Solution to Homework 5

1) Assume $\vec{\pi}_1$ and $\vec{\pi}_2$ are stationary distributions, and let

$$\vec{\pi}_3 = \lambda \vec{\pi}_1 + (1-\lambda)\vec{\pi}_2 , \qquad 0 \leq \lambda \leq 1.$$

Then,

- For all i,

$$\pi_3(i) = \lambda \pi_1(i) + (1-\lambda)\pi_2(i) \geq 0$$

since $\pi_1(i) \geq 0$, $\pi_2(i) \geq 0$, $0 \leq \lambda \leq 1$.

- $$\sum_{i=1}^{N} \pi_3(i) = \lambda \cancelto{1}{\sum_{i=1}^{N} \pi_1(i)} + (1-\lambda) \cancelto{1}{\sum_{i=1}^{N} \pi_2(i)}$$

$$= \lambda + (1-\lambda) = 1$$

- $$\vec{\pi}_3 P = [\lambda \vec{\pi}_1 + (1-\lambda)\vec{\pi}_2] P$$

$$= \lambda \vec{\pi}_1 P + (1-\lambda)\vec{\pi}_2 P$$

$$= \lambda \vec{\pi}_1 + (1-\lambda)\vec{\pi}_2 = \vec{\pi}_3$$

$\vec{\pi}_3$ satisfies the three conditions for being a stationary distribution.

2)

a) Let $\vec{1} = [1, 1, \dots, 1]^T$ be a column vector of ones. Since all rows of P sum to 1, we have

$$P\vec{1} = \vec{1}$$

So, 1 is an eigenvalue of P.

b) The Gershgorin Circle Theorem says that all the eigenvalues of a matrix A are contained in $G = \bigcup_{i=1}^{n} C_i$, where C_i is the circle with center at a_{ii} and radius $\sum_{j \neq i} |a_{ij}|$.

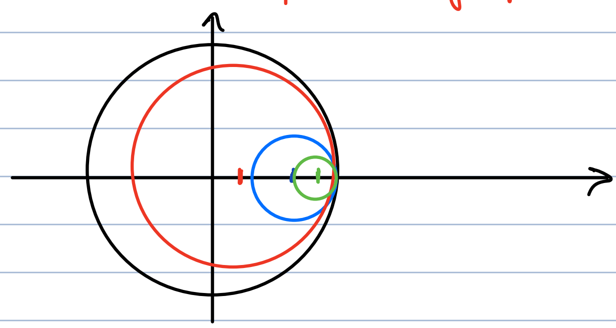
In other words, every eigenvalue λ of P satisfies

$$|\lambda - p_{ii}| \leq \sum_{j \neq i} p_{ij}$$ So

$$|\lambda| \leq |\lambda - p_{ii} + p_{ii}| \leq |\lambda - p_{ii}| + |p_{ii}|$$

$$\leq \sum_{j \neq i} p_{ij} + p_{ii} = \sum_{j=1}^{N} p_{ij} = 1$$

(In this problem we accept a graphical argument like this


)

3) Let $\vec{1}$ be a column vector of ones.
Note that

$$A \text{ is stochastic}$$

$$\Updownarrow$$

$$\text{rows and columns sum to } 1$$

$$\Updownarrow$$

$$A\vec{1} = \vec{1} \quad \text{and} \quad \vec{1}^T A = \vec{1}^T$$

So, suppose that A and B are stochastic. Then

$$(AB)\vec{1} = A(B\vec{1}) = A\vec{1} = \vec{1} \quad \text{and}$$

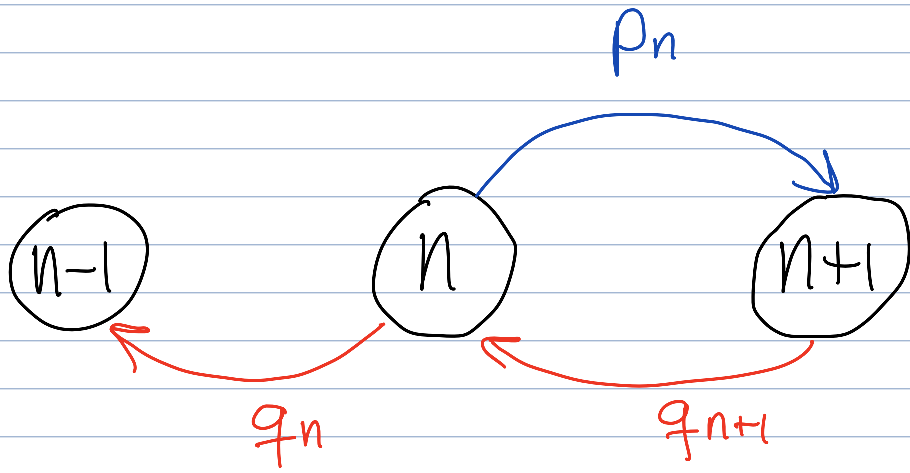
$$\vec{1}^T(AB) = (\vec{1}^T A)B = \vec{1}^T B = \vec{1}^T \quad \checkmark$$

So AB is also stochastic.

