CS 292C Computer-Aided Reasoning for Software

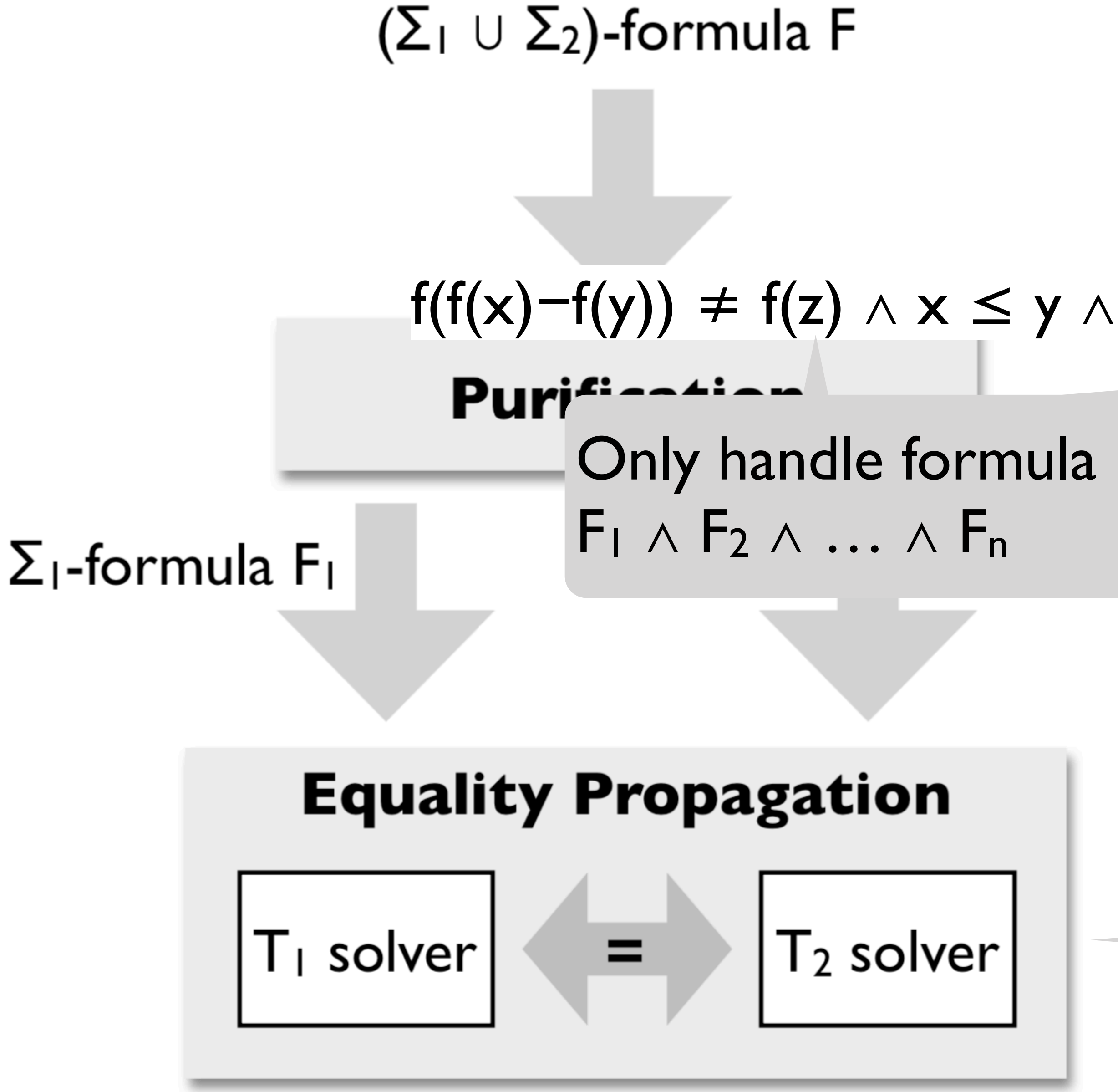
Lecture 9: The DPLL(T) Framework

Yu Feng
Fall 2020

Summary of previous lecture

- 4th paper review is out
- Proposal is due now
- Deciding a combination of theories
- The Nelson-Oppen algorithm

