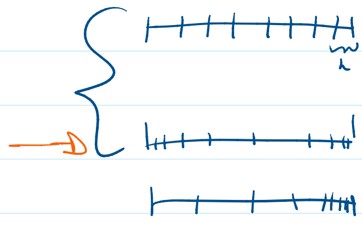
Least Squares approximations

Wednesday, October 25, 2023 1:26 PM

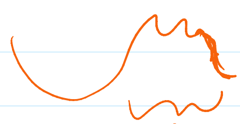
Warm-up: What was a good choice of interpolation nodes?



equispaced

Chebychev ← better in general
near optimal

local refinement



features - more data in this location is good.

We will see a connection between the chebychev nodes & $\mathcal{L}^2$-approximations

Chebychev polynomials

$$T_n(x) = \cos(n \cos^{-1}(x))$$

Also a 3 term recursion

$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

$$T_0 = 1 \quad T_1 = x$$

we know that roots of chebychev polynomials lead to clusting of nodes

- These are orthogonal polynomials wrt $w(x) = \frac{1}{\sqrt{1-x^2}}$ on $[-1,1]$

Def: - The $\mathcal{L}^2$-inner product on $[a,b]$ wrt weight function $w(x)$

$$\langle f,g \rangle_w = \int_a^b f(x)\,g(x)\,w(x)\,dx$$

- We say that $f(x)$ is <u>orthogonal</u> to $g(x)$ wrt $w(x)$ on $[a,b]$ if

$$\langle f,g\rangle_w = \int_a^b f(x)g(x)\,w(x)\,dx = 0.$$

Ex: weight functions | orthogonal polynomials

weight functions	orthogonal polynomials
$w(x) = 1/\sqrt{1-x^2}$	1^{st} kind Chebychev polynomials
$w(x) = 1$	Legendre
$w(x) = \sqrt{1-x^2}$	2^{nd} kind Chebychev polynomials
on $[-1,1]$	

Recall: ℓ^2-approx for data fitting $\{x_j\}_{j=0}^n$ $\{y_j\}_{j=0}^n$

our approx. $p(x) = \sum_{j=0}^n a_j x^j$

we minimized overall i $\quad |y_i - \sum_{j=0}^n a_j x_i^j|^2$

we found the coefficients by solving the normal eqn
or take derivatives wrt a_j & set $=0$.

Now we will create $\mathcal{L}^2$-approx. of functions using this as our guide.

Difference between ℓ^2 & $\mathcal{L}^2$

ℓ^2 - $x \in \mathbb{R}^n$ $\qquad \|x\|_2 = \left(\sum_{j=0}^n x_j^2\right)^{1/2}$

$\mathcal{L}^2$ $\quad f \in \mathcal{L}^2(a,b)$ if $\underbrace{\|f\|_2^2 = \int_a^b (f(x))^2 w(x)\,dx \geq 0}$

Our problem (Goal):
Give $f(x)$ & a set of basis functions $\{\phi_j\}_{j=0}^n$ plus weight function $w(x)$
approx f to minimize the following norm over (a,b)

$$\| f(x) - p(x) \|_2^2 = \int_a^b (f(x) - p(x))^2 \omega(x)\, dx$$

where $p(x) = \sum_{i=0}^{n} a_i \phi_i(x)$

So we need to find $\{a_i\}_{i=0}^n$ st the norm is minimized

Ex: say $n = 1$

$$\| f(x) - p(x) \|_2^2 = \int_a^b (f(x) - a_0\phi_0(x) - a_1\phi_1(x))^2 \omega(x)\, dx$$

$$= \int_a^b \left[(f(x))^2 - 2f(x)(a_0\phi_0(x) + a_1\phi_1(x)) + (a_0\phi_0(x) + a_1\phi_1(x))^2 \right] \omega(x)\, dx$$

Our goal is to find

$$\min_{a_i} \underbrace{\int_a^b w(x) \left(f(x) - \sum_{j=0}^n a_j \phi_j(x) \right)^2 dx}_{E(a_0, \dots a_n)}$$

This is a quadratic that is minized when

$$\frac{dE}{da_j} = 0 \quad \forall j$$

$$0 = \frac{dE}{da_k} = -2 \int_a^b \left(f(x) - \sum_{j=0}^n a_j \phi_j(x) \right) \phi_k(x) w(x)\, dx$$

$$= -2 \left[\int_a^b f(x)\phi_k(x) w(x)\, dx - \sum_{j=0}^n a_j \int_a^b \phi_j(x)\phi_k(x) w(x)\, dx \right]$$

