Orders

Alon Gurny

May 31, 2025

Contents

1 Preorders

We begin by studying the properties of preorders. Basically, we define a *property* as a class which is close under isomorphism. We then define the sum operation on preorders. This will be used to create new properties from old ones.

Definitions 1.1 (Preorder)**.** *A* (labeled) preorder *is a a set M together with a binary relation $\leq$ on M such that $\leq$ is reflexive and transitive, possibly endowed with monadic predicates (labels) over some first-order monadic signature.*

Definition 1.2 (Property of preorders)**.** *A* property $\mathbf{P}$ *of preorders is a class of preorders which is closed under isomorphism.*

Definition 1.3. *A property $\mathbf{P}$ of preorders is* monotone *if for every preorder M, $M \in \mathbf{P}$ implies that every suborder of M is in $\mathbf{P}$.*

Definition 1.4. *Let M be a preorder.*
Then M^* is the dual/reverse preorder of M.

Definition 1.5 (Sum of preorders)**.** *Let I be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders over a disjoint signature (i.e., for every $i \in I$, I and M_i have disjoint sets of labels).*
The sum $M = \sum_{i\in I} M_i$ is defined as follows:
The domain is $M = \biguplus_{i\in I} M_i$ (a disjoint union).
Let $\leq_i$ be the preorder on M_i.
Let $x \in M_i$ and $y \in M_j$.
Then we define $x \leq y$ iff either $i = j$ and $x \leq_i y$ or $i < j$.
The labels are inherited from either I or the M_i's.
If $I = 2$, we define $M_1 + M_2 := \sum_{i\in 2} M_i$.

Lemma 1.6. *Let I be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders, over a disjoint signature.*
Then $M = \sum_{i\in I} M_i$ is a preorder.

Proof. Reflexivity is clear.
For transitivity, suppose $x \leq y$ and $y \leq z$.
Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.
Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.
Otherwise, $i < k$, and thus $x \leq z$, as required. $\square$

Definition 1.7. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be properties of preorders.*
Then we define

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

The labels are inherited from either $\mathbf{P}_1$ or $\mathbf{P}_2$.

Definition 1.8. *A property $\mathbf{P}$ of preorders is an* additive property *if for every preorders M_1 and M_2, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

Definition 1.9 (Kleene plus)**.** *Let* $\mathbf{P}$ *be a property of preorders.*

We define its Kleene plus as the smallest property of preorders $\mathbf{P}^+$ *which contains* $\mathbf{P}$ *and is closed under finite sums.*

That is, $1^+ = \{1, 2, \ldots\}$*, and* $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$*.*

Definition 1.10 (Sum of a property over a preorder)**.** *Let* I *be a preorder.*

Let $\mathbf{Q}$ *be a property of preorders.*

Then we define

$$\sum_I \mathbf{Q} := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q} \right\}$$

Definition 1.11 (Sum of a family of properties over a preorder)**.** *Let* I *be a preorder.*

Let $\{\mathbf{Q}_i\}_{i \in I}$ *be a family of properties of preorders over a disjoint signature.*

Then we define

$$\sum_{i \in I} \mathbf{Q}_i := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q}_i \right\}$$

The labels are inherited from either I *or the* $\mathbf{Q}_i$*'s.*

Note 1.12. *By the previous two definitions, if* I *has no labels,*

$$\sum_I \mathbf{Q} = \sum_{i \in I} \mathbf{Q}$$

Definition 1.13 (Sum of a property over a property)**.** *Let* $\mathbf{P}$ *be a property of preorders.*

Let $\mathbf{Q}$ *be a property of preorders over a disjoint signature.*

Then we define,

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_I \mathbf{Q} : I \in \mathbf{P} \right\}$$

2 Linear Orders

Definitions 2.1 (Linear order)**.** *A* linear order *a preorder which is symmetric and total.*

Definition 2.2 (Property of linear orders)**.** *A* property $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 2.3. *Subintervals Let M be a linear order, and let $x, y \in M$, such that $x \leq y$.*

Then we define the bounded subintervals $[x, y]$, $(x, y]$, $[x, y)$ *and* (x, y) *as usual.*

We also define the semi-bounded subintervals $(-\infty, x]$, $[x, \infty)$, $(-\infty, x)$ *and* (x, ∞) *as usual.*

We also define the unbounded subinterval $(-\infty, \infty)$ *as the whole linear order M, as usual.*

A subinterval *is either a bounded subinterval, a semi-bounded subinterval or the unbounded subinterval.*

If $x > y$ then we define the intervals as follows:

$$[x, y] := [y, x]$$
$$(x, y] := (y, x]$$
$$[x, y) := [y, x)$$
$$(x, y) := (y, x)$$

Definition 2.4. *Let M be a linear order.*

A set $A \subseteq M$ is left cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $y < x$.*

A set $A \subseteq M$ is right cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $x < y$.*

A set $A \subseteq M$ is bi-directionally cofinal *in M if it is both left and right cofinal.*

Lemma 2.5. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let $M \in \mathbf{P}$ be a linear order.

