Online Bidding in Repeated Non-Truthful Auctions under Budget and ROI Constraints

Matteo Castiglioni * 1 Andrea Celli * 2 Christian Kroer 3

Abstract

Online advertising platforms typically use auction mechanisms to allocate ad placements. Advertisers participate in a series of repeated auctions, and must select bids that will maximize their overall rewards while adhering to certain constraints. We focus on the scenario in which the advertiser has budget and return-oninvestment (ROI) constraints. We investigate the problem of budget- and ROI-constrained bidding in repeated non-truthful auctions, such as firstprice auctions, and present a best-of-both-worlds framework with no-regret guarantees under both stochastic and adversarial inputs. By utilizing the notion of *interval regret*, we demonstrate that our framework does not require knowledge of specific parameters of the problem which could be difficult to determine in practice. Our proof techniques can be applied to both the adversarial and stochastic cases with minimal modifications, thereby providing a unified perspective on the two problems. In the adversarial setting, we also show that it is possible to loosen the traditional requirement of having a strictly feasible solution to the offline optimization problem at each round.

1. Introduction

Automatic bidding systems are critical for online advertising as they allow advertisers to efficiently manage and optimize their ad campaigns. These mechanisms automatically adjust bid prices for ad placements based on real-time data and performance metrics. This allows advertisers to reach their target audience, while simplifying the interaction with the platform. In particular, advertisers usually have to specify some parameters like the overall budget for the campaign and their targeting criteria.

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Then a *proxy bidder* operated by the platform places bids on the advertiser's behalf. A popular autobidding strategy is value maximization subject to a set of constraints encoding the target metrics of the campaign. Typically such constraints include *budget constraints* reflecting the advertisers' limited spending capabilities (Agarwal et al., 2014b; Conitzer et al., 2021; Balseiro et al., 2021a), and *return-on-investment* (ROI) constraints, which enforce a target return on the capital spent (Golrezaei et al., 2021b; Auerbach et al., 2008).

Recently, many advertising platforms have been transitioning from the second-price auction format toward a first-price format because of its simplicity and favorable properties (Akbarpour & Li, 2018; Paes Leme et al., 2020; Despotakis et al., 2021). This is the case, for example, for Google's Ad Manager and AdSense platforms (Bigler, 2019; Wong, 2021). While second-price auctions are a truthful mechanism, meaning that bidders can bid their true value and maximize their utility, this is not the case for first-price auctions. Much of the existing online learning literature leverages this truthfulness in their analysis. The widespread adoption of the first-price format leads us to the motivating question of the present paper:

What is an appropriate online bidding strategy for a bidder in a series of non-truthful auctions, considering budget and ROI constraints?

Surprisingly, it is unclear what the answer to this question is, as current approaches either concentrate on second-price auctions and heavily rely on truthfulness (Balseiro & Gur, 2019; Golrezaei et al., 2021a; Feng et al., 2022), or require an unrealistic amount of prior information to be available to the bidder (Castiglioni et al., 2022b).

Contributions. At each round t up to the time horizon T, the bidder observes the type of the item being auctioned at t, and has to submit a bid b_t to the auctioneer. Then, they receive reward $f_t(b_t) \in [0,1]$ and a cost $c_t(b_t) \in [0,1]$. Our framework does not require any particular assumption on the structure of f_t and c_t , so our results apply more

^{*}Equal contribution ¹Politecnico di Milano ²Bocconi University ³Columbia University. Correspondence to: Andrea Celli <andrea.celli2@unibocconi.it>.

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broadly to any single-parameter mechanism with a finite number of types (see Section 3). We present a best-of-bothworlds primal-dual framework with sublinear regret in the stochastic setting (i.e., (f_t, c_t) are i.i.d. samples from a fixed but unknown distribution), and constant-factor competitive ratio in the adversarial setting (i.e., (f_t, c_t) are selected by an oblivious adversary). In both settings, our framework guarantees vanishing cumulative ROI constraint violation, and cumulative expenditure less then or equal to the available budget. Unlike previous work following a similar primal-dual paradigm (see, e.g., Immorlica et al. (2022); Balseiro et al. (2020)), our analysis is based on the notion on interval regret (Hazan & Seshadhri, 2007). Our approach, which involves ensuring that primal and dual regret minimizers are weakly adaptive, improves upon previous work in three main directions: i) in the adversarial setting, Castiglioni et al. (2022b) assume the existence of a strictly feasible solution guaranteeing that, for each round t, all constraints are satisfied by a margin of at least $\alpha > 0$. Moreover, they also assume that this parameter is known beforehand to the bidder. Our framework eliminates the need for the latter assumption, and significantly relaxes the former; ii) it focuses on strict budget constraint satisfaction as opposed to "soft" constraints; iii) best-of-both-worlds algorithm for problems with long-term constraints typically require different proof techniques for the two different input models (see, e.g., Castiglioni et al. (2022a)). Our new approach unifies most of the analysis, with the only difference being in the characterization of a particular set of policies.

2. Related Work

The problem of online bidding in repeated auctions has been extensively studied using online learning approaches (see, e.g., Borgs et al. (2007); Weed et al. (2016); Nedelec et al. (2022)). In particular, online bidding under resource-consumption (i.e., packing) constraints has been studied in various settings. Balseiro & Gur (2019) and Ai et al. (2022) focus on utility-maximizing agents with one resource-consumption constraint. In the context of online allocation problems with an arbitrary number of constraints, Balseiro et al. (2020; 2022) propose a class of primal-dual algorithms attaining asymptotically optimal performance in the stochastic and adversarial case. In their setting, at each round, the input (f_t, c_t) is observed by the decision maker before they make a decision. This makes the problem substantially different from ours. In particular, their framework cannot handle non-truthful repeated auctions. Recent works have also examined settings similar to ours, involving bidders with constraints on their budget and ROI. Golrezaei et al. (2021a) propose a threshold-based framework that can manage "soft" budget and ROI constraints in repeated second-price auctions under a stochastic input model. The framework by Feng et al. (2022) can handle "hard" budget constraints, but crucially relies on truthfulness of second-price auctions, and on the stochasticity of the environment. Finally, the framework by Castiglioni et al. (2022b) allows for general "soft" constraints under both stochastic and adversarial inputs. Their framework cannot be applied in our setting for three reasons: i) we have hard budget constraints, ii) we don't make the stringent assumption of knowing the parameter α beforehand, and iii) we relax the assumption of having one strictly feasible solution for each round in the adversarial setting.

Another related line of works is the one studying the Bandits with Knapsacks (BwK) framework, which was introduced and optimally solved by Badanidiyuru et al. (2018). Other regret-optimal algorithms for Stochastic BwK have been proposed by Agrawal & Devanur (2019), and by Immorlica et al. (2022). The BwK framework has been subsequently extended along many directions e.g., Agrawal & Devanur (2019); Dudik et al. (2011);Badanidiyuru et al. (2014);Agarwal et al. (2014a); Agrawal et al. (2016); Sankararaman & Slivkins (2018)). The Adversarial Bandits with Knapsacks setting was first studied by Immorlica et al. (2022), who proved a $O(m \log T)$ competitive ratio. Recently, Kesselheim & Singla (2020) refined that analysis to obtain an $O(\log m \log T)$ competitive ratio for the adversarial setting, and Castiglioni et al. (2022a) proved a constant-factor competitive ratio in the regime $B = \Omega(T)$. We mention that further results have been obtained in the simplified setting with one constrained resource (Rangi et al., 2018; György et al., 2007; Tran-Thanh et al., 2010; 2012).

Another line of related work concerns online convex optimization with time-varying constraints (see, e.g., Mahdavi et al. (2012; 2013); Jenatton et al. (2016); Neely & Yu (2017); Chen & Giannakis (2018)), where it is usually assumed that the action set is a convex subset of \mathbb{R}^m , in each round rewards (resp., costs) are concave (resp., convex), and most importantly, resource constraints only apply at the last round. In contrast, in our setting, budget constraints apply in all rounds. Moreover, guarantees are usually provided either for stochastic constraints (Yu et al., 2017; Wei et al., 2020), or for adversarial constraints (Mannor et al., 2009; Sun et al., 2017; Liakopoulos et al., 2019), typically by employing looser notions of regret. In constrast, our framework will provide best-of-both-worlds guarantees.

3. Preliminaries

The set $\{1, \ldots, n\}$, with $n \in \mathbb{N}$, is compactly denoted as [n], and we let [0] be equal to the empty set. Moreover, given a discrete set \mathcal{X} , we denote by $\Delta_{\mathcal{X}}$ the $|\mathcal{X}|$ -simplex.

3.1. Model of the Repeated Interaction

We consider the problem faced by a bidder that takes part in a sequence of auctions. At each round $t \in [T]$, the bidder observes their valuation v_t extracted from a finite set $\mathcal{V} \subset [0,1]$ of n possible valuations. In ad auctions this models the fact that the auctioneer shares with the advertisers some targeting information about users. Then, the bidder chooses $b_t \in \mathcal{B}$, where $\mathcal{B} \subset [0,1]$ is a finite set of m possible bids. Then, the bidder observes utility $f_t(b_t)$ with $f_t: \mathcal{B} \to [0,1]$, and incurs a cost $c_t(b_t)$, with $c_t: \mathcal{B} \to [0,1]$. We denote as \mathcal{F} , respectively \mathcal{C} , the set of all the possible functions f_t , respectively c_t (e.g., \mathcal{F} and C may contain all the Lipschitz-continuous functions defined over \mathcal{B}). We assume that functions in \mathcal{F} and \mathcal{C} are measurable with respect to probability measures over \mathcal{B} . This ensures that expectations are well-defined, since the functions are assumed to be bounded above, and they are therefore integrable. Moreover, we assume the existence of a *void action* \varnothing such that, for any pair $(f, c) \in \mathcal{F} \times \mathcal{C}$, $f(\emptyset) = c(\emptyset) = 0$. This can be achieved in many common settings by just bidding 0.

The bidder has an overall budget $B \in \mathbb{R}_+$, which limits the total expenditure throughout the T rounds. We denote by $\rho>0$ the *per-iteration budget* defined as B/T. Moreover, the bidder has a target *return-on-investments* (ROI) $\iota>0$. In order to simplify the notation, throughout the paper we will assume $\iota\coloneqq 1$. This comes without loss of generality: whenever $\iota>1$ we can suitably scale down values of reward functions f_t . Then, the bidder has the goal of maximizing their cumulative utility $\sum_{t=1}^T f_t(b_t)$, subject to the following constraints:

- Budget constraints: $\sum_{t=1}^{T} c_t(b_t) \leq \rho T$. Such constraints should be satisfied "no matter what", so we refer to them as *hard* constraints.
- ROI constraints: $\sum_{t=1}^{T} (c_t(b_t) f_t(b_t)) \le 0$. We say ROI constraints are *soft* meaning that we allow, in expectation, for a small (*i.e.*, vanishing in the limit) cumulative violation across the T rounds.

Our model can be easily instantiated to describe various single-parameter mechanisms with finite types beyond the well-studied case of second-price auctions.

Example 3.1 (First-price auctions). Let β_t be the highest-competing bid at time t. The utility function of the bidder at time t is $f_t(b_t) := (v_t - \omega b_t) \mathbb{1}[b_t \geq \beta_t]$, and the cost function is $c_t(b_t) := b_t \mathbb{1}[b_t \geq \beta_t]$, where the indicator function $\mathbb{1}[b_t \geq \beta_t]$ specifies whether the bidder won the auction at time t. The parameter $\omega \in [0,1]$ represents the bidder's private capital cost which normalizes the bidder's accumulated valuation with the overall expenditure. This model includes the traditional quasi-linear set-up (i.e., $\omega = 1$), as well as the value-maximizing utility model (i.e., $\omega = 0$) (see, e.g., Babaioff et al. (2021); Balseiro et al. (2021b)).

Example 3.2 (Generalized second- and first-price auctions). At each time t the bidder can be assigned one of s available slots. In the traditional model of separable click probabilities (see, e.g., Edelman et al. (2007); Varian (2007)), each slot $j \in [s]$ is associated with a click-through rate r_j . At each t, if the bidder does not win any slot then $f_t(b_t) = c_t(b_t) = 0$ for any $b_t \in \mathcal{B}$. If at time t the bidder is assigned slot j, then their utility function is going to be $f_t : b_t \mapsto r_j v_t - c_t(b_t)$, where $c_t(b_t)$ is equal to b_t in the case of generalized first-price auctions, and to the highest bid which is less than or equal to b_t in the case of generalized second-price auctions.