Overview of Nelson-Oppen



$f(f(x)-f(y)) \neq f(z) \wedge x \leq y \wedge y + z \leq x \wedge 0 \leq z$

Phase 1

Only handle formula in CNF:
$F_1 \wedge F_2 \wedge \dots \wedge F_n$

Phase 2

Outline of this lecture

- Deciding arbitrary boolean combinations of theory constraints
- The DPLL (T) algorithm
- The last lecture about SMT/SAT

Boolean abstraction

CFG of SMT formula in theory T

- $F := a_T \mid F_1 \wedge F_2 \mid F_1 \vee F_2 \mid \neg F$

For each SMT formula, define a **boolean abstraction function**, that maps SMT formula to overapproximate SAT formula

- $B(a_T) = b$ (b fresh)
- $B(F_1 \wedge F_2) = B(F_1) \wedge B(F_2)$
- $B(F_1 \vee F_2) = B(F_1) \vee B(F_2)$
- $B(\neg F) = \neg B(F_1)$

$F : x = z \wedge ((y = z \wedge x < z) \vee \neg(x = z))$

$B(F) = b_1 \wedge ((b_2 \wedge b_3) \vee \neg b_1)$

Is B(F) satisfiable?
Is F satisfiable?

Off-line v.s. online

SAT solver may yield assignments that are not sat modulo T because boolean abstraction is an over-approximation

Need to learn theory conflict clauses

Two different approaches for learning theory conflict clauses

- Off-line (eager): Use SAT solver as black-box
- On-line (lazy): Integrate theory solver into the CDCL loop (adopted by mainstream SMT solvers)

Off-line version

```
Offline-DPLL(T-formula $\varphi$)
$\varphi_P \leftarrow B(\varphi)$
while (TRUE) do
    $\mu_P$, res $\leftarrow$ CDCL($\varphi_P$)
    if res = UNSAT then return UNSAT
    else
        T-res $\leftarrow$ T-solve($B^{-1}(\mu_P)$)
        if T-res = SAT then return SAT
        else $\varphi_P \leftarrow \varphi_P \wedge \neg \mu_P$
```

$F : x = z \wedge ((y = z \wedge x < z) \vee \neg(x = z))$

$B(F) = b_1 \wedge ((b_2 \wedge b_3) \vee \neg b_1)$

SAT assignment to B(F): $b_1 \wedge b_2 \wedge b_3$

$B^{-1}(F)$ is UNSAT

What is new boolean abstraction?

$b_1 \wedge ((b_2 \wedge b_3) \vee \neg b_1) \wedge \neg(b_1 \wedge b_2 \wedge b_3)$

Is this formula SAT?

$B^{-1}(F) = x = y \wedge x < y \wedge a_1 \wedge a_2 \wedge \ldots \wedge a_{2019}$

2^{2019} UNSAT assignments containing

$x = y \wedge x < y$ but $\neg A$ prevents only one of them

Theory conflict clause

Minimal UNSAT core

- Let φ be original unsatisfiable conjunct
- Drop one atom from φ, call this φ'
- If φ' is still unsat, $\varphi := \varphi'$
- Repeat this for every atom in φ
- resulting φ is minimal unsat core of original formula

$\varphi : x = y \wedge f(x)+z = 5 \wedge f(x) \neq f(y) \wedge y \leq 3$

Drop $x = y$ from φ. Is result UNSAT?

Drop $f(x)+z = 5$. Is result UNSAT?

New formula: $\varphi : x = y \wedge f(x) \neq f(y) \wedge y \leq 3$

Drop $f(x) \neq f(y)$. Is result UNSAT?

Drop $y \leq 3$. Is result UNSAT?

So, minimal UNSAT core is $x = y \wedge f(x) \neq f(y)$

Improvement on the off-line

```
Offline-DPLL(T-formula $\varphi$)
$\varphi_P \leftarrow B(\varphi)$
while (TRUE) do
    $\mu_P$, res ← CDCL($\varphi_P$)
    if res = UNSAT then return UNSAT
    else
        T-res ← T-solve($B^{-1}(\mu_P)$)
        if T-res = SAT then return SAT
        else
            t ← B(UNSATCORE($B^{-1}(\mu_P)$))
            $\varphi_P \leftarrow \varphi_P \wedge \neg t$
```

