

Orders

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Contents

1	Preorders	2
2	Linear Orders	4
3	General Hausdorff Rank	6
4	ω-Hausdorff rank	8
5	WO-Hausdorff rank	11
6	Decidability of the rank	12
7	Everything Better	17

1 Preorders

Definition 1.1 (Preorder). A relation \leq is a preorder if it is reflexive and transitive.

Definition 1.2 (Property of preorders). A property \mathbf{P} of preorders is a class of preorders which is closed under isomorphism.

Definition 1.3. A property \mathbf{P} of preorders is monotone if for every preorder M , $M \in \mathbf{P}$ implies that every suborder of M is in \mathbf{P} .

Definition 1.4. A property \mathbf{P} of preorders is an additive property if for every preorders M_1 and M_2 , $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$, i.e., if $\mathbf{P} + \mathbf{P} = \mathbf{P}$.

Definition 1.5. Let M be a preorder.

Then M^* is the dual - the reverse preorder of M .

Definition 1.6 (Sum of preorders). Let I be a preorder, and let $\{M_i\}_{i \in I}$ be a family of preorders.

The sum $M = \sum_{i \in I} M_i$ is defined as follows:

The domain is $M = \bigsqcup_{i \in I} M_i$ (a disjoint union).

Let \leq_i be the preorder on M_i .

The order is defined as follows:

$$x \leq y \iff \begin{cases} \exists i \in I. x, y \in M_i \wedge x \leq_i y \\ \exists i, j \in I. x \in M_i \wedge y \in M_j \wedge i < j \end{cases}$$

If $I = 2$, we define $M_1 + M_2 := \sum_{i \in 2} M_i$.

Lemma 1.7. Let I be a preorder, and let $\{M_i\}_{i \in I}$ be a family of preorders.

Then $M = \sum_{i \in I} M_i$ is a preorder.

Proof. Reflexivity is clear.

For transitivity, suppose $x \leq y$ and $y \leq z$.

Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.

Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.

Otherwise, $i < k$, and thus $x \leq z$, as required. \square

Definition 1.8. Let \mathbf{P}_1 and \mathbf{P}_2 be properties of preorders.

Then we define

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

Definition 1.9. Let \mathbf{P} and \mathbf{Q} be a property of preorders.

Then we define

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_{i \in I} M_i : I \in \mathbf{P} \wedge \forall i \in I. M_i \in \mathbf{Q} \right\}$$

Furthermore, if $\mathbf{P} = \{I\}$ is a singleton, we define $\sum_I \mathbf{Q} := \sum_{\mathbf{P}} \mathbf{Q}$.

Definition 1.10 (Kleene plus). *Let \mathbf{P} be a property of preorders.*

We define its Kleene plus as the smallest property of preorders \mathbf{P}^+ which contains \mathbf{P} and is closed under finite sums.

That is, $1^+ = \{1, 2, \dots\}$, and $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$.

2 Linear Orders

Definition 2.1. Let M be a linear order.

A set $A \subseteq M$ is left cofinal in M if for every $x \in M$, there exists $y \in A$ such that $y < x$.

A set $A \subseteq M$ is right cofinal in M if for every $x \in M$, there exists $y \in A$ such that $x < y$.

A set $A \subseteq M$ is bi-directionally cofinal in M if it is both left and right cofinal.

Lemma 2.2. Let \mathbf{P} be an additive property of linear orders.

Then $1 \in \mathbf{P}$.

Note 2.3. The above lemma is false if we do not restrict ourselves to linear orders.

For example, $(1 \uplus 1)^+$ is a property of preorders which is additive, but does not contain 1.

Proof. Let $M \in \mathbf{P}$ be any linear order.

Let $x \in M$. Then, $M = \{y \in M : y < x\} + \{x\} + \{y \in M : y > x\}$.

We conclude that $\{x\} \in \mathbf{P}$, thus $1 \in \mathbf{P}$. \square

Corollary 2.4. Let \mathbf{P} be an additive property of linear orders.

Let M be a linear order.

Let $x, y \in M$ be any two points in a linear order M . Then the following are equivalent:

1. $(x, y) \in \mathbf{P}$.
2. $(x, y] \in \mathbf{P}$.
3. $[x, y) \in \mathbf{P}$.
4. $[x, y] \in \mathbf{P}$.

Proof. This is just applying the definition of an additive property to the intervals $[x, y]$ and the order 1. \square

Definition 2.5. Let \mathbf{P} be a property of linear orders.

We define $\mathcal{B}[\mathbf{P}]$ to be the class of linear orders M such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in \mathbf{P} .

