n+m$ trials.

2) We have $\phi_X(s) = e^{\frac{1}{2}\sigma^2 s^2}$. Expanding in series,

$$\phi_X(s) = e^{\frac{1}{2}\sigma^2 s^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\sigma^2 s^2\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\sigma^{2n}}{2^n n!} s^{2n}$$

We know that
$$\phi_X(s) = \sum_{m=0}^{\infty} \frac{E[X^m]}{m!} s^m$$
$$= \underbrace{\sum_{n=0}^{\infty} \frac{E[X^{2n}]}{(2n)!} s^{2n}}_{\text{even}} + \underbrace{\sum_{k=0}^{\infty} \frac{E[X^{2k+1}]\, s^{2k+1}}{(2k+1)!}}_{\text{odd} \to 0}$$

So $\boxed{E[X^n] = 0 \quad \text{if } n \text{ is odd}}$. For the even moments, we get $\dfrac{E[X^{2n}]}{(2n)!} = \dfrac{\sigma^{2n}}{2^n n!}$, so

$$\boxed{E[X^{2n}] = \frac{(2n)!}{n!\, 2^n} \sigma^{2n}}$$

For example, $E[X^4] = \dfrac{4!}{2!\, 2^2} \sigma^4 = \dfrac{1 \times \cancel{2} \times 3 \times \cancel{4}}{\cancel{2} \times \cancel{4}} \sigma^4 = 3\sigma^4$

For APPM 5560/STAT 5100 students only:

We start from $E[X^n] = A_n \sigma^n$, so

$$A_n \sigma^n = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}} x^n \, dx$$

Now we differentiate with respect to σ, and use the product rule under the integral:

$$A_n n \sigma^{n-1} = \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \left(\frac{-1}{\sigma^2} \right) e^{-\frac{x^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} \left(\frac{x^2}{\sigma^3} \right) e^{-\frac{x^2}{2\sigma^2}} \right] x^n \, dx$$

Simplifying, we get

$$A_n n \sigma^{n-1} = \frac{-1}{\sigma} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} x^n \, dx + \frac{1}{\sigma^3} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} x^{n+2} \, dx$$

$$A_n n \sigma^{n-1} = -\frac{1}{\sigma} E[x^n] + \frac{1}{\sigma^3} E[x^{n+2}]$$

Using $E[X^n] = A_n \sigma^n$, we get

$$A_n n \sigma^{n-1} = -\frac{1}{\sigma} A_n \sigma^n + \frac{1}{\sigma^3} A_{n+2} \sigma^{n+2}$$

$$A_n n \cancel{\sigma^{n+2}} = -A_n \cancel{\sigma^{n+2}} + A_{n+2} \cancel{\sigma^{n+2}} \Rightarrow \boxed{A_{n+2} = A_n (n+1)}$$

3) We sort the probabilities from largest to smallest using a permutation σ.

$3 = \sigma(1)$ $\quad$ $p(3) = 0.9$

$2 = \sigma(2)$ $\quad$ $p(2) = 0.05$

$4 = \sigma(3)$ $\quad$ $p(4) = 0.04$

$1 = \sigma(4)$ $\quad$ $p(1) = 0.01$

Then we do what we did in class:

Step 1. Set $S = p(\sigma(1))$, $i = 1$

Step 2. Generate $U \sim U(0,1)$

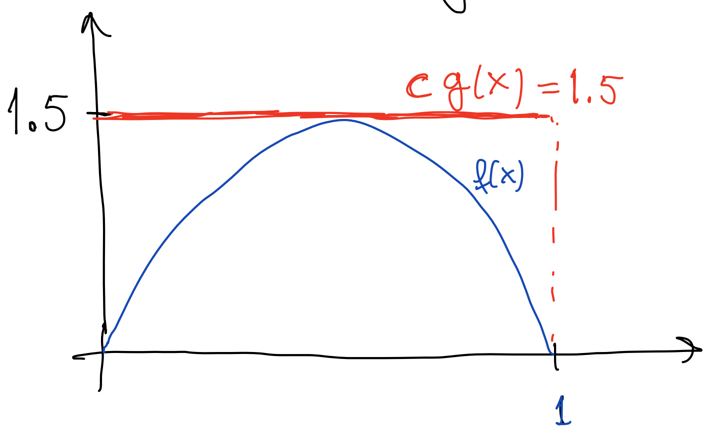
Step 3. If $U < S$, return $\sigma(i)$

Step 4. Else, set $i = i+1$, $S = S + p(\sigma(i))$, go to Step 3.

