QR Factorization

Wednesday, November 29, 2023 2:01 PM

What is QR?

$A \in \mathbb{C}^{n \times m}$ $\quad m \geq n$

$A = QR$ where $Q \in \mathbb{C}^{n \times n}$ is orthogonal

& $R \in \mathbb{C}^{n \times m}$ is upper triangular.

What do we use this for?

- Solve linear systems
- find eigenvalues.

Projects

$A = \text{rand}(n)$ $\quad [Q, R] = qr(A)$

$D =$ diagonal.

$B = Q D Q^*$

$C = \text{rand}(n)$ $\quad [Q_2, R] = qr(C)$

$M = Q D Q_2^*$

Q: How do we make the QR factorization?

A: Build Q or Q^* so that $Q^* A = R$

We will build Q to be a product of projectors.

Def: A <u>projector</u> is a square matrix $P \in \mathbb{C}^{m \times m}$ st $P^2 = P$. (idempotent)

Physical understanding of P.

let $S_1 = \text{Range}(P)$

$$\overset{P}{} \quad x \xrightarrow{f} \underset{\text{Range}}{y}$$

if $x \in \mathbb{C}^m$ the $v = Px \in S_1$

What is Pv?

$$Pv = P^2 x = Px = v$$

Q: if $S_1 = \text{Range}(P) = \mathbb{R}^n$
what is P?

$$P = I$$

What is $P(Px - x)$ for $x \in \mathbb{C}^n$?

$$P(Px - x) = P^2 x - Px = Px - Px = \bar{0}$$

$$\Rightarrow Px - x \in \text{Null}(P)$$

$$\text{Range}(P) \perp \text{Null}(P)$$

$$\underbrace{\text{Range}(P)}_{\text{Image}} \perp \underbrace{\text{Null}(P)}_{\text{kernel}}$$

- $Px - x = -$ residual
  $= -$ part of x not in the Range of P

<u>Ex</u>: let's look at the operator $Q = I - P$ where P is a projector.
Show that Q is also a projector.

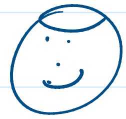
<u>Soln</u>: WTS $Q^2 = Q$

$$Q^2 = (I - P)(I - P) = I - 2P + \underset{\overset{\shortparallel}{P}}{P^2}$$

$$= I - P = Q$$

<u>Thm</u> If P is a projector then $Q = I - P$ is a projector &

(i) $PQ = 0$
(ii) $\text{Null}(P) = \text{Range}(Q)$
(iii) $\text{Null}(Q) = \text{Range}(P)$

Proof is HW. 

Ex: For any non-zero vector $v \in \mathbb{C}^n$

$$P = \frac{vv^*}{v^*v}$$ is an orthogonal projector.

Show this. $P^* = P$

$* = T$

Soln: $$P^* = \left(\frac{vv^*}{v^*v}\right)^* = \frac{1}{v^*v}(vv^*)^*$$

$$= \frac{vv^*}{v^*v} = P$$

$$P^*P = \frac{v(v^*}{v^*v}\,\frac{v)v^*}{v^*v} = \frac{vv^*(v^*v)}{(v^*v)^2} = \frac{vv^*}{v^*v} = P$$

If P is orthogonal projector then $Q = I - P$ is also an orthogonal projector.

<u>Lemma</u>: A projector P is orthogonal if $P^* = P$
i.e. P is Hermitian (for Real P - Symmetric)

Side. $M \in \mathbb{R}^{m \times n}$ $m < n$

$\text{Rank}(M) = k < m$

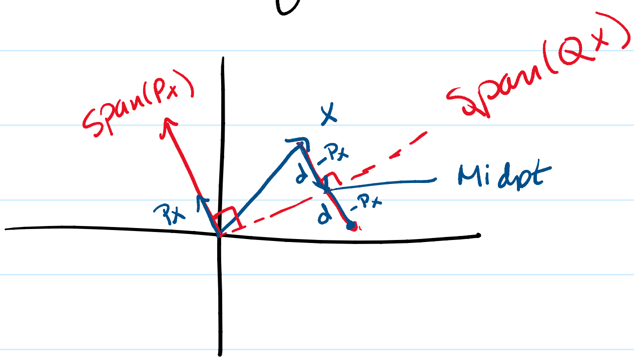
SVD: $U^* \Sigma V = M$

Back to QR

$$A = \begin{pmatrix} a_{11} & - & - & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{n1} & - & - & a_{nn} \end{pmatrix}$$

want to zero this out.

