Engineering mathematics (2015)

Q1. Fill in the blank :

(i) The value of $^nC_r + {^nC_{r-1}}$ ***is...............***

Sol. We have

$$^nC_r + {^nC_{r-1}} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\{n-(r-1)\}!}$$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\{n-(r-1)\}!}$$

$$= \frac{n!(n-r+1)}{r!(n-r+1)!} + \frac{n!r}{r!(n-r+1)!}$$

$$= \left\{\frac{n!}{r!(n-r+1)!}\right\}(h-r+1+r)$$

$$= \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!}$$

$$= {^{n+1}C_r}$$

(ii) The value of Determinate $\begin{vmatrix} a & b & a+b \\ b & c & b+c \\ c & a & c+a \end{vmatrix}$ = ...

Sol. We have

$$\Delta = \begin{vmatrix} a & b & a+b \\ b & c & b+c \\ c & a & c+a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} + \begin{vmatrix} a & b & b \\ b & c & c \\ c & a & a \end{vmatrix}$$

$$= 0 + 0 = 0$$

[$\because$ C_1 and C_3 are identical & C_2 and C_3 are identical].

(iii) Total number of terms in the expansion of $(a+x)^n$ ***is***

Sol. We have

$$(a+x)^n = {^nC_0}a^n + {^nC_1}a^{n-1}x + {^nC_2}a^{n-2}x^2 + \dots + {^nC_{n-1}}a_x^{\,n-1} + {^nC_n}x^n$$

So, The expansion of $(a+x)^n$ has (n + 1) terms.

(iv) In a triangle ABC, if A = tan⁻¹2, B = tan⁻¹C then ∠C = ?

Sol. We have In Δ ABC

$$A = \tan^{-1}2, \;\; B = \tan^{-1}C$$

Also we have in Δ ABC

$$A + B + C = \pi$$

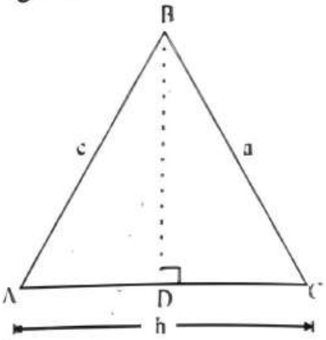
$$\tan^{-1}2 + \tan^{-1}C + C = \pi$$

$$\tan^{-1}\left(\frac{2+c}{1-2c}\right) + c = \pi$$

$$c = \pi - \tan^{-1}\left(\frac{2+c}{1-2c}\right)$$

(v) In $\Delta ABC, \cos A$ = ...

Sol. Let Δ ABC be given



From figure we have

$$BC^2 = BD^2 + DC^2$$

$$= BD^2 + (AC - AD)^2$$

$$= BD^2 + AD^2 + AC^2 - 2AC\,AD$$

$$= AB^2 + AC^2 - 2AC.AC\cos A$$

$$= C^2 + b^2 - 2bc\cos A$$

$$[\because BD^2 + AD^2 = AB^2 \;\&\; AD = AB\cos A]$$

$$\Rightarrow \quad a^2 = b^2 + c^2 - 2bc\cos A$$

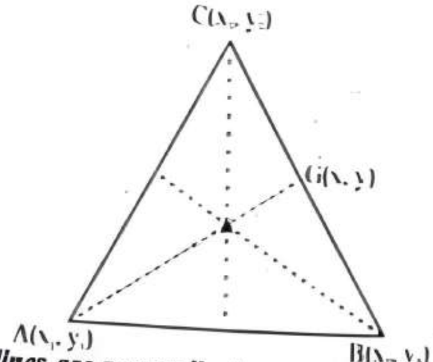
$$\Rightarrow 2bc\cos A = b^2 + c^2 - a^2$$

$$\Rightarrow \quad \boxed{\cos A = \frac{(b^2 + c^2 - a^2)}{2bc}}$$

(vi) If (x_1, y_1), ***B*** (x_2, y_2) ***and*** $C(x_3, y_3)$ ***be the vertices of a triangle ABC then co-ordinate of centroid =.....***

Sol. The co-ordinate of the centroid whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are given by

$$x = \frac{x_1 + x_2 + x_3}{3} \; and \;\; y = \frac{y_1 + y_2 + y_3}{3}$$



(vii) If two lines are perpendicular to each other having their slopes m_1 ***and*** m_2 ***then*** m_1m_2 = ...

