Orders

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Contents

1 Preorders

We begin by studying the properties of preorders. Basically, we define a *property* as a class which is close under isomorphism. We then define the sum operation on preorders. This will be used to create new properties from old ones.

Definitions 1.1 (Preorder)**.** *A* (labeled) preorder *is a a set M together with a binary relation $\leq$ on M such that $\leq$ is reflexive and transitive, possibly endowed with monadic predicates (labels) over some first-order monadic signature.*

Definition 1.2 (Property of preorders)**.** *A* property $\mathbf{P}$ *of preorders is a class of preorders which is closed under isomorphism.*

Definition 1.3. *A property $\mathbf{P}$ of preorders is* monotone *if for every preorder M, $M \in \mathbf{P}$ implies that every suborder of M is in $\mathbf{P}$.*

Definition 1.4. *Let M be a preorder.*
Then M^* is the dual/reverse preorder of M.

Definition 1.5 (Sum of preorders)**.** *Let I be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders over a disjoint signature (i.e., for every $i \in I$, I and M_i have disjoint sets of labels).*
The sum $M = \sum_{i\in I} M_i$ is defined as follows:
The domain is $M = \biguplus_{i\in I} M_i$ (a disjoint union).
Let $\leq_i$ be the preorder on M_i.
Let $x \in M_i$ and $y \in M_j$.
Then we define $x \leq y$ iff either $i = j$ and $x \leq_i y$ or $i < j$.
The labels are inherited from either I or the M_i's.
If $I = 2$, we define $M_1 + M_2 := \sum_{i\in 2} M_i$.

Lemma 1.6. *Let I be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders, over a disjoint signature.*
Then $M = \sum_{i\in I} M_i$ is a preorder.

Proof. Reflexivity is clear.
For transitivity, suppose $x \leq y$ and $y \leq z$.
Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.
Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.
Otherwise, $i < k$, and thus $x \leq z$, as required. □

Definition 1.7. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be properties of preorders.*
Then we define

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

The labels are inherited from either $\mathbf{P}_1$ or $\mathbf{P}_2$.

Definition 1.8. *A property $\mathbf{P}$ of preorders is an* additive property *if for every preorders M_1 and M_2, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

Definition 1.9 (Kleene plus)**.** *Let* $\mathbf{P}$ *be a property of preorders.*

We define its Kleene plus as the smallest property of preorders $\mathbf{P}^+$ *which contains* $\mathbf{P}$ *and is closed under finite sums.*

That is, $1^+ = \{1, 2, \ldots\}$*, and* $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$*.*

Definition 1.10 (Sum of a property over a preorder)**.** *Let* I *be a preorder.*

Let $\mathbf{Q}$ *be a property of preorders.*

Then we define

$$\sum_I \mathbf{Q} := \left\{ \sum_{i \in I} M_i : \forall i \in I. M_i \in \mathbf{Q} \right\}$$

Definition 1.11 (Sum of a family of properties over a preorder)**.** *Let* I *be a preorder.*

Let $\{\mathbf{Q}_i\}_{i \in I}$ *be a family of properties of preorders over a disjoint signature.*

Then we define

$$\sum_{i \in I} \mathbf{Q}_i := \left\{ \sum_{i \in I} M_i : \forall i \in I. M_i \in \mathbf{Q}_i \right\}$$

The labels are inherited from either I *or the* $\mathbf{Q}_i$*'s.*

Note 1.12. *Let* I *be a preorder, and let* $\mathbf{Q}$ *be a property of preorders.*

By the previous two definitions,

$$\sum_I \mathbf{Q} = \sum_{i \in I} \mathbf{Q}$$

Definition 1.13 (Sum of a property over a property)**.** *Let* $\mathbf{P}$ *be a property of preorders.*

Let $\mathbf{Q}$ *be a property of preorders over a disjoint signature.*

Then we define,

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_I \mathbf{Q} : I \in \mathbf{P} \right\}$$

2 Linear Orders

Linear orders are a central object of study in order theory, providing a natural setting for the analysis of sequences, intervals, and order types. In this chapter, we build upon the foundation of preorders to introduce linear orders, explore their properties, and develop tools for analyzing their structure. The concepts introduced here will be essential for understanding ranks, types, and decidability in later chapters.

Definitions 2.1 (Linear order). *A* linear order *a preorder which is symmetric and total.*

Definition 2.2 (Property of linear orders). *A* property $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 2.3 (Subintervals). *Let M be a linear order, and let $x, y \in M$, such that $x \leq y$.*

Then we define the bounded subintervals $[x, y]$, $(x, y]$, $[x, y)$ *and* (x, y) *as usual.*

We also define the semi-bounded subintervals $(-\infty, x]$, $[x, \infty)$, $(-\infty, x)$ *and* (x, ∞) *as usual.*

We also define the unbounded subinterval $(-\infty, \infty)$ *as the whole linear order M, as usual.*

A subinterval *is either a bounded subinterval, a semi-bounded subinterval or the unbounded subinterval.*

If $x > y$ then we define the intervals as follows:

$$\begin{aligned}
[x, y] &:= [y, x] \\
(x, y] &:= (y, x] \\
[x, y) &:= [y, x) \\
(x, y) &:= (y, x)
\end{aligned}$$

Definition 2.4. *Let M be a linear order.*

A set $A \subseteq M$ is left cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $y < x$.*

A set $A \subseteq M$ is right cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $x < y$.*

A set $A \subseteq M$ is bi-directionally cofinal *in M if it is both left and right cofinal.*

Lemma 2.5. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let $M \in \mathbf{P}$ be a linear order.

Let $x, y \in M$ be any two points in a linear order M.