4)

Step 1:	Simulate $U \sim \text{Uniform}(0,1)$
Step 2:	Set $i = 1$
Step 3:	If $U < \sum_{k=1}^{i} \mu(k)$ then $X_0 = i$, else $i = i+1$, and repeat
Step 4:	$t = 1$
Step 5:	While $t \leq n$ do Simulate $V \sim \text{Uniform}(0,1)$ $i = 1$ If $V \leq \sum_{k=1}^{i} p(X_{t-1}, i)$ then $X_t = i$, else $i = i+1$, and repeat $t = t+1$

5)



a)

The detailed balance condition is

$$\pi(n)\, p_n = \pi(n+1)\, q_{n+1} \quad , \quad \text{or}$$

$$\pi(n+1) = \pi(n)\,\frac{p_n}{q_{n+1}} = \pi(n)\,\frac{\beta e^{\alpha n}}{b\, e^{a n}}$$

$$\pi(n+1) = \pi(n)\left(\frac{\beta}{b}\right) e^{(\alpha-a)n}$$

To find $\vec{\pi}$, we start at $n=1$ and move up:

$\pi(1) = \pi(1)$ (found by normalization)

$$\pi(2) = \pi(1)\left(\frac{\beta}{b}\right) e^{(\alpha-a)}$$

$$\pi(3) = \pi(2)\left(\frac{\beta}{b}\right) e^{2(\alpha-a)} = \pi(1)\left(\frac{\beta}{b}\right)^2 e^{(\alpha-a)(1+2)}$$

the pattern gives

$$\pi(n) = \pi(1)\left(\frac{\beta}{b}\right)^{n-1} e^{(\alpha-a)(1+2+3+\cdots+(n-1))}$$

Now use $1+2+\cdots+n-1 = \frac{(n-1)n}{2}$

and $\left(\frac{\beta}{b}\right)^{n-1} = e^{(n-1)\ln(\beta/b)}$ to get

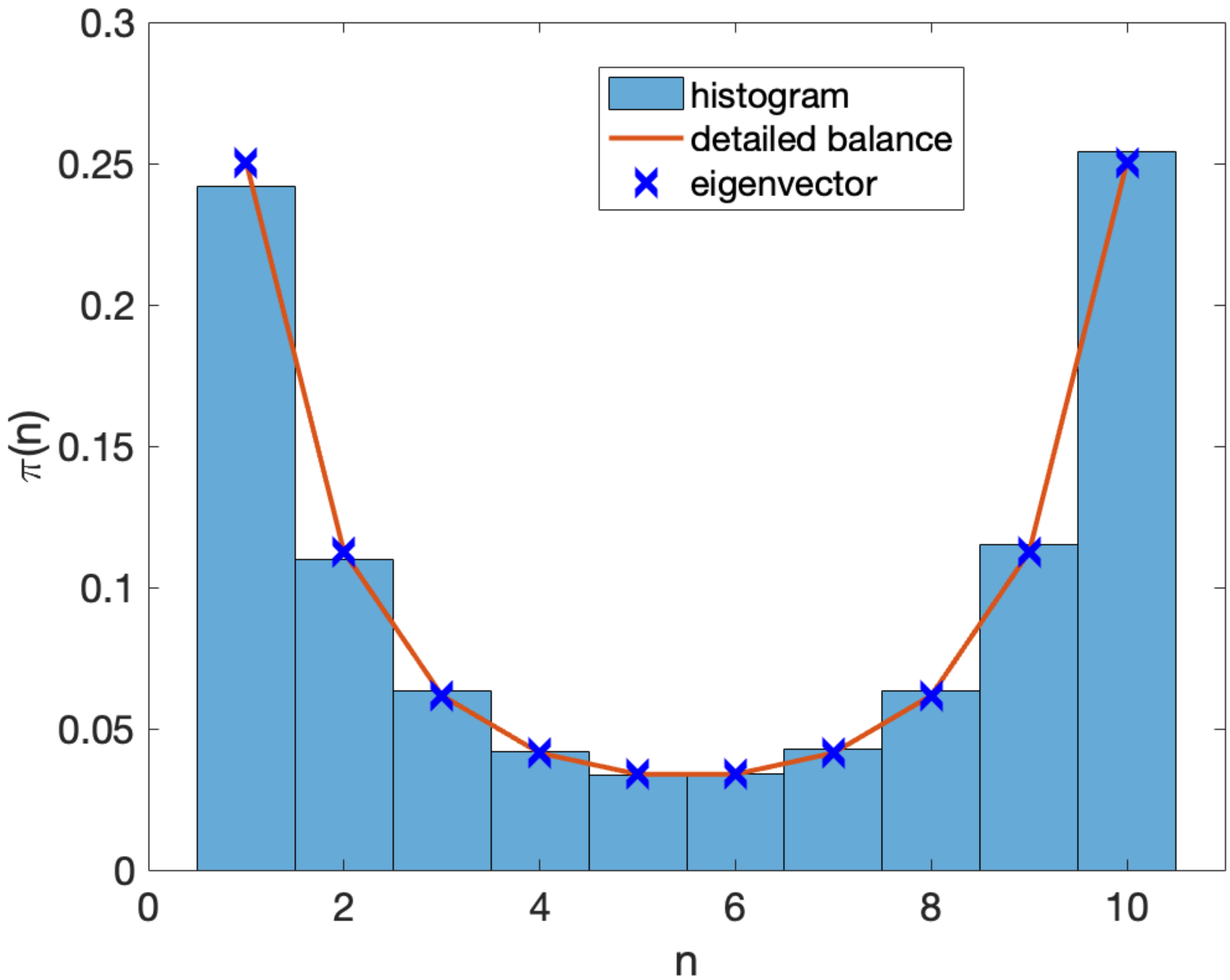
$$\boxed{\pi(n) = \pi(1)\, e^{(n-1)\left[\ln(\beta/b) + \frac{\alpha-a}{2} n\right]}}$$