Sol. Two lines are perpendicular to each other having their slopes m_1 and m_2 if product of their slope is equal to -1

i.e., $\boxed{m_1, m_2 = -1}$

(viii) Slope of straight line $ax + by + c = 0$ is equal to ....

Sol. We have to equation of line as

$$ax + by + c = 0 \quad \text{...............(i)}$$

(a) can be written as

$$by = -a_x - c$$

$$y = -\frac{a_x}{b} - \frac{c}{b} \quad \text{...............(ii)}$$

comparing equation (ii) with slope intercept from

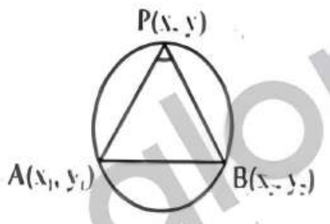
$y = mx + c$, we get

$$\boxed{\text{slope} = m = -\frac{a}{b}}$$

So, $\boxed{\text{slope} = m = -\frac{a}{b} = -\frac{\text{co-efficient of x}}{\text{co-efficient of y}}}$

(ix) Equation of circle whose ends of diameter are (x_1, y_1) and (x_2, y_2) is ....

Ans. Let P (x, y) be any point on the circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as ends of the diameter.



Then slope of line $AP = m_1 = \frac{y - y_1}{x - x_1}$

and slope of line $BP = m_2 = \frac{y - y_2}{x - x_2}$

But $AP \perp BP$

$\therefore \quad m_1, m_2 = -1$

$\Rightarrow \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$

$\Rightarrow (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

$\therefore \quad \boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0}$

Which is the required equation of circle in the diameter form.

(x) The centre of circle $x^2 + y^2 - 4x - 6y - 87 = 0$ is ....

Ans. We have the circle

$$x^2 + y^2 - 4x - 6y - 87 = 0 \quad \text{....(i)}$$

Also the general equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(ii)}$$

Comparing (i) and (ii), we get,

$2g = -4, \qquad 2f = -6$

$\therefore \quad g = -2, \qquad f = -3$

Centre $= (-g, -f) = (2, 3)$

So, the centre of circle (i) is (2, 3).

Q2.(a) If $\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y}$ then prove that $x^x \cdot y^y \cdot z^z = 1$.

Ans. We have $\frac{\log x}{y - z} = \frac{\log y}{z - x} = \frac{\log z}{x - y}$

From first two

$(z - x)\log x = (y - z)\log y \quad$ ...(i)

$(x - y)\log x = (y - z)\log z \quad$ ...(ii)

$(x - y)\log y = (z - x)\log z \quad$ ...(iii)

Adding (i) and (ii)

$(z - x + x - y)\log x = (y - z)[\log y + \log z]$

$\Rightarrow (z - y)\log x = (-y - z)(\log y + \log z)$

$\Rightarrow \log x + \log y + \log z = 0 \quad$ ...(iv)

Now consider,

$x\log x + y\log y + z\log z = x\log x$

$\quad + (y - x + x)\log y + (z - x + x)\log$

$= x(\log x + \log y + \log z) + (y - x)\log y + (z - x)\log z$

$= x \times 0 + (y - x)\log y + (z - x)\log z \quad$ [using (iv)]

$= (y - x)\log y + (x - y)\log y \quad$ [using (iii)]

$= -(x - y)\log y + (x - y)\log y = 0$

$\Rightarrow x\log x + y\log y + z\log z = 0$

$\Rightarrow \log x^x + \log y^y + \log z^z = 0 \quad \{n \log m = \log m^n\}$

$\Rightarrow \log x^x \cdot y^y \cdot z^z = 0 \quad \{\log n + \log m = \log mn\}$