Let $x, y \in M$ be any two points in a linear order M.

Then, $[x, y] \in \mathbf{P}$.

Proof. WLOG, suppose $x \leq y$.

Note that,

$$M = (-\infty, \infty) = (-\infty, x) + [x, y] + (y, \infty)$$

when $(-\infty, x)$ and/or (y, ∞) may be empty.

Since $\mathbf{P}$ is an additive property, we conclude that $[x, y] \in \mathbf{P}$. $\square$

Corollary 2.6. *Let $\mathbf{P}$ be a nontrivial additive property of linear orders.*

Then $1 \in \mathbf{P}$.

Proof. Let $M \in \mathbf{P}$ be any linear order and let $x \in M$ be any point in M.

Apply lemma 2.5 to the linear order M, and the points x and x, to conclude that $[x, x] \equiv 1 \in \mathbf{P}$. □

Note 2.7. *Note that corollary 2.6 is false if we do not restrict ourselves to linear orders.*

For example, $(1 \uplus 1)^+$ is a property of preorders which is additive, but does not contain 1.

Corollary 2.8. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let M be a linear order.

Let $x, y \in M$ be any two points in a linear order M. Then the following are equivalent:

1. $(x, y) \in \mathbf{P}$
2. $(x, y] \in \mathbf{P}$
3. $[x, y) \in \mathbf{P}$
4. $[x, y] \in \mathbf{P}$

Proof. This is just applying the definition of an additive property to the orders $[x, y]$ and 1. □

Corollary 2.9. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let M be a linear order.

Let $x, y, z \in M$ be any three points in a linear order M, such that $[x, y] \in \mathbf{P}$ and $[y, z] \in \mathbf{P}$.

Then $[x, z] \in \mathbf{P}$.

Proof. If $y \in [x, z]$, then $[x, z] = [x, y] + (y, z]$, and $(y, z] \in \mathbf{P}$ by corollary 2.8.

Otherwise, either $x \in [y, z]$ or $z \in [x, y]$. WLOG, suppose $z \in [x, y]$.

Then $[x, y] = [x, z] + (z, y]$, so $[x, z] \in \mathbf{P}$ by the fact that $\mathbf{P}$ is additive. □

Definitions 2.10. *Let $\mathbf{P}$ be a property of linear orders.*

We define the following properties of linear orders:

- $\mathcal{B}[\mathbf{P}]$ *is the class of linear orders M such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in $\mathbf{P}$.*
- $\mathcal{L}[\mathbf{P}]$ *is the class of linear orders M such that for every $x \in M$, the left-bounded ray $[x, \infty)$ is in $\mathbf{P}$.*
- $\mathcal{R}[\mathbf{P}]$ *is the class of linear orders M such that for every $x \in M$, the right-bounded ray $(-\infty, x]$ is in $\mathbf{P}$.*

Definition 2.11. *A property $\mathbf{P}$ of linear orders is a* star property *if for every linear orders M, and every family $\mathcal{F} \subseteq \mathbf{P}$ of subintervals of M such that $J_1 \cap J_2 \neq \emptyset$ for every $J_1, J_2 \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \in \mathbf{P}$.*

Lemma 2.12. *Let* $\mathbf{P}$ *be a star property.*

Then for every linear order M*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$*.*

Equivalently, we can define a convex equivalence relation $\sim_{\mathbf{P}}$ *on* M *such that* $x \sim_{\mathbf{P}} y$ *iff* $[x, y] \in \mathbf{P}$*.*

That is, $x \sim_{\mathbf{P}} y$ *iff* x *and* y *are in the same largest* $\mathbf{P}$*-subinterval.*

Proof. Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in $\mathcal{B}[\mathbf{P}]$, and by definition J is the largest $\mathbf{P}$-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. $\square$

Lemma 2.13 (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then the property $\mathcal{B}[\mathbf{P}]$ *is a star property.*

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by the definition of $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. Since $\mathbf{P}$ is additive, by corollary 2.9, we have $[x, y] \in \mathbf{P}$. $\square$

Lemma 2.14. *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then,

1. $\mathcal{L}[\mathbf{P}] = \{M : M + 1 \in \mathcal{B}[\mathbf{P}]\}$
2. $\mathcal{R}[\mathbf{P}] = \{M : 1 + M \in \mathcal{B}[\mathbf{P}]\}$
3. $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}] = \{M : 1 + M + 1 \in \mathcal{B}[\mathbf{P}]\}$

Proof. Let M be a linear order.

1. Suppose $M + \{\infty\} \in \mathcal{B}[\mathbf{P}]$. Then for every $x \in M$, we have $[x, \infty] \in \mathbf{P}$, and thus $[x, \infty) \in \mathbf{P}$. Therefore, $M \in \mathcal{L}[\mathbf{P}]$.

   Conversely, if $M \in \mathcal{L}[\mathbf{P}]$, let $x, y \in M$ be any two points in $M + 1$.