Definition 2.6. A property \mathbf{P} of linear orders is a star property if for every linear orders M , and every family $\mathcal{F} \subseteq \mathbf{P}$ of subintervals of M such that $J_1 \cap J_2 \neq \emptyset$ for every $J_1, J_2 \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \in \mathbf{P}$.

Lemma 2.7 (Star Lemma). Let \mathbf{P} be an additive property of linear orders.

Then the property $\mathcal{B}[\mathbf{P}]$ is a star property.

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of M .

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in \mathbf{P} .

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. However, \mathbf{P} is additive. Since $[x, y]$ is either the sum or difference of $[x, z]$ and $[z, y]$, we have that $[x, y] \in \mathbf{P}$. \square

Lemma 2.8. *Let \mathbf{P} be a star property.*

Then for every linear order M , and every point $x \in M$, there exists a largest subinterval $J \subseteq M$ such that $J \in \mathbf{P}$.

Equivalently, we can define a convex equivalence relation $\sim_{\mathbf{P}}$ on M such that $x \sim_{\mathbf{P}} y$ iff $[x, y] \in \mathbf{P}$.

That is, $x \sim_{\mathbf{P}} y$ iff x and y are in the same largest \mathbf{P} -subinterval.

Proof. Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$ -subintervals containing x . All such subintervals intersect at x .

Therefore, by the star lemma, J is in $\mathcal{B}[\mathbf{P}]$, and by definition J is the largest \mathbf{P} -subinterval containing x .

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. \square

Definition 2.9. *Let I be a linear order labeled with colors $\vec{C} = \{C_k\}_{k=1}^m$. Let $\gamma : I \rightarrow \vec{C}$ be the coloring function.*

Let $\vec{M} = \{M_k\}_{k=1}^m$ be a family of labeled linear orders.

Then we define the labeled sum,

$$\sum_I [\vec{C} \leftarrow \vec{M}] := \sum_{i \in I} M_{\gamma(i)}$$

Lemma 2.10 (Associativity of sum). *Let $\mathbf{P}_1, \mathbf{P}_2$ and \mathbf{P}_3 be properties.*

Then $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$.

Proof. It follows directly from the associativity of the sum operation on linear orders. Actually, it generalizes to any algebraic equation which holds on linear orders. \square

Lemma 2.11 (Sum over a union). *Let \mathcal{P} be a family of properties.*

Let \mathbf{Q} be a property.

Then $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$.

Proof. This is obvious from the definition of the sum operation. \square

Definition 2.12. *Let $\beta \geq \omega$ be a limit ordinal.*

We define $\Gamma_\beta := \{\gamma : \gamma \subseteq \beta^ + \beta\}^+$.*

Example 2.13.

$$\Gamma_\omega = \{1, \omega, \omega^*\}^+$$

Observation 2.14. *Let $\beta \geq \omega$ be a limit ordinal.*

Then Γ_β is a good property of linear orders.

3 General Hausdorff Rank

Definition 3.1. Let \mathbf{Q} be a good property of linear orders.

We define a property $\mathbf{Q}^{<\alpha}$ for every ordinal $\alpha > 0$ as follows:

- For $\alpha = 1$, $\mathbf{Q}^{<1} = \{1\}$.
- For $1 < \alpha = \gamma + 1$,

$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}$$

- For α a limit ordinal,

$$\mathbf{Q}^{<\alpha} = \bigcup_{\beta < \alpha} \mathbf{Q}^{<\beta}$$

We define further $\mathbf{Q}^{\leq \alpha} = \mathbf{Q}^{<\alpha+1}$ and $\mathbf{Q}^{=\alpha} = \mathbf{Q}^{\leq \alpha} - \mathbf{Q}^{<\alpha}$.

We call such an α , if it exists, the \mathbf{Q} -rank of a linear order M .

Observations 3.2. Let $\alpha, \beta > 0$ be ordinals.

Let \mathbf{Q} be a good property.

The following properties are immediate from the above definition:

- $\mathbf{Q}^{\leq 1} = \mathbf{Q}$.
- $\mathbf{Q}^{<\alpha}$ is a good property iff $\alpha > 1$.
- $\mathbf{Q}^{<\alpha} \subsetneq \mathbf{Q}^{<\beta}$ iff $\alpha < \beta$.

Lemma 3.3. Let $\alpha > 0$, $\delta \geq 0$ be ordinals. Let \mathbf{Q} be a good property.

Then,

$$\mathbf{Q}^{<\alpha+\delta} = \sum_{\mathbf{Q}^{<1+\delta}} \mathbf{Q}^{<\alpha}$$

Proof. We prove by induction on $\delta \geq 0$.

For $\delta = 0$, we need to show that $\mathbf{Q}^{<\alpha} = \sum_1 \mathbf{Q}^{<\alpha}$, which is obviously true.

