APPM 4600 — HOMEWORK # 9

1. Find the least squares solution to the overdetermined linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. Find the vector $\boldsymbol{x}$ that minimizes the quantity $E^2 = b_1^2 + 4b_2^2 + 25b_3^2 + 9b_4^2$, when it holds that

$$\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Hint: When we solve a linear system of equations $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, multiplication from the left with a nonsingular matrix will leave the solution unchanged. This is **not** the case when finding the least squares solution to an overdetermined system. Exploit this and multiply the system above with a suitable diagonal matrix, so that the problems becomes a regular least squares problem (for which we can apply the normal equation approach.)

Soln: We need to minimize E^2 and satisfy the equation involving x_1 and x_2.

Applying the hint, the diagonal matrix we need to left multiply the over determined system by is

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{\mathsf{A}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}_{\boldsymbol{b}} = \mathbf{0}$$

We need this equation and the one listed in the problem to be satisfied a the same time. By multiplying the equation in the problem statement by A we can eliminate the vector $\boldsymbol{b}$. Thus we need to solve the following over determined linear system.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \\ 6 & -1 \\ 4 & 0 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right) = \mathbf{0}$$

After multiplying and moving unknowns to be on left hand side alone, we get

$$\begin{bmatrix} 1 & 3 \\ 12 & -2 \\ 20 & 0 \\ 6 & 21 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 15 \\ 12 \end{bmatrix}$$

NOTE: if you did this problem via the way listed posted, you need to write each row equation in it's simplified form before moving forward to get the correct solution.)

Now we need to solve the over determined system. The normal equation is

$$\begin{bmatrix} 581 & 105 \\ 105 & 454 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 421 \\ 247 \end{bmatrix}$$

The solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.6536 \\ 0.3929 \end{bmatrix}$.

3. If a function is identically zero over an interval, all its derivatives must also be identically zero over the same interval. Based on this observation:

   (a) Prove that $\{1, x, x^2, \ldots, x^n\}$ are linearly independent.

   (b) Show that the function set

$$\{1, \cos(x), \cos(2x), \ldots, \cos(nx), \sin(x), \ldots, \sin(nx)\}$$

   is linearly independent (also over any interval).

Soln:

(a) We need to show that the only way for

$$\sum j = 0^n c_j x^j = 0 \tag{1}$$

is for the constants $c_j = 0$ for all j.
To prove this we will pick a point to evaluate (**??**) and $n-1$ of its derivatives. I am going to chose the point $x = 1$. This means that

$$\sum j = 0^n c_j = 0$$

$$\sum j = 1^n j c_j = 0$$

$$\sum j = 2^n j(j-1) c_j = 0$$

etc.
The result is an upper triangular linear system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & \cdots & \cdots & n-1 & n \\ 0 & 0 & 2(1) & \cdots & \cdots & (n-1)(n-2) & n(n-1) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & j! & \ddots & n(n-1)\cdots j \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & n! \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0}$$

Back solving the tridiagonal system, we find that all the coefficients must be 0.

(b) We will do the same thing starting with the functions $\{1, \cos(x), \sin(x)\}$ and evaluating at $x = 0$ to get an idea for how the proof works.
Creating the linear system corresponding to the equation $c_0 + c_1 \cos(x) + b_1 \sin(x) = 0$ and its first two derivatives evaluated at $x = 0$, we get

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to this equation is that $c_0 = c_1 = b_1 = 0$.

Next we will show that $\{1, \cos(x), \ldots, \cos(nx), \sin(x), \ldots, \sin(nx)\}$ are linearly independent.

We look at the equation $c_0 + \sum_{k=1}^{n} c_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx) = 0$ and take derivatives of it then evaluate at $x = 0$.