Then, $[x, y] \in \mathbf{P}$.

Proof. WLOG, suppose $x \leq y$.

Note that,

$$M = (-\infty, \infty) = (-\infty, x) + [x, y] + (y, \infty)$$

when $(-\infty, x)$ and/or (y, ∞) may be empty.

Since $\mathbf{P}$ is an additive property, we conclude that $[x, y] \in \mathbf{P}$. ☐

Corollary 2.6. *Let $\mathbf{P}$ be a nontrivial additive property of linear orders.*

Then $1 \in \mathbf{P}$.

Proof. Let $M \in \mathbf{P}$ be any linear order and let $x \in M$ be any point in M.

Apply lemma 2.5 to the linear order M, and the points x and x, to conclude that $[x, x] \equiv 1 \in \mathbf{P}$. ☐

Note 2.7. *Note that corollary 2.6 is false if we do not restrict ourselves to linear orders.*

For example, $(1 \uplus 1)^{+}$ is a property of preorders which is additive, but does not contain 1.

Corollary 2.8. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let M be a linear order.

Let $x, y \in M$ be any two points in a linear order M. Then the following are equivalent:

1. $(x, y) \in \mathbf{P}$
2. $(x, y] \in \mathbf{P}$
3. $[x, y) \in \mathbf{P}$
4. $[x, y] \in \mathbf{P}$

Proof. This is just applying the definition of an additive property to the orders $[x, y]$ and 1. ☐

Corollary 2.9. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let M be a linear order.

Let $x, y, z \in M$ be any three points in a linear order M, such that $[x, y] \in \mathbf{P}$ and $[y, z] \in \mathbf{P}$.

Then $[x, z] \in \mathbf{P}$.

Proof. If $y \in [x, z]$, then $[x, z] = [x, y] + (y, z]$, and $(y, z] \in \mathbf{P}$ by corollary 2.8.

Otherwise, either $x \in [y, z]$ or $z \in [x, y]$. WLOG, suppose $z \in [x, y]$.

Then $[x, y] = [x, z] + (z, y]$, so $[x, z] \in \mathbf{P}$ by the fact that $\mathbf{P}$ is additive. ☐

Definitions 2.10. *Let $\mathbf{P}$ be a property of linear orders.*

We define the following properties of linear orders:

- $\mathcal{B}[\mathbf{P}]$ *is the class of linear orders M such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in $\mathbf{P}$.*
- $\mathcal{L}[\mathbf{P}]$ *is the class of linear orders M such that for every $x \in M$, the left-bounded ray $[x, \infty)$ is in $\mathbf{P}$.*

- $\mathcal{R}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x \in M$*, the right-bounded ray* $(-\infty, x]$ *is in* $\mathbf{P}$.

Definition 2.11. *A property* $\mathbf{P}$ *of linear orders is a* star property *if for every linear orders* M*, and every family* $\mathcal{F} \subseteq \mathbf{P}$ *of subintervals of* M *such that* $J_1 \cap J_2 \neq \emptyset$ *for every* $J_1, J_2 \in \mathcal{F}$*, we have that* $\bigcup \mathcal{F} \in \mathbf{P}$.

Lemma 2.12. *Let* $\mathbf{P}$ *be a star property.*

Then for every linear order M*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$.

Equivalently, we can define a convex equivalence relation $\sim_{\mathbf{P}}$ *on* M *such that* $x \sim_{\mathbf{P}} y$ *iff* $[x, y] \in \mathbf{P}$.

That is, $x \sim_{\mathbf{P}} y$ *iff* x *and* y *are in the same largest* $\mathbf{P}$*-subinterval.*

Proof. Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in $\mathcal{B}[\mathbf{P}]$, and by definition J is the largest $\mathbf{P}$-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. $\square$

Lemma 2.13 (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then the property $\mathcal{B}[\mathbf{P}]$ *is a star property.*

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by the definition of $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. Since $\mathbf{P}$ is additive, by corollary 2.9, we have $[x, y] \in \mathbf{P}$. $\square$

Lemma 2.14. *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then,

1. $\mathcal{L}[\mathbf{P}] = \{M : M + 1 \in \mathcal{B}[\mathbf{P}]\}$
2. $\mathcal{R}[\mathbf{P}] = \{M : 1 + M \in \mathcal{B}[\mathbf{P}]\}$
3. $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}] = \{M : 1 + M + 1 \in \mathcal{B}[\mathbf{P}]\}$

Proof. Let M be a linear order.

1. Suppose $M + \{\infty\} \in \mathcal{B}[\mathbf{P}]$. Then for every $x \in M$, we have $[x, \infty] \in \mathbf{P}$, and thus $[x, \infty) \in \mathbf{P}$. Therefore, $M \in \mathcal{L}[\mathbf{P}]$.

   Conversely, if $M \in \mathcal{L}[\mathbf{P}]$, let $x, y \in M$ be any two points in $M + 1$.

   If $y < \infty$, then $[x, y] \subseteq [x, \infty)$. Since $[x, \infty) \in \mathbf{P}$, we conclude that $[x, y] \in \mathbf{P}$. Otherwise, if $y = \infty$, then $[x, y] = [x, \infty] = [x, \infty) + \{\infty\}$, and thus $[x, y] \in \mathbf{P}$.

2. The second case is dual to the first case.

3. We will show a triple inclusion.

   If $M \in \mathbf{P}$, then by additivity, $1 + M \in \mathbf{P}$ and $M + 1 \in \mathbf{P}$, and thus $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$.

   If $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, then by lemma 2.13, $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$.

   If $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$, then M is a bounded subinterval of $1 + M + 1$, so $M \in \mathcal{B}[\mathbf{P}]$.