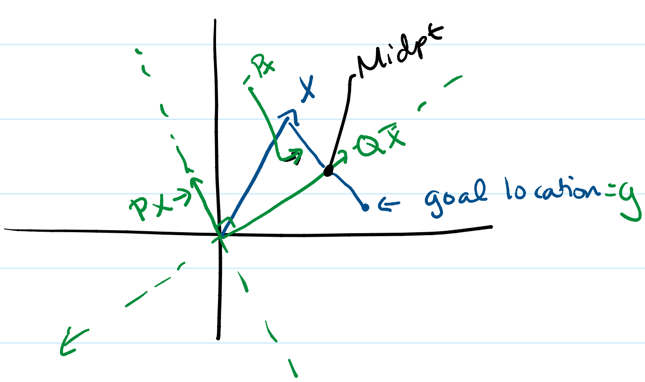
Towards Reflectors



$Q = I - P$

$\bar{d} = -Px$

Goal: find a Matrix st $Hx = x - 2Px$
$= (I - 2P)x$



P - projector
$Q = I - P$ - projector

How do we go from x to g?

$$y = x - 2Px = \underbrace{(I - 2P)}_{H} x$$

If $P = \frac{vv^*}{v^*v}$ then $H(v) = I - 2\frac{vv^*}{v^*v}$

__Def:__ $H(v) = I - 2\frac{vv^*}{v^*v}$ is called a __Householder reflector.__

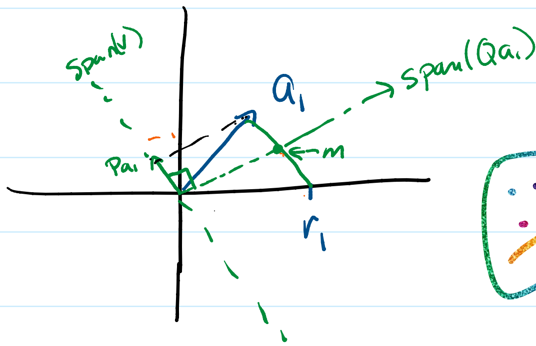
Back to QR

let $A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in \mathbb{C}^{n \times n}$

i.e. $a_1 = A(:,1) = 1^{st}$ column of A

__Goal:__
- zero out all but the 1^{st} entry in a_1 via a Householder.
- Find $v \in \mathbb{C}^n$ st $H(v)a_1 = \underbrace{\begin{pmatrix} \|a_1\|_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{r_1} = \|a_1\|_2 e_1$

What is v?



r_1 = Goal $= \begin{pmatrix} \|a\|_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$$\bar{m} = Q\bar{a}_1 = \frac{1}{2}\left(a_1 + \|a_1\|_2 e_1\right) \in v^\perp$$

Find v orthogonal to m

$$v = a_1 - \underbrace{\|a_1\|_2 e_1}_{r_1}$$

Check that v & m are orthogonal.

$$v^* m = (a_1 - \|a_1\|_2 e_1)^* \left(\tfrac{1}{2}(a_1 + \|a_1\|_2 e)\right)$$

$$= \tfrac{1}{2}\left(\underbrace{a_1^* a_1}_{\|a_1\|_2^2} - \|a_1\|_2^2 \underbrace{e_1^* e_1}_{1}\right) = 0 \checkmark$$

Now we know how to build QR.

Illustration: $A = \left[a_1^{(1)} \cdots a_n^{(1)}\right]$

superscript is the step in QR factorization process

The matrix $\hat{H}_1 = H(v_1)$ - zeros out the lower entries in a_1

$$\hat{H}_1 A = \begin{bmatrix} x & & & \\ 0 & & & \\ \vdots & a_2^{(2)} & \cdots & a_n^{(2)} \\ 0 & & & \end{bmatrix}$$

How do we eliminate the lower 3 to n entries in $a_2^{(2)}$?

$$\hat{H}_2 = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & H(v_2) \\ 0 & \end{pmatrix}$$

where v_2 is built for $a_2^{(2)}(2:n)$

$$a_2^{(2)} = \begin{pmatrix} a_{2,1} \\ \boxed{\begin{matrix} a_{2,2} \\ \vdots \\ a_{n,2} \end{matrix}} \end{pmatrix}$$

v_2 is built for this

$$\hat{H}_2 \hat{H}_1 A = \begin{bmatrix} x & x & & & \\ 0 & x & a_3^{(3)} & \cdots & a_n^{(3)} \\ & 0 & & & \\ 0 & & & & \end{bmatrix}$$

$$\hat{H}_3 = \begin{pmatrix} I_2 & 0 \\ 0 & \underbrace{H(v_3)}_{(n-2)\times(n-2)} \end{pmatrix}$$

where v_3 kills the 4 to n entries is $a_3^{(3)}$

$$a_3^{3} = \begin{pmatrix} x \\ x \\ \boxed{\begin{matrix} x \\ x \\ \vdots \\ x \end{matrix}} \end{pmatrix}$$

v_3 acts on this

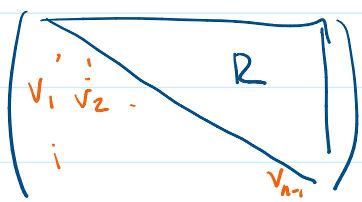
Continue...

$$\underbrace{\hat{H}_{n-1} \cdots \hat{H}_1}_{Q^*} A = R \quad = \text{upper triangular}$$

$$Q = \hat{H}_1 \cdots \hat{H}_{n-1}$$

Comments: Householder matrices are unitary. = Stable.

If you call QR in LAPACK. it overwrites A with R and the Householder vectors.



How to Apply H(v) efficiently?

$$H(v) = I - 2\frac{v v^*}{v^* v}$$

$$H(v)\,x = x - \frac{2\,v\,(v^* x)}{v^* v}$$ 2 $\leftarrow O(n)$

$$= x - \alpha v \qquad \text{where } \alpha = \frac{2 v^* x}{v^* v}$$

$O(n)$