   If $y < \infty$, then $[x, y] \subseteq [x, \infty)$. Since $[x, \infty) \in \mathbf{P}$, we conclude that $[x, y] \in \mathbf{P}$. Otherwise, if $y = \infty$, then $[x, y] = [x, \infty] = [x, \infty) + \{\infty\}$, and thus $[x, y] \in \mathbf{P}$.

2. The second case is dual to the first case.

3. We will show a triple inclusion.

   If $M \in \mathbf{P}$, then by additivity, $1 + M \in \mathbf{P}$ and $M + 1 \in \mathbf{P}$, and thus $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$.

   If $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, then by lemma 2.13, $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$.

   If $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$, then M is a bounded subinterval of $1 + M + 1$, so $M \in \mathcal{B}[\mathbf{P}]$.

□

Lemma 2.15. *Let* $\mathbf{P}$ *be an additive property of linear orders.*
Then,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = \mathbf{P} \\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) \\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}]) \\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Proof. By lemma 2.14, we conclude that $\mathcal{L}[\mathbf{P}], \mathcal{R}[\mathbf{P}] \subseteq \mathcal{B}[\mathbf{P}]$, since $M + 1 \in \mathbf{P}$ and $1 + M \in \mathbf{P}$ both imply $1 + M + 1 \in \mathbf{P}$.

Thus,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = (\mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]) \\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) \\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}]) \\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Since by lemma 2.14 $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, we conclude what we wanted to prove. □

Lemma 2.16. *Let* $\mathbf{P}$ *be an additive property of linear orders.*
Let M, M_1, M_2 *be linear orders such that* $M = M_1 + M_2$.
Then,

1. $M \in \mathcal{B}[\mathbf{P}] \iff M_1 \in \mathcal{L}[\mathbf{P}] \land M_2 \in \mathcal{R}[\mathbf{P}]$
2. $M \in \mathcal{L}[\mathbf{P}] \iff M_1 \in \mathcal{L}[\mathbf{P}] \land M_2 \in \mathbf{P}$
3. $M \in \mathcal{R}[\mathbf{P}] \iff M_1 \in \mathbf{P} \land M_2 \in \mathcal{R}[\mathbf{P}]$

Proof. From lemma 2.14, we know that

1. $$\begin{aligned}M \in \mathcal{B}[\mathbf{P}] &\iff M_1 + M_2 \in \mathcal{B}[\mathbf{P}] \\ &\iff M_1 + 1 \in \mathcal{B}[\mathbf{P}] \land 1 + M_2 \in \mathcal{B}[\mathbf{P}] \\ &\iff M_1 \in \mathcal{L}[\mathbf{P}] \land M_2 \in \mathcal{R}[\mathbf{P}]\end{aligned}$$

   where the second transition follows from the star lemma lemma 2.13.

2. TBC.

3. TBC.

□

Corollary 2.17. *Let* $\mathbf{P}$ *be an additive property of linear orders.*
Then,

$$\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]) = (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) + (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])$$

Lemma 2.18 (Associativity of sum)**.** *Let* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *be properties.*
Then $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$*.*

Proof. It follows directly from the associativity of the sum operation on linear orders. Actually, it generalizes to any algebraic equation which holds on linear orders. $\square$

Lemma 2.19 (Sum and union commute)**.** *Let* $\mathcal{P}$ *be a family of properties.*
Let $\mathbf{Q}$ *be a property.*
Then $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$*.*

Proof. This is obvious from the definition of the sum operation. $\square$

Definition 2.20. *We define* **CNT** *as the class of all countable linear orders.*

Definition 2.21. *Let* $\gamma \geq \omega$ *be a limit ordinal.*
We define $\Gamma_\gamma := \{\beta : \beta \subseteq \gamma^* + \gamma\}^+$*.*
We define $\Omega := \Gamma_\omega$*.*

Example 2.22.

$$\Omega = \{1, \omega, \omega^*\}^+$$

Observation 2.23. *Let* $\gamma \geq \omega$ *be a limit ordinal.*
Then Γ_γ *is a monotone, additive property of linear orders.*

3 General Hausdorff Rank

Definition 3.1. *Let $\mathbf{Q}$ be a property of linear orders.*
We define a property $\mathbf{Q}^{<\alpha}$ for every ordinal α as follows:

- *For $\alpha = 0$, $\mathbf{Q}^{<0} = \{1\}$.*
- *For $\alpha = \gamma + 1$,*
$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}$$
- *For α a limit ordinal,*
$$\mathbf{Q}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{Q}^{<\beta}$$

Example 3.2. *Let $\mathbf{Q}$ be a property of linear orders.*
Then $\mathbf{Q}^{<1} = \mathbf{Q}$.