For $\delta = \varepsilon + 1$, we have $\mathbf{Q}^{<\alpha+\delta} = \mathbf{Q}^{<\alpha+\varepsilon+1} = \sum_{\mathbf{Q}} \mathbf{Q}^{<\alpha+\varepsilon}$.

By the induction hypothesis, $\mathbf{Q}^{<\alpha+\varepsilon} = \sum_{\mathbf{Q}^{<1+\varepsilon}} \mathbf{Q}^{<\alpha}$, and thus we get

$$\mathbf{Q}^{<\alpha+\delta} = \sum_{\mathbf{Q}} \sum_{\mathbf{Q}^{<1+\varepsilon}} \mathbf{Q}^{<\alpha}$$

By associativity,

$$\mathbf{Q}^{<\alpha+\delta} = \sum_{\sum_{\mathbf{Q}} \mathbf{Q}^{<1+\varepsilon}} \mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}^{<1+\varepsilon+1}} \mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}^{<1+\delta}} \mathbf{Q}^{<\alpha}$$

For $\delta > 0$ a limit ordinal, note that $\alpha + \delta = \sup_{\varepsilon < \delta} \alpha + \varepsilon$, and $1 + \delta = \delta$ since δ is infinite.

Then,

$$\mathbf{Q}^{<\alpha+\delta} = \bigcup_{\varepsilon < \delta} \mathbf{Q}^{<\alpha+\varepsilon} = \bigcup_{\varepsilon < \delta} \sum_{\mathbf{Q}^{<1+\varepsilon}} \mathbf{Q}^{<\alpha} = \sum_{\bigcup_{\varepsilon < \delta} \mathbf{Q}^{<1+\varepsilon}} \mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}^{<1+\delta}} \mathbf{Q}^{<\alpha}$$

where the second equality follows by the induction hypothesis. \square

Lemma 3.4. *Let $\alpha > 0$ be an ordinal.*

Let \mathbf{Q} be a good property.

Then over countable linear orders, $\mathcal{B}[\mathbf{Q}^{<\alpha}] \subseteq \mathbf{Q}^{\leq\alpha}$.

Proof. Let $M \in \mathcal{B}[\mathbf{Q}^{<\alpha}]$ be a countable linear order.

Since M is countable, there exists some $I \subseteq \omega^* + \omega$ and an I -sequence $\{x_k\}_{k \in I} \subseteq M$, which is bi-directionally cofinal in M .

That is, $M = \sum_{k \in I} [x_k, x_{k+1})$.

Since $I \subseteq \omega^* + \omega \in \mathbf{Q}$, and since the rank of every interval $[x_k, x_{k+1})$ is $< \alpha$, $M \in \sum_{\mathbf{Q}} \mathbf{Q}^{<\alpha} = \mathbf{Q}^{\leq\alpha}$. \square

4 ω -Hausdorff rank

Lemma 4.1. *Let $\alpha > 0$ be an ordinal.*

Let M be a countable linear order.

Then $M \in \Gamma_\omega^{\leq \alpha}$ iff M is a finite sum of $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subintervals.

Proof. From the previous lemma, it is clear that if M is a finite sum of $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subintervals, then $M \in \Gamma_\omega^{\leq \alpha}$, since the rank bound is preserved under finite sums.

Conversely, suppose $M \in \Gamma_\omega^{\leq \alpha}$.

If $M = \sum_{i \in I} M_i$ for some $M_i \in \Gamma_\omega^{\leq \alpha}$ for $I \in \Gamma_\omega$, take $x, y \in M$. Then let $x \in M_{i_1}$ and $y \in M_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} M_i$.

But the distance between i_1 and i_2 is at most 1, so $\sum_{i \in [i_1, i_2]} M_i$ is a finite sum of $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subintervals.

So we have proven that each interval which is a Γ_ω -sum of linear orders of ω -rank $< \alpha$ is in $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$.

But generally, M is a finite sum of such $\omega^* + \omega$ -sums, so it is a finite sum of $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subintervals. \square

Definitions 4.2. *Let $\alpha > 0$ be an ordinal.*

Let M be a linear order.

