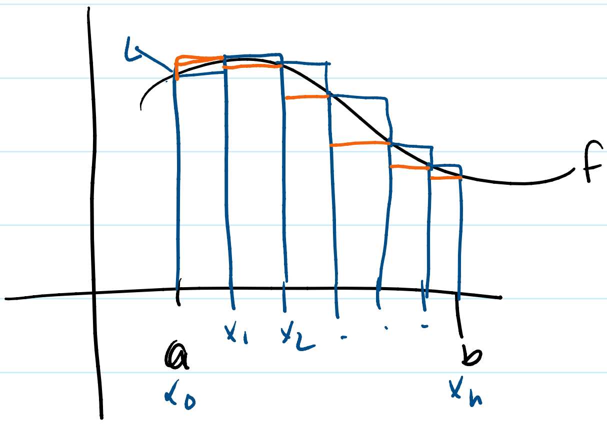
Newton-Cotes Quadrature

Wednesday, November 1, 2023 1:26 PM

Recall when you first learned how to integrate you used Reimann sums.



Left hand rule

$$I_L = \sum_{j=0}^{n-1} f(x_j)\,(x_{j+1} - x_j)$$

Right hand rule

$$I_R = \sum_{j=0}^{n-1} f(x_{j+1})(x_{j+1} - x_j)$$

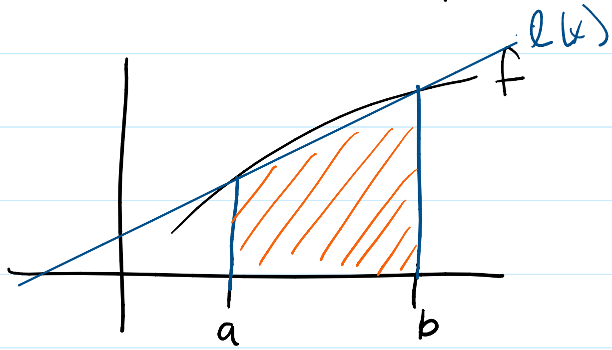
Midpt

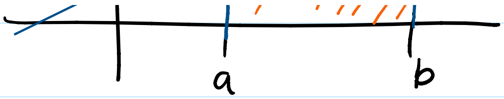
$$I_m = \sum_{j=0}^{n-1} f\left(\frac{x_{j+1} + x_j}{2}\right)(x_{j+1} - x_j)$$

These are called <u>quadratures</u>.

We will "fancy" quadratures using better approximations of $f(x)$.

<u>Ex</u>: Approx f w/a line & integrate.





$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\big(f(a)+f(b)\big)$$

How accurate is this approximation?

We know from interpolation

$$f(x) = \underbrace{f(a)\frac{(x-b)}{a-b} + f(b)\frac{(x-a)}{b-a}}_{P_1(x)} + \underbrace{\frac{f''(\eta)}{2}(x-a)(x-b)}_{\substack{\text{Remainder} \\ \text{for some } \eta\in(a,b)}}$$

$$\Rightarrow \int_a^b f(x)\,dx = \underbrace{\int_a^b P_1(x)\,dx}_{\text{Trapezoidal rule}} + \underbrace{\int_a^b \frac{f''(\eta)}{2}(x-a)(x-b)\,dx}_{\text{quadrature error} = E}$$

$$E = \int_a^b f''(\eta)\frac{(x-a)(x-b)}{2}\,dx = \frac{f''(\eta)}{2}\int_a^b (x-a)(x-b)\,dx$$

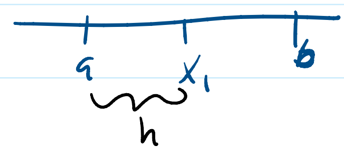
$$= \frac{f''(\eta)}{12}\underbrace{(b-a)^3}_{h^3}$$

h = length of interval $= b-a$

Error bound $\quad |E| \le \max\limits_{\eta \in [a,b]} |f''(\eta)| \dfrac{h^3}{12}$

Let's now approximate f w/ a quadratic.
This will give us Simpson's rule.

let $\quad x_0 = a, \quad x_1 = \dfrac{a+b}{2}, \quad x_2 = b$



$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$
$$+ f(x_2)$$
$$+ \text{Error}$$

Then integrate

$$\int_{x_0}^{x_2} f(x)\,dx = \underbrace{\frac{h}{3}\Big(f(x_0) + 4f(x_1) + f(x_2)\Big)}_{\text{Simpson's Rule}} - \underbrace{\frac{h^5 f^{(4)}(\eta)}{90}}_{\text{Error}}$$

for some $\eta \in (x_0, x_2)$, $\; h = \dfrac{b-a}{2}$

Why $f^{(4)}(\eta)$? $\qquad \displaystyle\int_a^b (x-x_0)(x-x_1)(x-x_2)\,dx = 0$

why $f'''(\eta)$. $\quad \int_a (x-x_0)(x-x_1)(x-x_2)\,dx = 0$

Error for Simpson's rule is ONE order higher then the approximation used to build it.

Ex: Power of Simpson's rule.
Consider $f(x) = x^3$ on $[0,1]$
Does the error result hold?

Soln:

$$\int_0^1 f(x)\,dx = \int_0^1 x^3\,dx = 1/4$$

Simpson's rule $\quad x_0=0, \; x_1=0.5, \; x_2=1$
$h=0.5$

$$S = \frac{\frac{1}{2}}{3}\left(f(0) + 4f(1/2) + f(1)\right)$$

$$= \frac{1}{6}\left(0^3 + 4\left(\frac{1}{8}\right) + 1^3\right) = \frac{1}{6}\left(\frac{1}{2}+1\right)$$

$$= \frac{1}{\cancel{6}_{2}}\left(\frac{\cancel{3}}{2}\right) = \frac{1}{4}$$

Generalization: Newton-Cotes quadratures

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Suppose we want to integrate $f(x)$ on $[a,b]$
with nodes $x_j = x_0 + jh$ for $j = 0, \cdots, n$
$x_0 = a \quad x_n = b \quad \Rightarrow \quad h = \frac{b-a}{n}$

Then we can approximate the integral as follows

$$\int_a^b f(x)\,dx \approx \sum_{j=0}^{n} w_j f(x_j)$$

where $w_j = \int_a^b \ell_j(x)\,dx \qquad \ell_j =$ Lagrange polynomial

<u>Thm 4.2</u>: Suppose that $\sum_{j=0}^{n} w_j f(x_j)$ denotes the $n+1$ pt Newton-Cotes formula then $\exists\ \eta \in (a,b)$ st

$$-\int_a^b f(x)\,dx = \sum_{j=0}^{n} w_j f(x_j) + \frac{h^{n+3} f^{(n+2)}(\eta)}{(n+2)!} \int_0^n t^2(t-1)\cdots(t-n)\,dt$$

if n is even & $f \in C^{(n+2)}(a,b)$

$$-\int_a^b f(x)\,dx = \sum_{j=0}^{n} w_j f(x_j) + \frac{h^{n+2} f^{(n+1)}(\eta)}{(n+1)!} \int_0^n t(t-1)\cdots(t-n)\,dt$$

for n odd & $f \in C^{(n+1)}(a,b)$

$\Psi(x) = (x - x_0) \cdots (x - x_n)$

Change of varible $t = x - x_0$ (I'll verify)