$\Rightarrow x^x \cdot y^y \cdot z^z = e^0$

$\Rightarrow \boxed{x^x y^y z^z = 1}$ = R.H.S

Q2.(b) Prove that :

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Ans. Let the given determinant be Δ. Then

$$\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} R_1 \to (R_1 + R_2 + R_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Taking common (a + b + c) from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$C_2 \to C_2 - C_1 ; C_3 \to C_3 - C_1$$

Expanding by R_1

$$= (a+b+c)\begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)[(a+b+c)^2] = (a+b+c)^3$$

Hence, $\Delta = (a+b+c)^3$ = R.H.S

Q3.(a) Resolve $\frac{2x+3}{(x-3)(x+1)}$ into partial fractions.

Ans. Let $\frac{2x+3}{(x-3)(x+1)} = \frac{A}{(x-3)} + \frac{B}{(x+1)}$ ....(i)

$$\Rightarrow \frac{(2x+3)}{(x-3)(x+1)} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

$$\Rightarrow 2x+3 = A(x+1)+B(x-3) \quad ...(ii)$$

For A put $x = 3$ in equation (ii)

$9 = 4A$ $\quad \boxed{A = 9/4}$

For B put $x = -1$ in equation (ii)

$1 = -4B$ $\quad \boxed{B = -1/4}$

$\therefore$ (i) becomes

$$\boxed{\frac{(2x+3)}{(x-3)(x+1)} = \frac{9}{4(x-3)} - \frac{1}{4(x+1)}}$$

Q3.(b) Find the 5th term of $(x + 1/x)^{13}$.

Ans. We know that in the expansion of $(a+b)^n$, we have

$(r+1)^{th}$ term $= t_{r+1} = {}^nC_r a^{n-r} b^r$

$\therefore$ In the expansion of $\left(x + \frac{1}{x}\right)^{13}$, we have

5^{th} term $= t_5 = t_{4+1} = {}^{13}C_4 x^{13-4}\left(\frac{1}{x}\right)^4$

[Here, $a = x$, $b = 1/x$, $r = 4$, $n = 13$]

$$= {}^{13}C_4 x^9 \frac{1}{x^4} = {}^{13}C_4 x^5$$

$$= \frac{13!}{4!(13-4)!}x^5 = \frac{13 \times 12 \times 11 \times 10(9)!}{4 \times 3 \times 2 \times 1(9)!}x^5$$

$$= \frac{\quad}{4 \times 3 \times 2}$$

So, the required 5th term is $715x^5$.

Q4.(a) Prove that :

$$\cos 20^0 \cdot \cos 40^0 \cdot \cos 60^0 \cdot \cos 80^0 = 1/60$$

Ans. We have :

$$\text{L.H.S} = \cos 20^0 \cdot \cos 40^0 \cdot \cos 60^0 \cdot \cos 80^0$$

$$= \frac{1}{2}(\cos 40^0 \cdot \cos 60^0) \cdot (2\cos 80^0 \cdot \cos 20^0)$$

$$= \frac{1}{2}(\cos 40^0 \cdot 1/2) \cdot [\cos 100^0 + \cos 60^0]$$

$$= \frac{1}{4}\cos 40^0(\cos 100^0 + 1/2)$$

$$= \frac{1}{8}\cos 40^0(2\cos 100^0 + 1)$$

$$= \frac{1}{8}\cos 40^0 + \frac{1}{8}(2\cos 100^0 \cdot \cos 40^0)$$

$$= \frac{1}{8}\cos 40^0 + \frac{1}{8}[\cos 140^0 + \cos 60^0]$$

$$= \frac{1}{8}\cos 40^0 + \frac{1}{8}\cos 140^0 + \frac{1}{8} \times \frac{1}{2}$$

$$= \frac{1}{8}\cos 40^0 + \frac{1}{8}\cos(180^0 - 40^0) + \frac{1}{16}$$

$$= \frac{1}{8}\cos 40^0 - \frac{1}{8}\cos 40^0 + \frac{1}{16}$$

$$= \frac{1}{8}\cos 40^0 - \frac{1}{8}\cos 40^0 + \frac{1}{16} = \frac{1}{16} = \text{R.H.S}$$