Since this holds for all k. We get a linear System

Since this holds for all k. We get a linear System

$$\underbrace{\begin{bmatrix} \langle \phi_0, \phi_0 \rangle_w & \langle \phi_0, \phi_1 \rangle_w & \cdots & \langle \phi_0, \phi_n \rangle_w \\ \vdots & & & \\ \langle \phi_n, \phi_0 \rangle_w & \cdots & \cdots & \langle \phi_n, \phi_n \rangle_w \end{bmatrix}}_{M} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle \phi_0, f \rangle_w \\ \langle \phi_1, f \rangle_w \\ \vdots \\ \langle \phi_n, f \rangle_w \end{bmatrix}$$

What happens if $\{\phi_j\}_{j=0}^{n}$ are orthogonal?

M is diagonal $\Rightarrow$ easy to find the coefficients.

$$\begin{bmatrix} \langle \phi_0, \phi_0 \rangle_w & & 0 \\ & \ddots & \\ 0 & & \langle \phi_n, \phi_n \rangle_w \end{bmatrix} \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle \phi_0, f \rangle_w \\ \vdots \\ \langle \phi_n, f \rangle_w \end{bmatrix}$$

$$\Rightarrow \quad a_j = \frac{\langle \phi_j, f(x) \rangle_w}{\langle \phi_j, \phi_j \rangle_w} = \frac{\int_a^b f(x)\, \phi_j(x) w(x)\, dx}{\int_a^b (\phi_j(x))^2 w(x)\, dx}$$

So our approximation is $p(x) = \sum_{j=0}^{n} a_j \phi_j(x)$

This is lovely! ♡

Warm-up:

Recall our goal is to construct an approximation

$p(x) = \sum^{n} \alpha_j \phi_j(x)$ s.t. $p(x)$ minimizes

$$p(x) = \sum_{j=0}^{n} \alpha_j \phi_j(x) \quad \text{st} \quad p(x) \text{ minimizes}$$

$$\int_a^b (f(x) - p(x))^2 w(x)\, dx.$$

What conditions must be satisfied for the minimizing function to have coefficients defined as

$$\alpha_j = \frac{\int_a^b f(x)\phi_j(x) w(x)\, dx}{\int_a^b (\phi_j(x))^2 w(x)\, dx} \quad ?$$

We need an orthogonal basis or family of functions

Given a weight function, we can construct a set of orthogonal polynomials via Gram-schmidt.

Process:

Always start with $\phi_0 = 1$

Set $\phi_1(x) = x - b_0 \phi_0(x)$

b_0 is unknown but we can find it.

Pick b_0 so that $\langle \phi_0, \phi_1 \rangle_w = 0$

$$\langle \phi_0, x \rangle_w - b_0 \langle \phi_0, \phi_0 \rangle_w = 0$$

$$\Rightarrow \quad b_0 = \frac{\langle \phi_0, x \rangle_w}{\langle \phi_0, \phi_0 \rangle_w}$$

You can continue.

$$\phi_j(x) = x^j - c_0 \phi_0 - c_1 \phi_1(x) - \cdots - c_{j-1} \phi_{j-1}(x)$$

Don't worry!

look at ϕ_2

$$\phi_2(x) = x^2 - c_0 \phi_0(x) - c_1 \phi_1(x)$$

What are the 2 conditions that must be satisfied?

$\langle \phi_2, \phi_0 \rangle = 0$ & $\langle \phi_2, \phi_1 \rangle = 0$

$$0 = \langle \phi_2, \phi_0 \rangle = \langle x^2, \phi_0 \rangle - c_0 \langle \phi_0, \phi_0 \rangle - c_1 \underbrace{\langle \phi_1, \phi_0 \rangle}_{= 0}$$

$$\Rightarrow \quad c_0 = \frac{\langle x^2, \phi_0 \rangle}{\langle \phi_0, \phi_0 \rangle}$$

Same approach get $c_1 = \dfrac{\langle x^2, \phi_1 \rangle}{\langle \phi_1, \phi_1 \rangle}$

So for general ϕ_j our coefficients are

$$c_i = \frac{\langle x^j, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}$$

Question: How do you normalize these polynomials?

$$\hat{\phi}_j(x) = \frac{\phi_j(x)}{\sqrt{\langle \phi_j, \phi_j \rangle_w}}$$

The set $\{\hat{\phi}_j\}_{j=0}^{n}$ are orthnormal polynomials.

Def: $x \in \mathbb{R}^n$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

$f \in \mathcal{L}^2$ $\quad \|f\|_2 = \left(\int_a^b (f(x))^2 w(x)\, dx \right)^{1/2}$