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APPM 4560
5/2/24

Homework 9

1. a. $$Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & \dots & n \\ \hline 0 & -\lambda & \lambda & 0 & \dots & 0 \\ 1 & 1/\mu & -1/\mu - \lambda/2 & \lambda/2 & \dots & 0 \\ 2 & 0 & 1/\mu & -1/\mu - \lambda/3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 0 & 0 & 0 & & -1/\mu - \lambda/(n+1) \end{array}$$

$\pi Q = 0$

By detailed balance, $\pi(i)P(i,j) = \pi(j)P(j,i)$

$\pi(n) \cdot \lambda/(n+1) = \pi(n+1) \cdot 1/\mu$

$$\frac{\pi(n+1)}{\pi(n)} = \frac{\mu\lambda}{n+1}$$

Total prob $= 1 = \pi(0) \sum_{n=0}^{\infty} (\lambda\mu)^n/(n+1)! = \pi(0) \cdot \frac{e^{\lambda\mu}-1}{\lambda\mu}$

$\pi(0) = (\lambda\mu)/(e^{\lambda\mu}-1)$

Thus, $\pi(n) = \pi(0) \cdot (\lambda\mu)^n/(n+1)! = \boxed{(\lambda\mu)^{n+1}/((e^{\lambda\mu}-1)(n+1)!)}$

b. Fraction of time when nobody in line $= \pi(0) = (\lambda\mu)/(e^{\lambda\mu}-1)$ as shown in part a

c. $$E[N] = \sum_{n=0}^{\infty} nP(n) = \sum_{n=0}^{\infty} n \cdot \frac{(\lambda\mu)^{n+1}}{(e^{\lambda\mu}-1)(n+1)!} = \frac{\lambda\mu e^{\lambda\mu}}{e^{\lambda\mu}-1} - 1$$

$Var[N] = E[N^2] - (E[N])^2$

$$E[N^2] = \sum_{n=0}^{\infty} n^2 \cdot \frac{(\lambda\mu)^{n+1}}{(e^{\lambda\mu}-1)(n+1)!} = \frac{1}{e^{\lambda\mu}-1} \sum_{n=0}^{\infty} (n^2+2n+1) \frac{(\lambda\mu)^{n+2}}{(n+2)!}$$

Thus, $$Var[N] = \frac{1}{e^{\lambda\mu}-1} \sum_{n=0}^{\infty} (n^2+2n+1) \frac{(\lambda\mu)^{n+2}}{(n+2)^2} - \left(\frac{\lambda\mu e^{\lambda\mu}}{e^{\lambda\mu}-1} - 1\right)^2$$

d. Rate of sale $= \pi(0) \cdot 0 + (1-\pi(0)) \cdot 1/\mu$

$$= \boxed{\left(1 - \frac{\lambda\mu}{e^{\lambda\mu}-1}\right)\left(\frac{1}{\mu}\right)}$$

2. a. 

b. $Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 3 & -3 \end{bmatrix} \end{array}$

$\pi Q = 0$

$-2\pi_0 + \pi_1 = 0 \quad \pi_1 = 2\pi_0$

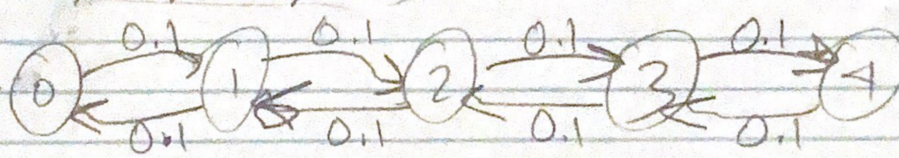
$2\pi_2 - 3\pi_3 = 0 \quad \pi_2 = 3\pi_3/2$

$2\pi_0 - 3\pi_1 + \pi_2 = 0 \quad \pi_1 - 3\pi_1 + \pi_2 = 0 \quad \pi_2 = 2\pi_1$

$\pi = [3c, 6c, 12c, 8c]$

$3c + 6c + 12c + 8c = 1$ so $c = 1/29$

$\boxed{\pi = [3/29, 6/29, 12/29, 8/29]}$

3. a. 

b. $\pi Q = 0$

$Q = \begin{bmatrix} -0.1 & 0.1 & 0 & 0 & 0 \\ 0.1 & -0.2 & 0.1 & 0 & 0 \\ 0 & 0.2 & -0.3 & 0.1 & 0 \\ 0 & 0 & 0.3 & -0.4 & 0.1 \\ 0 & 0 & 0 & 0.4 & -0.4 \end{bmatrix}$

$-0.1\pi_0 + 0.1\pi_1 = 0 \quad \pi_0 = \pi_1$

$0.1\pi_3 - 0.4\pi_4 = 0 \quad \pi_3 = 4\pi_4$

$0.1\pi_0 - 0.2\pi_1 + 0.2\pi_2 = 0 \quad 0.1\pi_1 - 0.2\pi_1 + 0.2\pi_2 = 0 \quad \pi_1 = 2\pi_2$

