Aloke Chatterjee, ARPM '09, "Home", Sept. 20, 2009

One of the most important considerations in a Markowitz mean variance optimization (or for that matter, any portfolio selection model) is the estimation of the covariance matrix. Practitioners typically take recent returns data and calculate covariances from this sample. Unfortunately, sample covariance data will produce a positive definite matrix but the actual (unobservable) matrix may indeed not be positive definite (hence, not invertible). From the Glivenko-Cantelli theorem, the empirical distribution of a set of IID variables tends to the true distribution as observations go to infinity (i.e. maximum likelihood in asymptotic statistics). For short sampling periods however, the benchmark estimators have high inefficiency although they exhibit low bias (close to the true, unobservable value in the location-dispersion framework). Stein showed weighted averages of the sample with a constant estimator (which has very high bias but no inefficiency) can outperform the standalone sample estimator. In this paper, three shrinkage estimators are calculated along with the sample covariance matrix alone (without shrinkage) to assess performance.

Methodology: The blended approach is refined by Ledoit and Wolf, who imposed structure by replacing the constant estimator with Sharpe's single factor market model (reference in the code). The authors also derive an expression for the shrinkage intensity (weights assigned to sample and other estimator) based upon the Frobenious norm. For purposes of comparison, they also consider shrinkage to the identity matrix, a constant correlation model (Elton) , principal components, a more complicated industry factors model (Fama French could work here), pseudo-inverse and the identity matrix. These differing estimators were subsequently used to calculate mean-variance optimal portfolios both in-sample and more importantly, out-of-sample, to examine performance (measured in their study by standard deviation).

In this paper, a portfolio was constructed with tracking error constraints for the same sample period and benchmark (Dow 30) to compare its standard deviation to the previous MV portfolios (no shrinkage). A second portfolio controlled for tracking error and total risk.

Data: DJ Industrial Average monthly linear returns for the 10yr period (August, 1999-August, 2009), with the first five years in-sample to construct the estimators. Kraft was excluded as its IPO was in 2001.

Assumptions: It is assumed that stock returns are IID, have a finite fourth moments and the residuals of the Sharpe index model are uncorrelated to each other. Also, the time series of returns are assumed to be long enough to preclude estimation error in the average correlation, i.e. absence of an estimation error improves analytical tractability and allows for the use of asymptotic statistics.

Annual Standard Deviation of the Min-Var Portfolios	*In-sample period*	Out-of-sample period	Out-of-sample with quarterly rebalancings
Shrinkage to Identity	15.39%	17.61%	18.29%
Constant Correlation	15.79%	17.53%	18.06%
Market Model	15.41%	17.34%	18.33%
Sample Covariance	15.42%	17.61%	18.29%
TE Constrained (No shrinkage est.)	*17.07%*		

Results: It is surprising that the differing estimators arrived at very similar results- in Ledoit and Wolf's study, the shrinkage to market model had 9.0% SD while constant correlation had 14.2%. One explanation is that their sample was N=909 to 1314; with a covariance matrix much larger than 29 by 29, estimation errors are magnified. Another issue may be the large and frequent market shocks this decade.