□

Lemma 2.15. *Let $\mathbf{P}$ be an additive property of linear orders.*
Then,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = \mathbf{P}&\\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}])\\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])\\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Proof. By lemma 2.14, we conclude that $\mathcal{L}[\mathbf{P}], \mathcal{R}[\mathbf{P}] \subseteq \mathcal{B}[\mathbf{P}]$, since $M + 1 \in \mathbf{P}$ and $1 + M \in \mathbf{P}$ both imply $1 + M + 1 \in \mathbf{P}$.

Thus,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = (\mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}])&\\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}])\\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])\\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Since by lemma 2.14 $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, we conclude what we wanted to prove. □

Lemma 2.16. *Let $\mathbf{P}$ be an additive property of linear orders.*
Let M, M_1, M_2 be linear orders such that $M = M_1 + M_2$.
Then,

1. $M \in \mathcal{B}[\mathbf{P}] \iff M_1 \in \mathcal{L}[\mathbf{P}] \wedge M_2 \in \mathcal{R}[\mathbf{P}]$

Proof. From lemma 2.14, we know that

1. $$\begin{aligned}M \in \mathcal{B}[\mathbf{P}] &\iff M_1 + M_2 \in \mathcal{B}[\mathbf{P}]\\ &\iff M_1 + 1 \in \mathcal{B}[\mathbf{P}] \wedge 1 + M_2 \in \mathcal{B}[\mathbf{P}]\\ &\iff M_1 \in \mathcal{L}[\mathbf{P}] \wedge M_2 \in \mathcal{R}[\mathbf{P}]\end{aligned}$$

□

Corollary 2.17. *Let* $\mathbf{P}$ *be an additive property of linear orders.*
Then,

$$\mathcal{B}\left[\mathbf{P}\right] \setminus \left(\mathcal{L}\left[\mathbf{P}\right] \cup \mathcal{R}\left[\mathbf{P}\right]\right) = \left(\mathcal{L}\left[\mathbf{P}\right] \setminus \mathcal{R}\left[\mathbf{P}\right]\right) + \left(\mathcal{R}\left[\mathbf{P}\right] \setminus \mathcal{L}\left[\mathbf{P}\right]\right)$$

Lemma 2.18 (Associativity of sum)**.** *Let* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *be properties.*
Then $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$*.*

Proof. It follows directly from the associativity of the sum operation on linear orders. Actually, it generalizes to any algebraic equation which holds on linear orders. □

Lemma 2.19 (Sum and union commute)**.** *Let* $\mathcal{P}$ *be a family of properties.*
Let $\mathbf{Q}$ *be a property.*
Then $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$*.*

Proof. This is obvious from the definition of the sum operation. □

Definition 2.20. *We define* **CNT** *as the class of all countable linear orders.*

Definition 2.21. *Let* $\gamma \geq \omega$ *be a limit ordinal.*
We define $\Gamma_\gamma := \{\beta : \beta \subseteq \gamma^* + \gamma\}^+$*.*
We define $\Omega := \Gamma_\omega$*.*

Example 2.22.

$$\Omega = \{1, \omega, \omega^*\}^+$$

Observation 2.23. *Let* $\gamma \geq \omega$ *be a limit ordinal.*
Then Γ_γ *is a monotone, additive property of linear orders.*

3 General Hausdorff Rank

The concept of rank provides a powerful tool for measuring the complexity of linear orders and related structures. In this chapter, we introduce the Hausdorff rank and its generalizations, which allow us to stratify classes of orders according to their structural depth. The results here lay the groundwork for the analysis of types and decidability in subsequent chapters.

Definition 3.1. *Let* $\mathbf{Q}$ *be a property of linear orders.*
We define a property $\mathbf{Q}^{<\alpha}$ *for every ordinal* α *as follows:*

- *For* $\alpha = 0$, $\mathbf{Q}^{<0} = \{1\}$.
- *For* $\alpha = \gamma + 1$,
$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}$$
- *For* α *a limit ordinal,*
$$\mathbf{Q}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{Q}^{<\beta}$$

Example 3.2. *Let* $\mathbf{Q}$ *be a property of linear orders.*
Then $\mathbf{Q}^{<1} = \mathbf{Q}$.

Lemma 3.3. *Let* $\mathbf{Q}$ *be a property of linear orders.*
Let α, δ *be ordinals.*
Then,
$$\mathbf{Q}^{<\alpha+\delta} = \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}$$

Proof. We shall prove this by induction on $\delta \geq 0$.
For $\delta = 0$ we need to prove

$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}^{0}} \mathbf{Q}^{<\alpha}.$$

Which is true by definition, since $\mathbf{Q}^{0} = \{1\}$.

For $\delta = \gamma + 1$, using the induction hypothesis,

$$\begin{aligned}
\mathbf{Q}^{<\alpha+\delta} &= \mathbf{Q}^{<\alpha+\gamma+1} \\
&= \sum_{\mathbf{Q}} \mathbf{Q}^{<\alpha+\gamma} \\
&= \sum_{\mathbf{Q}} \sum_{\mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\gamma+1}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}
\end{aligned}$$

For δ a limit ordinal, using the induction hypothesis,

$$\begin{aligned}
\mathbf{Q}^{<\alpha+\delta} &= \bigcup_{\gamma<\delta} \mathbf{Q}^{<\alpha+\gamma} \\
&= \bigcup_{\gamma<\delta} \sum_{\mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\bigcup_{\gamma<\delta} \mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}
\end{aligned}$$

