Composite Newton-Cotes

Wednesday, November 1, 2023 2:12 PM

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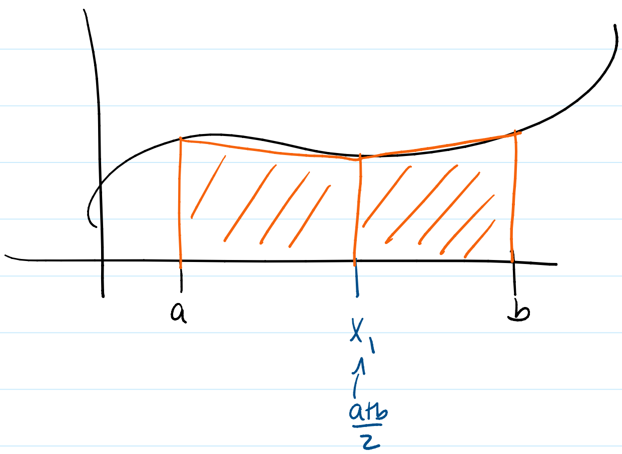


What was trapezoidal rule on 1 interval?

$$T = \frac{h}{2}\left(f(x_0)+f(x_1)\right)$$

On two intervals $h = \frac{b-a}{2}$

$$T = T_1 + T_2$$

$$= \frac{h}{2}\left(f(x_0)+f(x_1)\right) + \frac{h}{2}\left(f(x_1)+f(x_2)\right)$$

$$= \frac{h}{2}\left(f(x_0)+2f(x_1)+f(x_2)\right)$$

<u>Thm 4.5</u> let $f \in C^2[a,b]$. $h = \frac{b-a}{n}$ & $x_j = a+jh$ for $j=0,\dots,n$

then $\exists\, \mu \in (a,b)$

$$\int_a^b f(x)\,dx = \underbrace{\frac{h}{2}\left(f(a)+2\sum_{j=1}^{n-1} f(x_j) + f(b)\right)}_{\text{Composite trapezoidal rule.}}$$

$$- \frac{(b-a)h^2}{}\; f''(\mu)$$

$$\underbrace{\overbrace{- \frac{(b-a)h^2}{12} f''(\mu)}^{\text{Composite trapezoidal rule.}}}_{\text{error}}$$

- Trapezoidal is exact for lines
- the order of convergence is quadratic

Proof: $\int_a^b f(x)\,dx = \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} f(x)\,dx$

apply Trapezoidal to each interval

$$= \sum_{j=0}^{n-1} \left(\frac{h}{2} \left(f(x_j) + f(x_{j+1}) \right) - \frac{h^3}{12} f''(\eta_j) \right)$$

for some $\eta_j \in (x_j, x_{j+1})$

$$= \frac{h}{2} \left(f(x_0) + 2\sum_{j=0}^{n-1} f(x_j) + f(x_n) \right) - \underbrace{\frac{h^3}{12} \sum_{j=0}^{n-1} f''(\eta_j)}_{\text{error term}}$$

let's take a closer look at the error

$$-\frac{h^3}{12} \sum_{j=0}^{n-1} f''(\eta_j) = -\frac{h^3}{12} \frac{n}{n} \sum_{j=0}^{n-1} f''(\eta_j)$$

Since f'' is continuous it must attain all values between the min & max. i.e.

$$\exists\, \mu \in (a,b) \text{ st } f''(\mu) = \frac{1}{n} \sum_{j=0}^{n-1} f''(\eta_j)$$

i.e. IVT

Our error is now

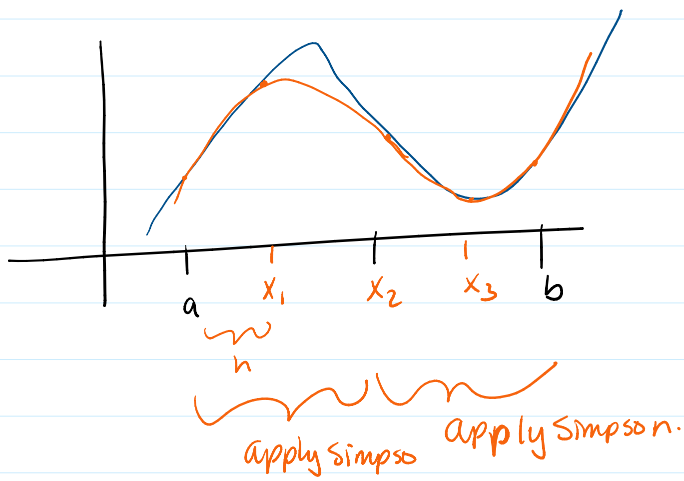
$$-\frac{h^3}{12}\, n \left(\frac{1}{n} \sum_{j=0}^{n-1} f''(\eta_j) \right) = \frac{-h^3}{12}\, n\, f''(\mu)$$

for some $\mu \in (a,b)$

we know $h = \frac{b-a}{n}$

$$\Rightarrow \text{Error} = -\frac{(b-a)h^2}{12} f''(\mu) \quad //$$

Composite simpson's rule.



Thm 4.4 let $f \in C^4(a,b)$, n even, $h = \frac{b-a}{n}$

$$x_j = a + jh \quad j = 0, \cdots, n$$

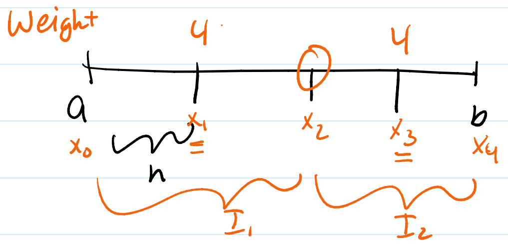
Then $\exists\, \mu \in (a,b)$ for which composite Simpson's rule for n intervals can be written as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a) + 2 \sum_{j=1}^{(n/2)-1} \underbrace{f(x_{2j})}_{\text{even}} + 4 \sum_{j=1}^{n/2} \underbrace{f(x_{2j-1})}_{\text{odd}} + f(b) \right)$$

$$- \frac{b-a}{180} h^4 f^{(4)}(\mu)$$

Weight 4 4

Ex:



$$S_1 = \frac{h}{3}\left(f(a) + 4f(x_1) + \underline{f(x_2)}\right)$$

$$+\quad S_2 = \frac{h}{3}\left(f(\underline{x_2}) + 4f(x_3) + f(b)\right)$$

$$S_1 + S_2 =$$

Composite Simpson is exact for polynomials of degree 3

Error: The order of convergence is $O(h^4)$, fourth order

$$\frac{1}{2^4} E(h) \approx E(h/2)$$

Special feature of Trapezoidal:

If $f(x)$ is periodic on interval of integration,

The convergence is exponential.

$$E(h) = O(e^{-h})$$

Ex: Determine the values of h needed to ensure an error less than 10^{-5} when approximating

$\int_0^{\pi} \sin x \, dx = 2$ via composite Trapeziodal & Simpsons rule

Soln:

Trapezoidal rule

$$E(h) = -\frac{(b-a)}{12} h^2 f''(\mu)$$

$$= -\frac{\pi}{12} h^2 (-\sin x)$$

$$|E(h)| = \frac{\pi}{12} h^2 |\sin x| \leq \frac{\pi}{12} h^2 < 10^{-5}$$

Solve for h: $h < 10^{-5/2} \frac{\sqrt{12}}{\pi} \sim 0.0087$