Homework 13

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8.2: 1. d. $\lambda^2 - 2\lambda + 3 = 0$

$\lambda = \frac{2 \pm \sqrt{4-12}}{2} = \boxed{1 \pm \sqrt{2}i}$

$\lambda = 1+\sqrt{2}i$: $\begin{bmatrix} -\sqrt{2}i & 2 \\ 1 & -\sqrt{2}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Let $x_1 = 1$, $x_2 = i/\sqrt{2}$

$\lambda = 1-\sqrt{2}i$: $\begin{bmatrix} \sqrt{2}i & 2 \\ 1 & \sqrt{2}i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Let $x_1 = 1$, $x_2 = -i/\sqrt{2}$

$\begin{bmatrix} 1 \\ i/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1 \\ -i/\sqrt{2} \end{bmatrix}$ for $\lambda = 1+\sqrt{2}i, 1-\sqrt{2}i$

j. $M = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ $A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$

$\lambda_a = \lambda^2 - 6\lambda - 7$ $\quad$ $\lambda_d = \lambda^2 - 6\lambda - 7$

$= -1, 7$ $\quad$ $\lambda_d = 7, -1$

$\lambda_a = -1$: $\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = x_2$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_a = 7$: $\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = -x_2$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_d = -1$: $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $2x_1 = 3x_2$ $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$\lambda_d = 7$: $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $2x_1 = x_2$ $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

eigenvectors: $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3/2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$ for $\lambda = -1, -1, 7, 7$

8.2: 4.6. $\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & -1 & 1 \\ 1/2 & -1 & 1/2 \\ 1 & 1 & -1 \end{bmatrix}$ check: $\det \begin{pmatrix} (-1-(-1)) & -1 & 1 \\ 1/2 & (-1-(-1)) & 1/2 \\ 1 & 1 & (-1-(-1)) \end{pmatrix}$

$\begin{vmatrix} 0 & -1 & 1 \\ 1/2 & 0 & 1/2 \\ 1 & 1 & 0 \end{vmatrix} = -(-1)\cdot 1/2 + 1\cdot(-1/2) = 0$ ✓

5. a.

1. $\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ \alpha & \beta & \gamma-\lambda \end{vmatrix}$

$-\lambda(-\lambda(\alpha-\lambda)-\beta) - (-\alpha) = 0$

$\boxed{-\lambda^3 + \lambda^2\gamma + \lambda\beta + \alpha = 0}$

b. plug a in for γ, b in for β and c in for α in the matrix and the characteristic eq is $-\lambda^3 + a\lambda^2 + b\lambda + c$.

9. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\lambda^2 - 2\lambda = 0$

$\lambda = 0, 2$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $(1-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) + (1-(1-\lambda)) - (1-\lambda-1)$

$-\lambda^3 + 3\lambda^2 = 0$

$\lambda = 0, 0, 3$

$\boxed{\lambda = \underbrace{0, 0, \ldots 0}_{n-1 \text{ times}}, n \quad \text{eigenvector for } \lambda = n: x_1 = x_2 = \ldots = x_n \quad \text{for } \lambda = 0: x_1 + x_2 + \ldots + x_n = 0}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $v_1 + v_2 = 0$ $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $v_1 - v_2 = 0$ $v_1 = v_2$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $v_1 + v_2 + v_3 = 0$ $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $v_1 = v_2 = v_3$

8.2. 13.a. $S(V+W)^T = S(V^T) + S(W^T)$

$S(v_1+w_1, v_2+w_2, \ldots, v_n+w_n)$

$= (v_2, v_3, \ldots v_n, v_1) + (w_2, w_3, \ldots, w_n, w_1)$

$= S(V^T) + S(W^T)$

$S(\alpha V)^T = (\alpha v_2, \alpha v_3, \ldots, \alpha v_n, \alpha v_1)$

$= \alpha(v_2, v_3, \ldots, v_n, v_1)$

$= \alpha S(V^T)$

So S is linear

Matrix: $\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}$

b. For any 2 columns of A

$\langle Ae_i, Ae_j \rangle = 0 + \ldots + 0 \cdot 1 + \ldots + 1 \cdot 0 + \ldots + 0 = 0$

$\|Ae_j\| = 0 + \ldots + 1 \cdot 1 + \ldots + 0 = 1$

So every column is an orthonormal vector

So A is an orthogonal matrix

17. Let A be an $n \times n$ upper or lower triangular matrix

$A - \lambda I$ must also be upper triangular because substracting λI only changes the entries on the diagonal

$$\begin{vmatrix} (A_{11}-\lambda) & A_{12} & A_{13} & \ldots & A_{1n} \\ 0 & (A_{22}-\lambda) & A_{23} & \ldots & A_{2n} \\ & & \vdots & & \\ & & \vdots & & \\ 0 & 0 & 0 & \ldots & (A_{nn}-\lambda) \end{vmatrix} = (A_{11}-\lambda)(A_{22}-\lambda)\cdots(A_{nn}-\lambda)$$

Thus, $\lambda = A_{11}, A_{22}, \ldots, A_{nn}$

$$\begin{vmatrix} A_{11}-\lambda & 0 & 0 & \ldots & 0 \\ A_{21} & (A_{22}-\lambda) & 0 & \ldots & 0 \\ & & \vdots & & \\ & & \vdots & & \\ A_{n1} & A_{n2} & A_{n3} & \ldots & (A_{nn}-\lambda) \end{vmatrix} = (A_{11}-\lambda)(A_{22}-\lambda)\cdots(A_{nn}-\lambda)$$