□

Definition 3.4. *Let* $\mathbf{Q}$ *be a property of linear orders.*
Let α, β *be ordinals with* $\alpha < \beta$.
We define,

1. $\mathbf{Q}^{\leq\alpha} := \mathbf{Q}^{<\alpha+1}$
2. $\mathbf{Q}^{=\alpha} := \mathbf{Q}^{\leq\alpha} \setminus \mathbf{Q}^{<\alpha}$
3. $\mathbf{Q}^{[\alpha,\beta)} := \mathbf{Q}^{<\beta} \setminus \mathbf{Q}^{<\alpha}$

Definition 3.5. *Let* $\mathbf{Q}$ *be a property of linear orders.*
Let M *be a linear order, such that* $M \in (\mathbf{Q}^{<\alpha})^+$ *for some ordinal* α.
We define the $\mathbf{Q}$-Hausdorff rank *of* M *as*

$$\mathbf{hrank}_{\mathbf{Q}}(M) = \sup\left\{\beta : M \notin \left(\mathbf{Q}^{<\beta}\right)^+\right\}$$

where the supremum is taken over all ordinals β. *(Recall that the supremum of the empty set is defined to be* 0.*)*

4 ω-Hausdorff rank

In this chapter, we focus on the special case of the Hausdorff rank associated with the ordinal ω. This case is of particular interest due to its connections with countable structures and its role in the classification of infinite linear orders. We introduce new families of properties and analyze their relationships, providing tools that will be essential for the study of types and decidability.

Definitions 4.1. *Let $\alpha > 0$ be an ordinal.*

We define:

1. $\mathcal{S}_\alpha^1 := \Omega^{<\alpha}$
2. $\mathcal{S}_\alpha^\omega := \mathcal{R}\left[\Omega^{<\alpha}\right] \setminus \mathcal{L}\left[\Omega^{<\alpha}\right]$
3. $\mathcal{S}_\alpha^{\omega^*} := \mathcal{L}\left[\Omega^{<\alpha}\right] \setminus \mathcal{R}\left[\Omega^{<\alpha}\right]$
4. $\mathcal{S}_\alpha^{\omega^*+\omega} := \mathcal{B}\left[\Omega^{<\alpha}\right] \setminus \left(\mathcal{L}\left[\Omega^{<\alpha}\right] \cup \mathcal{R}\left[\Omega^{<\alpha}\right]\right)$

The names will soon be justified.

Lemma 4.2. *Let $\alpha > 0$ be an ordinal.*

Then,

1. $\mathcal{R}\left[\Omega^{<\alpha}\right] = \sum_\omega \Omega^{<\alpha}$.
2. $\mathcal{L}\left[\Omega^{<\alpha}\right] = \sum_{\omega^*} \Omega^{<\alpha}$.
3. $\mathcal{B}\left[\Omega^{<\alpha}\right] = \sum_{\omega^*+\omega} \Omega^{<\alpha}$.

Proof. 1. Let us prove the first part. ($\supseteq$) Let $M \in \sum_\omega \Omega^{<\alpha}$ be a linear order.

Let $M = \sum_{i\in\omega} M_i$ be the decomposition of M, where $M_i \in \Omega^{<\alpha}$.

Let $x, y \in M$ be any two points in M. WLOG $x \leq y$.

Suppose $x \in M_i$ and $y \in M_j$ for $i, j \in \omega$.

Since i and j have a finite distance in ω, we conclude $[x, y] \subseteq M_i + \ldots + M_j$, and thus $[x, y] \subseteq \left(\Omega^{<\alpha}\right)^+ = \Omega^\alpha$.

($\subseteq$) Let $M \in \mathcal{R}\left[\Omega^{<\alpha}\right]$ be a linear order.

Since M is countable, let $\{x_i\}_{i\in\omega} M$ be a right cofinal ω-sequence in M.

Let $M_0 = (-\infty, x_0]$ and and $M_i = (x_{i-1}, x_i]$ for $i > 0$.

Then $M = \sum_{i\in\omega} M_i$.

But M_i is a right-bounded interval and thus $M_i \in \Omega^{<\alpha}$, so $M \in \sum_\omega \Omega^{<\alpha}$.

2. The second part is symmetric.

3. The third part follows from corollary 2.17:

$$\begin{aligned}
\mathcal{B}\left[\Omega^{<\alpha}\right] &= \mathcal{L}\left[\Omega^{<\alpha}\right] + \mathcal{R}\left[\Omega^{<\alpha}\right] \\
&= \sum_{\omega^*} \Omega^{<\alpha} + \sum_{\omega} \Omega^{<\alpha} \\
&= \sum_{\omega^*+\omega} \Omega^{<\alpha}
\end{aligned}$$

$\square$

An immediate corollary of lemma 4.2 is that:

Corollary 4.3. *Let $\alpha > 0$ be an ordinal.*
Then,

1. $\Omega^{\leq\alpha} = (\mathcal{B}[\Omega^{<\alpha}])^{+}$
2. $\Omega^{=\alpha} = \left(\mathcal{S}_{\alpha}^{\omega^*} \uplus \mathcal{S}_{\alpha}^{\omega}\right)^{+}$

Lemma 4.4. *Let α be an ordinal.*
Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$.
Then, we have the following:

$$\mathcal{S}_{\alpha+1}^{s} = \sum_{i\in s} \Omega^{=\alpha}$$

Proof. It is enough to prove the case $s = \omega$, since $s = \omega^*$ follows by symmetry, and $s = \omega^* + \omega$ follows by adding the previous two cases.

($\subseteq$) Let $M \in \mathcal{S}_{\alpha+1}^{\omega}$.