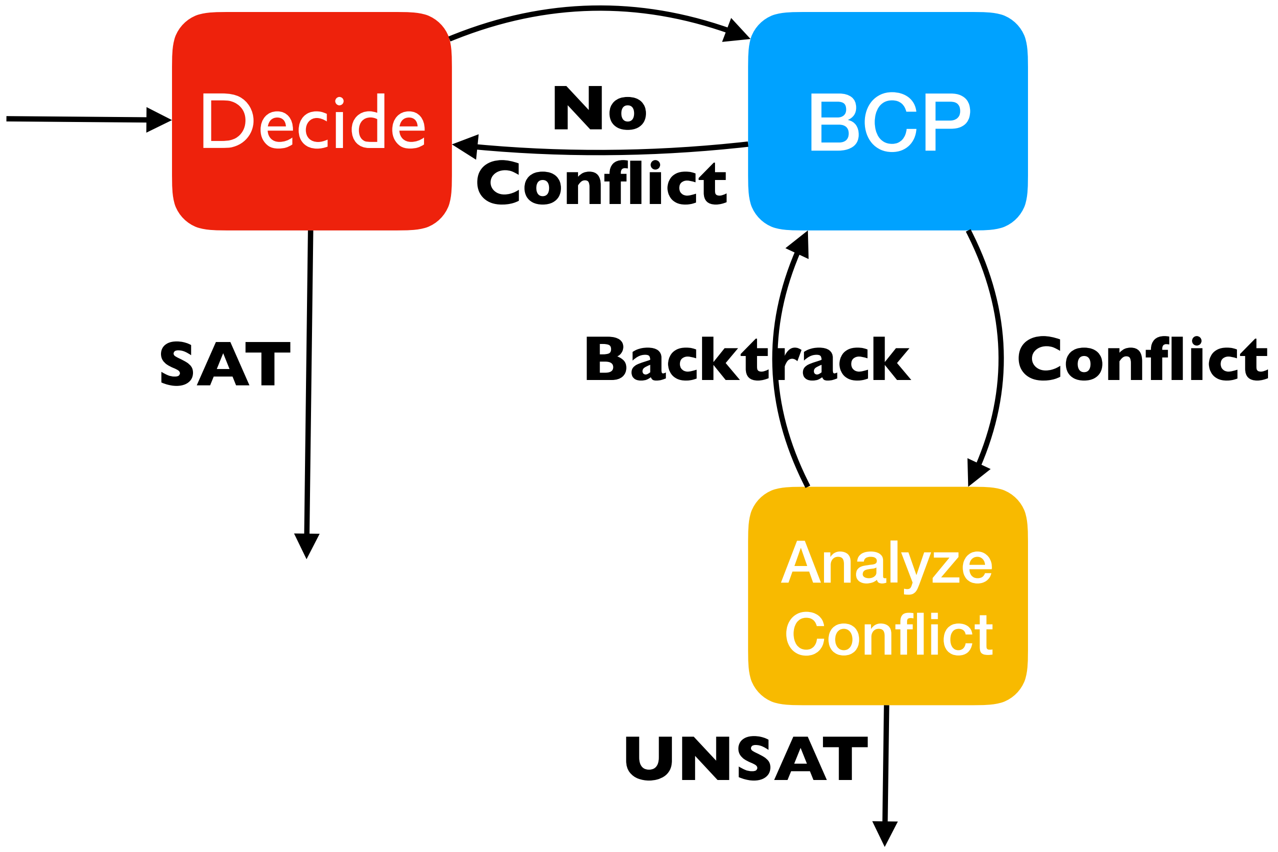
$B^{-1}(F) = x = y \wedge x < y \wedge a_1 \wedge a_2 \wedge ... \wedge a_{2019}$

$x = y$ and $x < y$ are overapproximated by boolean variables b_1 and b_2

we are doomed if both b_1 and b_2 are true.