Lemma 3.3. *Let $\mathbf{Q}$ be a property of linear orders.*
Let α, δ be ordinals.
Then,
$$\mathbf{Q}^{<\alpha+\delta} = \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}$$

Proof. We shall prove this by induction on $\delta \geq 0$.
For $\delta = 0$ we need to prove
$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}^{0}} \mathbf{Q}^{<\alpha}.$$

Which is true by definition, since $\mathbf{Q}^{0} = \{1\}$.
For $\delta = \gamma + 1$, using the induction hypothesis,
$$\begin{aligned}
\mathbf{Q}^{<\alpha+\delta} &= \mathbf{Q}^{<\alpha+\gamma+1} \\
&= \sum_{\mathbf{Q}} \mathbf{Q}^{<\alpha+\gamma} \\
&= \sum_{\mathbf{Q}} \sum_{\mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\gamma+1}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}
\end{aligned}$$

For δ a limit ordinal, using the induction hypothesis,

$$\begin{aligned}
\mathbf{Q}^{<\alpha+\delta} &= \bigcup_{\gamma<\delta} \mathbf{Q}^{<\alpha+\gamma} \\
&= \bigcup_{\gamma<\delta} \sum_{\mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\bigcup_{\gamma<\delta} \mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}
\end{aligned}$$

$\square$

Definition 3.4. *Let* $\mathbf{Q}$ *be a property of linear orders.*

Let α, β *be ordinals with* $\alpha < \beta$*.*

We define,

1. $\mathbf{Q}^{\leq\alpha} := \mathbf{Q}^{<\alpha+1}$
2. $\mathbf{Q}^{=\alpha} := \mathbf{Q}^{\leq\alpha} \setminus \mathbf{Q}^{<\alpha}$
3. $\mathbf{Q}^{[\alpha,\beta)} := \mathbf{Q}^{<\beta} \setminus \mathbf{Q}^{<\alpha}$

Definition 3.5. *Let* $\mathbf{Q}$ *be a property of linear orders.*

Let M *be a linear order, such that* $M \in \left(\mathbf{Q}^{<\alpha}\right)^{+}$ *for some ordinal* α*.*

We define the $\mathbf{Q}$-Hausdorff rank *of* M *as*

$$\mathbf{hrank}_{\mathbf{Q}}(M) = \sup\left\{\beta : M \notin \left(\mathbf{Q}^{<\beta}\right)^{+}\right\}$$

where the supremum is taken over all ordinals β*. (Recall that the supremum of the empty set is defined to be* 0*.)*

4 ω-Hausdorff rank

Definitions 4.1. *Let $\alpha > 0$ be an ordinal.*
We define:

1. $\mathcal{S}^1_\alpha := \Omega^{<\alpha}$
2. $\mathcal{S}^\omega_\alpha := \mathcal{R}\left[\Omega^{<\alpha}\right] \setminus \mathcal{L}\left[\Omega^{<\alpha}\right]$
3. $\mathcal{S}^{\omega^*}_\alpha := \mathcal{L}\left[\Omega^{<\alpha}\right] \setminus \mathcal{R}\left[\Omega^{<\alpha}\right]$
4. $\mathcal{S}^{\omega^*+\omega}_\alpha := \mathcal{B}\left[\Omega^{<\alpha}\right] \setminus \left(\mathcal{L}\left[\Omega^{<\alpha}\right] \cup \mathcal{R}\left[\Omega^{<\alpha}\right]\right)$

The names will soon be justified.

Lemma 4.2. *Let $\alpha > 0$ be an ordinal.*
Then,

1. $\mathcal{R}\left[\Omega^{<\alpha}\right] = \sum_\omega \Omega^{<\alpha}$.
2. $\mathcal{L}\left[\Omega^{<\alpha}\right] = \sum_{\omega^*} \Omega^{<\alpha}$.
3. $\mathcal{B}\left[\Omega^{<\alpha}\right] = \sum_{\omega^*+\omega} \Omega^{<\alpha}$.

Proof. 1. Let us prove the first part. ($\supseteq$) Let $M \in \sum_\omega \Omega^{<\alpha}$ be a linear order.
Let $M = \sum_{i\in\omega} M_i$ be the decomposition of M, where $M_i \in \Omega^{<\alpha}$.
Let $x, y \in M$ be any two points in M. WLOG $x \leq y$.
Suppose $x \in M_i$ and $y \in M_j$ for $i, j \in \omega$.
Since i and j have a finite distance in ω, we conclude $[x, y] \subseteq M_i + \ldots + M_j$, and thus $[x, y] \subseteq \left(\Omega^{<\alpha}\right)^+ = \Omega^\alpha$.
($\subseteq$) Let $M \in \mathcal{R}\left[\Omega^{<\alpha}\right]$ be a linear order.
Since M is countable, let $\{x_i\}_{i\in\omega} M$ be a right cofinal ω-sequence in M.
Let $M_0 = (-\infty, x_0]$ and and $M_i = (x_{i-1}, x_i]$ for $i > 0$.
Then $M = \sum_{i\in\omega} M_i$.
But M_i is a right-bounded interval and thus $M_i \in \Omega^{<\alpha}$, so $M \in \sum_\omega \Omega^{<\alpha}$.
2. The second part is symmetric.
3. The third part follows from corollary 2.17:

$$\begin{aligned}
\mathcal{B}\left[\Omega^{<\alpha}\right] &= \mathcal{L}\left[\Omega^{<\alpha}\right] + \mathcal{R}\left[\Omega^{<\alpha}\right] \\
&= \sum_{\omega^*} \Omega^{<\alpha} + \sum_{\omega} \Omega^{<\alpha} \\
&= \sum_{\omega^*+\omega} \Omega^{<\alpha}
\end{aligned}$$

∎

An immediate corollary of lemma 4.2 is that:

Corollary 4.3. *Let $\alpha > 0$ be an ordinal.*
Then $\Omega^{\leq\alpha} = \left(\mathcal{B}\left[\Omega^{<\alpha}\right]\right)^+$.