We define:

$$1. \mathcal{L}_\alpha := \{M \in \mathcal{B}[\Gamma_\omega^{\leq \alpha}] : 1 + M \in \mathcal{B}[\Gamma_\omega^{\leq \alpha}]\}$$

$$2. \mathcal{R}_\alpha := \{M \in \mathcal{B}[\Gamma_\omega^{\leq \alpha}] : M + 1 \in \mathcal{B}[\Gamma_\omega^{\leq \alpha}]\}$$

And then:

$$1. \mathcal{S}_\alpha^1 := \mathcal{L}_\alpha \cap \mathcal{R}_\alpha$$

$$2. \mathcal{S}_\alpha^\omega := \mathcal{L}_\alpha \setminus \mathcal{R}_\alpha$$

$$3. \mathcal{S}_\alpha^{\omega^*} := \mathcal{R}_\alpha \setminus \mathcal{L}_\alpha$$

$$4. \mathcal{S}_\alpha^{\omega^* + \omega} := \mathcal{B}[\Gamma_\omega^{\leq \alpha}] \setminus (\mathcal{L}_\alpha \cup \mathcal{R}_\alpha)$$

In particular, by the definition,

$$\mathcal{B}[\Gamma_\omega^{\leq \alpha}] = \mathcal{S}_\alpha^1 \uplus \mathcal{S}_\alpha^\omega \uplus \mathcal{S}_\alpha^{\omega^*} \uplus \mathcal{S}_\alpha^{\omega^* + \omega}$$

Let $M \in \mathcal{B}[\Gamma_\omega^{\leq \alpha}]$.

We define the α -shape of M to be the $s \in \{1, \omega, \omega^, \omega^* + \omega\}$ for which $M \in \mathcal{S}_\alpha^s$.*

Lemma 4.3.

$$\mathcal{S}_\alpha^1 = \Gamma_\omega^{\leq \alpha}$$

Proof. (\supseteq) Let $M \in \Gamma_\omega^{<\alpha}$. Since $\Gamma_\omega^{<\alpha}$ is additive, $1 + M$ and $M + 1$ are also in $\Gamma_\omega^{<\alpha}$. Therefore, $M \in \mathcal{L}_\alpha \cap \mathcal{R}_\alpha = \mathcal{S}_\alpha^1$.

(\subseteq) Let $M \in \mathcal{S}_\alpha^1$. Then $M \in \mathcal{L}_\alpha$ and $M \in \mathcal{R}_\alpha$, so $1 + M, M + 1 \in \mathcal{B}[\Gamma_\omega^{<\alpha}]$.

By lemma 2.7, $\mathcal{B}[\Gamma_\omega^{<\alpha}]$ is a star property. So $1 + M + 1 \in \mathcal{B}[\Gamma_\omega^{<\alpha}]$. Since it is by itself an interval, it is in $\Gamma_\omega^{<\alpha}$.

By monotonicity, $M \in \Gamma_\omega^{<\alpha}$. □

Corollary 4.4. *Let $\alpha > 0$, $\delta \geq 0$ be ordinals.*

Then,

$$\mathcal{S}_{\alpha+\delta}^1 = \sum_{\mathcal{S}_{1+\delta}^1} \Gamma_\omega^{<\alpha}$$

Proof. By lemma 4.3, we can prove, equivalently, that

$$\Gamma_\omega^{<\alpha+\delta} = \sum_{\Gamma_\omega^{<1+\delta}} \Gamma_\omega^{<\alpha}.$$

We can prove this by induction on $\delta \geq 0$.

For $\delta = 0$ we need to prove

$$\Gamma_\omega^{<\alpha} = \sum_{\Gamma_\omega^{<1}} \Gamma_\omega^{<\alpha}.$$

Which is true by definition, since $\Gamma_\omega^{<1} = \{1\}$.

For $\delta = \gamma + 1$, using the induction hypothesis,

$$\begin{aligned} \Gamma_\omega^{<\alpha+\gamma+1} &= \sum_{\Gamma_\omega} \Gamma_\omega^{<\alpha+\gamma} \\ &= \sum_{\Gamma_\omega} \sum_{\Gamma_\omega^{<1+\gamma}} \Gamma_\omega^{<\alpha} \\ &= \sum_{\sum_{\Gamma_\omega} \Gamma_\omega^{<1+\gamma}} \Gamma_\omega^{<\alpha} \\ &= \sum_{\Gamma_\omega^{<1+\gamma+1}} \Gamma_\omega^{<\alpha} \\ &= \sum_{\Gamma_\omega^{1+\delta}} \Gamma_\omega^{<\alpha} \end{aligned}$$

For δ a limit ordinal,

$$\begin{aligned}
\Gamma_{\omega}^{<\alpha+\delta} &= \bigcup_{\gamma < \delta} \Gamma_{\omega}^{<\alpha+\gamma} \\
&= \bigcup_{\gamma < \delta} \sum_{\Gamma_{\omega}^{<1+\gamma}} \Gamma_{\omega}^{<\alpha} \\
&= \sum_{\bigcup_{\gamma < \delta} \Gamma_{\omega}^{<1+\gamma}} \Gamma_{\omega}^{<\alpha} \\
&= \sum_{\Gamma_{\omega}^{<1+\delta}} \Gamma_{\omega}^{<\alpha}
\end{aligned}$$

□

Lemma 4.5. *Let $\alpha > 0$ be an ordinal. Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$.*

Suppose that $\alpha = \sup_{i \in s} (\alpha_i + 1)$ for ordinals $\alpha_i > 0$ for all $i \in s$.