4) Using the Acceptance - Rejection Method:

The maximum of $f(x) = 6x(1-x)$ occurs when $x = 1/2$, so $\max_{0<x<1} f(x) = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1.5$

So we can use $g \sim U(0,1)$ and $c = 1.5$



The pseudo-code is then:

```
Set running = 1
While (running = 1)
    Generate X ~ U(0,1)
    Generate U ~ U(0,1)
    If ( U <= 6X(1-X) / 1.5 )
        Return X, set running = 0
    End
End
```

5. **Simulation Assignment 1.** Write a code that simulates two independent exponential distributions with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$, and returns the variable $Z = Y/X$. Generate a lot of samples of Z (at least $10,000$) and plot a histogram of the values of Z. Plot this histogram and the theoretical distribution found in problem 6 of Homework 1 in the same plot (you might need to normalize the histogram so that the two are on top of each other).

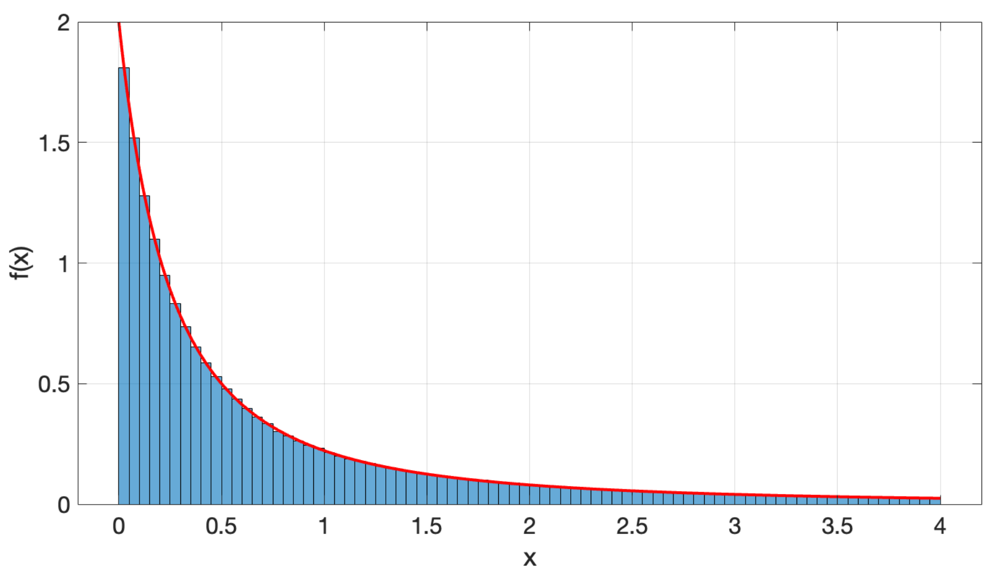
   We know that to generate an exponentially distributed number with parameter λ, we can generate a uniform random number in (0,1), U, and do $X = -\log(U)/\lambda$. Therefore, the pseudo-code for generating one sample is

   i. Generate two independent, uniform random variables $U_1, U_2 \sim U(0,1)$.
   ii. Create the exponential random variables, $X = -\log(U_1)$ and $Y = -\log(U_2)/2$.
   iii. Return $Z = Y/X$.

   The code "problem5.m", provided with these solutions, implements this in matlab. The histogram of generated values of Z is plotted along the theoretical distribution from homework 1,

$$f(z) = \frac{2}{(1+2z)^2}. \tag{1}$$

   When plotting the histogram, make sure you don't plot the whole domain of $Z's$, since there will be usually be a lot of huge outliers. The plot is shown below, with the red line being the PDF and the blue bars a normalized histogram of the generated values.