By lemma 4.2 and corollary 4.3

$$\mathcal{R}\left[\Omega^{<\alpha+1}\right] = \sum_{\omega} \Omega^{<\alpha+1} = \sum_{\omega} \Omega^{\leq\alpha} = \sum_{\omega} (\mathcal{B}[\Omega^{<\alpha}])^{+} = \sum_{\omega} \mathcal{B}[\Omega^{<\alpha}]$$

since by definitions 4.1, $M \in \mathcal{R}\left[\Omega^{<\alpha+1}\right]$, we conclude that $M = \sum_{i\in\omega} M_i$ for a sequence $\{M_i\}_{i\in\omega} \subseteq \mathcal{B}[\Omega^{<\alpha}]$.

If $M_i \in \Omega^{=\alpha}$ held for only finitely many $i \in \omega$, we would have $M \in \Omega^{\leq\alpha}$, which is a contradiction to $M \notin \mathcal{L}[\Omega^{<\alpha}]$.

Thus, $M_i \in \Omega^{=\alpha}$ holds for infinitely many $i \in \omega$, and thus (by adjoining $M_i \in \Omega^{<\alpha}$ to the next $\Omega^{=\alpha}$ one) we conclude $M \in \sum_{\omega} \Omega^{=\alpha}$.

($\supseteq$) Let $M \in \sum_{\omega} \Omega^{=\alpha}$.

Since $M \in \sum_{\omega} \Omega^{<\alpha+1}$, by lemma 4.2, $M \in \mathcal{R}\left[\Omega^{<\alpha+1}\right]$.

By corollary 4.3,

$$M \in \sum_{\omega} \Omega^{=\alpha} = \sum_{\omega} \left(\mathcal{S}_{\alpha}^{\omega^*} \uplus \mathcal{S}_{\alpha}^{\omega}\right)^{+} = \sum_{\omega} \left(\mathcal{S}_{\alpha}^{\omega^*} \uplus \mathcal{S}_{\alpha}^{\omega}\right)$$

Suppose $M = \sum_{i\in\omega} M_i$ where $M_i \in \mathcal{S}_{\alpha}^{s_i}$ for $s_i \in \omega^*, \omega$. By the pigeonhole principle, suppose eventually $s_i = s$. WLOG, eventually $s_i = \omega$. WLOG, $s_i = \omega$ for all $i \in \omega$, so $M \in \sum_{\omega} \mathcal{S}_{\alpha}^{\omega}$.

Suppose by contradiction $M = \Omega^{\leq\alpha} = (\mathcal{B}[\Omega^{<\alpha}])^{+}$. In particular, by the pigeonhole principle, there exists some $N \in \omega$ such that $\sum_{N\leq i<\omega} M_i \in \mathcal{B}[\Omega^{<\alpha}]$, which is a contradiction because it follows that $M_{N+1} \in \Omega{<}\ \alpha$ as it is bounded between M_N and M_{N+2}. □

Lemma 4.5. *Let $\{\alpha_i\}_{i\in\omega}$ be a strictly increasing ordinal sequence, and let $\alpha =$* $\sup_{\alpha_i}$.
Then,

$$\mathcal{S}_{\alpha}^{s} = \sum_{i\in s} \Omega^{[\alpha_i,\alpha)}$$

Proof. Again, it is enough to prove for $s = \omega$. ($\subseteq$) Let $M \in \mathcal{S}^{\omega}_{\alpha}$. Let $y_{i i<\omega}$ be a right cofinal ω-sequence in M.

Thus we can choose some x_0 far enough such that $(-\infty, x_0] \in \Omega^{[\alpha_0,\alpha)}$, and $x_0 > y_0$. Now by induction we choose x_1 such that $(x_0, x_1] \in \Omega^{[\alpha_1,\alpha)}$, and $x_1 > y_1$.

By iterating ω times we get an ω-sequence $\{M_i\}_{i\in\omega}$ such that $M = \sum_{i\in\omega} M_i$ and $M_i \in \Omega^{[\alpha_i,\alpha)}$, where $M_i = (x_{i-1}, x_i]$ (where $x_{-1} := -\infty$).

($\supseteq$) Let $M \in \sum_{i\in\omega} \Omega^{[\alpha_i,\alpha)}$. It is obvious that $M \in \mathcal{R}\left[\Omega^{<\alpha}\right]$ since every right-bounded ray is in $\Omega^{\leq\alpha_i}$ for some $i \in \omega$.

However, $M \notin \Omega^{<\alpha_i}$ for any $i \in \omega$, so $M \notin \Omega^{<\alpha}$.

$\square$

Lemma 4.6. *Let $\{\alpha_i\}_{i\in\omega}$ be a non-decreasing ordinal sequence, and let $\alpha =$* $\sup_{\alpha_i+1}$.

Then,

$$\mathcal{S}^{s}_{\alpha} = \sum_{i\in s} \Omega^{[\alpha_i,\alpha)}$$

Proof. It is just a way to write lemma 4.4 and lemma 4.5 together more succinctly. $\square$

Note 4.7. *In the proof of lemma 4.6, we actually use the fact that we work over $\Omega = \Gamma_{\omega}$. This proof would not have worked over Γ_{β} for $\beta > \omega$.*

5 Type Theory

Type theory provides a framework for analyzing the expressive power of logical languages over classes of structures. In this chapter, we introduce the notion of types for properties of preorders and study their computability. The results here connect the structural properties of orders with logical definability, setting the stage for the study of decidability in the next chapter.

Definition 5.1. *Let* $\mathbf{P}$ *be a property of preorders.*

Let $n \in \mathbb{N}$.

We define $\mathbf{type}_n[\mathbf{P}]$ *as the set of all* n*-types satisfiable in* $\mathbf{P}$.

