Ejercicio Control 9/10/2021

Esquema:

$$\begin{cases} y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2) \\ k_1 = f(t_n, y_n) \\ k_2 = f(t_n + h/2, y_n + h/2\, k_1) \end{cases}$$

Runge-Kutta de dos etapas. Tablero de Butcher

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

Coeficientes: $c_2 = a_{21} = 1/2 \qquad b_1 = b_2 = 1/2$

orden 2 si $b_2 c_2 = 1/2$ pero $b_2 c_2 = \frac{1}{2}\frac{1}{2} = \frac{1}{4} \neq \frac{1}{2}$

Por lo tanto el método es de dos etapas pero tiene orden 1 al ser $b_1 + b_2 = 1$ pero $b_2 c_2 \neq 1/2$.

a) Calculemos el error de consistencia local:

$$\ell_y(t;h) = y(t+h) - y(t) - \frac{h}{2}\left(k_1(y(t)) + k_2(y(t))\right)$$

Sabemos que $k_1(y(t)) = f(t, y(t)) = y'(t)$

$$\begin{aligned} k_2(y(t)) &= f(t + h/2, y(t) + h/2\, y'(t)) = \\ &= f(t, y(t)) + h/2\, \partial_t f + h/2\, y'(t)\, \partial_y f + \\ &\quad + (h/2)^2 \frac{1}{2!} \partial_{tt} f + \frac{h}{2} \cdot \frac{h}{2} y'(t)\, \partial_t \partial_y f + \left(\frac{h}{2} y'(t)\right)^2 \frac{1}{2!} f_{yy} \\ &\quad + O(h^3) \end{aligned}$$

luego $k_2(y(t)) = y'(t) + h/2\, y''(t) + \frac{h^2}{4}\underbrace{\left[\frac{1}{2}\partial_{tt} f + f \cdot f_{ty} + \frac{1}{2} f^2 \cdot f_{yy}\right]}_{(*)} + O(h^3)$

Entonces

$$\ell_y(t;h) = y(t+h) - y(t) - \tfrac{h}{2}y'(t) - \tfrac{h}{2}y'(t) - \tfrac{h^2}{4}y''(t) - \tfrac{h^3}{8}[\circledast] + O(h^4)$$
$$= y(t+h) - y(t) - h\,y'(t) - \tfrac{h^2}{4}y''(t) - \tfrac{h^3}{8}[\circledast] + O(h^4)$$
$$= \left[y(t) + y'(t)h + y''(t)\tfrac{h^2}{2} + y'''(t)\tfrac{h^3}{3!} + \dots\right) - y(t) - y'(t)h - \tfrac{h^2}{4}y''(t) - \dots$$
$$= \tfrac{1}{4}y''(t)h^2 + h^3\left[\tfrac{1}{3!}y'''(t) - \tfrac{1}{8}\left(\tfrac{1}{2}f_{tt} + f f_{ty} + \tfrac{1}{2}f^2 f_{yy}\right)\right] + O(h^4)$$

Entonces $\ell_y(t;h) = O(h^2)$ ya que $y''(t) \neq 0$ en general.

Por lo tanto, el método posee error local de consistencia de orden 2, con respecto a h.

b) Estabilidad:

$$y_{n+1} - z_{n+1} = y_n - z_n + \tfrac{h}{2}\left(k_1(y_n) - k_1(z_n) + k_2(y_n) - k_2(z_n)\right)$$

$$|k_1(y_n) - k_1(z_n)| = |f(t_n, y_n) - f(t_n, z_n)| \leq L_f |y_n - z_n|$$

$$|k_2(y_n) - k_2(z_n)| = |f(t_n + \tfrac{h}{2}, y_n + \tfrac{h}{2}k_1(y_n)) - f(t_n + \tfrac{h}{2}, z_n + \tfrac{h}{2}k_1(z_n))| \leq$$
$$\leq L_f\, |y_n + \tfrac{h}{2}k_1(y_n) - (z_n + \tfrac{h}{2}k_1(z_n))| \leq L_f |y_n - z_n| + L_f^2 \tfrac{h}{2} |y_n - z_n|$$

de donde $|k_2(y_n) - k_2(z_n)| \leq \left(L_f + \tfrac{h}{2}L_f^2\right)|y_n - z_n|$