3.2. Regret Minimization

A regret minimizer (RM) for a set \mathcal{X} is an abstract model for a decision maker repeatedly interacting with a blackbox environment. At each t, a RM performs two operations: (i) NEXTELEMENT(), which outputs an element $x_t \in \mathcal{X}$; and (ii) OBSERVEUTILITY(·), which updates the internal state of the RM using the feedback received from the environment. This is either a utility function $\ell_t: \mathcal{X} \to [a,b] \subseteq \mathbb{R}$ (full feedback), or only the value $\ell_t(x_t)$ (bandit feedback), with ℓ_t possibly depending adversarially on x_1,\ldots,x_{t-1} . The objective of the RM is to output a sequence $(x_t)_{t=1}^T$ of points in \mathcal{X} so that its cumulative regret, defined as $\sup_{x \in \mathcal{X}} \sum_{t=1}^T (\ell_t(x) - \ell_t(x_t))$, grows asymptotically sublinearly in T. See Cesa-Bianchi & Lugosi (2006) for a review of the various RM available in the literature.

4. Baselines

Let Π be the set of randomized bidding policies. In particular, each policy is $\pi \in \Pi$ is a mapping $\pi : \mathcal{V} \to \Delta_{\mathcal{B}}$. We denote by $\pi(v)_b$ the probability of selecting b under valuation v. Given valuation $v \in \mathcal{V}$, reward function f, and a cost function c, let $g: \Pi \ni \pi \mapsto \mathbb{E}_{\pi}[c(b)] - \rho$ be the expected gap between the cost for policy π and the periteration budget ρ , and $h: \Pi \ni \pi \mapsto \mathbb{E}_{\pi}[c(b) - f(b)]$ be the expected ROI constraint violation for policy π . We will denote by g_t , resp. h_t , the constraints defined for the pair (f_t, c_t) observed at round t. The bid b_t selected for round t is drawn from $\pi_t(v_t)$. Moreover, we will denote by $f_t(b)$ (equivalently, we will write $g_t(b), h_t(b)$) the value of the reward function under the deterministic bidding policy always playing $b \in \mathcal{B}$ for each valuation.

Auxiliary LP. Let \mathcal{P} be an arbitrary probability measure over the space of possible inputs $\mathcal{F} \times \mathcal{C}$. Then, we define the linear program $\mathbb{LP}_{\mathcal{P}}$ as follows:

$$\mathsf{OPT}_{\mathcal{P}} \coloneqq \begin{cases} \sup_{\pi \in \Pi} & \mathbb{E}_{f \sim \mathcal{P}} f(\pi) \\ \text{s.t.} & \mathbb{E}_{\mathcal{P}} g(\pi) \le 0 \\ & \mathbb{E}_{\mathcal{P}} h(\pi) \le 0 \end{cases}$$
 (LP_{\mathcal{P}})

The above LP selects the bidding policy π that maximizes expected reward according to \mathcal{P} , while guaranteeing that constraints g and h encoded by \mathcal{P} are satisfied in expectation (both g and h are defined by $(f,c) \sim \mathcal{P}$).

The Lagrangian function $\mathcal{L}_{\mathcal{P}}: \Pi \times \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ of $\mathbb{LP}_{\mathcal{P}}$ is defined as $\mathcal{L}_{\mathcal{P}}(\pi, \lambda, \mu) := \mathbb{E}_{(f,c) \sim \mathcal{P}}[f(\pi) - \lambda g(\pi) - \mu h(\pi)].$

Baselines. Our goal is designing online algorithms that output a sequence of policies π_1,\ldots,π_T such that i) the cumulative regret with respect to the performance of the baseline grows sublinearly in T, ii) the budget constraint is (deterministically) satisfied, i.e., $\sum_{t=1}^T c_t(b_t) \leq B$, and iii) the cumulative ROI constraint violation $\sum_{t=1}^T h_t(\pi_t)$ grows sublinearly in the number of rounds T. The cumulative regret of the algorithm is defined as $R^T \coloneqq T \text{ OPT} - \sum_{t=1}^T f_t(b_t)$, where the baseline OPT depends on how the input sequence $\gamma \coloneqq (f_t, c_t)_{t=1}^T$ is generated. We consider two settings for which we define an appropriate value of the baseline, and a suitable problem-specific parameter $\alpha \in \mathbb{R}$ which is related to the feasibility of the offline optimization problem:

- Stochastic setting: at each $t \in [T]$, the pair (f_t, c_t) is independently drawn according to a fixed but unknown distribution \mathcal{P} over $\mathcal{F} \times \mathcal{C}$. The baseline is $\mathsf{OPT}_{\mathcal{P}}$, which is the standard baseline for stochastic BwK problems since its value is guaranteed to be closed to that of the best dynamic policy (Badanidiyuru et al., 2018, Lemma 3.1). In this setting, let $\alpha := -\inf_{\pi \in \Pi} \max\{\mathbb{E}_{\mathcal{P}}g(\pi), \mathbb{E}_{\mathcal{P}}h(\pi)\}$.
- Adversarial setting: the sequence of inputs γ is selected by an oblivious adversary. Given γ , we define the following distribution over inputs: for any pair $(f,c) \in \mathcal{F} \times \mathcal{C}, \bar{\gamma}[f,c] = \sum_{t=1}^T \mathbb{1}[f_t = f,c_t = c]/T.$ Then, the baseline is the solution of $\mathrm{LP}_{\bar{\gamma}}$ (i.e., $\mathrm{OPT}_{\bar{\gamma}}$), which is the standard baseline for the adversarial setting (see, e.g., Balseiro et al. (2022); Immorlica et al. (2022)). Therefore, the baseline is obtained by solving the offline problem initialized with the average of the realization observed over the T rounds. Moreover, our results will also hold with respect to the best unconstrained policy. We define α as $\alpha := -\inf_{\pi} \max_{t \in [T]} \max\{g_t(\pi), h_t(\pi)\}$. In this setting, α represents the "worst-case feasiblity" with respect to functions observed up to T.

We start by developing our analysis under the following standard assumption on the existence of a "safe" policy. In Section 8, we show how this requirement can be relaxed.

Assumption 4.1. In the adversarial (resp., stochastic) setting, γ (resp., \mathcal{P}) is such that $\alpha > 0$.

This means that $LP_{\mathcal{P}}$ and $LP_{\bar{\gamma}}$ satisfy (stochastic) Slater's condition. In particular, in the adversarial setting we are

requiring the existence of a randomized policy that, in expectation, strictly satisfies the constraints for each t. This is a frequent assumption in works focusing on settings similar to ours (see, e.g., Chen et al. (2017); Neely & Yu (2017); Yi et al. (2020); Castiglioni et al. (2022b)). Section 8 shows how this assumption can be relaxed through our framework.

When studying primal-dual algorithms, a key implication of Slater's condition is the existence and boundedness of Lagrange multipliers (see, e.g., Nedić & Ozdaglar (2009)). Therefore, when $\alpha > 0$ is known, the set of dual multipliers can be safely bounded by requiring the ℓ_1 -norm of the multiplier to be less than or equal to $1/\alpha$ (see, e.g., Balseiro et al. (2022)). This is the case, for example, for problems with only budget constraints, in which $\alpha = \rho > 0$, which is achieved by bidding the void action \(\times \) at each round. However, ROI constraints complicate the problem as the bidder does *not* know α beforehand. Therefore, previous frameworks like the one by Castiglioni et al. (2022b) cannot be applied in this setting, as Lagrange multipliers cannot be easily bounded without knowledge of α . This begets the question of how to keep the size of dual multipliers under control, even without knowing α .

5. Primal-Dual Framework

Our framework assumes access to two regret minimizers with the following characteristics. The first one, which we denote by $\mathcal{R}^{\mathbb{P}}$, is the *primal regret minimizer* which outputs policies in Π , and receives as feedback the loss $\ell^{\mathbb{P}}_{t}(b_{t}) \coloneqq -f_{t}(b_{t}) + \lambda_{t}g_{t}(b_{t}) + \mu_{t}h_{t}(b_{t}) + 1 + \lambda_{t} + \mu_{t}$, with b_{t} sampled according to the bidding policy computed. Therefore, the primal regret minimizer has to work under bandit feedback. The primal loss function is obtained from the Lagrangian relaxation of the problem at time t, plus the additive term $1 + \lambda_{t} + \mu_{t}$ to ensure $\ell^{\mathbb{P}}_{t}(\cdot) \in \mathbb{R}_{+}$. The second regret minimizer, which we denote by $\mathcal{R}^{\mathbb{D}}$, is the dual regret minimizer. It outputs points in the space of dual variables $\mathbb{R}^{2}_{\geq 0}$, and receives as feedback the linear utility $u^{\mathbb{D}}_{t} : \mathbb{R}^{2}_{\geq 0} \to \mathbb{R}$ such that, for each $(\lambda, \mu) \in \mathbb{R}^{2}_{\geq 0}$, $u^{\mathbb{D}}_{t}(\lambda, \mu) := \lambda g_{t}(b_{t}) + \mu h_{t}(b_{t})$. The dual regret minimizer $\mathcal{R}^{\mathbb{D}}$ has full feedback by construction.

In order to prove our results, our framework requires $\mathcal{R}^{\mathbb{P}}$ and $\mathcal{R}^{\mathbb{D}}$ to be *weakly adaptive*, that is, they should guarantee sublinear *adaptive* (a.k.a. interval) regret (see, e.g., , Hazan & Seshadhri (2007); Luo et al. (2018)). This notion of regret is stronger than "standard" external regret (see Hazan et al. (2016, Chapter 10)), and it will be essential in our analysis. Let $t_1, t_2 \in [T], t_1 \leq t_2$. We denote by $\mathcal{I} \coloneqq [t_1, t_2]$ the set $\{t_1, t_1 + 1, \dots, t_2\}$, and we call \mathcal{I} the *time interval* starting from round t_1 to round t_2 . The primal regret minimizer must be such that, for $\delta \in (0, 1]$, with probability at least $1 - \delta$ it holds that, for any $\pi \in \Pi$ and

for any interval \mathcal{I} ,

$$\sum_{t \in \mathcal{I}} \left(\ell_t^{\mathbb{P}}(b_t) - \ell_t^{\mathbb{P}}(\pi(v_t)) \right) \le M_{\mathcal{I}}^2 \mathcal{E}_{T,\delta}^{\mathbb{P}}, \tag{1}$$

where $M_{\mathcal{I}}$ is the maximum absolute value of the losses $\ell^{\mathbb{P}}_t$ observed in interval \mathcal{I} , and $\mathcal{E}^{\mathbb{P}}_{T,\delta}$ is a term of order $\tilde{O}(\sqrt{T})$ (see Theorem 5.2). We require a similar property for the dual regret minimizer. However, since the dual regret minimizer works under full-information feedback we can use a regret minimizer with deterministic regret guarantees. In particular, \mathcal{R}^{D} should guarantee that, for any time interval $\mathcal{I} = [t_1, t_2]$, and for any pair of dual variables $(\lambda,\mu) \in \mathbb{R}^2_{>0}$ it holds

$$\sum_{t \in \mathcal{T}} u_t^{\mathrm{D}}(\lambda, \mu) - u_t^{\mathrm{D}}(\lambda_t, \mu_t) \leq \nu(T) (\mu - \mu_{t_1})^2 + \mathcal{E}_T^{\mathrm{D,B}} + \mathcal{E}_T^{\mathrm{D,R}},$$

where $\nu(T) \geq 0$ is such that $\nu(T) = o(T)$, and $\mathcal{E}_T^{\mathrm{D,B}}$ (resp., $\mathcal{E}_T^{\mathbb{D},\mathbb{R}}$) is a term sublinear in T related to the budget (resp., ROI) constraint. Sections 5.1 and 5.2 will provide precise bounds for our setting, and show that we can build efficient primal and dual regret minimizers with the required features.