Q4.(b) If $\tan\theta = \frac{a}{b}$ then prove that

$$a\sin 2\theta + b\cos 2\theta = b.$$

Ans. We have :

$$\tan\theta = \frac{a}{b}$$

To prove that $a\sin 2\theta + b\cos 2\theta = b$

We know that

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

L.H.S $= a\sin 2\theta + b\cos 2\theta$

$$= a\left(\frac{2\tan\theta}{1+\tan 2\theta}\right) + b\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{2a\tan\theta}{1+\tan^2\theta} + \frac{b(1-\tan^2\theta)}{1+\tan^2\theta}$$

$$= \frac{2a\cdot\frac{a}{b}}{1+a^2/b^2} + \frac{b(1-a^2/b^2)}{1+a^2/b^2} \quad [\because \tan\theta = a/b]$$

$$= \frac{2a^2/b + b\left(1-a^2/b^2\right)}{\left(a^2+b^2\right)/b^2}$$

$$= b^2\frac{[2a^2/b + b(b^2-a^2)/b^2]}{a^2+b^2}$$

$$= \frac{b^2[2a^2/b + (b^2-a^2)/b]}{(a^2+b^2)}$$

$$= b\frac{[2a^2+b^2-a^2]}{a^2+b^2} = b\frac{(a^2+b^2)}{(a^2+b^2)} = b.$$

Q5. (a) Prove that :

$$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$$

Ans. Let $\tan\theta = x$

Then, $\sin^{-1}\cdot\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$

$$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$$

$$\Rightarrow \boxed{2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}} \quad \text{.............(i)}$$

Again, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$= \cos^{-1}(\cos 2\theta) = 2\theta$$

$$\boxed{\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1}x} \quad \text{.............(ii)}$$

Also, $\tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\left(\frac{2+\tan\theta}{1-\tan^2\theta}\right)$

$$= \tan^{-1}(\tan 2\theta) = 2\theta$$

$$\boxed{\tan^{-1}\frac{2x}{1+x^2} = 2\tan^{-1}x} \quad \text{.............(iii)}$$

From (i), (ii) and (iii) we get

$$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}.$$

Q5.(b) In any ΔABC, show that :

$$\sin A/2 = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

Ans. We know that

$$2\sin^2\frac{A}{2} = (1-\cos A) \quad \text{....(i)}$$

Also we have cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\therefore \quad \cos A = \frac{(b^2+c^2-a^2)}{2bc}$$

$\therefore$ (i) becomes,

$$2\sin^2 A/2 = \left[1 - \frac{(b^2+c^2-a^2)}{2bc}\right]$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc} \quad \text{....(ii)}$$

$$\{\because a^2-b^2 = (a+b)(a-b)\}$$

Also, In Δ ABC

$$a+b+c = 2s$$

$$\therefore \quad a+b-c = 2(s-c)$$

$$a-b+c = 2(s-b)$$

$\therefore$ (ii) becomes,

$$2\sin^2 A/2 = \frac{2(s-c)\times 2(s-b)}{2bc}$$

$$\sin^2 A/2 = \frac{(s-c)(s-b)}{bc}$$

$$\Rightarrow \quad \boxed{\sin\frac{A}{2} = \sqrt{\frac{(s-c)(s-b)}{2bc}}}$$

Q6. (a) Two vertices of a triangle are (–4, 6), (2, –2) and its centroid is (0, 3) find the third vertex.