Then, we have the following:

$$\mathcal{S}_{\alpha}^s = \sum_{i \in s} \Gamma_{\omega}^{<\alpha_i}$$

Proof.

□

Corollary 4.6. *Let $\alpha > 0$, $\delta \geq 0$ be ordinals.*

Let $s \in \{\omega, \omega^, \omega^* + \omega\}$*

Then,

$$\mathcal{S}_{\alpha+\delta}^s = \sum_{\mathcal{S}_{1+\delta}^s} \Gamma_{\omega}^{<\alpha}$$

Proof. Suppose that $\delta = \sup_{i \in s} (\delta_i + 1)$.

Then $\alpha + \delta = \sup_{i \in s} (\alpha_i + 1 + \delta_i)$.

$$\mathcal{S}_{\alpha+\delta}^s = \sum_{i \in s} \mathcal{S}_{\alpha+\delta_i}^s = \sum_{i \in s} \sum_{\Gamma_{\omega}^{<1+\delta_i}} \Gamma_{\omega}^{<\alpha} = \sum_{\sum_{i \in s} \Gamma_{\omega}^{<1+\delta_i}} \Gamma_{\omega}^{<\alpha} = \sum_{\mathcal{S}_{1+\delta}^s} \Gamma_{\omega}^{<\alpha}$$

□

5 WO-Hausdorff rank

TBC.

6 Decidability of the rank

Definition 6.1. A property \mathbf{P} of preorders is if a computable property if $\mathbf{type}_n[\mathbf{P}]$ is computable as a function of n .

Theorem 6.2 (Decomposition theorem). *There exists a computable translation \mathcal{T} from MSO formulae $\varphi(\vec{X})$,*

such that for any $M = \sum_{i \in I} M_i$, and formula φ , if n is the quantifier-depth of φ , then

$$M, \vec{X} \models \varphi \iff I, \Pi \models \mathcal{T}\varphi$$

where $\Pi(i) = \mathbf{type}_n[M_i]$.

Corollary 6.3. *Let \mathbf{P} be a property of linear orders labeled with colors $\vec{C} = \{C_k\}_{k=1}^m$.*

Let $\{\mathbf{Q}_k\}_{k=1}^m$ be a family of (possibly labeled) properties.

Let $n \in \mathbb{N}$.

We can compute the type $\mathbf{type}_n[\sum_{\mathbf{P}} [\vec{C} \leftarrow \vec{\mathbf{Q}}]]$ from the types $\mathbf{type}_n[\mathbf{P}]$ and $\mathbf{type}_n[\mathbf{Q}_k]$ for all k .

Proof. Let τ be an n -type of the appropriate signature.

Assume we have $\mathbf{type}_n[\mathbf{P}]$ and $\mathbf{type}_n[\mathbf{Q}_k]$ for all k . □

Lemma 6.4. *There exists a global computable function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that the following holds.*

Let \mathbf{P} be a property of linear orders.

Let $\{\mathbf{Q}_i\}_{i=1}^k$ be a finite set of properties of linear orders.

Let $\{C_i\}_{i=1}^k$ be a finite set of colors.

Then $\mathbf{type}_n[\sum_{\mathbf{P}} [\vec{C} \leftarrow \vec{\mathbf{Q}}]]$ is a computable function of $\mathbf{type}_{h(n)}[\mathbf{P}]$ and $\mathbf{type}_n[\vec{\mathbf{Q}}] = \{\mathbf{type}_n[\mathbf{Q}_i]\}_{i=1}^k$.

Proof. □

Lemma 6.5. *Let \mathbf{Q} be a good property of linear orders.*

There exists a computable function $f_{\mathbf{Q}} = f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, and all $a \in \mathbb{N}$ such that $a \geq f(n)$, $\mathbf{type}_n[\mathbf{Q}^{\leq a}] = \mathbf{type}_n[\mathbf{Q}^{\leq f(n)}]$.

Equivalently, every linear order of finite rank is n -equivalent to some linear order of rank $\leq f(n)$.

Proof. Since there are only finitely many n -types, and the ω -sequence

$$\{\mathbf{type}_n[\mathbf{Q}^{\leq k}]\}_{k \in \omega}$$

is monotone, there must be some k where the sequence stabilizes.