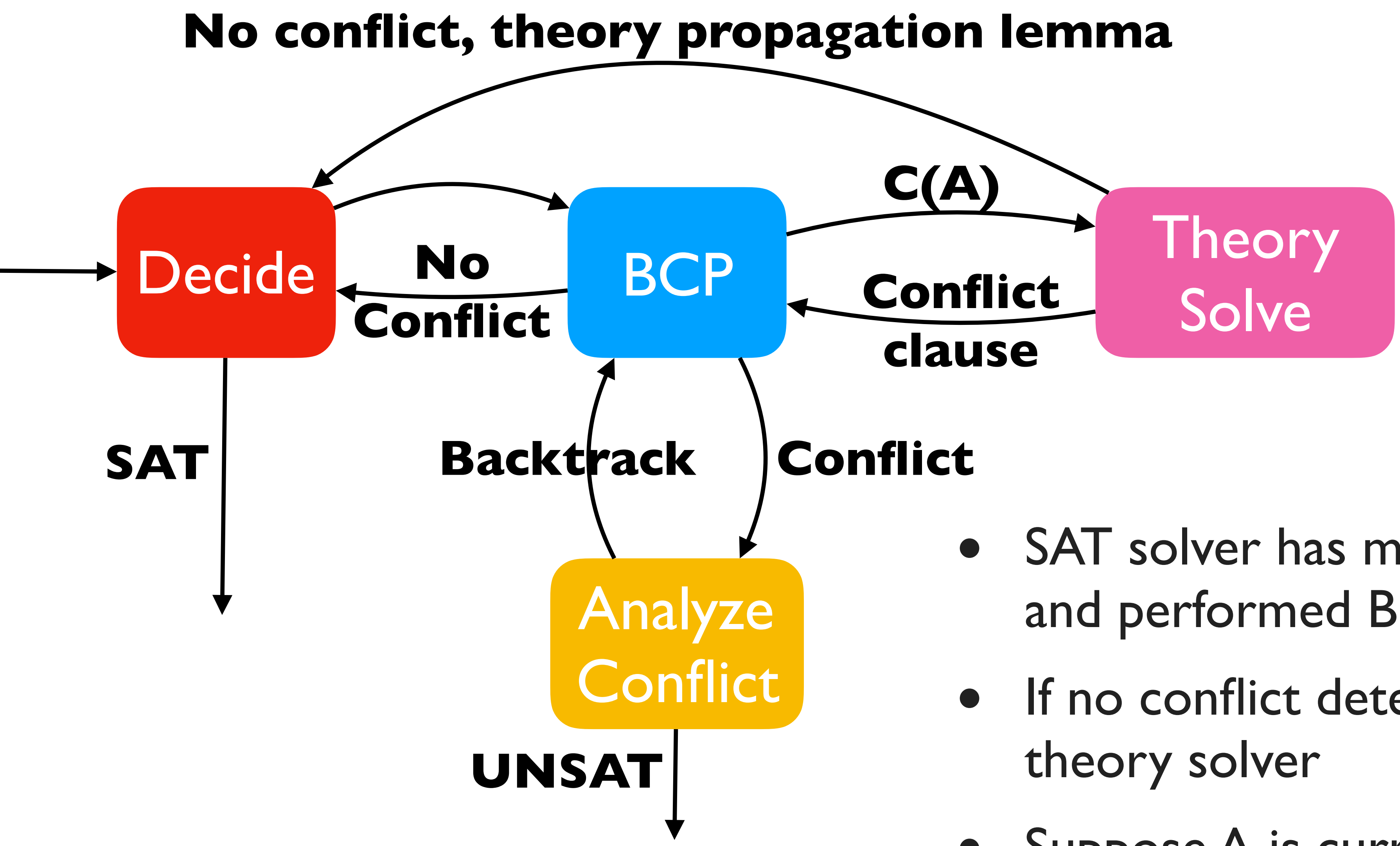
Better but still need a *full assignment* to the boolean abstraction in order to generate a conflict clause.