Lemma 4.4. *Let $\alpha > 0$ be an ordinal.*
Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$.
Suppose that $\alpha = \limsup_{i \in s} (\alpha_i + 1)$ for ordinals $\{\alpha_i\}_{i \in s}$.
Then, we have the following:

$$\mathcal{S}^s_\alpha = \sum_{i \in s} \Omega^{[\alpha_i, \alpha)}$$

Proof. It is enough to prove the case $s = \omega$, since $s = \omega^*$ follows by symmetry, and $s = \omega^* + \omega$ follows by adding the previous two cases.

($\supseteq$) Let $M \in \sum_{i \in \omega} \Omega^{[\alpha_i, \alpha)}$ be a linear order. Then obviously $M \in \sum_\omega \Omega^{<\alpha} = \mathcal{R}[\Omega^{<\alpha}]$.

Suppose for the contrary that $M \in \mathcal{L}[\Omega^{<\alpha}]$. TBC.

($\subseteq$) Let $M \in \mathcal{S}^\omega_\alpha$ be a linear order.

By lemma 4.2, $M = \sum_\omega M_i$ for some $M_i \in \Omega^{<\alpha}$.

We will define by induction a new sequence $\{N_i\}_{i \in \omega}$ such that $M = \sum_{i \in \omega} N_i$, where $N_i \in \Omega^{[\alpha_i, \alpha)}$.

Suppose that N_j is defined for all $j < i$.

Since $M \notin \mathcal{L}[\Omega^{<\alpha}]$, $[M_i, \infty) \notin \Omega^{<\alpha}$. TBC. □

Note 4.5. *For the proof of lemma 4.4, we actually use the fact that we work over Ω. This proof would not have worked over Γ_β for $\beta > \omega$.*

Corollary 4.6. *Let $\alpha, \delta > 0$ be limit ordinals.*
Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$
Then,

$$\mathcal{S}^s_{\alpha+\delta} = \sum_{\mathcal{S}^s_\delta} \Omega^{=\alpha}$$

Proof. Suppose that $\delta = \limsup_{i \in s} \delta_i$.

Then $\alpha + \delta = \limsup_{i \in s} (\alpha_i + \delta_i)$.

$$\mathcal{S}^s_{\alpha+\delta} = \sum_{i \in s} \mathcal{S}^s_{\alpha+\delta_i} = \sum_{i \in s} \sum_{\Omega^{=\delta_i}} \Omega^{=\alpha} = \sum_{\sum_{i \in s} \Omega^{=\delta_i}} \Omega^{=\alpha} = \sum_{\mathcal{S}^s_\delta} \Omega^{=\alpha}$$

□

5 Type Theory

Definition 5.1. *Let* $\mathbf{P}$ *be a property of preorders.*

Let $n \in \mathbb{N}$.

We define $\mathbf{type}_n[\mathbf{P}]$ *as the set of all* n*-types satisfiable in* $\mathbf{P}$.

Definition 5.2. *A property* $\mathbf{P}$ *of preorders is* computable *if* $n \mapsto \mathbf{type}_n[\mathbf{P}]$ *is a computable function.*

Lemma 5.3. *Let* $\mathbf{Q}$ *be a property of preorders.*

There exists a computable function $f_{\mathbf{Q}} = f : \mathbb{N} \to \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and every ordinal* $\alpha \geq f(n)$, $\mathbf{type}_n[\mathbf{Q}^{<\alpha}] = \mathbf{type}_n\left[\mathbf{Q}^{<f(n)}\right]$.

Proof. Since there are only finitely many n-types, and the ordinal sequence

$$\left\{\mathbf{type}_n\left[\mathbf{Q}^{<\kappa}\right]\right\}_\kappa$$

is monotone, there must be some minimal $\kappa_0 \in \omega$ where the sequence stabilizes.

This κ_0 is computable as a function of n, because $\mathbf{type}_n[\mathbf{Q}^{<\kappa}]$ is computable for every finite κ. □

Lemma 5.4. *There exist global computable functions* $a, b : \mathbb{N} \to \mathbb{N}$ *such that for all* $n, c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c_1}] = \mathbf{type}_n[\mathbf{Q}^{=c_2}]$$

Proof. Let $n \in \mathbb{N}$.