This point k is computable as a function of n , because $\mathbf{type}_n[\mathbf{Q}^{\leq k}]$ is computable for every finite k . □

Lemma 6.6. *There exist global computable functions $a, b : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n, c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 \geq a(n)$ and $c_1 \equiv c_2 \pmod{b(n)}$,*

$$\mathbf{type}_n[\mathbf{Q}^{=c_1}] = \mathbf{type}_n[\mathbf{Q}^{=c_2}]$$

Equivalently, the sequence $\{\mathbf{type}_n[\mathbf{Q}^k]\}_{k \in \omega}$ is ultimately periodic for all $n \in \mathbb{N}$. Furthermore, the starting point and the period itself can be computed as a function of n .

Proof. Let $n \in \mathbb{N}$.

Since there are only finitely many sets of n -types, there exist (and can be computed) some $a(n) > f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n[\mathbf{Q}^{=a(n)}] = \mathbf{type}_n[\mathbf{Q}^{=a(n)+b(n)}]$$

holds for every s .

We shall prove by induction that for all $c \geq a(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c}] = \mathbf{type}_n[\mathbf{Q}^{=c+b(n)}]$$

This will complete the proof.

The base case $c = a(n)$ has been proven in the beginning.

Suppose the induction hypothesis holds for c .

Let M be of rank $c + 1$.

Write $M = \sum_{i \in I} M_i$ where $M_i \in \Gamma_{\omega}^{<c+1}$, and $M_i \in \Gamma_{\omega}^{=c}$ infinitely many times.

By the induction hypothesis, if $M_i \in \Gamma_{\omega}^{=c}$, we can find $N_i \equiv_n M_i$ with $N_i \in \Gamma_{\omega}^{=c+b(n)}$. Setting $N_i := M_i$ for all other i , we conclude that $N := \sum_{i \in I} N_i$ is n -equivalent to M .

However, clearly $N \in \Gamma_{\omega}^{=c+b(n)+1}$. So overall,

$$\mathbf{type}_n[\mathbf{Q}^{=c+1}] \subseteq \mathbf{type}_n[\mathbf{Q}^{=c+b(n)+1}]$$

Conversely, suppose M is of rank $c + b(n) + 1$. Write $M = \sum_{i \in I} M_i$ where $M_i \in \Gamma_{\omega}^{<c+b(n)+1}$, and $M_i \in \Gamma_{\omega}^{=c+b(n)}$ infinitely many times.

By the induction hypothesis, we can find for all i such that $M_i \in \Gamma_{\omega}^{=c+b(n)}$ some $N_i \equiv_n M_i$ with $N_i \in \Gamma_{\omega}^{=c}$. Furthermore, since $c \geq a(n) > f(n)$, we can find $N_i \equiv_n M_i$ with $N_i \in \Gamma_{\omega}^{\leq f(n) < c}$ for all other i .

We conclude that $N := \sum_{i \in I} N_i$ is n -equivalent to M . However, clearly $N \in \Gamma_{\omega}^{=c+1}$. So overall,

$$\mathbf{type}_n[\mathbf{Q}^{=c+b(n)+1}] \subseteq \mathbf{type}_n[\mathbf{Q}^{=c+1}]$$

So we have proven the induction step, and the lemma follows. \square

Corollary 6.7. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Let $s \in \{1, \omega, \omega^, \omega^* + \omega\}$ be a shape.*

Then there exists a computable function $b(n)$ such that for all $c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 > a(n)$ and $c_1 \equiv c_2 \pmod{b(n)}$, we have

$$\mathbf{type}_n[\mathcal{S}_{c_1}^s] = \mathbf{type}_n[\mathcal{S}_{c_2}^s]$$

Proof. For $s = 1$, it follows from lemma 6.5, since $\mathcal{S}_c^1 = \mathbf{Q}^{<c} = \mathbf{Q}^{\leq c-1}$ by lemma 4.3. and $c > a(n) \geq f(n)$ so $c-1 \geq f(n)$ for $c \in \{c_1, c_2\}$.

For $s \in \{\omega, \omega^*, \omega^* + \omega\}$, it follows easily from lemma 4.5 and lemma 6.6. \square

Lemma 6.8. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Let $s \in \{1, \omega, \omega^, \omega^* + \omega\}$ be a shape.*

$$\mathbf{type}_n[\mathcal{S}_\alpha^s] = \bigcup_{c < b(n)} \mathcal{S}_{a(n)+c}^s$$

In particular, $\mathbf{type}_n[\mathcal{S}_\alpha^s]$ can be computed, and is independent of the choice $\alpha \geq \omega$.

