Solution to Homework 3

1) Suppose there was such a constant c. Then we would have, for $x > 0$,

$$\frac{e^{-x}}{\frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}} \leq c \quad , \quad \text{or}$$

$$e^{\frac{x^2}{2} - x} \leq \frac{2c}{\sqrt{2\pi}} \qquad \text{for all } x > 0.$$

Since $\frac{x^2}{2} - x \longrightarrow \infty$ as $x \to \infty$, $e^{\frac{x^2}{2} - x} \longrightarrow \infty$ as $x \longrightarrow \infty$, so there must be some $x > 0$ where the above inequality is false. Therefore, such a c can not exist.

2)

a) Using the Inverse Transform Method:

$$F(x) = \int_{-\frac{\pi}{2}}^{x} \frac{\cos(y)}{2}\, dy = \left.\frac{\sin(y)}{2}\right|_{-\frac{\pi}{2}}^{x}$$

$$= \frac{1}{2}\left[\sin(x) - \sin\left(-\frac{\pi}{2}\right)\right] = \frac{1}{2}\left[\sin(x) + 1\right]$$

Now we invert F: $\quad u = \frac{1}{2}\left[\sin(x) + 1\right]$

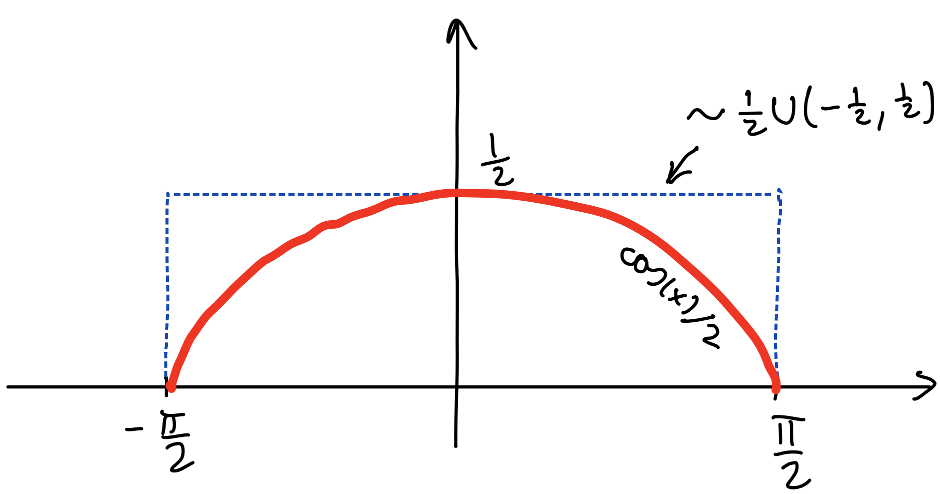
$$\Rightarrow \sin(x) = 2u - 1 \Rightarrow x = \arcsin(2u-1)$$

So the pseudo-code would be:

1. Generate $U \sim U(0,1)$
2. Return $X = \arcsin(2U-1)$

b) Using Acceptance-Rejection.

We can use a uniform random variable in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



We use $c = \frac{1}{2}$ and g uniform in $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Then the pseudo-code will be

$$\boxed{\begin{array}{l} 1.\ \text{Generate } X \sim U(-\frac{\pi}{2}, \frac{\pi}{2}) \\ 2.\ \text{Generate } U \sim U(0,1) \\ 3.\ \text{If } U \leq \frac{\cos(x)/\cancel{2}}{1/\cancel{2}} = \cos(x),\ \text{return } X. \\ 4.\ \text{Else, go back to } 1. \end{array}}$$

3)

a) We need to find the maximum of

$$h(x) = \frac{f(x)}{g(x)} = \frac{\pi}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}(1+x^2)$$