Since there are only finitely many sets of n-types, there exist (and can be computed) some $a(n) \geq f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$$

By induction if follows that for all $c \geq a(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c}] = \mathbf{type}_n\left[\mathbf{Q}^{=c+b(n)}\right]$$

since $\mathbf{Q}^{=c+1} = \sum_{\mathbf{Q}} \mathbf{Q}^{=c}$. □

Corollary 5.5. *Let* $n \in \mathbb{N}$, *and let* $\alpha \geq \omega$ *be an ordinal.*

Let $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$ *be a shape.*

Then there exists a computable function $b(n)$ *such that for all* $c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$, *we have*

$$\mathbf{type}_n\left[\mathcal{S}^s_{c_1}\right] = \mathbf{type}_n\left[\mathcal{S}^s_{c_2}\right]$$

Proof. For $s = 1$, it follows from lemma 5.3, since $\mathcal{S}^1_c = \mathbf{Q}^{<c}$ and $c \geq a(n) \geq f(n)$ for $c \in \{c_1, c_2\}$.

For $s \in \{\omega, \omega^*, \omega^* + \omega\}$, it follows easily from lemma 4.4 and lemma 5.4. □

Lemma 5.6. *For every $n \in \mathbb{N}$ and for every ordinal $\alpha \geq \omega$,*

$$\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right] = \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]$$

In particular, $\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.

Proof. By induction on $\alpha \geq \omega$.

Let $\{\alpha_i\}_{i\in\omega}$ be an ω-sequence of ordinals such that $a(n) \leq \alpha_i$ for all $i \in \omega$, and $\limsup_{i\in\omega}(\alpha_i + 1) = \alpha$.

Then $\mathbf{Q}^{=\alpha} = \sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{Q}^{=\alpha_i}$ and thus,

$$\begin{aligned}
\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right] &= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{Q}^{=\alpha_i}\right] \\
&= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \sum_{\mathbf{Q}} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c+1}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]
\end{aligned}$$

where the last transition is because $\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$. ∎

Corollary 5.7. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$ be a shape.

$$\mathbf{type}_n\left[\mathcal{S}^s_\alpha\right] = \mathbf{type}_n\left[\sum_s \bigcup_{c<b(n)} \Omega^{=a(n)+c}\right]$$

In particular, $\mathbf{type}_n\left[\mathcal{S}^s_\alpha\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.

Proof. There exists a sequence $\{\alpha_i\}_{i\in s}$ such that $a(n) \leq \alpha_i$ for all $i \in s$, and $\limsup_{i\in s}(\alpha_i + 1) = \alpha$.

Then $\mathcal{S}^s_\alpha = \sum_{i\in s} \Omega^{=\alpha_i}$, and thus,

$$\begin{aligned}
\mathbf{type}_n\left[\mathcal{S}^s_\alpha\right] &= \mathbf{type}_n\left[\sum_{i\in s}\Omega^{=\alpha_i}\right] \\
&= \mathbf{type}_n\left[\sum_{s}\bigcup_{c<b(n)}\Omega^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\sum_{s}\Omega^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\mathcal{S}^s_{a(n)+c+1}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\mathcal{S}^s_{a(n)+c}\right]
\end{aligned}$$

where the last transition is by corollary 5.5.

□

6 Decidability of the rank

Definition 6.1. *Let* $\mathbf{Q}$ *be a property of linear orders.*

Let M *be a linear order.*

We define the predicate $\mathbf{Int_Q}(J)$ *as true in* M *iff* J *is a* $\mathbf{Q}$*-subinterval of* M*.*

Lemma 6.2. *Let* $\alpha > 0$ *be an ordinal.*

Then predicates $\mathbf{Int}_{\Omega^{\leq\alpha}}$*,* $\mathbf{Int}_{\Omega^{=\alpha}}$ *are expressible in* $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$*.*

Proof. Obviously,

$$\mathbf{Int}_{\Omega^{=\alpha}} \iff \mathbf{Int}_{\Omega^{\leq\alpha}} \wedge \neg\mathbf{Int}_{\Omega^{<\alpha}}$$

So it is enough to express $\mathbf{Int}_{\Omega^{\leq\alpha}}$.

Now, J is a $\Omega^{\leq\alpha}$-subinterval of M iff $J \in \sum_{\Omega} \Omega^{<\alpha}$.

But this can be expressed in **MSO** since it is expressible to check whether an arbitrary subset is in Ω. $\square$

Definition 6.3. *Let* $\alpha > 0$ *be an ordinal.*

Let M *be a linear order and* $x \in M$*.*

We define the convex equivalence relation:

$$\sim_\alpha := \sim_{\mathcal{B}[\Omega^{<\alpha}]}$$

and $[x]_\alpha := [x]_{\mathcal{B}[\Omega^{<\alpha}]}$*.*

That is, $[x]_\alpha$ *is the largest* $\mathcal{B}[\Omega^{<\alpha}]$*-subinterval containing* x *in* M*.*

We define $\sigma_\alpha(x)$ *as the* α*-shape of* $[x]_\alpha$*.*

We define $L_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{L}[\Omega^{<\alpha}]}$ *and* $R_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{R}[\Omega^{<\alpha}]}$*.*