Algorithm 1 summarizes the structure of our primaldual framework. For each t, the algorithm first computes primal and dual actions at time t by invoking $\mathcal{R}^{\mathbb{P}}$.NEXTELEMENT() and $\mathcal{R}^{\mathbb{D}}$.NEXTELEMENT(). The bid at time t is going to be $b_t \sim \pi_t(v_t)$ unless the available budget B_t is less than 1, in which case we set b_t equal to the void action \varnothing . Then, $\ell_t^{\mathbb{P}}(b_t)$ and $u_t^{\mathbb{D}}$ are observed, and the budget consumption is updated according to the realized cost c_t . Finally, the internal state of the two regret minimizer is updated on the basis of the last value observed v_t , and the feedback specified by $\ell_t^{\mathbb{P}}, u_t^{\mathbb{D}}$ (see the invocation of OBSERVEUTILITY()). The algorithm terminates when the time horizon T is reached. We will denote by $\tau \in [T]$ the time in which the budget is fully depleted and the bidder starts playing the void action \emptyset .

5.1. Design of the Primal Regret Minimizer

In order to provide a suitable primal regret minimizer for Algorithm 1, we start by proving some useful properties of the EXP3-SIX algorithm by Neu (2015) in the simpler setting of multi-armed bandits (i.e., when the number of possible valuations is n = 1). In particular, we derive guarantees characterizing the interval regret of the algorithm. As a byproduct of this, Corollary 5.3 provides the guarantee for the case of multiple possible valuations (i.e., n > 1). The resulting primal regret minimizer is described in Algorithm 2. At each round t, the algorithm maintains a set of weights $\boldsymbol{w}_t \in [0,1]^{n,m}$. The probability of playing b under valuation v_t is proportional to the weight $w_{t,v_t,b}$. After drawing b_t , the primal regret minimizer observes $\ell_t^{\mathbb{P}}(b_t)$ and builds

Algorithm 1 Primal-Dual framework.

Input: parameters B, T, δ ; regret minimizers $\mathcal{R}^{\mathbb{P}}$ and

Initialization: $B_1 \leftarrow B$; initialize $\mathcal{R}^{\mathbb{P}}, \mathcal{R}^{\mathbb{D}}$

for t = 1, 2, ..., T do

Dual decision: $(\lambda_t, \mu_t) \leftarrow \mathcal{R}^{\text{D}}.\text{NEXTELEMENT}()$ **Primal decision:** $\Pi \ni \pi_t \leftarrow \mathcal{R}^{P}$.NEXTELEMENT()

Observe valuation v_t and **select bid** as

$$b_t \leftarrow \begin{cases} b_t \sim \pi_t(v_t) & \text{if } B_t \ge 1\\ \varnothing & \text{otherwise} \end{cases}.$$

Observe cost: observe $c_t(b_t)$ and update available resources: $B_{t+1} \leftarrow B_t - c_t(b_t)$

Primal update:

- $\ell_t^{\mathbb{P}}(b_t) \leftarrow -f_t(b_t) + \lambda_t g_t(b_t) + \mu_t h_t(b_t) + 1 + \lambda_t + \mu_t$
- $\mathcal{R}^{\mathbb{P}}$.OBSERVEUTILITY $(\ell_t^{\mathbb{P}}(b_t), v_t)$

Dual update:

- $\begin{array}{l} \bullet \ u_t^{\mathrm{D}} : \mathbb{R}^2_{\geq 0} \ni (\lambda, \mu) \mapsto \lambda g_t(b_t) + \mu h_t(b_t) \\ \bullet \ \mathcal{R}^{\mathrm{D}}.\mathsf{OBSERVEUTILITY}(u_t^{\mathrm{D}}) \end{array}$

end for

the estimated loss $\tilde{\ell}_t^{\mathbb{P}}$, where $\xi > 0$ is the implicit exploration term. Then, the update of weights w is inspired by the Fixed Share algorithm by Herbster & Warmuth (1998).

In order to proceed with the analysis, let $p_{t+1} \in [0,1]^m$ be the vector of *pre-weights* for time t+1, which is defined as

$$p_{t+1,b} \coloneqq \frac{\pi_t(v_t)_b \, e^{-\eta \hat{\ell}_t^{\tilde{\nu}}(b)}}{\sum_{b' \in \mathcal{B}} \pi_t(v_t)_{b'} \, e^{-\eta \hat{\ell}_t^{\tilde{\nu}}(b')}} \qquad \text{for all } b \in \mathcal{B}.$$

Then, we have the following intermediate result (all the omitted proofs can be found in the appendix).

Lemma 5.1. Let $\eta > 0$ be s.t. $\eta \mathbb{E}_{\pi} \tilde{\ell}_{t}^{\mathbb{P}}(b) < 1$ for all $t \in [T]$ and $\pi \in \Pi$. Then, for any $t \in [T]$, and $b' \in \mathcal{B}$, it holds

$$\mathbb{E}_{\pi_t(v_t)} \Big[\tilde{\ell}_t^{\mathbb{P}}(b) \Big] - \tilde{\ell}_t^{\mathbb{P}}(b') \le \frac{1}{\eta} \log \left(\frac{p_{t+1,b'}}{\pi_t(v_t)_{b'}} \right) + \frac{\eta}{2} \mathbb{E} \Big[\tilde{\ell}_t^{\mathbb{P}}(b)^2 \Big].$$

The above lemma allows us to prove the following regret guarantees when focusing on a single valuation.

Theorem 5.2. Consider the case in which the set of valuations is a singleton (i.e., n=1). Let $\eta := 1/\sqrt{mT}$, $\xi := 1/(2\sqrt{mT}), \ \sigma := 1/T, \ and \ assume \ that \ \eta \le 1/M_{[T]}.$ Then, for any $\delta > 0$, EXP3-SIX guarantees that, with probability at least $1 - \delta$, for any interval $\mathcal{I} = [t_1, t_2]$, and for any bid $b \in \mathcal{B}$,

$$\sum_{t \in \mathcal{T}} \left(\ell_t^{\scriptscriptstyle P}(b_t) - \ell_t^{\scriptscriptstyle P}(b) \right) \le M_{\mathcal{I}}^2 \mathcal{E}_{T,\delta}^{\scriptscriptstyle P},$$

$$\mathcal{E}_{T,\delta}^{\mathbb{P}} := \left(\frac{3}{2} + \frac{4}{M_{\mathcal{T}}} \log \left(\frac{mT}{\delta}\right) + \frac{\log(T) + 1}{M_{\mathcal{T}}^2}\right) \sqrt{mT}.$$

Algorithm 2 Primal regret minimizer.

Input: parameters $\eta > 0, \xi > 0, \sigma > 0$ **Initialization:** $[0,1]^{n \times m} \ni \boldsymbol{w}_1 \leftarrow \boldsymbol{1}$

for t = 1, 2, ..., T do

• **Observe** valuation $v_t \in \mathcal{V}$

- Set $\pi(v_t)_b \leftarrow w_{t,v_t,b} / \sum_{b' \in \mathcal{B}} w_{t,v_t,b'}, \ \forall b \in \mathcal{B}$
- Bid $b_t \sim \pi$
- **Observe** loss $\ell_t^{P}(b_t)$
- $\ell_t^{\mathbb{P}}(b) \leftarrow \ell_t^{\mathbb{P}}(b) \mathbb{1}[b=b_t]/(\pi_b + \xi)$ for each $b \in \mathcal{B}$
- For each $b \in \mathcal{B}$ set

$$\begin{aligned} w_{t+1,v_t,b} \leftarrow (1-\sigma) w_{t,v_t,b} \cdot e^{-\eta \tilde{\ell}_t^{\tilde{\nu}}(b)} \\ &+ \frac{\sigma}{m} \sum_{b' \in \mathcal{B}} w_{t,v_t,b'} \cdot e^{-\eta \tilde{\ell}_t^{\tilde{\nu}}(b')} \end{aligned}$$

end for

Theorem 6.2 will show that in our framework the assumption $\eta \leq 1/M_{[T]}$ is always satisfied for our choice of η . This immediately yields the following result for the case in which one independent instance of EXP3-SIX is employed for each valuation in \mathcal{V} .

Corollary 5.3. Suppose Algorithm 2 is run with the same fixed choice of parameters and assumptions of Theorem 5.2. Then, for any time interval $\mathcal{I} = [t_1, t_2]$, the regret accumulated by the algorithm over \mathcal{I} is upper bounded by $M_{\mathcal{I}}^p \sqrt{n} \, \mathcal{E}_{T,\delta}^p$ with probability at least $1 - n\delta$.

5.2. Design of the Dual Regret Minimizer

As a dual regret minimizer we employ the standard online gradient descent algorithm (OGD) (see, e.g., Zinkevich (2003)) on each of the two Lagrangian multipliers λ and μ . We initialize the algorithm by letting $\mu_1 = \lambda_1 = 0$. We employ two separate learning rates η_B and η_R , which will be specified in Lemma 5.4. At each round, the dual regret minimizer updates the Lagrangian multipliers as $\lambda_{t+1} =$ $P_{[0,1/\rho]}(\lambda_t + \eta_B h_t(b_t)), \text{ and } \mu_{t+1} = P_{\mathbb{R}_+}[\mu_t + \eta_R h_t(b_t)]$ for each $t \in [T]$, where P denotes the projection operator. The former update performs a gradient step and then projects the result on the interval $[0, 1/\rho]$. This is possible because we know that playing the void action \infty would satisfy the budget constraints by a margin of at least ρ , and therefore we can safely consider as the set of λ the interval $[0, 1/\rho]$ (Castiglioni et al., 2022a). On the other hand, the update of μ performs a grandient step and then ensures that the value is in \mathbb{R}_+ . Since the bidder does not know the feasibility parameter of ROI constraints, bounding μ becomes more complex, and we show how to do it in Section 6.

Given a time interval $\mathcal{I} = [t_1, t_2]$, and $\delta \in [0, 1]$, we let

$$\mathcal{E}_{T,\delta}^{\mathcal{I}} := \left\{ \begin{array}{ll} 2\sqrt{(t_2-t_1)\log(2T/\delta)} & \text{if } \delta \in (0,1] \\ 0 & \text{if } \delta = 0 \end{array} \right.,$$

and, when clear from context, we drop the dependency on $\mathcal I$ to denote $\mathcal E^{[T]}_{T,\delta}$. Let $\mathcal E^{\mathtt D,\mathtt B}_T$ be a term of order $O(T^{1/2}/\rho)$, and $\mathcal E^{\mathtt D,\mathtt R}_T$ be a term of order $O(T^{1/2})$. The regret guarantees of the dual regret minimizer follow from standard results on OGD (see, *e.g.*, Hazan et al. (2016, Chapter 10)).

Lemma 5.4. Let $\lambda_1=0$ and $\mu_1=0$. Let $\eta_B\coloneqq 1/(\rho T^{1/2})$, and $\eta_R\coloneqq \left(6+T^{1/2}+\mathcal{E}^{\mathrm{D},\mathrm{B}}_T+6\mathcal{E}^{\mathrm{T}}_{T,\delta}+16\mathcal{E}^{\mathrm{P}}_{T,\delta}\right)^{-1}$. Then, online gradient descent guarantees that, for any interval $\mathcal{I}=[t_1,t_2]$, it holds

•
$$\sum_{t \in \mathcal{I}} \mu_t h_t(b_t) \leq \frac{1}{\eta_R} (\mu - \mu_{t_1})^2 + \mathcal{E}_T^{D,R} \text{ for all } \mu \in \mathbb{R}_+,$$

•
$$\sum_{t \in \mathcal{I}}^{\mathbb{D}, \mathbb{B}} \lambda_t g_t(b_t) \leq \mathcal{E}_T^{\mathbb{D}, \mathbb{B}} \text{ for all } \lambda \in R_+.$$

The dependency on δ in the construction of η_R is resolved by Algorithm 1 taking δ in input as a parameter, the final guarantees of the framework will be parametrized on δ .

Now, we prove the following simple result characterizing the growth of μ variables.

Lemma 5.5. *For all* $t_1, t_2 \in [T]$,

$$\mu_{t_2} \ge \eta_{\mathbb{R}} \sum_{t' \in [t_1, t_2 - 1]} h_{t'}(b_{t'}) + \mu_{t_1}.$$

6. Bounding the Lagrange Multipliers

Previous work on online optimization with long-term constraints usually assume to know, either exactly or via some upper bound, the parameter α characterizing a strictly feasible solution of the problem (see, e.g., Balseiro et al. (2020); Immorlica et al. (2022)). This information is then used to bound the magnitude of dual multipliers. In our setting, the decision maker has no knowledge of the gap that the strictly feasible solution guarantees for the ROI constraint, which renders the traditional approach not viable.