Definition 5.2. *A property* $\mathbf{P}$ *of preorders is* computable *if* $n \mapsto \mathbf{type}_n[\mathbf{P}]$ *is a computable function.*

Equivalently, satisfiability of$\mathbf{MSO}$ *over* $\mathbf{P}$ *is decidable.*

Lemma 5.3. *Let* $\mathbf{Q}$ *be a property of preorders.*

There exists a computable function $f_{\mathbf{Q}} = f : \mathbb{N} \to \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and every ordinal* $\alpha \geq f(n)$, $\mathbf{type}_n[\mathbf{Q}^{<\alpha}] = \mathbf{type}_n\left[\mathbf{Q}^{<f(n)}\right]$.

Proof. Since there are only finitely many n-types, and the ordinal sequence

$$\left\{\mathbf{type}_n\left[\mathbf{Q}^{<\kappa}\right]\right\}_\kappa$$

is monotone, there must be some minimal $\kappa_0 \in \omega$ where the sequence stabilizes.

This κ_0 is computable as a function of n, because $\mathbf{type}_n[\mathbf{Q}^{<\kappa}]$ is computable for every finite κ. □

Lemma 5.4. *There exist global computable functions* $a, b : \mathbb{N} \to \mathbb{N}$ *such that for all* $n, c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c_1}] = \mathbf{type}_n[\mathbf{Q}^{=c_2}]$$

Proof. Let $n \in \mathbb{N}$.

Since there are only finitely many sets of n-types, there exist (and can be computed) some $a(n) \geq f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$$

By induction if follows that for all $c \geq a(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c}] = \mathbf{type}_n\left[\mathbf{Q}^{=c+b(n)}\right]$$

since $\mathbf{Q}^{=c+1} = \sum_{\mathbf{Q}} \mathbf{Q}^{=c}$. □

Corollary 5.5. *Let* $n \in \mathbb{N}$, *and let* $\alpha \geq \omega$ *be an ordinal.*

Let $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$ *be a shape.*

Then there exists a computable function $b(n)$ *such that for all* $c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$, *we have*

$$\mathbf{type}_n\left[\mathcal{S}^s_{c_1}\right] = \mathbf{type}_n\left[\mathcal{S}^s_{c_2}\right]$$

Proof. For $s = 1$, it follows from lemma 5.3, since $\mathcal{S}_c^1 = \Omega^{<c}$ and $c \geq a(n) \geq f(n)$ for $c \in \{c_1, c_2\}$.

For $s \in \{\omega, \omega^*, \omega^* + \omega\}$, it follows easily from lemma 4.6 and lemma 5.4. $\square$

Lemma 5.6. *For every $n \in \mathbb{N}$ and for every pair of ordinals $\alpha \geq \omega$, $\beta > \alpha$,*

$$\mathbf{type}_n\left[\mathbf{Q}^{[\alpha,\beta)}\right] = \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]$$

In particular, $\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.

Proof. It is enough to prove that

$$\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right] = \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]$$

We thus proceed by induction on $\alpha \geq \omega$.

Let $\{\alpha_i\}_{i\in\omega}$ be an increasing ω-sequence of ordinals such that $a(n) \leq \alpha_i$ for all $i \in \omega$, and $\sup_{i\in\omega}(\alpha_i + 1) = \alpha$.

Then $\mathbf{Q}^{=\alpha} = \sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{Q}^{[\alpha_i,\alpha)}$ and thus,

$$\begin{aligned}
\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right] &= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{Q}^{[\alpha_i,\alpha)}\right] \\
&= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \sum_{\mathbf{Q}} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c+1}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]
\end{aligned}$$

where the last transition is because $\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$. $\square$

Corollary 5.7. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$ be a shape.

$$\mathbf{type}_n \left[\mathcal{S}^s_\alpha\right] = \mathbf{type}_n \left[\sum_s \bigcup_{c<b(n)} \Omega^{=a(n)+c}\right]$$

In particular, $\mathbf{type}_n \left[\mathcal{S}^s_\alpha\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.

Proof. There exists an increasing s-sequence $\{\alpha_i\}_{i\in s}$ such that $a(n) \leq \alpha_i$ for all $i \in s$, and $\sup_{i\in s} (\alpha_i + 1) = \alpha$.

Then $\mathcal{S}^s_\alpha = \sum_{i\in s} \Omega^{=\alpha_i}$, and thus,

$$\begin{aligned}
\mathbf{type}_n \left[\mathcal{S}^s_\alpha\right] &= \mathbf{type}_n \left[\sum_{i\in s} \Omega^{=\alpha_i}\right] \\
&= \mathbf{type}_n \left[\sum_s \bigcup_{c<b(n)} \Omega^{=a(n)+c}\right] \\
&= \mathbf{type}_n \left[\bigcup_{c<b(n)} \sum_s \Omega^{=a(n)+c}\right] \\
&= \mathbf{type}_n \left[\bigcup_{c<b(n)} \mathcal{S}^s_{a(n)+c+1}\right] \\
&= \mathbf{type}_n \left[\bigcup_{c<b(n)} \mathcal{S}^s_{a(n)+c}\right]
\end{aligned}$$

where the last transition is by corollary 5.5.

□

Lemma 5.8. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Then there exists a computable function $e(n)$ such that for all $c_1, c_2 \in \mathbb{N}$ with $c_1, c_2 \geq e(n)$,

$$\mathbf{type}_n \left[\mathbf{Q}^{=c_1\cdot\alpha}\right] = \mathbf{type}_n \left[\mathbf{Q}^{=c_2\cdot\alpha}\right]$$

Proof. Let $n \in \mathbb{N}$.