The derivatives of the equation for $j > 0$ are

$$\frac{\partial^j}{\partial x^j}\left(c_0 + \sum_{k=1}^{n} c_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)\right) = 0$$

$$0 + \sum_{k=1}^{n} c_k k^j (-1)^{(j+1)/2} \sin(kx) + \sum_{k=1}^{n} b_k (-1)^{2(j+1)/2+1} k^j \cos(kx) = 0 \quad \text{for } j \text{ odd}$$

$$0 + \sum_{k=1}^{n} c_k k^j (-1)^{j/2} \cos(kx) + \sum_{k=1}^{n} b_k (-1)^{j+1} k^j \sin(kx) = 0 \quad \text{for } j \text{ even}$$

Evaluated at $x = 0$, we get

$$0 + \sum_{k=1}^{n} c_k(0) + \sum_{k=1}^{n} b_k (-1)^{2(j+1)/2+1} k^j = 0 \quad \text{for } j \text{ odd}$$

$$0 + \sum_{k=1}^{n} c_k k^j (-1)^{j/2} + \sum_{k=1}^{n} b_k(0) = 0 \quad \text{for } j \text{ even}$$

The row equations are linearly independent thus the resulting linear matrix only has a trivial null space and the coefficients all must be 0.

4. Prove the three-term recursion formula for orthogonal polynomials:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x)$$

where

$$b_k = \frac{< x\phi_{k-1}, \phi_{k-1} >}{< \phi_{k-1}, \phi_{k-1} >} \quad c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}$$

Hint: Since $\phi_k(x)$ is a polynomial of degree k and of the form $\phi_k = x^k + \{\text{lower order terms}\}$, we can clearly select b_k and c_k so that the right hand side (RHS) of (1) matches $\phi_k(x)$ for powers x^k , x^{k-1} and x^{k-2} . We have no obvious reason to expect that the two sides will match the other lower order terms. Hence, we would expect to need to include a lot more terms in the RHS to get the two sides to become equal:

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + \cdots + a_0\phi_0(x)\} \quad (2)$$

We now need to show that all these a's are in fact are zero. To show that $a_j = 0$, $j \leq k-3$, we form the scalar product of (**??**) with $\phi_j(x)$ for $j = 0, \dots, k-1$. You need to show that everything in (**??**) apart from $a_j < \phi_j, \phi_j >$ then vanishes, thereby showing that $a_j = 0$, $j \leq k-3$. After that, it remains to determine the values of b_k and c_k . These coefficients follow by again forming suitable scalar products.

5. One of the many formulas for computing the Chebychev polynomials $T_n(x)$ is

$$T_n(x) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right), \quad (3)$$

where z is implicitly defined through x via $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Confirm that the formula (**??**) indeed generates the same polynomials as the standard definition of the Chebychev polynomials.
Hint: One way would be to verify that it produces the correct result for T_0 and T_1 and that it satisfies the 3 term recursion.

Soln: We begin with the equation

$$\phi_k(x) = (x - b_k)\phi_{k-1}(x) - c_k\phi_{k-2}(x) - \{a_{k-3}\phi_{k-3}(x) + a_{k-4}\phi_{k-4}(x) + \cdots + a_0\phi_0(x)\}$$

and take the inner production with ϕ_j for $0 \leq j \leq k-3$.
We know that ϕ_k should be orthogonal to ϕ_j so

$$0 =< \phi_k(x), \phi_j >=< (x-b_k)\phi_{k-1}(x), \phi_j > -c_k < \phi_{k-2}(x), \phi_j > -\{a_{k-3} < \phi_{k-3}(x), \phi_j > +a_{k-4} < \phi_{k-4}(x), \phi$$

Now ϕ_j is orthogoal to all the other phi_l's so all that remains is

$$0 =< x\phi_{k-1}, \phi_j > -a_j < \phi_j, \phi_j > .$$

The property of inner products (or because this inner product is defined as integral allows us to move the x into the second arguement.

$$< x\phi_{k-1}, \phi_j >=< \phi_{k-1}, x\phi_j >$$

Now ϕ_{k-1} is orthogonal to all polynomials of degree less than $k-1$ thus it is orthogonal to $x\phi_j$ (a polynomial of max degree $k-2$.

This means that $0 = -a_j < \phi_j, \phi_j >$. The only way for this to be true is for $a_j = 0$.

Now we will find the coefficents b_k and c_k via the same process.

We know that ϕ_k should be orthogonal to ϕ_{k-2} and $< \phi_{k-2}, \phi_{k-1} >= 0$ thus

$$0 =< \phi_k, \phi_{k-2} >=< x\phi_{k-1}, \phi_{k-2} > -c_k < \phi_{k-2}, \phi_{k-2} >$$

Solving for c_k we get

$$c_k = \frac{< x\phi_{k-1}, \phi_{k-2} >}{< \phi_{k-2}, \phi_{k-2} >}.$$

The same technique results in the value for b_k.