Proof. Draft:

\square

Definition 6.9. *Let $\alpha \geq \omega$ be an ordinal.*

Let M be a linear order and $x \in M$.

We define the convex equivalence relation:

$$\sim_\alpha := \sim_{\mathcal{B}[\Gamma_\omega^{\leq \alpha}]}$$

and $[x]_\alpha := [x]_{\mathcal{B}[\Gamma_\omega^{\leq \alpha}]}$ (that is, $[x]_\alpha$ is the largest $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subinterval containing x in M).

We define $\sigma_\alpha(M)$ as the α -shape of M .

Lemma 6.10. *The property $\Gamma_\omega^{\leq \alpha}$ is expressible over intervals in $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$.*

That is, there exists a formula $\varphi_\alpha(\Pi, \Xi)$ such that for every linear order M and every $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subinterval I of M , we have

$$M, \Pi, \Xi \models \varphi_\alpha(\Pi, \Xi) \iff I = \sum_{i \in I} M_i \text{ where } M_i \in \Gamma_\omega^{\leq \alpha} \text{ for all } i$$

Proof. It is equivalent to being a sum of \sim_α -subintervals, of which at least one has $\sigma_\alpha \neq 1$. \square

Theorem 6.11. *There is an oracle reduction from SAT for $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$, to SAT for \mathbf{MSO} .*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ formula φ of quantifier-depth n . outputs an \mathbf{MSO} formula $\psi(\Pi)$ such that...

Let φ be an $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ formula, and let n be the quantifier-depth of φ .

WLOG, assume that φ is a sentence.

First, let us calculate the sets:

$$T_s := \mathbf{type}_n[\mathcal{S}_\alpha^s]$$

for every shape s .

Now we create the formulae:

$$\theta_s(\Pi, \Xi) := \left\{ i : \bigvee_{\tau \in S_s} \Xi(\Pi(i)) = s \right\}$$

$$L(\Pi, \Xi) := \theta_\omega(\Pi, \Xi) \vee \theta_{\omega^* + \omega}(\Pi, \Xi)$$

$$R(\Pi, \Xi) := \theta_{\omega^*}(\Pi, \Xi) \vee \theta_{\omega^* + \omega}(\Pi, \Xi)$$

We create the formula $\chi(\Pi, \Xi)$ as follows:

$$\chi := \Pi = \text{domain}(\Xi) \wedge \forall i, i'. i' = i + 1 \implies i \in R(\Pi, \Xi) \vee i' \in L(\Pi, \Xi)$$

Now we claim that φ is satisfiable in $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ iff $\psi \wedge \chi$ is satisfiable in \mathbf{MSO} .

If φ is satisfiable, then there exists a model M of φ .

Let $M = \sum_{i \in I} M_i$ be the decomposition of M where $I = \sim_\alpha$ and M_i are the \sim_α -equivalence classes.

By the decomposition theorem, Ψ holds in $I, \Pi := \mathbf{type}_n[\cdot]$.

We claim that χ holds in $I, \Pi := \mathbf{type}_n[\cdot]$.

It follows from the star property of \sim_α that the constraint holds.

Conversely, suppose $\psi \wedge \chi$ is satisfiable in \mathbf{MSO} .

Let $I, \Pi := T$ be a model of $\psi \wedge \chi$.

Let us take a model M_i with the appropriate type. Now define $M := \sum_{i \in I} M_i$.

We claim that each M_i is a *maximum* $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subinterval of M .

Suppose $[M_i, M_j]$ is a $\mathcal{B}[\Gamma_\omega^{\leq \alpha}]$ -subinterval of M .

In particular, it has a rank, so it is scattered. So in particular, $[i, j] \subseteq I$ is a scattered interval.

If $i = j$ we are done. Otherwise, let i', j' be such that $i \leq i' < j' \leq j$, and $j' = i' + 1$. But it cannot be the case by the constraint.

□

Theorem 6.12. Suppose $\alpha_k > \alpha_{k-1} > \dots > \alpha_0 > 0$ are ordinals.

Then there is an oracle reduction from SAT for $\mathbf{MSO}[\sim_{\alpha_k}, \sigma_{\alpha_k}, \dots, \sim_{\alpha_0}, \sigma_{\alpha_0}]$,

to SAT for $\mathbf{MSO}[\sim_{\delta_{k-1}}, \sigma_{\delta_{k-1}}, \dots, \sim_{\delta_0}, \sigma_{\delta_0}]$.

where $\alpha_0 + \delta_i = \alpha_{i+1}$ for $0 \leq i < k$.

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[\sim_{\alpha_k}, \sigma_{\alpha_k}, \dots, \sim_{\alpha_0}, \sigma_{\alpha_0}]$ formula φ of quantifier-depth n . outputs an \mathbf{MSO} formula $\psi(\Pi)$ such that...