In this section, we show that even without a priori information on α , our framework guarantees that, with high probability, the Lagrange multiplier μ_t is bounded by $2/\alpha$ throughout the entire time horizon. We start by providing a general condition that we will prove to be satisfied both in the stocastich and adversarial setting.

Definition 6.1. Given $\delta \in (0,1]$, a policy π° is δ -safe if, for any interval $\mathcal{I} := [t_1, t_2]$, with $t_1, t_2 \in [T]$, $t_1 < t_2$, it holds

$$\sum_{t \in \mathcal{T}} \lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ) \le \left(\mu_{\mathcal{I}} + \frac{1}{\alpha}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - \alpha \sum_{t \in \mathcal{T}} \mu_t,$$

where $\mu_{\mathcal{T}}$ is the largest multiplier in the interval \mathcal{I} .

A safe policy gives to the primal regret minimizer a way to limit the realized penalties imposed by the dual regret minimizer. In particular, we can show that if the dual regret minimizer increased the value of Lagrange multipliers μ_t too much, then the primal regret minimizer could "fight back" by playing the safe policy π° , thereby preventing the dual player from being no-regret. Indeed, the next result shows that whenever there exists a safe policy the Lagrange multipliers must be bounded.

Theorem 6.2. Suppose that there exists a safe policy and that the primal regret minimizer has regret at most $M_{\mathcal{I}} \mathcal{E}_{T,\delta}^{P}$ for any time interval \mathcal{I} . Then, the Lagrange multipliers μ_{t} are such that $\mu_{t} \leq 2/\alpha$ for each $t \in [\tau]$.

Then, we show that both in the stochastic and in the adversarial setting there exists a safe policy with high probability, which implies that with high probability the Lagrange multipliers are bounded under both input models.

Lemma 6.3. Assume that inputs (f_t, c_t) are drawn i.i.d. from \mathbb{P} , and that there exists a policy π such that $\mathbb{E}_{\mathbb{P}}g(\pi) \leq -\alpha$ and $\mathbb{E}_{\mathbb{P}}h(\pi) \leq -\alpha$. Then, there exists a δ -safe policy with probability at least $1 - \delta$.

Lemma 6.4. Assume that the sequence of inputs (f_t, g_t) is selected by an oblivious adversary, and that there exists a policy π such that $g_t(\pi) \leq -\alpha$ and $h_t(\pi) \leq -\alpha$ for each $t \in [T]$. Then, there exists a δ -safe policy for any $\delta \in (0, 1]$.

7. Regret and Violations Guarantees

In this section, we describe the guarantees provided by Algorithm 1 in both the stochastic and the adversarial inputs setting. Interestingly, we prove best-of-both-worlds guarantees through a unified argument which captures both settings. This is not the case in previous work, where the analysis of the stochastic setting typically requires to study convergence to a Nash equilibrium of the *expected Lagrangian game* (see., *e.g.*, Immorlica et al. (2022); Castiglioni et al. (2022a)), which is not well defined in the adversarial setting.

We introduce the following event that holds with high probability. Most of our results will hold deterministically given this event.

Definition 7.1. We denote with **E** the event in which Algorithm 1 satisfies the following two conditions:

- the primal regret minimizer has regret upper bounded by $(3/\alpha + 1)\mathcal{E}_{T}^{P}$ for all time intervals \mathcal{I} ;
- the dual multipliers due to the ROI constraint are such that $\mu_t \leq 2/\alpha$ for each $t \in [T]$.

Then, by applying a union bound to the events of Theorem 5.2 and Lemma 6.3 or Lemma 6.4, we can invoke Theorem 6.2 to obtain the following result.

Lemma 7.2. Event **E** holds with probability at least $1-2\delta$.

We start by observing that the cumulative violation of ROI constraints must be sublinear in T with high probability, under both input models. This is a direct consequence of properties of the dual regret minimizer (Section 5.2), and the bound on dual multipliers implied by Lemma 7.2.

Lemma 7.3. Assume that event **E** holds and let τ be the round in which the budget is fully depleted. Then, it holds that $\sum_{t \in [\tau]} h_t(\pi_t) \leq 1 + 2/(\eta_R \alpha)$.

Then, we define the following class of policies.

Definition 7.4. Given $\delta \in (0,1), q \in (0,1)$, a sequence of T inputs $\{(f_t,c_t)\}_{t=1}^T$, and the value of a baseline OPT, we say that a policy π is (δ,q,OPT) -optimal, if

- $\sum_{t \in [T]} f_t(\pi) \ge q \cdot T \cdot \text{OPT} \mathcal{E}_{T,\delta}$, and
- $\sum_{t \in [t']} \lambda_t g_t(\pi) + \mu_t h_t(\pi) \leq (\mu_{[t']} + \frac{1}{\alpha}) \mathcal{E}_{T,\delta}$ for each $t' \in [T]$, where $\mu_{[t']}$ is the largest multiplier μ_t observed up to t'.

In other words, a $(\delta, q, \mathsf{OPT})$ -optimal policy guarantees a reward which is a fraction q of that of the baseline up to a sublinear term, and guarantees that the cumulative value of the penalty due to the Lagrangian relaxation of the problem is vanishing in time.

First, we need the following result that holds both in the stochastic and adversarial setting.

Lemma 7.5. Algorithm 1 guarantees that

$$\sum_{t \in [\tau]} \lambda_t g(b_t) \ge T - \tau - \frac{1}{\rho} - \mathcal{E}_T^{\scriptscriptstyle D,B}.$$

Then, we prove that the existence of a (δ,q,OPT) -optimal policy implies the following regret bound with respect to the generic baseline OPT .

Lemma 7.6. Suppose that event **E** holds and that there exists a (δ, q, OPT) -optimal policy. Then,

$$\sum_{t \in [\tau]} f_t(b_t) \ge qT \operatorname{OPT} - \mathcal{C}(T, \alpha, \delta),$$

where

$$\mathfrak{C}(T,\alpha,\delta) \coloneqq \frac{1}{\alpha} + \left(\frac{3}{\alpha} + 1\right) \left(\mathcal{E}_{T,\delta}^{\mathsf{P}} + \mathcal{E}_{T,\delta}\right) + \mathcal{E}_{T}^{\mathsf{D},\mathsf{R}} + \mathcal{E}_{T}^{\mathsf{D},\mathsf{B}}.$$

Next, we show that a suitable $(\delta, q, \mathsf{OPT})$ -optimal policy exists with high probability both under the stochastic and the adversarial input model.

Lemma 7.7. In the stochastic setting, with probability at least $1 - 2\delta$ there exists a $(\delta, 1, OPT_{\mathcal{P}})$ -optimal policy (where $OPT_{\mathcal{P}}$ is the optimal value of $LP_{\mathcal{P}}$).

This is saying that, fixed an underlying distribution \mathcal{P} , there exist with high probability a policy satisfying Definition 7.4. Stochasticity of the environment (*i.e.*, pairs (f_t, c_t)

are i.i.d. samples from \mathcal{P}) is used to prove that the solution to $LP_{\mathcal{P}}$ satisfies the second condition of Definition 7.4 for all $t \in [T]$. If we tried a similar approach in the adversarial setting, the solution to ${\tt LP}_{\bar{\gamma}}$ would guarantee that the second condition is satisfied over the whole time horizon, but not necessarily at earlier time steps t < T. Moreover, feasibility in expectation has no implications on feasibility of a policy under the adversarial sequence $(\lambda_t, f_t, \mu_t, c_t)$, in which dual variables are optimized to "punish violations". However, it is possible to show the existence of a policy satisfying Definition 7.4 even in the adversarial setting by following a different approach. In particular, we show that it is possible to build a suitable convex combination between a strictly feasible policy π° guaranteeing that all constraints are satisfied by at least a margin $\alpha > 0$ for each $t \in [T]$, and the optimal unconstrained policy π^* maximizing $\sum_{t \in [T]} f_t(\pi)$. The following lemma employs a policy $\hat{\pi}$ such that, for all $v \in \mathcal{V}, b \in \mathcal{B}$, $\hat{\pi}(v)_b := \pi^{\circ}(v)_b/(1+\alpha) + \alpha \pi^*(v)_b/(1+\alpha).$

Lemma 7.8. *In the adversarial setting, there always exists* $a(0, \alpha/(1+\alpha), OPT_{\bar{\gamma}})$ -optimal policy.

Now, we provide the overall guarantees of Algorithm 1.

Theorem 7.9 (Stochastic setting). For each $t \in [T]$, let (f_t, c_t) be i.i.d. samples from a fixed but unknown distribution \mathcal{P} over the set of possible inputs. For $\delta > 0$, with probability at least $1 - 4\delta$, Algorithm 1 guarantees

$$TOPT_{\mathcal{P}} - \sum_{t \in [T]} f_t(b_t) \le \mathcal{C}(T, \alpha, \delta).$$

Moreover, we have

$$\sum_{t \in [T]} h_t(b_t) \le 1 + \frac{2}{\eta_R \alpha}, \quad \text{and} \quad \sum_{t \in [T]} c_t(b_t) \le B.$$

Proof. The regret upper bound holds since event **E** holds with probability at least $1 - 2\delta$ (Lemma 7.2), and by combining Lemma 7.7 and Lemma 7.6. The ROI constrain is upper bounded by Lemma 7.3, and the budget constraint is strictly satisfied by construction of Algorithm 1.

Notice that if we initialize primal and dual regret minimizers as detailed in Section 5.1 we have that the cumulative regret and the cumulative ROI constraint violation are of order $\tilde{O}(\sqrt{T})$, while the budget constrained is strictly satisfied.

Analogously, by exploiting Lemma 7.8, we have the following guarantees for the adversarial setting, in which we show that it is possible to achieve a $\alpha/(1+\alpha)$ fraction of the cumulative reward of the baseline, while guaranteeing that constraint violations are under control.

Theorem 7.10 (Adversarial setting). Suppose the sequence of inputs $\gamma = (f_t, c_t)_{t=1}^T$ is selected by an oblivious adversary. Then, for $\delta > 0$, with probability at least $1 - 2\delta$, Algorithm 1 guarantees

$$\frac{\alpha}{1+\alpha} \operatorname{OP} T_{\bar{\gamma}} - \sum_{t \in [T]} f_t(b_t) \le \mathfrak{C}(T, \alpha, \delta).^2$$

Moreover.

$$\sum_{t \in [T]} h_t(b_t) \leq 1 + \frac{2}{\eta_{\mathbb{R}} \alpha} \quad \text{and} \ \sum_{t \in [T]} c_t(b_t) \leq B.$$

Our competitive ratio matches that of Castiglioni et al. (2022b), and, in the case in which we only have budget constraints, it yields the state-of-the-art $1/\alpha$ competitive ratio of Castiglioni et al. (2022a).

8. Relaxing the Safe-Policy Assumption

In the adversarial online setting with long-term constraints, the usual assumption for recovering Slater's condition is that there exist a policy guaranteeing that constraints are satisfied by at least $\alpha>0$ for each t (Chen et al., 2017; Yi et al., 2020; Castiglioni et al., 2022b). Our analysis, up to this point, made the same assumption (Assumption 4.1), except that, unlike those past works, we do not need to know the value of α . Now, we show that our analysis carries over with the following looser requirement.

Assumption 8.1. There exists a policy $\pi^{\circ} \in \Pi$ such that, for each interval $\mathcal{I} = [t_1, t_2]$ with $t_2 - t_1 = k$, we have $\sum_{t \in \mathcal{I}} g_t(\pi^{\circ}) \leq -\alpha k$ and $\sum_{t \in \mathcal{I}} h_t(\pi^{\circ}) \leq -\alpha k$.

The traditional assumption of requiring a safe policy for $each\ t$ would require the bidder to have an action yielding expected ROI which is strictly above their target for each round t. This may not hold practice. For example, if we assume one ad placement is being allocated at each t then the bidder would be priced out by other bidders for at least some time steps (the traditional assumption is more natural if we consider a set of auctions per t, instead of a single one, since bidders typically have user segments for which they have good ROI). Next, we show that if the size of the intervals k is not too big (i.e., if there exists a "safe" policy frequently enough), there exist the following policies.