Lemma 6.4. *Let* M *be a linear order and* $\alpha > 0$ *an ordinal.*

Let $J \subseteq M$ *be an interval.*

Then $J \in \Omega^{<\alpha}$ *iff it is contained in a single* $\sim_\alpha$*-equivalence class* K*, such that:*

- *Either* $K \in \mathcal{L}[\Omega^{<\alpha}]$ *or there exists some* $x \in K$ *such that* $x < J$*.*
- *Either* $K \in \mathcal{R}[\Omega^{<\alpha}]$ *or there exists some* $x \in K$ *such that* $x > J$*.*

Proof. Suppose $J \in \Omega^{<\alpha}$. Then obviously J is contained in a single $\sim_\alpha$-equivalence class K.

We will show the first condition, the second is symmetric.

Suppose that for all $x \in K$, $J \leq x$. Then we can write $K = J + J'$. Since $J \in \Omega^{<\alpha}$, it follows that $K \in \mathcal{L}[\Omega^{<\alpha}]$. $\square$

Corollary 6.5. *Let* $\alpha > 0$ *be an ordinal.*

The predicate $\mathbf{Int}_{\Omega^{<\alpha}}$ *is* **MSO***-expressible over* $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$*.*

Theorem 6.6. *Let* $\mathbf{P}$ *be a computable property of linear orders of some finite signature.*

Let $\{\mathbf{Q}_i\}_{i\in I}$ *be a* finite *family of computable properties of linear orders over some finite signature which is disjoint from the signature of* $\mathbf{P}$*.*

Then $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$ *is a computable property of linear orders.*

Proof. We will use the decomposition theorem. Let $\tau(X_1, \ldots, X_m)$ be an n-type.

Then we can compute a formula $\psi(\xi)$ (where ξ has the type of a coloring whose range is the set of n-types) such that for any linear order $M = \sum_{i \in I} M_i$, and any given $A_1, \ldots, A_m \subseteq M$,

$$M \models \tau(A_1, \ldots, A_m) \iff I \models \psi(\Xi)$$

where Ξ is the coloring assigning $i \in I$ the n-type of M_i.

TBC. $\square$

Lemma 6.7. *Let α be an ordinal.*

Let P, L and R be first-order unary predicates.

Let C be the class of all countable linear orders labeled with P, L and R, such that P represents $\sim_\alpha$, $L_\alpha(x) \iff [x]_\alpha \in \mathcal{L}[\Omega^{<\alpha}]$ and $R_\alpha(x) \iff [x]_\alpha \in \mathcal{R}[\Omega^{<\alpha}]$.

Let $\mathbf{G}$ be the class of all countable linear orders I, labeled with a P, L and R, such that for every pair $i, i' \in I$ such that i' is the successor of i, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.

Let $\sigma(i) \in \{1, \omega, \omega^*, \omega^* + \omega\}$ be such that $L(i) = 1$ iff $\sigma(i) \in \{1, \omega\}$ and $R(i) = 1$ iff $\sigma(i) \in \{1, \omega^*\}$.

Then, $C = \bigcup_{I \in \mathbf{G}} \sum_{i \in I} \mathcal{S}_\alpha^{\sigma(i)}$.

Proof. ($\subseteq$) Let M be a countable linear order labeled with P, L and R as above.

Let $I = M / \sim_\alpha$ be the quotient of M by the equivalence relation $\sim_\alpha$.

Then $M = \sum_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ are the $\sim_\alpha$-equivalence class of I.

Then for each $i \in I$, $M_i \in \mathcal{B}[\Omega^{<\alpha}]$, and by definition $\sigma(i) = \sigma_\alpha(M_i)$.

Let i' be the successor of i in I.

Then $P(i) \neq P(i')$ since P represents $\sim_\alpha$.

Furthermore, suppose $R(i) = L(i') = 1$ holds. Then $M_i \in \mathcal{R}[\Omega^{<\alpha}]$ and $M_{i'} \in \mathcal{L}[\Omega^{<\alpha}]$. so M_i and $M_{i'}$ are the same $\sim_\alpha$-equivalence class of M, which is a contradiction.

Thus either $R(i) = 0$ or $L(i') = 0$.

($\supseteq$) Let $M = \sum_{i \in I} M_i$ be a linear order such that $I \in \mathbf{G}$ and $M_i \in \mathcal{S}_\alpha^{\sigma(i)}$ for each $i \in I$.

In particular $M_i \in \mathcal{B}[\Omega^{<\alpha}]$ for each $i \in I$, so it is contained in a single $\sim_\alpha$-equivalence class of M.

Suppose that there exist distinct $j, k \in I$ such that $j < k$, and M_j, M_k are in the same $\sim_\alpha$-equivalence class.

Let $x \in M_j$ and $y \in M_k$. Then $[x, y] \in \Omega^{<\alpha}$, and thus $[j, k] \in \Omega^{<\alpha}$, and in particular it is sparse.

