QR Factorization

Wednesday, November 29, 2023 2:01 PM

What is QR?

$A \in \mathbb{C}^{n \times m}$ $m \geq n$

$A = QR$ where $Q \in \mathbb{C}^{n \times n}$ is orthogonal
& $R \in \mathbb{C}^{n \times m}$ is upper triangular.

What do we use this for?

- Solve linear systems
- find eigenvalues.

Projects
$$\begin{cases} A = \text{rand}(n) & [Q, R] = qr(A) \\ D = \text{diagonal.} & \\ B = Q D Q^* & \\ C = \text{rand}(n) & [Q_2, R] = qr(C) \\ M = Q D Q_2^* & \end{cases}$$

Q: How do we make the QR factorization?

A: Build Q or Q^* so that $Q^* A = R$

We will build Q to be a product of projectors.

Def: A <u>projector</u> is a square matrix $P \in \mathbb{C}^{m \times m}$ st $P^2 = P$. (idempotent)

Physical understanding of P.

P

$x \xrightarrow{f} y$

Range

let $S_1 = \text{Range}(P)$

if $x \in \mathbb{C}^m$ the $v = Px \in S_1$

What is Pv?

$$Pv = P^2 x = Px = v$$

Q: if $S_1 = \text{Range}(P) = \mathbb{R}^n$

what is P?

$$P = I$$

What is $P(Px - x)$ for $x \in \mathbb{C}^n$?

$$P(\underbrace{Px - x}) = P^2 x - Px = Px - Px = \bar{0}$$

$$\Rightarrow \quad Px - x \in \text{Null}(P)$$

$$\text{Range}(P) \perp \text{Null}(P)$$

$$\underbrace{\text{Range}(P)}_{\text{Image}} \perp \underbrace{\text{Null}(P)}_{\text{kernel}}$$

- $Px - x$ = − residual
  = − part of x not in the Range of P

Ex: let's look at the operator $Q = I - P$
where P is a projector.
Show that Q is also a projector.

Soln: WTS $Q^2 = Q$

$$Q^2 = (I-P)(I-P) = I - 2P + \underbrace{P^2}_{= P}$$

$$= I - P = Q$$

Thm If P is a projector then $Q = I - P$ is a projector &

(i) $PQ = 0$

(ii) $\text{Null}(P) = \text{Range}(Q)$

(iii) $\text{Null}(Q) = \text{Range}(P)$

(ii) Null (Q) = Range (P)

Proof is HW.

Ex: For any non-zero vector $v \in \mathbb{C}^n$

$$P = \frac{vv^*}{v^*v} \quad \text{is an orthogonal projector.}$$

Show this. $P^* = P$

$* = T$

Soln: $$P^* = \left(\frac{vv^*}{v^*v}\right)^* = \frac{1}{v^*v}(vv^*)^*$$

$$= \frac{vv^*}{v^*v} = P$$

$$P^*P = \frac{v(v^*}{v^*v}\frac{v)v^*}{v^*v} = \frac{vv^*(v^*v)}{(v^*v)^2} = \frac{vv^*}{v^*v} = P$$

If P is orthogonal projector then $Q = I - P$ is also an orthogonal projector.

<u>Lemma</u>: A projector P is <u>orthogonal</u> if $P^* = P$
i.e. P is Hermitian (for Real P - Symmetric)

Side. $M \in \mathbb{R}^{m \times n}$ $\quad m < n$

Rank$(M) = k < m$

SVD : $U^* \Sigma V = M$

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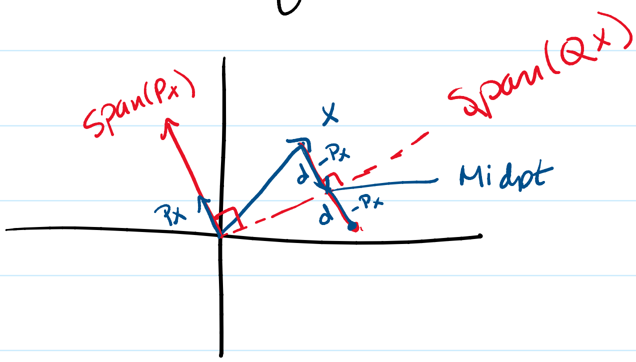
Back to QR

$$A = \begin{pmatrix} a_{11} & - & - & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & - & - & a_{nn} \end{pmatrix}$$

want to zero this out.

Towards Reflectors



$Q = I - P$

$\bar{d} = -Px$

Goal: find a Matrix st $Hx = x - 2Px$
$= (I - 2P)x$