Lemma 8.2. Suppose Assumption 8.1 holds with $k < \mathcal{E}_{T,\delta}/(2T\eta_{\mathbb{B}})$. Then, for $\delta > 0$, there exists a δ -safe and a $(\delta, \alpha/(1+\alpha), OPT_{\overline{\gamma}})$ -optimal policy.

This result allows us to balance the tightness of the required assumption with the final regret guarantees, by suitably choosing the learning rates rates η_B and η_R . When

²We observe that the same guarantees would hold with respect to the optimal unconstrained policy maximizing $\sum f_t(\pi)$.

Assumption 8.1 holds for $k = \log T$, we recover exactly the bounds of Theorems 7.9 and 7.10. As a further example, if $k = T^{1/4}$, then we can obtain regret guarantees of order $\tilde{O}(T^{3/4})$ by setting $\eta_{\rm B} = O(T^{-3/4})$ and by suitably updating the definition of $\mathcal{E}_{T,\delta}$. In the context of auctions, this allows us to make the milder assumption of requiring that the bidder sees an auction with strictly positive ROI at least every k steps, instead of at every step.

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A. Proofs for Section 5

Lemma 5.1. Let $\eta > 0$ be s.t. $\eta \mathbb{E}_{\pi} \tilde{\ell}_{t}^{\mathbb{P}}(b) < 1$ for all $t \in [T]$ and $\pi \in \Pi$. Then, for any $t \in [T]$, and $b' \in \mathcal{B}$, it holds

$$\mathbb{E}_{\pi_t(v_t)} \left[\tilde{\ell}_t^{\mathcal{P}}(b) \right] - \tilde{\ell}_t^{\mathcal{P}}(b') \le \frac{1}{\eta} \log \left(\frac{p_{t+1,b'}}{\pi_t(v_t)_{b'}} \right) + \frac{\eta}{2} \mathbb{E} \left[\tilde{\ell}_t^{\mathcal{P}}(b)^2 \right].$$

Proof. Let $y := \pi_t(v_t) \in \Delta_m$. By the fact that, for any $x \ge 0$, $e^{-x} \le 1 - x + x^2/2$, we have that

$$-\log \mathbb{E}_{\boldsymbol{y}} \left[e^{-\eta \tilde{\ell}_{t}^{\mathbb{P}}(b)} \right] \ge -\log \left(1 - \eta \mathbb{E}_{\boldsymbol{y}} \left[\tilde{\ell}_{t}^{\mathbb{P}}(b) \right] + \frac{\eta^{2}}{2} \mathbb{E}_{\boldsymbol{y}} \left[\ell_{t}^{\mathbb{P}}(b)^{2} \right] \right)$$

$$\log \mathbb{E}_{\boldsymbol{y}} \left[e^{-\eta \tilde{\ell}_{t}^{\mathbb{P}}(b)} \right] \le -\eta \mathbb{E}_{\boldsymbol{y}} \left[\tilde{\ell}_{t}^{\mathbb{P}}(b) \right] + \frac{\eta^{2}}{2} \mathbb{E}_{\boldsymbol{y}} \left[\tilde{\ell}_{t}^{\mathbb{P}}(b)^{2} \right],$$

where the second inequality holds since, by assumption, $\eta \mathbb{E}_{\boldsymbol{y}} \left[\tilde{\ell}_t^{\tilde{p}}(b) \right] < 1$, which implies that the argument of the logarithm is strictly greater than 0. Then, by definition of the preweights \boldsymbol{p}_{t+1} , we have that, for any $b' \in \mathcal{B}$,

$$\begin{split} \eta \mathbb{E}_{\boldsymbol{y}} \Big[\tilde{\ell}_t^{\mathbb{P}}(b) \Big] &\leq -\log \mathbb{E}_{\boldsymbol{y}} \Big[e^{-\eta \tilde{\ell}_t^{\mathbb{P}}(b)} \Big] + \frac{\eta^2}{2} \mathbb{E}_{\boldsymbol{y}} \Big[\tilde{\ell}_t^{\mathbb{P}}(b)^2 \Big] \\ &= -\log \Bigg(\frac{\pi_{t,b'} e^{-\eta \tilde{\ell}_t^{\mathbb{P}}(b')}}{p_{t+1,b'}} \Bigg) + \frac{\eta^2}{2} \mathbb{E}_{\boldsymbol{y}} \Big[\tilde{\ell}_t^{\mathbb{P}}(b)^2 \Big]. \end{split}$$

This yields

$$\mathbb{E}_{\pi_t(v_t)} \left[\tilde{\ell}_t^{\mathbb{P}}(b) \right] - \tilde{\ell}_t^{\mathbb{P}}(b') \le \frac{1}{\eta} \log \left(\frac{p_{t+1,b'}}{\pi_{t,b'}(v_t)} \right) + \frac{\eta}{2} \mathbb{E}_{\boldsymbol{y}} \left[\tilde{\ell}_t^{\mathbb{P}}(b)^2 \right],$$

for any possible alternative bid $b' \in \mathcal{B}$.

Theorem 5.2. Consider the case in which the set of valuations is a singleton (i.e., n=1). Let $\eta:=1/\sqrt{mT}$, $\xi:=1/(2\sqrt{mT})$, $\sigma:=1/T$, and assume that $\eta\leq 1/M_{[T]}$. Then, for any $\delta>0$, EXP3-SIX guarantees that, with probability at least $1-\delta$, for any interval $\mathcal{I}=[t_1,t_2]$, and for any bid $b\in\mathcal{B}$,

$$\sum_{t \in \mathcal{I}} \left(\ell_t^{\scriptscriptstyle P}(b_t) - \ell_t^{\scriptscriptstyle P}(b) \right) \le M_{\mathcal{I}}^2 \mathcal{E}_{T,\delta}^{\scriptscriptstyle P},$$

where

$$\mathcal{E}_{T,\delta}^{\mathbb{P}} := \left(\frac{3}{2} + \frac{4}{M_{\mathcal{I}}} \log \left(\frac{mT}{\delta}\right) + \frac{\log(T) + 1}{M_{\mathcal{I}}^2}\right) \sqrt{mT}.$$

Proof. In order to increase the readability, we will write π_t in place of $\pi_t(v)$ since $v \in \mathcal{V}$ is constant throughout the proof.

By definition of $\ell_t^{\mathbb{P}}$, we have that for any $b \in \mathcal{B}$ and π , $\mathbb{E}\ell_t^{\mathbb{P}}(b) \leq \mathbb{E}[\mathbb{1}[b=b_t]\ell_t^{\mathbb{P}}(b)/\pi_b] = \ell_t^{\mathbb{P}}(b)$. Therefore, since by assumption we have $\eta < 1/M_{[T]}$, where $M_{[T]}$ is the maximum range of the loss functions $\ell_t^{\mathbb{P}}$ over the time horizon, the assumption of Lemma 5.1 holds. Then, for any interval $[t_1, t_2]$, with $t_1, t_2 \in [T]$, $t_1 < t_2$, by Lemma 5.1 we have that for any $b' \in \mathcal{B}$,

$$\sum_{t \in [t_1, t_2]} \left(\mathbb{E}_{\boldsymbol{\pi}_t} \left[\tilde{\ell}_t^{\mathbb{P}}(b) \right] - \tilde{\ell}_t^{\mathbb{P}}(b') \right) \le \sum_{t \in [t_1, t_2]} \left(\frac{1}{\eta} \log \left(\frac{p_{t+1, b'}}{\pi_{t, b'}} \right) + \frac{\eta}{2} \mathbb{E}_{\boldsymbol{\pi}_t} \left[\tilde{\ell}_t^{\mathbb{P}}(b)^2 \right] \right).$$

Moreover we have that

$$\begin{split} \sum_{t \in [t_1, t_2]} \log \left(\frac{p_{t+1, b'}}{\pi_{t, b'}} \right) &= \log \left(\frac{1}{\pi_{t_1, b'}} \right) + \sum_{t \in [t_1 + 1, t_2]} \log \left(\frac{p_{t, b'}}{\pi_{t, b'}} \right) + \log(p_{t_2 + 1, b'}) \\ &\leq \log \left(\frac{m}{\sigma} \right) + \sum_{t \in [t_1 + 1, t_2]} \log \left(\frac{1}{1 - \sigma} \right). \end{split}$$

The last inequality holds since, for any $t \in [T]$ and $b \in \mathcal{B}$,

$$\pi_{t+1,b} = \frac{(1-\sigma)w_{t,b}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(b)} + \sigma/m\sum_{i\in\mathcal{B}}w_{t,i}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(i)}}{\sum_{i\in\mathcal{B}}\left((1-\sigma)w_{t,i}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(i)} + \sigma/m\sum_{j\in\mathcal{B}}w_{t,j}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(j)}\right)}$$

$$= \frac{(1-\sigma)w_{t,b}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(b)} + \sigma/m\sum_{i\in\mathcal{B}}w_{t,i}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(i)}}{\sum_{i\in\mathcal{B}}w_{t,i}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(i)}}$$

$$\geq (1-\sigma)\frac{w_{t,b}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(b)}}{\sum_{i\in\mathcal{B}}w_{t,i}}\frac{\sum_{i\in\mathcal{B}}w_{t,i}}{\sum_{i\in\mathcal{B}}w_{t,i}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(i)}}$$

$$= (1-\sigma)\frac{\pi_{t,b}e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(b)}}{\mathbb{E}_{\pi_{t}}\left[e^{-\eta\tilde{\ell}_{t}^{\tilde{p}}(i)}\right]}$$

$$= (1-\sigma)p_{t+1,b},$$

where we used the definition of π_t and p_t as per Algorithm 2.

Then,

$$\sum_{t \in [t_1, t_2]} \left(\mathbb{E}_{\boldsymbol{\pi}_t} \tilde{\ell}_t^{\tilde{\mathbb{P}}}(b) - \tilde{\ell}_t^{\mathbb{P}}(b') \right) \le \frac{1}{\eta} \left(\log \left(\frac{m}{\sigma} \right) + (t_2 - t_1 - 1) \log \left(\frac{1}{1 - \sigma} \right) \right) + \frac{\eta}{2} \sum_{t \in [t_1, t_2]} \mathbb{E}_{i \sim \boldsymbol{\pi}_t} \left[\tilde{\ell}_t^{\mathbb{P}}(b)^2 \right]. \tag{2}$$

Neu (2015, Lemma 1) states that given a fixed non-increasing sequence (ξ_t) with $\xi_t \geq 0$, and by letting $\beta_{t,i}$ be a nonnegative random variable such that $\beta_{t,i} \leq 2\xi_t$ for all t and $i \in \mathcal{B}$, then with probability at least $1 - \delta$,

$$\sum_{t \in [T]} \sum_{i \in \mathcal{B}} \beta_{t,i} \Big(\ell_t^{\tilde{\mathbf{p}}}(i) - \ell_t^{\mathbf{p}}(i) \Big) \leq \log(1/\delta).$$

Then, for any bid $i \in \mathcal{B}$, by setting

$$\beta_{t,j} = \left\{ \begin{array}{ll} 2\xi_t \mathbbm{1}[i=j] & \text{if } t \in \mathcal{I} \\ 0 & \text{otherwise} \end{array} \right.,$$

and by applying a union bound we obtain that, with probability at least $1 - \delta$,

$$\sum_{t \in \mathcal{I}} \left(\tilde{\ell}_t^{\mathbb{P}}(i) - \ell_t^{\mathbb{P}}(i) \right) \le \frac{M_{\mathcal{I}} \log(m/\delta)}{2\xi},\tag{3}$$

where $M_{\mathcal{I}}$ is the maximum range of the loss functions ℓ_t^{p} over time interval \mathcal{I} .