Use calculus:

$$h'(x) = \cancel{\frac{\pi}{\sqrt{2\pi}}}\left[\cancel{e^{-\frac{x^2}{2}}}(-\cancel{x})(1+x^2) + \cancel{e^{-\frac{x^2}{2}}}(2\cancel{x})\right] = 0$$

$$\Rightarrow -1-x^2+2=0 \Rightarrow x = \pm 1$$

The maximum is $h(\pm 1) = \frac{2\pi}{\sqrt{2\pi}} e^{-\frac{1}{2}} = c$

b) As done in class, the pseudo-code would be

1. Generate $U \sim U(0,1)$
2. Return $Y = \tan(\frac{1}{2}U - 1))$

c) Putting all together, the pseudo-code would be

1. Generate $W \sim U(0,1)$
2. $Y = \tan\left(\frac{1}{2}(W-1)\right)$
3. Generate $U \sim U(0,1)$
4. If $U \leq \dfrac{\cancel{\frac{1}{\sqrt{2\pi}}} e^{-Y^2/2}}{\frac{2\cancel{\pi}}{\cancel{\sqrt{2\pi}}} e^{-\frac{1}{2}} \cancel{\frac{1}{\pi}} \frac{1}{(1+Y^2)}} = \frac{1}{2}(1+Y^2)e^{\frac{1}{2}(1-Y^2)}$,

Return Y

(Although not required, a code implementing this is included in the solutions.)

4) For APPM 5560/STAT 5100 students only.

If the process described was a Markov Chain it would satisfy:

$$P(X_3 = 1 \mid X_2 = 1, X_2 = 0) \stackrel{?}{=} P(X_3 = 1 \mid X_2 = 1)$$

However, if we know $X_2 = 1$ and $X_2 = 0$, we know there are 49 red balls and 49 white balls remaining when we draw the third ball, so

$$P(X_3 = 1 \mid X_2 = 1, X_2 = 0) = \frac{1}{2}.$$

On the other hand, we have, using cases

$$P(X_3 = 1 \mid X_2 = 1) = P(X_3 = 1 \mid X_2 = 1, X_1 = 0)\, P(X_1 = 0) + P(X_3 = 1 \mid X_2 = 1, X_1 = 1)\, P(X_1 = 1)$$

By the same reasoning, we have

$$P(X_3 = 1 \mid X_2 = 1, X_1 = 0) = \frac{1}{2} \quad \text{and}$$

$$P(X_3 = 1 \mid X_2 = 1, X_1 = 1) = \frac{48}{98} = \frac{\#\text{ remaining whites}}{\#\text{ remaining balls}}$$

So $$P(X_3 = 1 \mid X_2 = 1) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{48}{98} \neq \frac{1}{2},$$

which shows the process is NOT Markovian.

5)

a) The number of possible working lightbulbs is 0 to 100. So,

$$S = \{0, 1, 2, \dots, 100\}$$

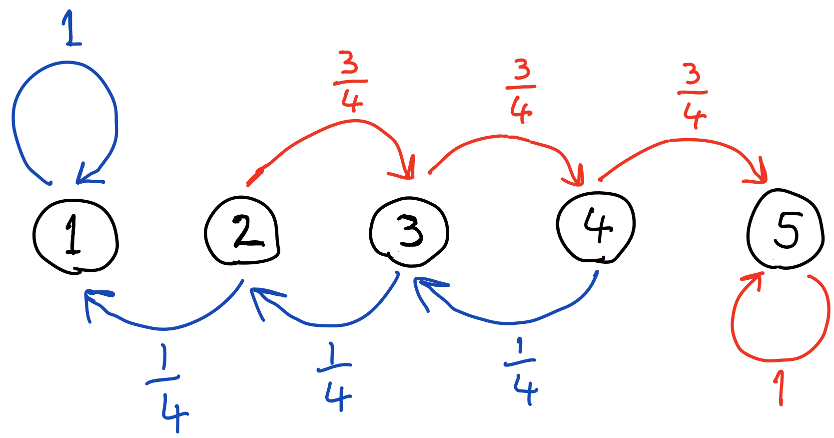
b) Each month there are X_t working lightbulbs, and the number of working lightbulbs in the next month depends on how many of these X_t fail, not on what happened before.

c) To go from i working lightbulbs to j working lightbulbs we need that $i \geq j \geq 0$, and that exactly $i-j$ lightbulbs fail in that month. Since each fails independently, this probability is binomial, as follows

$$P(X_{t+1} = j \mid X_t = i) = P(i-j \text{ lightbulbs fail out of } i)$$

$$= \begin{cases} \binom{i}{i-j} p^{i-j}(1-p)^{j}, & 0 \leq j \leq i \leq 100 \\ 0, & \text{else.} \end{cases}$$