DPLL-based SAT solver



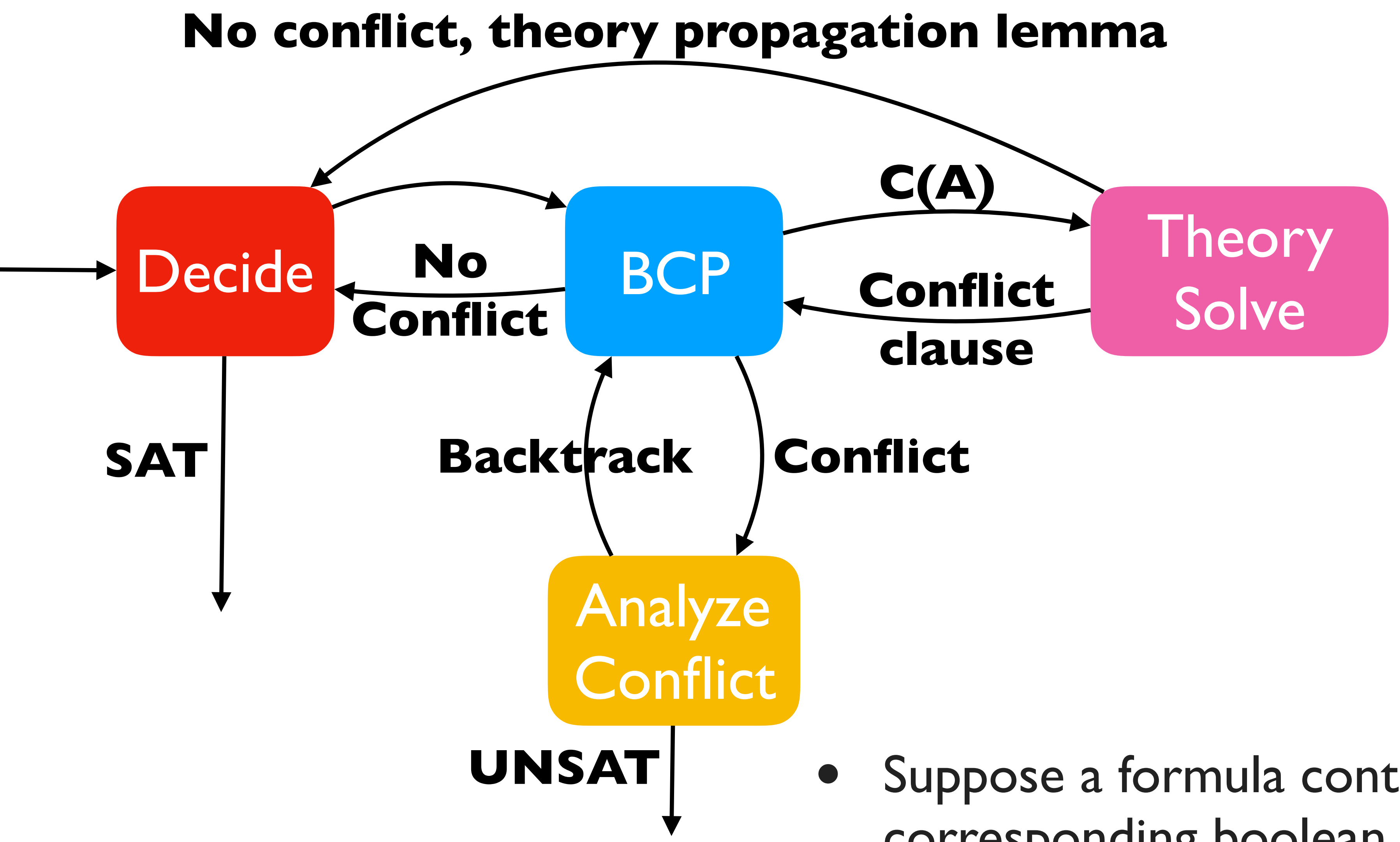
Integrate theory solver right into this SAT solving loop!

From DPLL to DPLL(T)



