APPM 4650 — HOMEWORK # 6

1. In this problem we find the the polynomial $p(x) = c_n + c_{n-1}x + c_{n-3}x^2 + \cdots + c_1 x^{n-1}$ that interpolates the data $(x_j, y_j) = (x_j, f(x_j)), j = 1, \ldots, n$.

   (a) Assume $(x_j, y_j), j = 1, \ldots, n$ are given. Derive the system $V\mathbf{c} = \mathbf{y}$ that determines the coefficients $\mathbf{c} = [c_1, \ldots, c_n]^T$ (here $\mathbf{y} = [y_1, y_2, \ldots, y_n]^T$), that is, find how the matrix V looks like.

   To solve the system of equations you can simply use the backslash operator, i.e. `c = V\y;`. Since we have numbered the coefficients in the polynomial in the same way that matlab does we can use the built in function `polyval` to evaluate the polynomial (do `help polyval` to see how to use it).

   (b) Find the polynomial (i.e. the coefficients $\mathbf{c}$) that interpolates

   $$f(x) = \frac{1}{1 + (10x)^2},$$

   in the points $x_i = -1 + (i-1)h, h = \frac{2}{N-1}, i = 1, \ldots, N$. Plot data points as circles (`plot(x,f,'o')`) and, in the same plot, plot the polynomial and $f(x)$ on a finer grid (still on $x \in [-1, 1]$), say with 1001 points. Observe what happens when you increase N. Try $N = 2, 3, 4, \ldots$ and continue until the maximum value of $p(x)$ is about 100 (should be for $N \sim 17 - 20$). As you can see the polynomial behaves badly near the endpoints of the interval due to Runge's phenomena.

Soln:

(a) The matrix V has the form

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & \cdots & x_n^{n-1} \end{bmatrix}$$

(b) When $n = 20$, $cond(V) = 2.722408234739040e{+}08$; When $n = 50$, $cond(V) = 4.234975467822553e{+}18$. The condition number of$V$ indicates how big is the effect of error in $\mathbf{y}$ to error in $\mathbf{c}$, given the system $V\mathbf{c} = \mathbf{y}$. Since the floating point system will inevitably carry error at the 16th digit and after, for condition number very large, the calculated $\mathbf{c}$ will have distinguishable error.

The following plots shows when $n = 50$, the interpolating polynomial obtained using the Vandermonde matrix no longer interpolates the given function.

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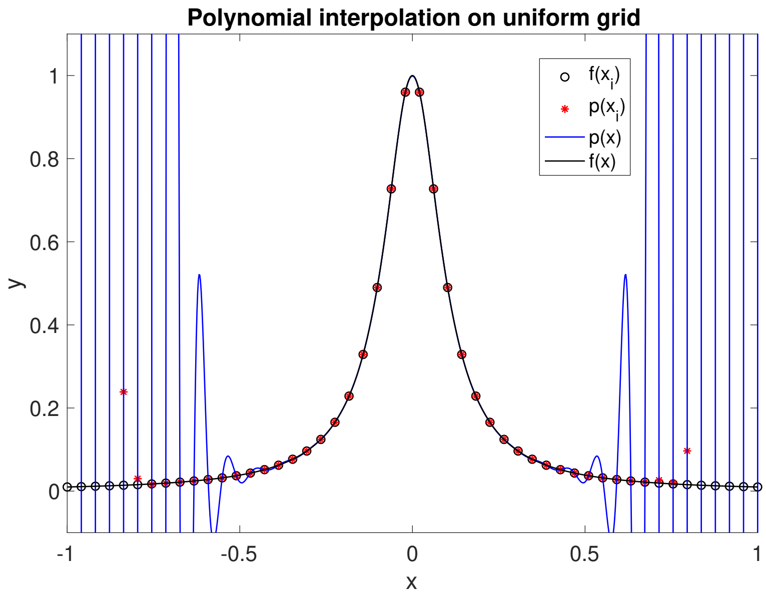


2. Solving the interpolation problem using the monomial basis (as above) is notoriously ill-conditioned, in fact it is possible to show `cond(V)`$\sim \pi^{-1}e^{\pi/4}(3.1)^n$. A better way of interpolating is to use either of the barycentric Lagrange interpolation formulas:

$$p(x) = \Phi_n(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f(x_j).$$

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j}{x-x_j} f(x_j)}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}}, \quad x \neq x_j.$$

