1591 Métodos Numéricos para las Ecuaciones Diferenciales. Curso 2021-2022.
Grado en Matemáticas. Universidad de Murcia.
Date: 2022/06/22, from 09:30 to 13:45

Remark: Read all the exam before start. From 10:00 on the result counts on your records.

Total score: $0.7\times T + 0.3\times P$ as long as both $T \geq 4$ and $P \geq 4$ hold.

Name: **ID:**

Theory, total of 10 points

1. Let $f \in C^0(\mathbb{R})$ be globally Lipschitz. Given the Cauchy problem

$$\left\{ \begin{array}{lcl} y'(t) & = & f(y(t)) \text{ en } [t_0, t_0+T], \\ y(t_0) & = & \alpha, \end{array} \right.$$

we approximate the solution with the following numerical method

$$M_\theta \equiv \left\{ \begin{array}{lcl} y_{n+1} & = & y_n + h\, f\left(y_n + \theta h f(y_n)\right) \\ y_0 & = & \alpha \end{array} \right.$$

where $t_n = t_0 + nh$, $h = T/N$ and $\theta \in [0,1]$ is a parameter to choose.

Se pide:

a) **(0.5 points)** Give the geometrical meaning of M_θ.

b) **(0.5 points)** Prove that M_θ is 0-estable for any θ.

c) **(1 point)** Determine the consistency order of M_θ in terms of θ.

d) **(1 point)** Bound the global error for the best possible choice of θ. This choice gives the **modified Euler method**.

2. **(2 points)** Prove that the explicit Euler method applied to

$$w'(t) = 1 + t^2,\ w(0) = 0$$

with $y_0 = 0$ and $t_n = n\,h$ is of order 1 with respect to h but it can not be of order 2 with respect to h.

3. **(3 points)** Give the geometrical meaning of this explicit Runge-Kutta method

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 2/3 & 2/3 & 0 & 0 \\ 2/3 & 1/3 & 1/3 & 0 \\ \hline & 1/6 & 2/6 & 3/6 \end{array}$$

and apply it to the problem $y'(t) = y(t),\ y(0) = 1$. Obtain the global error estimation for this problem and explain your result.

(turn over)

4. **(2 points)** Find the values of α such that

$$y_{n+2} + 2\,\alpha\, y_{n+1} - (2\alpha + 1)\, y_n \quad = \quad h[(\alpha + 2)\, f_{n+1} + \alpha\, f_n],\; n \geq 0$$

is 0-estable, consistent and determine the largest possible order.

Computation, over 10 points: from 12:15 to 13:45

Instructions:

- Send codes and plots to **eliseo@um.es**. Names of codes and plots must follow the format `Apellido_DNI_Ejercicio_X.m` donde $X = 1, 2, 3$.
- Codes must indicate whithin comment lines your name and id. The main steps should also be clearly indicated in comment lines. If you reuse another code clear off all that does not belong to the problem.
- If your code does not compile the mark will be cero.

1. **(3 points)** Computationally determine the convergence order of the Runge-Kutta method

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h}{7}\,(k_1 + 3\,k_2 + 3\,k_3) \\
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{1}{2}h,\, y_n + \frac{1}{2}h\,k_1) \\
k_3 &= f(t_n + \frac{2}{3}h,\, y_n + \frac{1}{3}h\,k_1 + \frac{1}{3}h\,k_2)
\end{aligned}
$$

approximating the solution of the differencial equation

$$
\left\{
\begin{array}{rcl}
y'(t) &=& -4t^3\,y(t)^2, \quad t \in (-10, 0], \\
y(-10) &=& 1/10001.
\end{array}
\right.
$$

with exact solution $y(t) = 1/(t^4 + 1)$. **Send by mail code and plot with the slope line contrasting the order** of the method. Use legends to each of the lines obtained Title your figure with your name and estimated order p: **Yourname: problem 1, order p**

(turn over)

2. **(4 points)** Given $\alpha \in (-\pi/2, \pi/2)$ the pendulum equation is given by

$$y''(t) + \sin(y(t)) = 0, \quad y(0) = \alpha, \quad y'(0) = 0.$$

Use a Runge-Kutta method of your choice to approximate the solution. The first time when $y'(t) = 0$ is given by the expresion

$$\pi\,(1 + \frac{1}{4}\sin^2(\frac{\alpha}{2}) + \frac{9}{64}\sin^4(\frac{\alpha}{2})).$$

Use this result to validate the accuracy of your method with respect to h for the value $\alpha = \pi/4$.
Send code and plot of (y,y'). Title the plot with: **Yourname: Problem 2, phase plane**

3. **(3 points)** Chek on the 0-stability with the method

$$y_{n+2} - (1+a)y_{n+1} + a\,y_n \;=\; \frac{h}{2}[(3-a)f(t_{n+1}, y_{n+1}) - (1+a)f(t_n, y_n)]$$

to approximate the solution $y(t) = (t^2+1)^2$ on $[0,1]$ of the equation $y' = 4ty^{1/2}$ with $y(0) = 1$. Use the values $a = 0$ and $a = -5$ with steps $h = 0.1, 0.05, 0.025$.
Send code and plots. Title with the format **Yourname: Problem 3,** $a = ..., h = ...$