Moreover, from the definition of $\tilde{\ell}_t^{\mathbb{P}}$ (see Algorithm 2), we have that

$$\sum_{t \in \mathcal{T}} \mathbb{E}_{b \sim \pi_t} \tilde{\ell}_t^{\tilde{\mathbb{P}}}(b) = \sum_{t \in \mathcal{T}} \left(\ell_t^{\mathbb{P}}(b_t) - \sum_{b \in \mathcal{B}} \xi \tilde{\ell}_t^{\tilde{\mathbb{P}}}(b) \right). \tag{4}$$

Finally, given $t \in \mathcal{I}$, we observe that

$$\mathbb{E}_{b \sim \pi_t} \ell_t^{\tilde{\mathbb{P}}}(b)^2 = \sum_{b \in \mathcal{B}} (\pi_{t,b} \, \ell_t^{\tilde{\mathbb{P}}}(b)) \, \ell_t^{\tilde{\mathbb{P}}}(b) \le M_{\mathcal{I}} \sum_{b \in \mathcal{B}} \ell_t^{\tilde{\mathbb{P}}}(b). \tag{5}$$

Finally, we conclude by showing that, for any $b \in \mathcal{B}$, with probability at least $1 - \delta$,

$$\begin{split} \sum_{t \in [t_1, t_2]} \left(\ell_t^{\mathbb{P}}(b_t) - \ell_t^{\mathbb{P}}(b) \right) &\leq \sum_{t \in [t_1, t_2]} \ell_t^{\mathbb{P}}(b_t) + \frac{M_{\mathcal{I}} \log(m/\delta)}{2\xi} - \sum_{t \in [t_1, t_2]} \tilde{\ell}_t^{\mathbb{P}}(b) \end{split} \qquad \text{by Equation (3)} \\ &= \frac{M_{\mathcal{I}} \log(m/\delta)}{2\xi} + \sum_{t \in [t_1, t_2]} \sum_{i \in \mathcal{B}} \xi \tilde{\ell}_t^{\mathbb{P}}(i) + \sum_{t \in [t_1, t_2]} \left(\mathbb{E}_{\pi_t} \tilde{\ell}_t^{\mathbb{P}} - \tilde{\ell}_t^{\mathbb{P}}(b) \right) \qquad \text{by Equation (4)} \\ &\leq \frac{M_{\mathcal{I}} \log(m/\delta)}{2\xi} + \sum_{t \in [t_1, t_2]} \sum_{i \in \mathcal{B}} \xi \tilde{\ell}_t^{\mathbb{P}}(i) \qquad \qquad \text{by Equation (2) and Equation (5)} \\ &+ \frac{1}{\eta} \left(\log \left(\frac{m}{\sigma} \right) + (t_2 - t_1 - 1) \log \left(\frac{1}{1 - \sigma} \right) \right) + \frac{\eta}{2} M_{\mathcal{I}} \sum_{t \in [t_1, t_2]} \sum_{i \in \mathcal{B}} \tilde{\ell}_t^{\mathbb{P}}(i) \\ &\leq \frac{M_{\mathcal{I}} \log(m/\delta)}{2\xi} + (t_2 - t_1) m \xi M_{\mathcal{I}} + \frac{1}{\eta} \left(\log \left(\frac{m}{\sigma} \right) + (t_2 - t_1 - 1) \log \left(\frac{1}{1 - \sigma} \right) \right) \\ &+ (t_2 - t_1) \frac{\eta M_{\mathcal{I}}^2 m}{2}. \end{split}$$

By setting $\eta = \frac{1}{\sqrt{mT}}$, $\xi = \frac{1}{2\sqrt{mT}}$, with probability at least $1 - \delta$,

$$\sum_{t \in [t_1, t_2]} \left(\ell_t^{\mathbb{P}}(b_t) - \ell_t^{\mathbb{P}}(b) \right) \leq \frac{3}{2} M_{\mathcal{I}}^2(t_2 - t_1) \sqrt{\frac{m}{T}} + \sqrt{mT} \left(M_{\mathcal{I}} \log \left(\frac{m}{\delta} \right) + \log \left(\frac{m}{\sigma} \right) + (T - 1) \log \left(\frac{1}{1 - \sigma} \right) \right) \\
\leq \frac{3}{2} M_{\mathcal{I}}^2(t_2 - t_1) \sqrt{\frac{m}{T}} + \sqrt{mT} \left(2M_{\mathcal{I}} \log \left(\frac{m}{\delta} \right) - \log \left(\sigma (1 - \sigma)^{T - 1} \right) \right).$$

By letting $h(x) := -x \log x - (1-x) \log (1-x)$ be the binary entropy function for $x \in [0,1]$, we have that, for $x \in [0,1]$, $h(x) \le x \log (e/x)$ (see, e.g., Cesa-Bianchi et al. (2012, Corollary 1)). Then, for $\sigma = 1/T$, we have that $-\log \sigma (1-\sigma)^{T-1} \le \log (eT)$. This yields

$$\sum_{t \in [t_1, t_2]} \left(\ell_t^{\mathbb{P}}(b_t) - \ell_t^{\mathbb{P}}(b) \right) \le \frac{3}{2} M_{\mathcal{I}}^2(t_2 - t_1) \sqrt{\frac{m}{T}} + \sqrt{mT} \left(2M_{\mathcal{I}} \log \left(\frac{m}{\delta} \right) + \log(eT) \right).$$

By using the union bound over all possible intervals $[t_1, t_2]$ we obtain that, with probability at least $1 - \delta$,

$$\begin{split} \sum_{t \in [t_1, t_2]} \left(\ell_t^{\mathbb{P}}(b_t) - \ell_t^{\mathbb{P}}(b) \right) &\leq \frac{3}{2} M_{\mathcal{I}}^2(t_2 - t_1) \sqrt{\frac{m}{T}} + \sqrt{mT} \left(4M_{\mathcal{I}} \log \left(\frac{mT}{\delta} \right) + \log(T) + 1 \right) \\ &\leq M_{\mathcal{I}}^2 \left(\frac{3}{2} + \frac{4}{M_{\mathcal{I}}} \log \left(\frac{mT}{\delta} \right) + \frac{\log(T) + 1}{M_{\mathcal{I}}^2} \right) \sqrt{mT}, \end{split}$$

which proves our statement.

Lemma 5.5. *For all* $t_1, t_2 \in [T]$,

$$\mu_{t_2} \ge \eta_{\mathbb{R}} \sum_{t' \in [t_1, t_2 - 1]} h_{t'}(b_{t'}) + \mu_{t_1}.$$

Proof. We prove the result by induction. First, it's easy to see that the result holds for t=1, since $\mu_1=0$. Then, suppose

that the statement holds for round t. Then,

$$\begin{split} \mu_{t+1} &= \left[\mu_t + \eta_{\mathbb{R}} \, h_t(b_t) \right]^+ \\ &\geq \mu_t + \eta_{\mathbb{R}} \, h_t(b_t) \\ &\geq \eta_{\mathbb{R}} \sum_{t' \in [t-1]} \eta_{\mathbb{R}} \, h_{t'}(b_{t'}) + \eta_{\mathbb{R}} \, h_t(b_t) \\ &= \eta_{\mathbb{R}} \sum_{t' \in [t]} \eta h_{t'}(b_{t'}), \end{split}$$

where $[x]^+ := \max\{x, 0\}$. This concludes the proof.

B. Proofs for Section 6

Theorem 6.2. Suppose that there exists a safe policy and that the primal regret minimizer has regret at most $M_{\mathcal{I}}\mathcal{E}_{T,\delta}^{\mathcal{P}}$ for any time interval \mathcal{I} . Then, the Lagrange multipliers μ_t are such that $\mu_t \leq 2/\alpha$ for each $t \in [\tau]$.

Proof. We consider two cases.

Case 1: $\alpha \leq 10/\sqrt{T}$. By construction of the dual regret minimizer, and by the choice of $\eta_{\mathbb{R}}$, the dual variable μ_t can reach at most value $\eta_{\mathbb{R}}T \leq \sqrt{T}/16$. Therefore, we have $\mu_t \leq \sqrt{T}/16 < 2/\alpha$.

Case 2: $\alpha > 10/\sqrt{T}$. Let $\mathcal{I} = [t_1, t_2]$, with $t_1, t_2 \in [T]$, $t_1 \leq t_2$. Moreover, assume that there exists a safe policy π° . We show that, if the Lagrangian multiplier μ_t is greater than $2/\alpha$, we reach a contradiction.

Suppose, by contradiction, that there exists a round t_2 such that $\mu_{t_2} \ge 2/\alpha$. Let t_1 be the first round such that $\mu_t \ge 1/\alpha$ for any $t \in [t_1, t_2]$. Notice that the structure of the dual regret minimizer (see Section 5.2) implies that

$$\mu_{t_1} \le 1/\alpha + \eta_{\mathbb{R}} \quad \text{and} \quad \mu_{t_2} \le 2/\alpha + \eta_{\mathbb{R}},$$
 (6)

since the dual losses are in [-1,1]. Therefore, we can upperbound the primal loss function as $M_{[T]} \le (1+4/\alpha)$ (i.e., we use $\mu_t \le 3/\alpha$). This implies that, for $m \ge 2$,

$$\eta M_{[T]} = \frac{1}{\sqrt{mT}} \left(1 + \frac{4}{\alpha} \right) < \frac{1}{\sqrt{mT}} \left(1 + \frac{2\sqrt{T}}{5} \right) \le 1.$$

Therefore, Theorem 5.2 applies and the primal regret minimizer satisfies the bound on the adaptive regret of Equation (1). Then, by the no-regret property of the primal we get:

$$\sum_{t \in \mathcal{I}} (f_t(b_t) - \lambda_t g_t(b_t) - \mu_t h_t(b_t)) \ge \sum_{t \in \mathcal{I}} (f_t(\pi^{\circ}) - \lambda_t g_t(\pi^{\circ}) - \mu_t h_t(\pi^{\circ})) - M_{\mathcal{I}}^2 \mathcal{E}_{T,\delta}^{\mathsf{P}}
\ge \alpha \sum_{t \in \mathcal{I}} \mu_t - \left(\mu_{\mathcal{I}} + \frac{1}{\alpha}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - M_{\mathcal{I}}^2 \mathcal{E}_{T,\delta}^{\mathsf{P}}$$
(by Definition 6.1)
$$\ge (t_2 - t_1) - \left(\mu_{[t_1, t_2 - 1]} + \frac{3}{\alpha} + \eta_{\mathbb{R}}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - M_{\mathcal{I}}^2 \mathcal{E}_{T,\delta}^{\mathsf{P}}$$
(by Def. of t_1 and Equation (6))
$$\ge (t_2 - t_1) - \left(\frac{5}{\alpha} + \eta_{\mathbb{R}}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - \left(1 + \frac{2}{\alpha} + \frac{1}{\alpha}\right)^2 \mathcal{E}_{T,\delta}^{\mathsf{P}}$$
(by Def. of $M_{\mathcal{I}}$ and ℓ_t^{P})
$$\ge (t_2 - t_1) - \left(\frac{5}{\alpha} + \eta_{\mathbb{R}}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - \frac{16}{\alpha^2} \mathcal{E}_{T,\delta}^{\mathsf{P}}.$$
(7)

By Lemma 5.5 it holds

$$\mu_{t_2} \ge \eta_{\mathbb{R}} \sum_{t' \in [t_1, t_2 - 1]} h_{t'}(b_{t'}) + \mu_{t_1}.$$

Hence, by definition of t_2 and Equation (6),

$$\sum_{t' \in [t_1, t_2 - 1]} h_{t'}(b_{t'}) \ge \frac{\mu_{t_2} - \mu_{t_1}}{\eta_{\mathbb{R}}} \ge \frac{1}{\alpha \eta_{\mathbb{R}}} - 1.$$

Then, by the regret bound of the dual with respect to $\mu = \mu_t$ and $\lambda = 0$, we get

$$\sum_{t \in [t_{1}, t_{2} - 1]} (f_{t}(b_{t}) - \lambda_{t} g_{t}(b_{t}) - \mu_{t} h_{t}(b_{t})) \leq \sum_{t \in [t_{1}, t_{2} - 1]} (f_{t}(b_{t}) - \mu_{t} h_{t}(b_{t})) + \mathcal{E}_{T}^{D,R} + \mathcal{E}_{T}^{D,B}$$

$$\leq (t_{2} - t_{1}) - \frac{1}{\alpha} \sum_{t \in [t_{1}, t_{2} - 1]} h_{t}(b_{t}) + \mathcal{E}_{T}^{D,R} + \mathcal{E}_{T}^{D,B}$$

$$\leq (t_{2} - t_{1}) - \frac{1}{\alpha^{2} \eta} + \frac{1}{\alpha} + \mathcal{E}_{T}^{D,R} + \mathcal{E}_{T}^{D,B}.$$
(8)