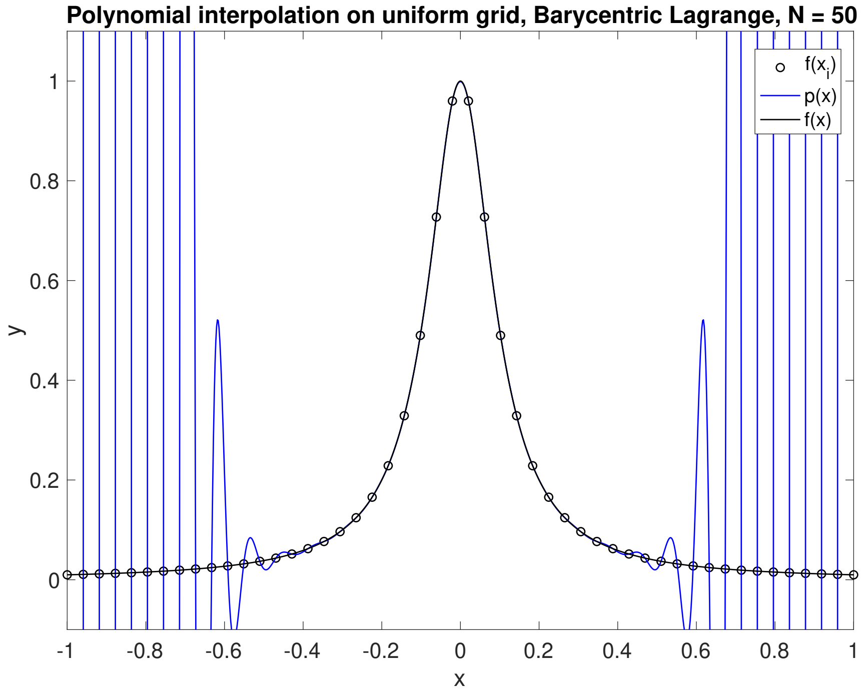
Where

$$\Phi_n(x) = \prod_{i=0}^{n} (x - x_i), \quad w_j = \frac{1}{\prod_{i=0, i\neq j}^{n} (x_j - x_i)}.$$

Using either of the above formulas try again to interpolate $f(x)$. Show with some pictures that you still get the same bad behavior close to the endpoints (this is a property of the function $f(x)$ and the distribution of the grid points not of the form of interpolation) but that the approximation is well behaved for small x for very large n.

Soln:

The following plots shows when $n = 50$, the approximation is well behaved for small x for very large n. We still observe the Runge phenomenon near the interpolating interval endpoints.



3. It is much better to interpolate on a grid made up of points that are clustered towards the endpoints. Try to interpolate $f(x)$ in the Chebyshev points

$$x_j = \cos\frac{(2j-1)\pi}{2N}, \quad i = 1, \ldots, N,$$

using either of the methods above. Can you get the interpolation to fail now?

Soln: With Chebyshev nodes, the Runge phenomenon is almost non-existent. But the interpolation is still possible to fail when the condition number of V is large if using the Vandermonde matrix to obtain the interpolating polynomial. The approximation is hard to fail when using the barycentric Lagrange interpolation.