$\lambda = A_{11}, A_{22}, \ldots, A_{nn}$

8.2. 21. a. False

$A\mathbf{v} = \lambda \mathbf{v}$ $B\mathbf{v} = \lambda \mathbf{v}$ $(A+B)\mathbf{v} = 2\lambda \mathbf{v}$

b. True

$A\mathbf{v} = \lambda_1 \mathbf{v}$ $B\mathbf{v} = \lambda_2 \mathbf{v}$ $(A+B)\mathbf{v} = (\lambda_1 + \lambda_2)\mathbf{v}$

22. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$\lambda_a = \pm 1$ $\lambda_b = 1$ $\lambda_c = (1 \pm \sqrt{5})/2$

$\boxed{\text{False}}$

30. a. True

32. a. $B = S^{-1}AS$

$P_B(\lambda) = \det(B - I\lambda)$

$= \det(S^{-1}AS - \lambda S^{-1}S)$

$= \det(S^{-1})\det(A - \lambda I) \cdot \det(S)$

$= \det(A - \lambda I)$

$= P_A(\lambda)$

b. Similar matrices have the same characteristic polynomials as proven in (a) and the eigenvalues are the roots of this polynomial which are the same

8.3: 1. b. $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$ $\lambda = 2$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = 0$ eigenspace $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ This is a complete eigenvalue. The dimension of the eigenspace is 1

8.3. 1. d. $\det(A-\lambda I) = -1\cdot(1-(1-\lambda))+(2-\lambda)((1-\lambda)(1-\lambda))-1)$

$= -\lambda^3+4\lambda^2-5\lambda$

$= -\lambda(\lambda^2-4\lambda+5)$

$\lambda = \frac{4\pm\sqrt{16-20}}{2}, 0 = 2\pm i, 0$

$\boxed{\text{1 isn't an eigenvalue}}$

2. f. $\det(A-\lambda I) = (-6-\lambda)(2-\lambda)(6-\lambda) - 8\cdot(-4(2-\lambda))$

$= -\lambda^3+12\lambda-12\lambda+36\lambda-6\lambda^2+6\lambda^2+2\lambda^2-72+64-32\lambda$

$= -\lambda^3+2\lambda^2+4\lambda-8 = (-\lambda+2)(\lambda^2-4)$ $\lambda=\pm 2$

2: $\begin{bmatrix} -8 & 0 & -8 \\ -4 & 0 & -4 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $x_1+x_3=0$ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

-2: $\begin{bmatrix} -4 & 0 & -8 \\ -4 & 4 & -4 \\ 4 & 0 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$ $x_1+2x_3=0$ $-x_1+x_2-x_3=0$ $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

Eigenspace basis $= \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$, complete

6. a. True

Same algebraic multiplicity as geometric multiplicity

b. False

A.M. and G.M. are different $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ A.M. = 2, G.M. = 1

8.3. 13. d. $(-2-\lambda)(3-\lambda)(1-\lambda)=0 \quad -\lambda^3+2\lambda^2+5\lambda-6$

$\lambda=-2,3,1$

$\lambda=1$: $\begin{bmatrix}-3&3&1\\0&0&-1\\0&0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$

$\lambda=-2$: $\begin{bmatrix}0&3&1\\0&3&-1\\0&0&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$\lambda=3$ $\begin{bmatrix}-5&3&1\\0&-2&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1\\-5\\10\end{bmatrix}$

$S^{-1}AS=\begin{bmatrix}0&1&1/2\\1&-1&-2/5\\0&0&1/10\end{bmatrix}\begin{bmatrix}-2&3&1\\0&1&-1\\0&0&3\end{bmatrix}\begin{bmatrix}1&1&-1\\1&0&5\\0&0&10\end{bmatrix}=\begin{bmatrix}1&0&0\\0&-2&0\\0&0&3\end{bmatrix}$

14. $\lambda^2-\lambda-1=0$

$\lambda=\frac{1\pm\sqrt{5}}{2}$

$\lambda=\frac{1+\sqrt{5}}{2}$: $\begin{bmatrix}1/2-\sqrt{5}/2&1\\1&-\sqrt{5}/2-1/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}1&-\sqrt{5}/2-1/2\\1&-\sqrt{5}/2-1/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1/2+\sqrt{5}/2\\1\end{bmatrix}$

$\lambda=\frac{1-\sqrt{5}}{2}$: $\begin{bmatrix}1/2+\sqrt{5}/2&1\\1&-1/2+\sqrt{5}/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}1&-1/2+\sqrt{5}/2\\1&-1/2+\sqrt{5}/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \quad \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1/2-\sqrt{5}/2\\1\end{bmatrix}$

$S^{-1}AS=\begin{bmatrix}-1/\sqrt{5}&\frac{1}{2\sqrt{5}}+\frac{1}{2}\\1/\sqrt{5}&-\frac{1}{2\sqrt{5}}+\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}\frac{1}{2}-\frac{\sqrt{5}}{2}&\frac{1}{2}+\frac{\sqrt{5}}{2}\\1&1\end{bmatrix}=\begin{bmatrix}\frac{1-\sqrt{5}}{2}&0\\0&\frac{1+\sqrt{5}}{2}\end{bmatrix}$

8.3. 23. True

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & 0 & & \vdots \\ & & & & a_{nn} \end{bmatrix}$$

For λ_1
$$\begin{bmatrix} a_{11}-\lambda_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda_1 & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33}-\lambda_1 & \cdots & a_{3n} \\ \vdots & & 0 & & \vdots \\ & & & & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The nullspace of this matrix is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

And for v_2 it's and for λ_n it's

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$