By putting Equation (7) and Equation (8) together we have that

$$(t_2-t_1)-\frac{1}{\alpha^2\eta_{\mathbb{R}}}+\frac{3}{\alpha}+2+\mathcal{E}_T^{\mathrm{D},\mathbb{R}}+\mathcal{E}_T^{\mathrm{D},\mathbb{B}}\geq (t_2-t_1)-\left(\frac{5}{\alpha}+\eta_{\mathbb{R}}\right)\mathcal{E}_{T,\delta}^{\mathcal{I}}-\frac{16}{\alpha^2}\mathcal{E}_{T,\delta}^{\mathrm{P}}.$$

We observe that in Lemma 5.4 we set $\eta_{\mathbb{R}} \coloneqq \left(6 + \mathcal{E}_T^{\mathbb{D},\mathbb{R}} + \mathcal{E}_T^{\mathbb{D},\mathbb{B}} + 6\mathcal{E}_{T,\delta}^{\mathcal{I}} + 16\mathcal{E}_T^{\mathbb{P}}\right)^{-1}$. Then, from the inequality above we have

$$\frac{1}{\alpha^2 \eta_{\mathrm{R}}} \leq \frac{3}{\alpha} + 2 + \mathcal{E}_T^{\mathrm{D,R}} + \mathcal{E}_T^{\mathrm{D,B}} + \left(\frac{5}{\alpha} + 1\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} + \frac{16}{\alpha^2} \mathcal{E}_{T,\delta}^{\mathrm{P}}.$$

However, we reach a contradiction since

$$\frac{1}{\alpha^2\eta_{\mathbb{R}}} \geq \frac{4}{\alpha} + 2 + \mathcal{E}_T^{\mathrm{D,R}} + \mathcal{E}_T^{\mathrm{D,B}} + \left(\frac{5}{\alpha} + 1\right)\mathcal{E}_{T,\delta}^{\mathcal{I}} + \frac{16}{\alpha^2}\mathcal{E}_T^{\mathbb{P}} > \frac{3}{\alpha} + 2 + \mathcal{E}_T^{\mathrm{D,R}} + \mathcal{E}_T^{\mathrm{D,B}} + \left(\frac{5}{\alpha} + 1\right)\mathcal{E}_{T,\delta}^{\mathcal{I}} + \frac{16}{\alpha^2}\mathcal{E}_T^{\mathbb{P}},$$

where we used the fact that $\alpha \in (0,1]$ by assumption $(\alpha > 0)$, and by boundedness of g_t and h_t for all $t \in [T]$. This concludes the proof.

Lemma 6.4. Assume that the sequence of inputs (f_t, g_t) is selected by an oblivious adversary, and that there exists a policy π such that $g_t(\pi) \le -\alpha$ and $h_t(\pi) \le -\alpha$ for each $t \in [T]$. Then, there exists a δ -safe policy for any $\delta \in (0, 1]$.

Proof. By assumption there exists a policy π such that $g_t(\pi) \leq -\alpha$ and $h_t(\pi) \leq -\alpha$ for each $t \in [T]$. Then, for each $t_1, t_2 \in [T]$, with $t_1 < t_2$, it holds $\sum_{t \in [t_1, t_2]} (\lambda_t g_t(\pi) + \mu_t h_t(\pi)) \leq -\alpha \sum_{t \in [t_1, t_2]} (\lambda_t + \mu_t) \leq -\alpha \sum_{t \in [t_1, t_2]} \mu_t$, which implies that π is δ -safe for any $\delta \in (0, 1]$.

Lemma 6.3. Assume that inputs (f_t, c_t) are drawn i.i.d. from \mathbb{P} , and that there exists a policy π such that $\mathbb{E}_{\mathbb{P}}g(\pi) \leq -\alpha$ and $\mathbb{E}_{\mathbb{P}}h(\pi) \leq -\alpha$. Then, there exists a δ -safe policy with probability at least $1 - \delta$.

Proof. By the definition of α , there exists a policy π such that $\mathbb{E}_{\mathcal{P}}g(\pi) \leq -\alpha$ and $\mathbb{E}_{\mathcal{P}}h(\pi) \leq -\alpha$. Then, given a time interval $\mathcal{I} = [t_1, t_2], t_1, t_2 \in [T]$, by appling the Azuma–Hoeffding inequality to the martingale difference sequence W_1, \ldots, W_T with

$$W_t := \lambda_t g_t(\pi) + \mu_t h_t(\pi) - \lambda_t \mathbb{E}_{\mathcal{P}} g(\pi) - \mu_t \mathbb{E}_{\mathcal{P}} h(\pi),$$

we obtain that

$$\left| \sum_{t \in \mathcal{I}} W_t \right| \le \left(\mu_{\mathcal{I}} + \frac{1}{\alpha} \right) \sqrt{2(t_2 - t_1) \log \left(\frac{2}{\delta} \right)}$$

holds with probability at least $1-\delta$. By applying a union bound we get that the inequalities for each time interval $\mathcal I$ hold simultaneously with probability at least $1-T^2\delta$. Let $\mathcal E_{T,\delta}^{\mathcal I}:=2\sqrt{(t_2-t_1)\log(\frac{2T}{\delta})}$ as per Definition 6.1. Then, with

probability at least $1 - \delta$ it holds

$$\sum_{t \in \mathcal{I}} (\lambda_t g_t(\pi) + \mu_t h_t(\pi)) \leq \left(\mu_{\mathcal{I}} + \frac{1}{\alpha}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} + \sum_{t \in \mathcal{I}} (\lambda_t \mathbb{E}_{\mathcal{P}} g(\pi) + \mu_t \mathbb{E}_{\mathcal{P}} h(\pi))$$

$$\leq \left(\mu_{\mathcal{I}} + \frac{1}{\alpha}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - \alpha \sum_{t \in \mathcal{I}} (\lambda_t + \mu_t)$$

$$\leq \left(\mu_{\mathcal{I}} + \frac{1}{\alpha}\right) \mathcal{E}_{T,\delta}^{\mathcal{I}} - \alpha \sum_{t \in \mathcal{I}} \mu_t.$$

This concludes the proof.

C. Proofs for Section 7

Lemma 7.2. Event **E** holds with probability at least $1 - 2\delta$.

Proof. By Lemmas 6.4 and 6.3, we have that in both settings there exists a safe policy with probability at least $1 - \delta$. Moreover, by Corollary 5.3 with probability at least $1 - \delta$ the regret of the primal is upperbounded by $M_{\mathcal{I}}\mathcal{E}_T^{\mathbb{P}}$ for each interval $\mathcal{I} = [t_1, t_2], t_1, t_2 \in [T]$. Applying a union bound sufficies to show that the two events hold simultaneously with probability at least $1 - 2\delta$. Then, the statement directly follows from Theorem 6.2.

Lemma 7.3. Assume that event **E** holds and let τ be the round in which the budget is fully depleted. Then, it holds that $\sum_{t \in [\tau]} h_t(\pi_t) \leq 1 + 2/(\eta_R \alpha)$.

Proof. By the definition of event **E** we have that $\mu_{\tau} \leq 2/\alpha$. Moreover, by Lemma 5.5 it holds that $\mu_{\tau} \geq \eta_{\mathbb{R}} \sum_{t \in [\tau-1]} h_t(\pi_t)$. Hence, $\sum_{t \in [\tau]} h_t(\pi_t) \leq \mu_{\tau}/\eta_{\mathbb{R}} + 1 \leq 2/(\eta_{\mathbb{R}}\alpha) + 1$.

Lemma 7.5. Algorithm 1 guarantees that

$$\sum_{t \in [\tau]} \lambda_t g(b_t) \ge T - \tau - \frac{1}{\rho} - \mathcal{E}_T^{\text{D,B}}.$$

Proof. We consider two cases.

• If $\tau = T$, then

$$\sum_{t \in [\tau]} \lambda_t g_t(b_t) \ge -\mathcal{E}_T^{\mathsf{D},\mathsf{B}} \ge T - \tau - \frac{1}{\rho} - \mathcal{E}_T^{\mathsf{D},\mathsf{B}}.$$

• Otherwise, if $\tau < T$,

$$\begin{split} \sum_{t \in [\tau]} \lambda_t g_t(b_t) &\geq \frac{1}{\rho} \sum_{t \in [\tau]} g_t(b_t) - \mathcal{E}_{\tau}^{\text{D,B}} \\ &= \frac{1}{\rho} \sum_{t \in [\tau]} (c_t(b_t) - \rho) - \mathcal{E}_{\tau}^{\text{D,B}} \\ &= \frac{1}{\rho} (B - 1 - \tau \rho) - \mathcal{E}_{\tau}^{\text{D,B}} \\ &= \left(T - \tau - \frac{1}{\rho} \right) - \mathcal{E}_{\tau}^{\text{D,B}}. \end{split}$$

where the first inequality follows by the no-regret guarantee of the dual regret minimizer with respect to the fixed choice of $\lambda = 1/\rho$, and then we use the definition of g_t and the fact that τ is the time at which the budget is depleted, that is the round in which the available budget becomes strictly smaller than 1 (see Algorithm 1).

This concludes the proof. \Box

Lemma 7.6. Suppose that event **E** holds and that there exists a (δ, q, OPT) -optimal policy. Then,

$$\sum_{t \in [\tau]} f_t(b_t) \ge qT \operatorname{OPT} - \mathcal{C}(T, \alpha, \delta),$$

where

$$\mathfrak{C}(T,\alpha,\delta) \coloneqq \frac{1}{\alpha} + \left(\frac{3}{\alpha} + 1\right) \left(\mathcal{E}_{T,\delta}^{\mathsf{P}} + \mathcal{E}_{T,\delta}\right) + \mathcal{E}_{T}^{\mathsf{D},\mathsf{R}} + \mathcal{E}_{T}^{\mathsf{D},\mathsf{B}}.$$

Proof. Let π^* be a (δ, q, OPT) -optimal policy. Then, we have that

$$\begin{split} \sum_{t \in [\tau]} f_t(b_t) &\geq \sum_{t \in [\tau]} (f_t(\pi^*) - \lambda_t g_t(\pi^*) - \mu_t h_t(\pi^*) + \lambda_t g_t(b_t) + \mu_t h_t(b_t)) - \left(\frac{3}{\alpha} + 1\right) \mathcal{E}_{\tau,\delta}^{\mathsf{P}} \\ &\geq \sum_{t \in [\tau]} (f_t(\pi^*) + \lambda_t g_t(b_t) + \mu_t h_t(b_t)) - \frac{3}{\alpha} \mathcal{E}_{\tau,\delta} - \left(\frac{3}{\alpha} + 1\right) \mathcal{E}_{\tau,\delta}^{\mathsf{P}} \\ &\geq \sum_{t \in [\tau]} f_t(\pi^*) + \sum_{t \in [\tau]} \lambda_t g_t(b_t) - \frac{3}{\alpha} \mathcal{E}_{\tau,\delta} - \left(\frac{3}{\alpha} + 1\right) \mathcal{E}_{\tau,\delta}^{\mathsf{P}} - \mathcal{E}_{\tau}^{\mathsf{D},\mathsf{R}} \\ &\geq \sum_{t \in [\tau]} f_t(\pi^*) + T - \tau - \frac{1}{\rho} - \frac{3}{\alpha} \mathcal{E}_{\tau,\delta} - \left(\frac{3}{\alpha} + 1\right) \mathcal{E}_{\tau,\delta}^{\mathsf{P}} - \mathcal{E}_{\tau}^{\mathsf{D},\mathsf{R}} - \mathcal{E}_{T}^{\mathsf{D},\mathsf{B}} \\ &\geq \sum_{t \in [T]} f_t(\pi^*) - \frac{1}{\rho} - \frac{3}{\alpha} \mathcal{E}_{\tau,\delta} - \left(\frac{3}{\alpha} + 1\right) \mathcal{E}_{\tau,\delta}^{\mathsf{P}} - \mathcal{E}_{\tau}^{\mathsf{D},\mathsf{R}} - \mathcal{E}_{T}^{\mathsf{D},\mathsf{B}} \\ &\geq q T \mathsf{OPT} - \frac{1}{\rho} - \left(\frac{3}{\alpha} + 1\right) \left(\mathcal{E}_{T,\delta}^{\mathsf{P}} + \mathcal{E}_{T,\delta}\right) - \mathcal{E}_{T}^{\mathsf{D},\mathsf{R}} - \mathcal{E}_{T}^{\mathsf{D},\mathsf{B}}, \end{split}$$

where the first inequality comes from the regret bound of the primal regret minimizer, the second follows by the definition of (δ,q,OPT) -optimal policy, the third follows by the no-regret guarantee of the dual regret minimizer with respect to action $\mu=0$, the fourth one follows from Lemma 7.5. Finally, the fifth inequality follows from the fact that $f_t(\cdot)\in[0,1]$, and the last one is by definition of (δ,q,OPT) -optimal policy. This proves our statement. \square

Lemma 7.7. In the stochastic setting, with probability at least $1 - 2\delta$ there exists a $(\delta, 1, OPT_{\mathcal{P}})$ -optimal policy (where $OPT_{\mathcal{P}}$ is the optimal value of $LP_{\mathcal{P}}$).