Where $\pi(1)$ is solved from

$$1 = \sum_{n=1}^{10} \pi(n), \quad \text{so} \quad \pi(1) = \frac{1}{\sum_{n=1}^{10} e^{(n-1)\left[\ln(\beta/b) + \frac{\alpha-a}{2} n\right]}}$$

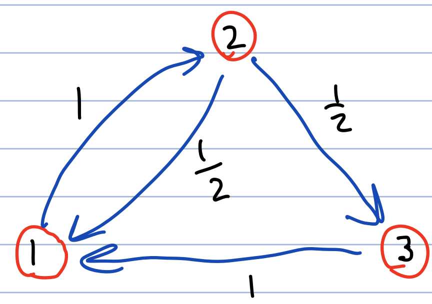
The rest of the problem is done in the attached matlab file called problem5.m.

The plots of the histogram run with 10^6 steps, the detailed-balance result, and the numerically calculated eigenvector are shown below.



histogram
detailed balance
eigenvector
0.3
0.25
0.2
0.15
0.1
0.05
0
π(n)
0
2
4
6
8
10
n

6) a)



We have $p^2(1,1) > 0$ and $p^3(1,1) > 0$, so $2, 3 \in I_1$. Since 2 and 3 are mutually prime, the period of 1 is 1. Since the chain is irreducible, every state has period 1 $\Rightarrow$ $\boxed{\text{aperiodic}}$

Stationary Distribution:

Set up equations:

$$\pi(1) = \pi(3) + \tfrac{1}{2}\pi(2)$$
$$\pi(2) = \pi(1)$$
$$\pi(3) = \tfrac{1}{2}\pi(2)$$

$\Rightarrow$ Let $\pi(3) = c$ Then

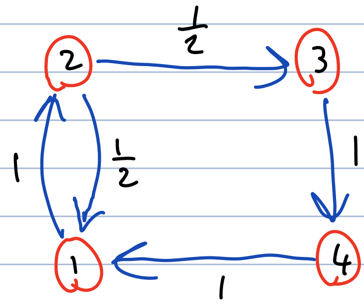
$$\pi(2) = 2c$$
$$\pi(1) = c + \tfrac{1}{2}2c = 2c$$

So $\vec{\pi} = [2c, 2c, c]$

By normalization, $c = 1/5$

$$\boxed{\vec{\pi} = \left[\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}\right]}$$

b)



One can only come back to node 1 following the path $1 \to 2 \to 1$ or $1 \to 2 \to 3 \to 4 \to 1$, both of which have an even number of steps.

$I_1 = \{2, 4, 6, 8, \ldots\}$ so period of $1 = GCD(I_1) = 2$

Since the chain is irreducible, everyone has period 2.