6)



a) The possible paths are
$3 \to 2 \to 1 \to 1 \to 1$
$3 \to 4 \to 3 \to 2 \to 1$
$3 \to 2 \to 3 \to 2 \to 1$

Adding these up gives

$$P(X_4 = 1 \mid X_0 = 3) = p(3,2)p(2,1)p(1,1)p(1,1) + p(3,4)p(4,3)p(3,2)p(2,1) + p(3,2)p(2,3)p(3,2)p(2,1)$$

$$= \frac{1}{4}\frac{1}{4}\,1\,1 + \frac{3}{4}\frac{1}{4}\frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\frac{3}{4}\frac{1}{4}\frac{1}{4} = \frac{16}{256} + \frac{3}{256} + \frac{3}{256}$$

$$= \frac{22}{256} = \boxed{\frac{11}{128}}$$

b) The transition probability matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

Calculating P^4 (don't do it by hand) we get:

$$P^4 = \begin{bmatrix} 1 & \frac{19}{64} & 0 & 0 & 0 \\ \frac{19}{64} & \frac{9}{128} & 0 & \frac{27}{128} & \frac{27}{64} \\ \frac{11}{128} & 0 & \frac{9}{64} & 0 & \frac{99}{128} \\ \frac{1}{64} & \frac{3}{128} & 0 & \frac{9}{128} & \frac{57}{64} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And we have $\boxed{P^4(3,1) = \frac{11}{128}}$

✓ ✓

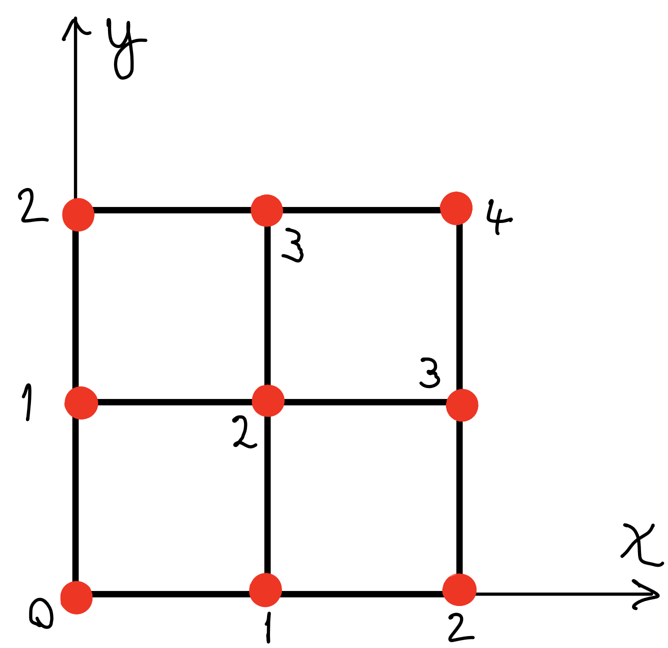
d) We have

$$P^{100} \approx \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.325 & 0 & 0 & 0 & 0.675 \\ 0.1 & 0 & 0 & 0 & 0.9 \\ 0.025 & 0 & 0 & 0 & 0.975 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{100}(3,1) \approx 0.1$$

This should be a good estimate of eventually ending in state 1 starting from 3, since by time 100 the probability of not ending in states 1 or 5 should be very small.

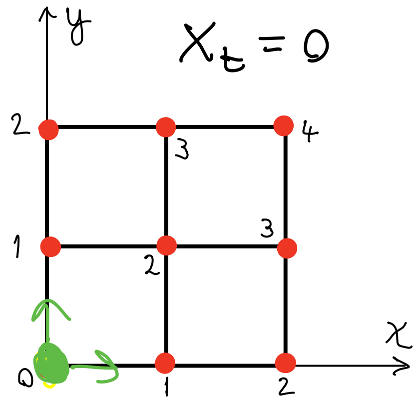
7) Let's add the value of $x+y$ to each red dot:



a) The possible values of X_t are

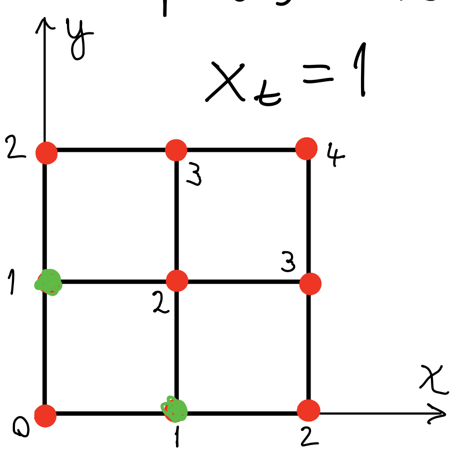
$$S = \{0, 1, 2, 3, 4\}$$

If $X_t = 0$, we must be at the lower left corner, and we can only go to $X_{t+1} = 1$



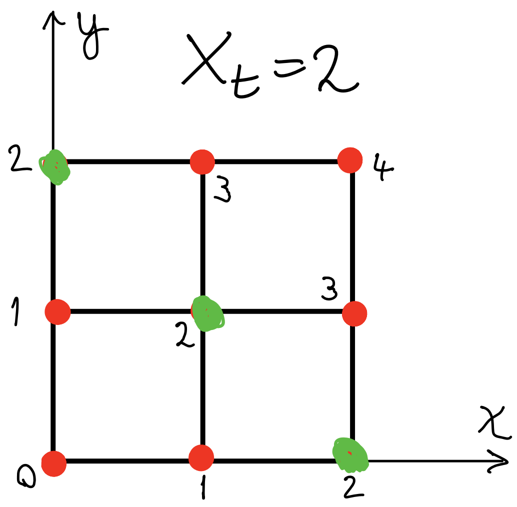
$$p(0,1) = 1$$

If $X_t = 1$, we must be at one of the two points marked below. In either case, there are two ways of going to $X_t = 2$ and one way of going to $X_t = 1$. Since the paths are chosen with equal probability,



$$p(1,0) = \frac{1}{3},$$

$$p(1,2) = \frac{2}{3}.$$



If $X_t = 2$, as seen below, it is equally likely to go to $X_t = 1$ or $X_t = 3$, so

$$p(2,1) = \frac{1}{2},$$
$$p(2,3) = \frac{1}{2}.$$

We can see that, by symmetry

$$p(3,2) = p(1,2) = \frac{2}{3},$$
$$p(3,4) = p(1,0) = \frac{1}{3},$$
$$p(4,3) = p(0,1) = 1,$$

while all the other entries are 0.

So

$$p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

c) The shortest possible paths have length 4. So, we want to find

$p^4(0,4)$. We get

$$p^4 = \begin{bmatrix} \frac{2}{9} & 0 & \frac{2}{3} & 0 & \frac{1}{9} \\ 0 & \frac{5}{9} & 0 & \frac{4}{9} & 0 \\ \frac{1}{6} & 0 & \frac{2}{3} & 0 & \frac{1}{6} \\ 0 & \frac{4}{9} & 0 & \frac{5}{9} & 0 \\ \frac{1}{9} & 0 & \frac{2}{3} & 0 & \frac{2}{9} \end{bmatrix}$$

The answer is then $p^4(0,4) = \frac{1}{9}$

8) a) The matrix is

$$p = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{10} & \frac{7}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{4}{5} & 0 \end{bmatrix}$$

b) We want $P(X_7 = 1 \mid X_0 = 1)$

We get

$$p^7 \approx \begin{bmatrix} 0.1084 & 0.6975 & 0.1942 \\ 0.1085 & 0.6977 & 0.1938 \\ 0.1086 & 0.6978 & 0.1935 \end{bmatrix}$$

The probability is approximately 0.1084

c) We get

$$P^{50} \approx \begin{bmatrix} 0.1085 & 0.6977 & 0.1938 \\ 0.1085 & 0.6977 & 0.1938 \\ 0.1085 & 0.6977 & 0.1938 \end{bmatrix}$$

No matter what dish we start from (i.e., which row we look at) the probability of having dish 2 is approximately 0.6977.