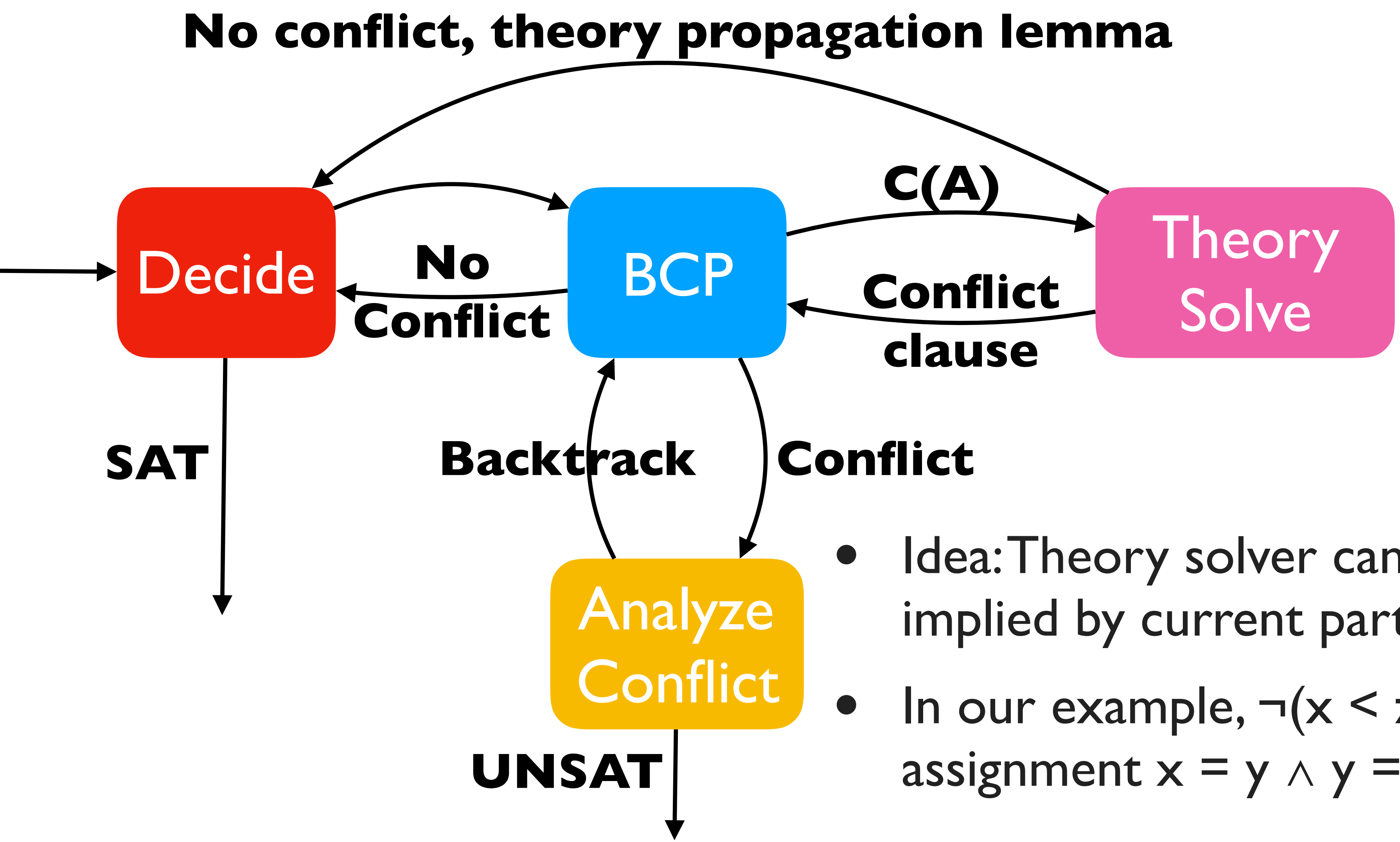
- SAT solver has made assignment in Decide step and performed BCP
- If no conflict detected, immediately invoke theory solver
- Suppose A is current partial assignment to boolean abstraction
- Use theory solver to decide if $B^{-1}(A)$ is UNSAT
- If $B^{-1}(A)$ UNSAT, add theory conflict clause ¬A to clause database

Theory Propagation Lemmas



- Suppose a formula contains x = y, y = z, x < z with corresponding boolean variables b_1, b_2, b_3
- Suppose SAT solver makes partial assignment b_1: $\top$, b_2: $\top$
- In next Decide step, free to assign b_3: $\top$ or b_3: $\bot$
- But assignment b_3: $\top$ is stupid, why?

Theory Propagation Lemmas



- Idea: Theory solver can communicate which literals are implied by current partial assignment
- In our example, ¬(x < z) implied by current partial assignment x = y ∧ y = z
- Thus, safely add $b_1 \wedge b_2 \rightarrow \neg b_3$ to clause database
- The clauses implied by theory are theory propagation lemmas

The lemmas prevents bad assignments to boolean abstraction

Theory Propagation Lemmas

- Which theory propagation lemmas do we add?
  - Option #1 (exhaustive): Figure out and add all literals implied by current partial assignment
  - Option #2 (heuristics): Only figure out literals "obviously" implied by current partial assignment
- Exhaustive theory propagation can be very expensive
- There isn't much of a science behind which literals are "obviously" implied
- Solvers use different heuristics to obtain simple-to-find implications

Modern SMT solvers

- All competitive SMT solvers today are based on the on-line version
- Many existing off-the-shelf SMT solvers: Z3, CVC4, Yices, MathSAT, etc.
- Lots of on-going research on SMT, esp. related to quantifier support
- Annual competition SMT-COM

TODOs by next lecture

- Start to work on your final report