$0.1\pi_1 - 0.3\pi_2 + 0.3\pi_3 = 0 \quad 0.2\pi_2 - 0.3\pi_2 + 0.3\pi_3 = 0 \quad \pi_2 = 3\pi_3$

$\pi = [24c, 24c, 12c, 4c, c]$

$24c + 24c + 12c + 4c + c = 0 \quad c = 1/65$

$\boxed{\pi = [24/65, 24/65, 12/65, 4/65, 1/65]}$

3.c. E[ions] $= 10^5 \cdot (0 \cdot \pi(0) + 1 \cdot \pi(1) + 2 \cdot \pi(2) + 3 \cdot \pi(3) + 4 \cdot \pi(4))$

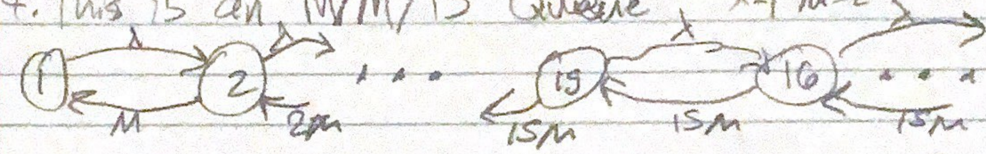
$= 10^5 \cdot (24/65 + 2 \cdot 12/65 + 3 \cdot 4/65 + 4 \cdot 1/65) = 64/65 \cdot 10^5$

E[Molecules with no calcium] $= \pi(0) \cdot 10^5 = 24/65 \cdot 10^5$

d. There is a free site 64/65 of the time $(1 - \pi(4))$

E[Bindings in 100 s] $= 100\,s \cdot 0.1$ bindings/s $\cdot\, 64/65 = \boxed{640/65}$

4. This is an M/M/15 Queue $\lambda = 1 \quad \mu = 2$



P(0 cars at M/M/15 Queue) $= \left(\sum_{k=0}^{15-1} \frac{(\lambda/\mu)^k}{k!} + \frac{(\lambda/\mu)^{15}}{15!(1 - \lambda/15\mu)} \right)^{-1}$

$= \boxed{0.607}$

5.a.

As shown in the Markov chain above, states 0 (Loss) and 4 (Win) are recurrent, while all levels 1, 2, and 3 are transient.

6. Let E_1 be the expected number of steps to the end from level 1 and E_2 and E_3 the same for levels 2 and 3

$E_1 = 0.1 \cdot 1 + 0.9 \cdot (E_2 + 1) = 1 + 0.9E_2$

$E_2 = 0.5(E_1 + 1) + 0.5(E_3 + 1) = 1 + 0.5E_3 + 0.5E_1$

$E_3 = 0.2 \cdot 1 + 0.8(E_2 + 1) = 1 + 0.8E_2$

$$\left[\begin{array}{ccc|c} 1 & -0.9 & 0 & 1 \\ -0.5 & 1 & -0.5 & 1 \\ 0 & -0.8 & 1 & 1 \end{array}\right] + \tfrac{1}{2}R1 \quad \left[\begin{array}{ccc|c} 1 & -0.9 & 0 & 1 \\ 0 & 0.55 & -0.5 & 1.5 \\ 0 & -0.8 & 1 & 1 \end{array}\right] + \tfrac{1}{2}R3 \quad \left[\begin{array}{ccc|c} 1 & -0.9 & 0 & 1 \\ 0 & 0.15 & 0 & 2 \\ 0 & -0.8 & 1 & 1 \end{array}\right]$$

$0.15E_2 = 2 \qquad E_2 = 13.333$

We want E_1 since you start at level 1

$E[N] = E_1 = 1 + 0.9(13.333) = \boxed{13}$

5. c. $Q = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.03 & -0.3 & 0.27 & 0 & 0 \\ 2 & 0 & 0.2 & -0.4 & 0.2 & 0 \\ 3 & 0 & 0 & 0.16 & -0.2 & 0.04 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{array}$

d. Let E_1, E_2 and E_3 represent the expected time to finish the game from Levels 1, 2, and 3 respectively.

$$E_1 = 1/0.3 + 0.9E_2$$
$$E_2 = 1/0.4 + 0.5E_1 + 0.5E_3$$
$$E_3 = 1/0.2 + 0.8E_2$$

$$\left[\begin{array}{ccc|c} 1 & -0.9 & 0 & 3.33 \\ -0.5 & 1 & -0.5 & 2.5 \\ 0 & -0.8 & 1 & 5 \end{array}\right] + 1/2 R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & -0.9 & 0 & 3.33 \\ 0 & 0.55 & -0.5 & 4.167 \\ 0 & -0.8 & 1 & 5 \end{array}\right] + 1/2 R_3$$

$$\left[\begin{array}{ccc|c} 1 & -0.9 & 0 & 3.33 \\ 0 & 0.15 & 0 & 6.66 \\ 0 & -0.8 & 1 & 5 \end{array}\right]$$

$0.15E_2 = 6.666 \rightarrow E_2 = 44.44$

We want E_1 since the player starts at 1

$E[T] = E_1 = 1/0.3 + 0.9(48.44) = \boxed{43.33 \text{ minutes}}$