Let φ be an $\mathbf{MSO}[\sim_{\alpha_k}, \sigma_{\alpha_k}, \dots, \sim_{\alpha_0}, \sigma_{\alpha_0}]$ formula, and let n be the quantifier-depth of φ .

WLOG, assume that φ is a sentence.

First, let us calculate the sets $\mathbf{type}_n[\Gamma_\omega^{<\alpha}]$, and

Now we create the formulae:

$$\theta_s(\Pi, \Xi) := \left\{ i : \bigvee_{\tau \in S_s} \Xi(\Pi(i)) = s \right\}$$

$$L(\Pi, \Xi) := \theta_\omega(\Pi, \Xi) \vee \theta_{\omega^* + \omega}(\Pi, \Xi)$$

$$R(\Pi, \Xi) := \theta_{\omega^*}(\Pi, \Xi) \vee \theta_{\omega^* + \omega}(\Pi, \Xi)$$

We create the formula $\chi(\Pi, \Xi)$ as follows:

$$\chi := \Pi = \text{domain}(\Xi) \wedge \forall i, i'. i' = i + 1 \implies i \in R(\Pi, \Xi) \vee i' \in L(\Pi, \Xi)$$

Now we claim that φ is satisfiable in $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ iff $\psi \wedge \chi$ is satisfiable in \mathbf{MSO} .

If φ is satisfiable, then there exists a model M of φ .

Let $M = \sum_{i \in I} M_i$ be the decomposition of M where $I = \sim_\alpha$ and M_i are the \sim_α -equivalence classes.

By the decomposition theorem, Ψ holds in $I, \Pi := \mathbf{type}_n[\cdot]$.

We claim that χ holds in $I, \Pi := \mathbf{type}_n[\cdot]$.

It follows from the star property of \sim_α that the constraint holds.

Conversely, suppose $\psi \wedge \chi$ is satisfiable in \mathbf{MSO} .

Let $I, \Pi := T$ be a model of $\psi \wedge \chi$.

Let us take a model M_i with the appropriate type. Now define $M := \sum_{i \in I} M_i$.

We claim that each M_i is a *maximum* $\mathcal{B}[\Gamma_\omega^{<\alpha}]$ -subinterval of M .

Suppose $[M_i, M_j]$ is a $\mathcal{B}[\Gamma_\omega^{<\alpha}]$ -subinterval of M .

In particular, it has a rank, so it is scattered. So in particular, $[i, j] \subseteq I$ is a scattered interval.

If $i = j$ we are done. Otherwise, let i', j' be such that $i \leq i' < j' \leq j$, and $j' = i' + 1$. But it cannot be the case by the constraint.

□

7 Everything Better

Theorem 7.1. *Let \mathcal{C} be a computable property of labeled linear orders, such that \mathcal{C} is closed under taking subintervals, projections and inverse-projections (i.e., of one of the colors), and all finite-sums and \mathcal{C} -sums.*

Let $\mathbf{P}_1, \dots, \mathbf{P}_k \subseteq \mathcal{C}$ be computable properties of labeled linear orders.

Let $\mathbf{MSO}[P_1, \dots, P_k]$ be monadic second order logic of order over \mathcal{C} , with P_1, \dots, P_k as monadic predicates whose semantics are: $P_i(X)$ holds iff X is a subinterval which satisfies \mathbf{P}_i .

Given ϕ a formula of $\mathbf{MSO}[P_1, \dots, P_k]$ (possibly with free variables) we define

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

(Note that M above may be a labeled linear order.)

Then \mathcal{C}_ϕ is a computable property of linear orders.

Proof. By structural induction on ϕ .

Suppose ϕ is an atomic formula. If ϕ is of the form $X \subseteq Y$ or $X \leq Y$,

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

and thus,

$$\mathbf{type}_n[\mathcal{C}_\phi] = \{\tau \in \mathbf{type}_n[\mathcal{C}] : \tau \models \phi\}$$

which is computable since $\mathbf{type}_n[\mathcal{C}]$ is computable, and we can then compute whether $\tau \models \phi$ for each $\tau \in \mathbf{type}_n[\mathcal{C}]$.

If ϕ is of the form $P_i(X)$, then

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models P_i(X)\}$$

and thus,

$$\mathbf{type}_n[\mathcal{C}_\phi] = \mathbf{type}_n[\mathbf{P}_i]$$

which is computable since \mathbf{P}_i is computable.

If $\phi = \neg\phi_1$, then

$$\mathcal{C}_\phi = \mathcal{C} \setminus \mathcal{C}_{\phi_1}$$

□