Proof. Let π^* be an optimal solution to LP_{γ} . We show that, with probability at least $1 - \delta$, the policy π^* is $(\delta, 1, OPT_{\gamma})$ -optimal, proving the statement.

First, by Azuma-Hoeffding inequality we have that, for $t' \in [T]$, with probability at least $1 - \delta$

$$\sum_{t \in [t']} (\lambda_t g_t(\pi^*) + \mu_t h_t(\pi^*) - \lambda_t \mathbb{E}_{\mathcal{P}} g(\pi^*) - \mu_t \mathbb{E}_{\mathcal{P}} h(\pi^*)) \le \left(\mu_{[t']} + \frac{1}{\alpha}\right) \sqrt{2T \log\left(\frac{1}{\delta}\right)},$$

where $\mu_{[t']}$ is the largest dual multiplier μ_t observed up to t'. Notice that we cannot upper bound it righ away as $2/\alpha$ because here we are not requiring event \mathbf{E} (Definition 7.1) to hold. Then, assuming T>2, by taking a union bound over all possible rounds t', we get that the following inequality holds with probability at least $1-\delta$ simultaneously for all $t' \in [T]$,

$$\sum_{t \in [t']} (\lambda_t g_t(\pi^*) + \mu_t h_t(\pi^*) - \lambda_t \mathbb{E}_{\mathcal{P}} g(\pi^*) - \mu_t \mathbb{E}_{\mathcal{P}} h(\pi^*)) \le \left(\mu_{[t']} + \frac{1}{\alpha}\right) \sqrt{2T \log\left(\frac{T}{\delta}\right)} \le \left(\mu_{[t']} + \frac{1}{\alpha}\right) \mathcal{E}_{T,\delta}.$$

Similarly, we can prove that

$$\left| \sum_{t \in [T]} (f_t(\pi^*) - \mathbb{E}_{\mathcal{P}} f(\pi^*)) \right| \le 2\sqrt{T \log\left(\frac{2T}{\delta}\right)} = \mathcal{E}_{T,\delta}$$

holds with probability at least $1 - \delta$. Then,

$$\sum_{t \in [T]} f_t(\pi^*) \geq \sum_{t \in [T]} \mathbb{E}_{\mathcal{P}} f(\pi^*) - \mathcal{E}_{T,\delta}^{\mathcal{I}} = \operatorname{OPT}_{\gamma} - \mathcal{E}_{T,\delta}.$$

Assuming T > 2 and applying an union bound, the statement follows.

Lemma 7.8. In the adversarial setting, there always exists a $(0, \alpha/(1+\alpha), OPT_{\bar{\gamma}})$ -optimal policy.

Proof. Let π° be a strictly feasible policy such that $\alpha = -\max_{t \in [T]} \max\{g_t(\pi^{\circ}), h_t(\pi^{\circ})\}$, with $\alpha > 0$, and let $\pi^* \in \arg\max_{\pi \in \Pi} \sum_{t \in [T]} f_t(\pi)$ be an optimal unconstrained policy. It holds $\sum_{t \in [T]} f_t(\pi^*) \geq T \text{ optimal unconstrained policy}$, which is a solution to $\text{LP}_{\bar{\gamma}}$.

Then, consider the policy $\hat{\pi}$ such that, for each $v \in \mathcal{V}$, $b \in \mathcal{B}$,

$$\hat{\pi}(v)_b = \frac{1}{1+\alpha} \pi^{\circ}(v)_b + \frac{\alpha}{1+\alpha} \pi^{*}(v)_b,$$

where, given a policy π , we denote by $\pi(v)_b$ the probability of bidding b under valuation v.

At each iteration we have that both the budget and the ROI constraints are satisfied by the policy $\hat{\pi}$ (in expectation with respect to $\hat{\pi}$). Indeed, for each $t \in [T]$, we have thath $g_t(\hat{\pi}) = \frac{1}{1+\alpha}g_t(\pi^\circ) + \frac{\alpha}{1+\alpha}g_t(\pi^*) \leq \frac{-\alpha}{1+\alpha} + \frac{\alpha}{1+\alpha} \leq 0$. Similarly, we can prove that for each $t \in [T]$ it holds $h_t(\hat{\pi}) \leq 0$. Then, the policy $\hat{\pi}$ satisfies the condition $\sum_{t \in [t']} (\lambda_t g_t(\hat{\pi}) + \mu_t h_t(\hat{\pi})) \leq 0$ for each $t' \in [T]$. Moreover,

$$\sum_{t \in [T]} f_t(\hat{\pi}) = \sum_{t \in [T]} \left(\frac{1}{1+\alpha} f_t(\pi^\circ) + \frac{\alpha}{1+\alpha} f_t(\pi^*) \right) \ge \sum_{t \in [T]} \frac{\alpha}{1+\alpha} f_t(\pi^*) \ge \frac{\alpha}{1+\alpha} \mathsf{OPT}_{\bar{\gamma}},$$

which satisfies the first condition of Definition 7.4. This concludes the proof.

D. Proofs for Section 8

Lemma 8.2. Suppose Assumption 8.1 holds with $k < \mathcal{E}_{T,\delta}/(2T\eta_{\mathbb{B}})$. Then, for $\delta > 0$, there exists a δ -safe and a $(\delta, \alpha/(1 + \alpha), OPT_{\overline{\gamma}})$ -optimal policy.

Proof. First, we need to show that there exists a δ -safe policy. In particular, we show that there exists a policy π° such that, for any time interval $\mathcal{I} = [t_1, t_2]$, it holds

$$\sum_{t \in \mathcal{I}} (\lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ)) \le \mathcal{E}_{T,\delta} - \alpha \sum_{t \in \mathcal{I}} (\mu_t + \lambda_t). \tag{9}$$

To do that, we show that the interval \mathcal{I} can be split in smaller intervals of length k, and for each of such smaller intervals \mathcal{I}' , it holds

$$\sum_{t \in \mathcal{T}'} (\lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ)) \le 2k^2 \eta_B - \alpha \sum_{t \in \mathcal{T}'} (\mu_t + \lambda_t).$$

We show that this holds for any \mathcal{I}' of length k in Lemma D.1. Then, the cumulative sum on the original interval \mathcal{I} is at most

$$\sum_{t \in \mathcal{I}} (\lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ)) \le \left\lceil \frac{|\mathcal{I}|}{k} \right\rceil \left(2k^2 \eta_B - \alpha \sum_{t \in \hat{\mathcal{I}}} (\mu_t + \lambda_t) \right) \le 2Tk \eta_B - \alpha \sum_{t \in \mathcal{I}} (\mu_t + \lambda_t) \le \mathcal{E}_{T,\delta} - \alpha \sum_{t \in \mathcal{I}} (\mu_t + \lambda_t),$$

where we set $\hat{\mathcal{I}} \in \arg\max_{\mathcal{I}':|\mathcal{I}'|=k} \sum_{t\in\mathcal{I}'} (\mu_t + \lambda_t)$. This shows that Equation (9) holds for any interval \mathcal{I} .

Then, we can show that a $(\delta, \alpha/(1+\alpha), OPT_{\bar{\gamma}})$ -optimal policy exists. In particular, by defining a policy $\hat{\pi}$ as in the proof of Lemma 7.8, we have

$$\sum_{t \in \mathcal{I}} (\lambda_t g_t(\hat{\pi}) + \mu_t h_t(\hat{\pi})) = \frac{1}{1+\alpha} \left(\sum_{t \in \mathcal{I}} (\lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ)) \right) + \frac{\alpha}{1+\alpha} \left(\sum_{t \in [t']} (\lambda_t g_t(\pi^*) + \mu_t h_t(\pi^*)) \right)$$

$$\leq \mathcal{E}_{T,\delta} - \frac{\alpha}{1+\alpha} \sum_{t \in \mathcal{I}} (\mu_t + \lambda_t) + \frac{\alpha}{1+\alpha} \sum_{t \in \mathcal{I}} (\lambda_t + \mu_t)$$

$$\leq \mathcal{E}_{T,\delta} \leq \frac{3}{\alpha} \mathcal{E}_{T,\delta},$$

where the first inequality is by Equation (9). The first condition of Definition 7.4 can be shown to hold with the same steps of Lemma 7.8. This concludes the proof.

Lemma D.1. For any time interval \mathcal{I} of length k, there exist a policy $\pi^{\circ} \in \Pi$ for which it holds

$$\sum_{t \in \mathcal{I}} (\lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ)) \le 2k^2 \eta_B - \alpha \sum_{t \in \mathcal{I}} (\mu_t + \lambda_t).$$

Proof. Given an interval \mathcal{I} of length k, let $(\bar{\lambda}, \bar{\mu})$ be the largest Lagrangian multipliers in the interval \mathcal{I} , and let $(\underline{\lambda}, \underline{\mu})$ be the smallest Lagrangian multipliers in such interval. Let $G := \max\{\bar{\lambda} - \underline{\lambda}, \bar{\mu} - \underline{\mu}\}$. Then, we have $G \leq k\eta_{\mathbb{B}}$ since $\eta_{\mathbb{B}}$ is more aggressive than $\eta_{\mathbb{R}}$ (see Section 5.2), and there are at most k gradient updates in the interval. Then,

$$\sum_{t \in \mathcal{I}} (\lambda_t g_t(\pi^\circ) + \mu_t h_t(\pi^\circ)) \leq \sum_{t \in \mathcal{I}: g_t(\pi^\circ) > 0} \bar{\lambda} g_t(\pi^\circ) + \sum_{t \in \mathcal{I}: g_t(\pi^\circ) \leq 0} \underline{\lambda} g_t(\pi^\circ) \\
+ \sum_{t \in \mathcal{I}: h_t(\pi^\circ) > 0} \bar{\mu} h_t(\pi^\circ) + \sum_{t \in \mathcal{I}: h_t(\pi^\circ) \leq 0} \underline{\mu} g_t(\pi^\circ) \\
\leq -\bar{\lambda} \left(\sum_{t \in \mathcal{I}: g_t(\pi^\circ) \leq 0} g_t(\pi^\circ) + \alpha k \right) + \sum_{t \in \mathcal{I}: h_t(\pi^\circ) \leq 0} \underline{\lambda} g_t(\pi^\circ) \\
- \bar{\mu} \left(\sum_{t \in \mathcal{I}: h_t(\pi^\circ) \leq 0} h_t(\pi^\circ) + \alpha k \right) + \sum_{t \in \mathcal{I}: h_t(\pi^\circ) \leq 0} \underline{\mu} g_t(\pi^\circ) \\
\leq kG - \alpha k \bar{\lambda} + kG - \alpha k \bar{\mu} \\
\leq 2kG - \alpha \sum_{t \in \mathcal{I}} (\lambda_t + \mu_t) \\
\leq 2k^2 \eta_{\mathbb{B}} - \sum_{t \in \mathcal{I}} (\lambda_t + \mu_t),$$

where the second inequality comes from $\sum_{t \in \mathcal{I}} g_t(\pi^\circ) \leq -\alpha k$ and $\sum_{t \in \mathcal{I}} h_t(\pi^\circ) \leq -\alpha k$. This concludes the proof. \square