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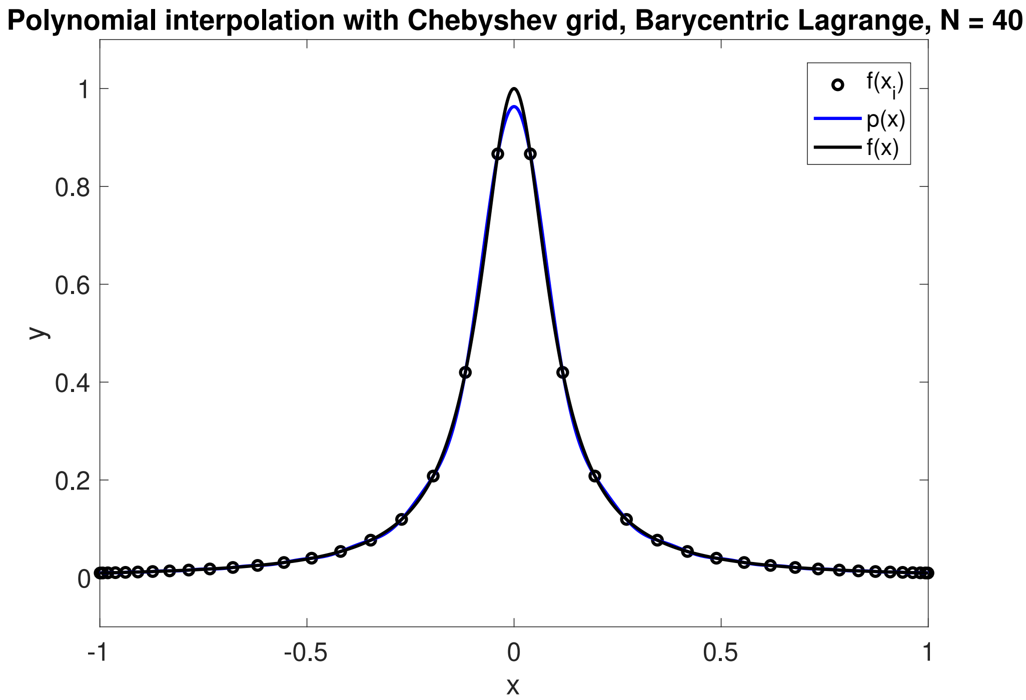

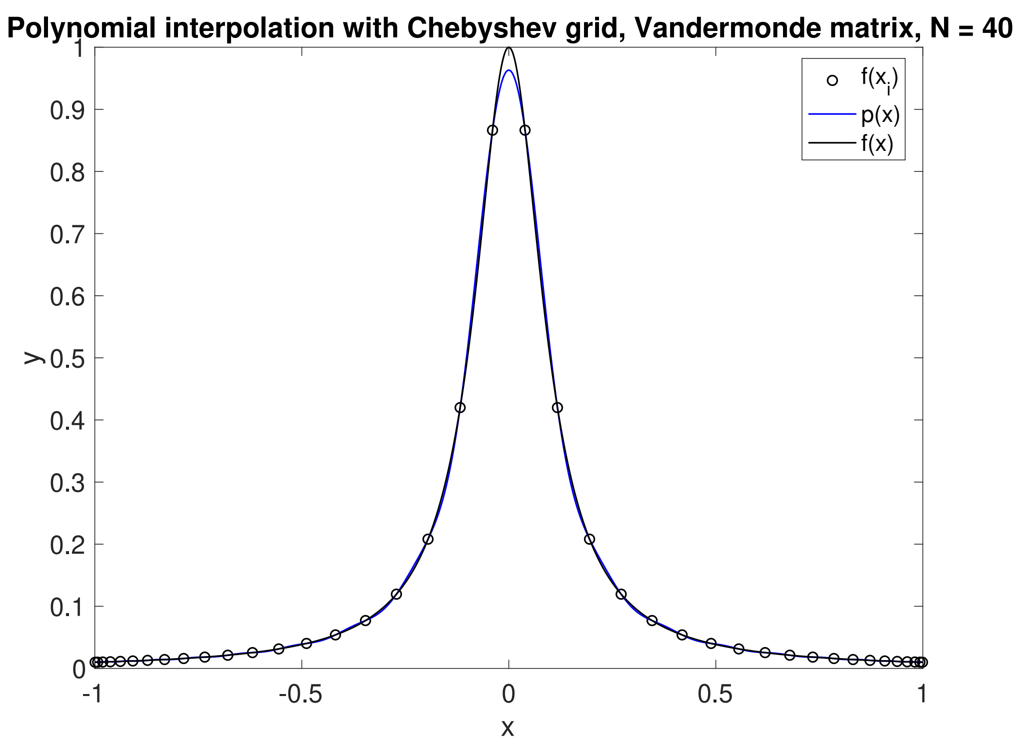

Extra fun! Not for credit

Finally, for the functions

$$f_1(x) = \sin(x), \quad f_2(x) = |x|, \quad f_3(x) = \sqrt{|x|},$$

plot (use log-log scale) the maximum error,

$$\max_{x \in [-1,1]} |f(x) - p(x)|,$$

as a function of N (make sure you sample the error on a fine grid) where $p(x)$ is constructed via the methods in Problems 1, 2 and 3.. What is the rate of convergence for the interpolation error for the three different functions?

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