Then there exist some $j', k' \in I$ such that $j < j' < k' < k$, and k' is the successor of j' in I.

Then $M_{j'}$ and $M_{k'}$ are in the same $\sim_\alpha$-equivalence class. Thus it must be the case that $M_{j'} \in \mathcal{R}[\Omega^{<\alpha}]$ and $M_{k'} \in \mathcal{L}[\Omega^{<\alpha}]$, which implies $R(j') = L(k') = 1$, which is a contradiction.

Thus $\{M_i\}_{i\in I}$ are pairwise distinct $\sim_\alpha$-equivalence classes, and obviously the conditions holds, so $M \in C$ and we are done. $\square$

Corollary 6.8. *Let C be defined as in lemma 6.7.*

Then C is a computable property.

Lemma 6.9. *Let $\alpha_k > \ldots > \alpha_1 > 0$ be ordinals, such that $\alpha_{j+1} \geq \alpha_j + \omega$ for all $j < k$.*

For each $j = 1, \ldots, k$, let P_j, L_j and R_j be first-order unary predicates, and let α_j be the corresponding ordinal.

Let C be the class of all countable linear orders labeled with P_j, L_j and R_j for $j = 1, \ldots, k$, such that P_j represents $\sim_{\alpha_j}$, $L_j(x) \iff [x]_{\alpha_j} \in \mathcal{L}[\Omega^{<\alpha_j}]$ and $R_j(x) \iff [x]_{\alpha_j} \in \mathcal{R}[\Omega^{<\alpha_j}]$ for each j.

Let $\mathbf{G}$ be the class of all countable linear orders I, labeled with a P, L and R, such that for every pair $i, i' \in I$ such that i' is the successor of i, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.

Let $\sigma(i) \in \{1, \omega, \omega^*, \omega^* + \omega\}$ be such that $L(i) = 1$ iff $\sigma(i) \in \{1, \omega\}$ and $R(i) = 1$ iff $\sigma(i) \in \{1, \omega^*\}$.

Then, $C = \bigcup_{I\in\mathbf{G}} \sum_{i\in I} \mathcal{S}_\alpha^{\sigma(i)}$.

Proof. Actually the proof is the same as in lemma 6.7, as we may assume there that the classes come with labels, and the arguments carry over verbatim. $\square$

Theorem 6.10. *Let $\alpha > 0$ be an ordinal.*

Satisfiability of $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$ over all countable linear orders is decidable.

Proof. First, by corollary 6.5, we can convert any formula in $\mathbf{MSO}[\mathbf{int}_{\Omega^{<\alpha}}]$ to an equivalent formula φ in $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$.

Now, we shall replace every occurrence of $[\cdot]_\alpha$ in φ with P, every occurrence of L_α with L, and every occurrence of R_α with R, getting a new formula φ'.

Then, satisfiability of φ over all countable linear orders, amounts to satisfiability of φ' over C, which is computable by corollary 6.8.

Thus we can compute $\mathbf{type}_n[C]$ and $\mathbf{type}_n[\varphi']$, and thus we can compute whether φ is satisfiable over all countable linear orders, by seeing if these sets intersect. $\square$

7 Everything Better

Theorem 7.1. *Let $\mathcal{C}$ be a computable property of linear orders, such that $\mathcal{C}$ is closed under taking subintervals, projections and inverse-projections (i.e, of one of the colors), and all finite-sums and $\mathcal{C}$-sums.*

Let $\mathbf{P}_1, \ldots, \mathbf{P}_k \subseteq \mathcal{C}$ be computable properties of linear orders.

Let $\mathbf{MSO}[P_1, \ldots, P_k]$ be monadic second order logic of order over $\mathcal{C}$, with $P_1, \ldots, P_k$ as monadic predicates whose semantics are: $P_i(X)$ holds iff X is a subinterval which satisfies $\mathbf{P}_i$.

Given ϕ a formula of $\mathbf{MSO}[P_1, \ldots, P_k]$ (possibly with free variables) we define

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

Then $\mathcal{C}_\phi$ is a computable property of linear orders.

Proof. By structural induction on ϕ.

Suppose ϕ is an atomic formula. If ϕ is of the form $X \subseteq Y$ or $X \leq Y$,

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

and thus,

$$\mathbf{type}_n [\mathcal{C}_\phi] = \{\tau \in \mathbf{type}_n [\mathcal{C}] : \tau \models \phi\}$$

which is computable since $\mathbf{type}_n [\mathcal{C}]$ is computable, and we can then compute whether $\tau \models \phi$ for each $\tau \in \mathbf{type}_n [\mathcal{C}]$.

If ϕ is of the form $P_i(X)$, then

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models P_i(X)\}$$

and thus,

$$\mathbf{type}_n [\mathcal{C}_\phi] = \mathbf{type}_n [\mathbf{P}_i]$$

which is computable since $\mathbf{P}_i$ is computable.

If $\phi = \neg\phi_1$, then

$$\mathcal{C}_\phi = \mathcal{C} \setminus \mathcal{C}_{\phi_1}$$

□