Finalmente:

$$|y_{n+1} - z_{n+1}| \leq |y_n - z_n| + \tfrac{h}{2}L_f|y_n - z_n| + \tfrac{h}{2}\left(L_f + \tfrac{h}{2}L_f^2\right)|y_n - z_n|$$
$$= \left(1 + \frac{hL_f}{2} + \frac{hL_f}{2} + \frac{h^2L_f^2}{2}\right)|y_n - z_n|$$

luego $|y_n - z_n| \leq \left(1 + hL_f + \dfrac{h^2L_f^2}{2}\right)^n |y_0 - z_0| \leq$
$$\leq e^{n\left(hL_f + \frac{h^2L_f^2}{2}\right)} |y_0 - z_0| \leq$$
$$\leq e^{TL_f} |y_0 - z_0|$$

luego si $|y_0 - z_0| \underset{h\to 0}{\longrightarrow} 0 \implies |y_n - z_n| \to 0 \quad \forall \underset{n\to\infty}{h\to 0}$ tq $nh \leq T$

c) Convergencia

$$y(t_n+h) - y_{n+1} = \underbrace{y(t_n+h) - \tilde{y}_{n+1}}_{\text{consistencia}} + \underbrace{\tilde{y}_{n+1} - y_{n+1}}_{\text{estabilidad}}$$

$$\leq O(h^2) + \left(1+hL_f+\frac{h^2L_f^2}{2}\right)|y(t_n)-y_n|$$

Entonces $|y(t_n+h)-y_{n+1}| = E_{n+1}$, $\Lambda_f = L_f + \frac{hL_f^2}{2}$

$$E_{n+1} \leq (1+h\Lambda_f)E_n + cte \cdot h^2$$

de donde

$$E_{n+1} \leq (1+h\Lambda_f)^n E_0 + \frac{(1+h\Lambda_f)^n - 1}{1+h\Lambda_f-1} cte\, h^2$$

$$\leq e^{T\Lambda_f} E_0 + \frac{e^{T\Lambda_f}-1}{\Lambda_f} cte \cdot h$$

$$= Cte(T,\Lambda_f)(E_0+h)$$

Luego si $E_0 = |y_0 - y(t_0)| = O(h)$ hay convergencia con órden 1 en h.

d) $\begin{cases} y'(t) = t \\ y(0)=0 \end{cases} \rightarrow f(t,y)=t$

$$k_1 = f(t_n, y_n) = t_n, \quad k_2 = f(t_n+h/2, y_n + h/2\, k_1) = t_n + h/2$$

$$t_n + t_n + h/2 = nh + nh + h/2 = 2nh + h/2$$

$$y_{n+1} = y_n + h/2\,(2nh + h/2) = y_n + \frac{h^2}{4}(4n+1)$$

$$y_{n+1} - y_n = \frac{h^2}{4}(4n+1)$$

$$\sum_{n=0}^{N-1}(y_{n+1}-y_n) = y_N - y_0 = \frac{h^2}{4}\left(4\frac{N(N+1)}{2} + N\right)$$

Como $y_0 = 0$ $\quad y_N = h^2\frac{N(N+1)}{2} + \frac{h^2}{4}N = \frac{h^2N^2}{2} + \frac{h^2N}{2} + \frac{h^2N}{4}$

$$y_N = \frac{t_N^2}{2} + h\, t_N \frac{3}{4} \Rightarrow \underline{y_N - y(t_N) = \frac{3}{4} t_N \cdot h}$$