By lemma 5.3, there exists $e(n)$ such that for all $c \geq e(n)$,

$$\mathbf{type}_n \left[\mathbf{Q}^{=c\cdot\alpha}\right] = \mathbf{type}_n \left[\mathbf{Q}^{=c\cdot\alpha+1}\right]$$

Since α is a limit ordinal, $\mathbf{Q}^{=c\cdot\alpha+1} = \sum_{\beta<\alpha} \mathbf{Q}^{=c\cdot\beta}$.

Thus, by the stability of types under summation,

$$\mathbf{type}_n\left[\mathbf{Q}^{=c\cdot\alpha+1}\right] = \mathbf{type}_n\left[\sum_{\beta<\alpha}\mathbf{Q}^{=c\cdot\beta}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=c}\right]$$

for all $c \geq e(n)$. $\square$

6 Decidability of the rank

Decidability questions lie at the heart of mathematical logic and theoretical computer science. In this chapter, we investigate the decidability of rank-related properties for linear orders, connecting the structural results of previous chapters with algorithmic considerations. We introduce key predicates and equivalence relations, and show how they can be expressed and manipulated in logical frameworks.

Definition 6.1. *Let* $\mathbf{Q}$ *be a property of linear orders.*

Let M *be a linear order.*

We define the predicate $\mathbf{Int}_{\mathbf{Q}}(J)$ *as true in* M *iff* J *is a* $\mathbf{Q}$*-subinterval of* M*.*

Lemma 6.2. *Let* $\alpha > 0$ *be an ordinal.*

Then predicates $\mathbf{Int}_{\Omega^{\leq\alpha}}$*,* $\mathbf{Int}_{\Omega^{=\alpha}}$ *are expressible in* $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$*.*

Proof. Obviously,

$$\mathbf{Int}_{\Omega^{=\alpha}} \iff \mathbf{Int}_{\Omega^{\leq\alpha}} \wedge \neg\mathbf{Int}_{\Omega^{<\alpha}}$$

So it is enough to express $\mathbf{Int}_{\Omega^{\leq\alpha}}$.

Now, J is a $\Omega^{\leq\alpha}$-subinterval of M iff $J \in \sum_{\Omega} \Omega^{<\alpha}$.

But this can be expressed in $\mathbf{MSO}$ since it is expressible to check whether an arbitrary subset is in Ω. □

Definition 6.3. *Let* $\alpha > 0$ *be an ordinal.*

Let M *be a linear order and* $x \in M$*.*

We define the convex equivalence relation:

$$\sim_\alpha := \sim_{\mathcal{B}[\Omega^{<\alpha}]}$$

and $[x]_\alpha := [x]_{\mathcal{B}[\Omega^{<\alpha}]}$*.*

That is, $[x]_\alpha$ *is the largest* $\mathcal{B}[\Omega^{<\alpha}]$*-subinterval containing* x *in* M*.*

We define $\sigma_\alpha(x)$ *as the* α*-shape of* $[x]_\alpha$*.*

We define $L_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{L}[\Omega^{<\alpha}]}$ *and* $R_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{R}[\Omega^{<\alpha}]}$*.*

Lemma 6.4. *Let* M *be a linear order and* $\alpha > 0$ *an ordinal.*

Let $J \subseteq M$ *be an interval.*

Then $J \in \Omega^{<\alpha}$ *iff it is contained in a single* $\sim_\alpha$*-equivalence class* K*, such that:*

- *Either* $K \in \mathcal{L}[\Omega^{<\alpha}]$ *or there exists some* $x \in K$ *such that* $x < J$*.*
- *Either* $K \in \mathcal{R}[\Omega^{<\alpha}]$ *or there exists some* $x \in K$ *such that* $x > J$*.*

Proof. Suppose $J \in \Omega^{<\alpha}$. Then obviously J is contained in a single $\sim_\alpha$-equivalence class K.

We will show the first condition, the second is symmetric.

Suppose that for all $x \in K$, $J \leq x$. Then we can write $K = J + J'$. Since $J \in \Omega^{<\alpha}$, it follows that $K \in \mathcal{L}[\Omega^{<\alpha}]$. □

Corollary 6.5. *Let $\alpha > 0$ be an ordinal.*

The predicate $\mathbf{Int}_{\Omega^{<\alpha}}$ *is* **MSO***-expressible over* $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$.

Theorem 6.6. *Let* $\mathbf{P}$ *be a computable property of linear orders of some finite signature, including* $C_1, \ldots, C_k$.

Let $\mathcal{Q}$ *be a* finite *set of computable properties of linear orders over some finite signature which is disjoint from the signature of* $\mathbf{P}$.

Let $F : 2^k \to \mathcal{Q}$ *be any function.*

Then $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} F(C_1(i), \ldots, C_k(i))$ *is a computable property of linear orders.*

Proof. We will use the decomposition theorem. Let φ be a formula of quantifier depth n. WLOG, φ is a sentence.

Then we can compute a formula $\psi(\xi)$ (where ξ has the type of a coloring whose range is the set of n-types) such that for any linear order $M = \sum_{i\in I} M_i$,

$$M \models \varphi \iff I \models \psi(\Xi)$$

where Ξ is the coloring assigning $i \in I$ the n-type of M_i.

Thus, there is some $M \in \bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$, such that $M \models \varphi$ iff there exists some $I \in \mathbf{P}$, and assignment Ξ of n-types, such that $\Xi(i)$ is satisfiable in $\mathbf{Q}_i$ for all $i \in I$, and $I \models \psi(\Xi)$.