$\Rightarrow$ $\boxed{\text{period } 2}$

Stationary Distribution:

$\pi(1) = \frac{1}{2}\pi(2) + \pi(4)$

$\pi(2) = \pi(1)$

$\pi(3) = \frac{1}{2}\pi(2)$

$\pi(4) = \pi(3)$

Let $\pi(4) = c$

$\pi(3) = c$

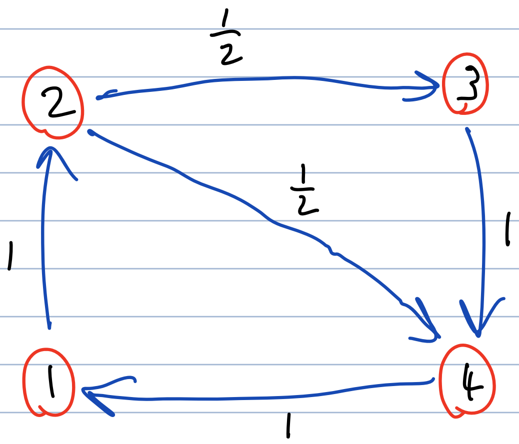
$\pi(2) = 2c$

$\pi(1) = \frac{1}{2}(2c) + c = 2c$

$\Rightarrow \vec{\pi} = [2c, 2c, c, c]$ By normalization, $c = \frac{1}{6}$

$$\boxed{\vec{\pi} = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{6}\right]}$$

c.)



State 1 belongs to a cycle of length 3 and another of length 4. Since 3, 4 are mutually prime and $3, 4 \in I_1$, then $GCD(I_1) =$ period of $1 = 1$

Since the chain is irreducible, everyone has period 1.

$\Rightarrow$ $\boxed{\text{aperiodic}}$

Stationary Distribution:

$$\pi(1) = \pi(4)$$
$$\pi(2) = \pi(1)$$
$$\pi(3) = \tfrac{1}{2}\pi(2)$$
$$\pi(4) = \pi(3) + \tfrac{1}{2}\pi(2)$$

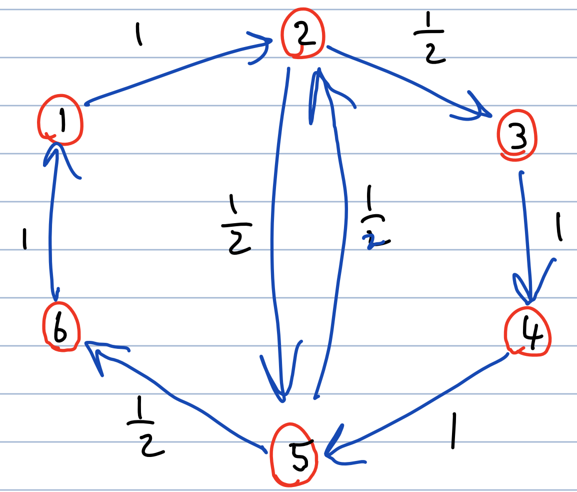
Let $\pi(1) = C \rightarrow \pi(2) = C \rightarrow \pi(3) = C/2$

$\rightarrow \pi(4) = C/2 + \frac{1}{2}C = C$

$\rightarrow \vec{\pi} = [C, C, \frac{C}{2}, C]$ By normalization, $C = \frac{2}{7}$

$$\boxed{\vec{\pi} = [\tfrac{2}{7}, \tfrac{2}{7}, \tfrac{1}{7}, \tfrac{2}{7}]}$$

d)



One can only come back to state 2 using the paths $2 \to 5 \to 2$, $2 \to 3 \to 4 \to 5 \to 6 \to 1 \to 2$, $2 \to 3 \to 4 \to 5 \to 2$, all of which have even length, and the smallest of which has length 2.

So period of $2 = 2 \underset{\text{irreducible}}{\Longrightarrow}$ $\boxed{\text{period } 2}$

Stationary distribution

By symmetry, we see that $\pi(2) = \pi(5)$

$$\pi(3) = \pi(6)$$

$$\pi(4) = \pi(1)$$

We have

$$\pi(1) = \pi(6) \to \boxed{\pi(1) = \pi(3)}$$

$$\pi(2) = \pi(1) + \tfrac{1}{2}\pi(5) \to \boxed{\pi(2) = \pi(1) + \tfrac{1}{2}\pi(2)}$$

$$\boxed{\pi(3) = \tfrac{1}{2}\pi(2)}$$

Let $\pi(1) = c$. Then $\pi(3) = c$

$\pi(2) = 2c$

So $\vec{\pi} = [c, 2c, c, c, 2c, c]$

By normalization, $c = \frac{1}{8}$, so

$$\boxed{\vec{\pi} = \left[\frac{1}{8}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right]}$$