Ans. Let A(–4, 6) and B(2, –2) be the two vertices of Δ ABC

Let the 3rd vertex be C(x_3, y_3) and G(0, 3) be the centroid of Δ ABC. Then we get

$$x_1 = -4, \quad x_2 = 2$$

$$y_1 = 6, \quad y_2 = -2$$

Also by formula of centroid we have

$$x = \frac{x_1+x_2+x_3}{3} \qquad y = \frac{y_1+y_2+y_3}{3}$$

$$\Rightarrow \quad 0 = \frac{-4+2+x}{3} \qquad 3 = \frac{y_1+y_2+y_3}{3}$$

$$\Rightarrow \quad -2+x_3 = 0 \qquad 9 = 4+y_3$$

$$\Rightarrow \quad x_3 = 2 \qquad y_3 = 5$$

So, the 3rd vertex is C(2, 3).

$$= a\left(\frac{2\tan\theta}{1+\tan 2\theta}\right) + b\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{2a\tan\theta}{1+\tan^2\theta} + \frac{b(1-\tan^2\theta)}{1+\tan^2\theta}$$

$$= \frac{2a\cdot\frac{a}{b}}{1+a^2/b^2} + \frac{b(1-a^2/b^2)}{1+a^2/b^2}\ [\because \tan\theta = a/b]$$

$$= \frac{2a^2/b + b\left(1-a^2/b^2\right)}{\left(a^2+b^2\right)/b^2}$$

$$= b^2\frac{[2a^2/b + b(b^2-a^2)/b^2]}{a^2+b^2}$$

$$= \frac{b^2[2a^2/b + (b^2-a^2)/b]}{(a^2+b^2)}$$

$$= b\frac{[2a^2+b^2-a^2]}{a^2+b^2} = b\frac{(a^2+b^2)}{(a^2+b^2)} = b = \text{R.H.S}$$

Q5. (a) Prove that :

$$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2} = \tan^{-1}\frac{2x}{1-x^2}$$

Ans. Let $\tan\theta = x$

Then, $\sin^{-1}\cdot\frac{2x}{1+x^2} = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$

$= \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}x$

$$\Rightarrow \boxed{2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}} \quad \text{.......(i)}$$

Again, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \cos^{-1}(\cos 2\theta) = 2\theta$

$$\boxed{\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1}x} \quad \text{.......(ii)}$$

Also, $\tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\left(\frac{2+\tan\theta}{1-\tan^2\theta}\right)$

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From (i), (ii) and (iii) we get

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Ans. We know that

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Also we have cosine formula

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$\therefore$ (i) becomes,

$$\sin^2 A/2 = \left[1-\frac{(b^2+c^2-a^2)}{2bc}\right]$$

$$= \frac{2bc-b^2-c^2+a^2}{2bc} = \frac{a^2-(b-c)^2}{2bc}$$

$$= \frac{(a+b-c)(a-b+c)}{2bc} \quad \text{....(ii)}$$

$$\{\because a^2-b^2 = (a+b)(a-b)\}$$

Also, In Δ ABC

$a+b+c = 2s$

$\therefore \quad a+b-c = 2(s-c)$

$a-b+c = 2(s-b)$

$\therefore$ (ii) becomes,

$$2\sin^2 A/2 = \frac{2(s-c)\times 2(s-b)}{2bc}$$

$$\sin^2 A/2 = \frac{(s-c)(s-b)}{bc}$$

$$\Rightarrow \boxed{\sin\frac{A}{2} = \sqrt{\frac{(s-c)(s-b)}{2bc}}}$$

Q6. (a) Two vertices of a triangle are (–4, 6), (2, –2) and its centroid is (0, 3) find the third vertex.

Ans. Let A(–4, 6) and B(2, –2) be the two vertices of Δ ABC

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Also by formula of centroid we have

$$x = \frac{x_1+x_2+x_3}{3} \qquad y = \frac{y_1+y_2+y_3}{3}$$

$$\Rightarrow 0 = \frac{-4+2+x}{3} \qquad 3 = \frac{y_1+y_2+y_3}{3}$$

$$\Rightarrow -2+x_3 = 0 \qquad 9 = 4+y_3$$

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So, the 3rd vertex is C(2, 3).