6. **Simulation Assignment 2.** Use the inverse transform method to simulate $10,000$ samples of a power-law distributed random variable with PDF given by

$$f(x) = \begin{cases} 3x^{-4} & 1 < x, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Calculate numerically the mean of your samples and compare it with the theoretical mean obtained from the PDF.

The CDF is given by

$$F(x) = \int_1^x 3y^{-4} dy = 1 - x^-3. \tag{3}$$

Solving $1 - x^{-3} = u$ for x we get $x = (1-u)^{-1/3}$. Since $1 - U \sim U(0,1)$ we can replace $1 - U$ by another $U(0,1)$ random variable. Therefore the algorithm to generate one sample is

i. Generate $W \sim U(0,1)$.
ii. Create the power-law distributed random variables, $X = W^{-1/3}$.
iii. Return X.

This algorithm is implemented in the code "problem6.m", which is provided as part of the solutions. Generating 10^6 samples and taking the average the code gives an average of 1.4993, The theoretical value is

$$E[X] = \int_1^\infty 3x^{-4} x dx = (-3/2)x^{-2}|_1^\infty = 3/2, \tag{4}$$

which agrees well with the average from the simulated values.

7) First we simulate Y, then we simulate X exponential with parameter Y^2.

To simulate Y, we use the Inverse Transform Method

Invert $F(y)$: $\quad F(y) = u$

$$\frac{e^y}{1+e^y} = u \Rightarrow e^y = u + u e^y$$

$$\Rightarrow \quad e^y(1-u) = u \Rightarrow e^y = \frac{u}{1-u}$$

$$\Rightarrow \quad y = \ln\left(\frac{u}{1-u}\right)$$

So the algorithm is

Step 1. Generate $U \sim U(0,1)$

Step 2. Set $Y = \ln\left(\frac{U}{1-U}\right)$

Generate Y

Step 3. Generate $W \sim U(0,1)$

Step 4. Return $X = -\frac{1}{Y^2} \ln W$

Generate X given Y

8)

The first method is as we did in class (without sorting the probabilities or optimizing)

1. Define $p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

2. Set $k = 0$, $S = p(0)$

3. Generate $U \sim U(0,1)$

4. If $(U < S)$, Return k

5. Else, $k = k+1$, $S = S + p(k)$, go to 4.

The second method uses that a binomial (n, p) is the sum of n Bernoulli's with parameter p:

Set $x = 0$

For $(i = 1$ to $i = n)$

Generate $U \sim U(0,1)$

If $(U < p)$ $x = x + 1$ (Counts success)

End

Return x

9)

a) Dish number 2 has the highest amount of "probability coming in", plus it can be eaten in consecutive days. It seems dish 2 would be the most commonly eaten dish in the long run.

b) We can condition as follows:

$$P(\text{dish 1 on day } t+1) =$$

$$P(\text{dish 1 on day } t+1 \mid \text{dish 1 on day } t)\, P(\text{dish 1 on day } t)$$
$$+ P(\text{dish 1 on day } t+1 \mid \text{dish 2 on day } t)\, P(\text{dish 2 on day } t)$$
$$+ P(\text{dish 1 on day } t+1 \mid \text{dish 3 on day } t)\, P(\text{dish 3 on day } t)$$

the conditional probabilities are the probabilities given in the problem. Using these, we can rewrite this as

$$P^{t+1}(1) = 0.1\, P^t(2) + 0.2\, P^t(3)$$
$$P^{t+1}(2) = 0.5\, P^t(1) + 0.7 P^t(2) + 0.8\, P^t(3)$$
$$P^{t+1}(3) = 0.5 P^t(1) + 0.2\, P^t(2)$$

c) In matrix notation, we get

$$\begin{bmatrix} p^{t+1}(1) \\ p^{t+1}(2) \\ p^{t+1}(3) \end{bmatrix} = \begin{bmatrix} 0 & 0.1 & 0.2 \\ 0.5 & 0.7 & 0.8 \\ 0.5 & 0.2 & 0 \end{bmatrix} \begin{bmatrix} p^{t}(1) \\ p^{t}(2) \\ p^{t}(3) \end{bmatrix}$$