Homework 7

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1. $P(A \text{ exceeds } t \text{ hrs}) = e^{-2t}$ $CDF(A) = 1-e^{-2t}$
$P(B \text{ exceeds } t \text{ hrs}) = e^{-3t}$ $CDF(B) = 1-e^{-3t}$
$P(\text{both have at least one sale at } t) = (1-e^{-2t})(1-e^{-3t})$
$F(t) = (1-e^{-2t})(1-e^{-3t})$
$f(t) = F'(t) = 2e^{-2t}+3e^{-3t}-5e^{-5t}$
$E[t] = \int_0^\infty t(2e^{-2t}+3e^{-3t}-5e^{-5t})dt$
$= -te^{-2t} - \int -e^{-2t}dt - te^{-3t} - \int -e^{-3t}dt + te^{-5t} - \int e^{-5t}dt \Big|_0^\infty$
$= 1/2 + 1/3 - 1/5 = 0.633 \text{ hrs} = \boxed{38 \text{ minutes}}$

2. a. $E[\text{Carol sees teller}] = 1/(\lambda_A+\lambda_B) = 1/(1/4+1/4) = 1/\tfrac{1}{2} = 2$
$E[\text{Carol finished}] = 2+4 = 6$

b. $E[\text{Alice}] = 4$ $E[\text{Bob}] = 4$ $E[\text{Carol}] = 6$
However, one of Alice or Bob must finish before Carol can start, so only $E[\max(\text{Alice}/\text{Bob})]$ and $E[\text{Carol}]$ matter.
$E[\max(\text{Alice}/\text{Bob})] = \int_0^\infty t \frac{d}{dt}((1-e^{-t/4})^2)dt$
$= \int_0^\infty t(-e^{-t/2}/2 + e^{-t/4}/2)dt$
$= te^{-t/2} - \int e^{-t/2}dt - 2te^{-t/4} - \int -2e^{-t/4}dt \Big|_0^\infty$
$= te^{-t/2} + 2e^{-t/2} - 2te^{-t/4} - 8e^{-t/4} \Big|_0^\infty$
$= -2+8 = 6$
Thus, $E[\max(\text{Alice}/\text{Bob})] = 6$ minutes

$E[\max(\max(\text{Alice}/\text{Bob}), \text{Carol}] = \int_0^\infty d/dt[(1-e^{-t/6})^2]dt$
$= \int_0^\infty t(1/3 e^{-t/6} - 1/3 e^{-t/3})dt$
$= -2te^{-t/6} - \int -2e^{-t/6}dt + te^{-t/3} - \int e^{-t/3}dt \Big|_0^\infty$
$= -2te^{-t/6} - 12e^{-t/6} + te^{-t/3} - 3e^{-t/3} \Big|_0^\infty$
$= 12 - 3 = \boxed{9 \text{ minutes}}$

It will take 9 minutes on average for everyone to leave

2.c. Since E[max(Alice/Bob)] = 6 mins and E[Carol] = 6 mins, the Probability of Carol being last out is $\boxed{0.5}$

3. $\lambda_{car} = 6 \cdot 1/3 = 2$ per hour that will take him

$P(T_{car} < T_{bus}) = \int_0^1 (1 - F_{bus}) f_{car} \, dt$

$= \int_0^1 (1-t) \cdot 2e^{-2t} \, dt$

$= -e^{-2} + 1 + e^{-2} + e^{-2}/2 - 1/2 = 0.568$

$P(T_{bus} \le T_{car}) = 1 - P(T_{car} < T_{bus}) = \boxed{0.432}$

4. a. E[Time between trains] = $(1+2)/2 = 1.5$

E[Passengers per train] = $1.5 \cdot 24 = \boxed{36}$

b. Var[Passengers per hour] = $\lambda = 24$

Var[time between trains] = $1/12 \, (2-1)^2 = 1/12$

Var[passengers on train] = $E[24 \cdot E[\text{time between trains}]] + 24^2 \cdot 1/12 = 36 + 48 = \boxed{84}$

5. a. $P(\text{Dogs}(T_{3\,cats}) \ge 2) = \int_0^\infty P(\text{Dogs}(t) \ge 2) \, f_{T_{3\,cats}}(t) \, dt$

$= \int_0^\infty (1 - e^{-\lambda_D t} - \lambda_D t e^{-\lambda_D t}) \, \lambda_C^3 t^2 e^{-\lambda_C t}/2 \, dt$

$= \boxed{1 - 1/(1+\lambda_D/\lambda_C)^3 - 3\lambda_D/\lambda_C (1+\lambda_D/\lambda_C)^4}$

b. $P[\text{Dogs}(1) + \text{Cats}(1) = 5] = \boxed{e^{-(\lambda_D+\lambda_C)} (\lambda_D + \lambda_C)^5 / 5!}$ based on Poisson distribution.

6. $P(N(t) = 1) = e^{-\lambda t}(\lambda t)$

$P(T_1 \le s \mid N(t) = 1) = P(T_1 \le s \cap N(t) = 1) / P(N(t) = 1)$

$P(T_1 \le s \cap N(t) = 1) = P(T_1 \le s) \cdot P(N(t) - N(s) = 0)$

$P(T_1 \le s \cap N(t) = 1) = (1 - e^{-s\lambda}) \cdot e^{-\lambda(t-s)} = e^{-\lambda t} \lambda s$

Thus, $P(T_1 \le s \mid N(t) = 1) = \dfrac{e^{-\lambda t} \lambda s}{e^{-\lambda t} \lambda t} = \dfrac{s}{t}$, the CDF of a uniform distribution at time s.

7. a. For all a_i in $i = 1, 2, \ldots, n-2$,

$a_i = \sum_{j=1}^{N} p_{ij} a_j$

Since you can't get to N from N-1, then equation can be represented as $a_i = \sum_{j=1}^{N-2} p_{ij} a_j + p_{iN} + 0 \cdot p_{iN-1}$

b. Lower bound = 0 because a_i represents the liklihood of transitioning to state N, and none of the transition probabilities can be negative so $a_i \geq 0$

Upper bound = 1 because a_i includes all possible paths from i to N, which can't exceed one because all of the possible paths represent mutually exclusive outcomes in the same Markov chain

c. Let A be transition probability matrix without the Nth and N-1th rows and columns

Let $v = p_{iN}$ for $i = 1, 2, \ldots, N-2$

$a = Aa + v$

$(I - A)a = v$

$a = (I - A)^{-1} v$

8. a. $q_i = 1 + \sum_{j=1}^{N-1} p_{ij} q_{ij}$

This is summing through N-1 not N because $q_N = 0$.

$q_{N-1} = \infty$

b. Let B be the probability transition matrix p without the Nth row and Nth column.

Let z be a vector of 1s to add one for each step

$q = Bq + z$

$(I - B)q = z$

$q = (I - B)^{-1} z$