Equivalently, φ is satisfiable over $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$ iff

$$\begin{aligned} \exists\xi.\psi(\xi) \land \xi \text{ is a coloring with } n\text{-types} \\ \land \forall i.\xi(i) \in \mathbf{type}_n\left[F(C_1(i), \ldots, C_k(i))\right] \end{aligned}$$

is satisfiable over $\mathbf{P}$.

Since $\mathcal{Q}$ has only computable properties, We can pre-compute $\mathbf{type}_n\left[F(\vec{c})\right]$ for any value $\vec{c} \in 2^k$ so we can actually write the formula above in **MSO**. Furthermore, since $\mathbf{P}$ is computable, we can check whether it is satisfiable over $\mathbf{P}$. So we are done. ∎

Lemma 6.7. *Let α be an ordinal.*

Let P, L and R be first-order unary predicates.

Let C be the class of all countable linear orders labeled with P, L and R, such that P represents $\sim_\alpha$, $L_\alpha(x) \iff [x]_\alpha \in \mathcal{L}[\Omega^{<\alpha}]$ and $R_\alpha(x) \iff [x]_\alpha \in \mathcal{R}[\Omega^{<\alpha}]$.

Let $\mathbf{G}$ *be the class of all countable linear orders I, labeled with a P, L and R, such that for every pair $i, i' \in I$ such that i' is the successor of i, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.*

Let $\sigma(i) \in \{1, \omega, \omega^*, \omega^* + \omega\}$ be such that $L(i) = 1$ iff $\sigma(i) \in \{1, \omega\}$ and $R(i) = 1$ iff $\sigma(i) \in \{1, \omega^*\}$.

Then, $C = \bigcup_{I\in\mathbf{G}} \sum_{i\in I} \mathcal{S}_\alpha^{\sigma(i)}$.

Proof. ($\subseteq$) Let M be a countable linear order labeled with P, L and R as above.

Let $I = M/\sim_\alpha$ be the quotient of M by the equivalence relation $\sim_\alpha$.

Then $M = \sum_{i\in I} M_i$, where $\{M_i\}_{i\in I}$ are the $\sim_\alpha$-equivalence class of I.

Then for each $i \in I$, $M_i \in \mathcal{B}\left[\Omega^{<\alpha}\right]$, and by definition $\sigma(i) = \sigma_\alpha\left(M_i\right)$.

Let i' be the successor of i in I.

Then $P(i) \neq P(i')$ since P represents $\sim_\alpha$.

Furthermore, suppose $R(i) = L(i') = 1$ holds. Then $M_i \in \mathcal{R}\left[\Omega^{<\alpha}\right]$ and $M_{i'} \in \mathcal{L}\left[\Omega^{<\alpha}\right]$. so M_i and $M_{i'}$ are the same $\sim_\alpha$-equivalence class of M, which is a contradiction.

Thus either $R(i) = 0$ or $L(i') = 0$.

($\supseteq$) Let $M = \sum_{i \in I} M_i$ be a linear order such that $I \in \mathbf{G}$ and $M_i \in \mathcal{S}_\alpha^{\sigma(i)}$ for each $i \in I$.

In particular $M_i \in \mathcal{B}\left[\Omega^{<\alpha}\right]$ for each $i \in I$, so it is contained in a single $\sim_\alpha$-equivalence class of M.

Suppose that there exist distinct $j, k \in I$ such that $j < k$, and M_j, M_k are in the same $\sim_\alpha$-equivalence class.

Let $x \in M_j$ and $y \in M_k$. Then $[x, y] \in \Omega^{<\alpha}$, and thus $[j, k] \in \Omega^{<\alpha}$, and in particular it is sparse.

Then there exist some $j', k' \in I$ such that $j < j' < k' < k$, and k' is the successor of j' in I.

Then $M_{j'}$ and $M_{k'}$ are in the same $\sim_\alpha$-equivalence class. Thus it must be the case that $M_{j'} \in \mathcal{R}\left[\Omega^{<\alpha}\right]$ and $M_{k'} \in \mathcal{L}\left[\Omega^{<\alpha}\right]$, which implies $R(j') = L(k') = 1$, which is a contradiction.

Thus $\left\{M_i\right\}_{i \in I}$ are pairwise distinct $\sim_\alpha$-equivalence classes, and obviously the conditions holds, so $M \in C$ and we are done. ☐

Corollary 6.8. *Let $\alpha > 0$ be an ordinal.*

Let C be defined as in lemma 6.7.

Then C is a computable property of linear orders.

Proof. Since $\gg$ is clearly computable, it follows from combining theorem 6.6 and lemma 6.7. ☐

Theorem 6.9. *Let $\alpha > 0$ be an ordinal.*

Satisfiability of $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$ over all countable linear orders is decidable.

Proof. First, by corollary 6.5, we can convert any formula in $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$ to an equivalent formula φ in $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$.

Now, we shall replace every occurrence of $[\cdot]_\alpha$ in φ with P, every occurrence of L_α with L, and every occurrence of R_α with R, getting a new formula φ'.

Then, satisfiability of φ over all countable linear orders, amounts to satisfiability of φ' over C, which is computable by corollary 6.8. ☐

Lemma 6.10. *Let $\alpha_0, \ldots, \alpha_k$, such that $\alpha_i \geq \alpha_{i-1} + \omega$ for $i > 0$.*

Let $P_1, L_1, R_1, \ldots, P_k, L_k, R_k$. first-order unary predicates.

Let C be the class of all countable linear orders labeled with P_j, L_j and R_j, such that for every quadruple α, P, L, R, P represents $\sim_\alpha$, $L_\alpha(x) \iff [x]_\alpha \in \mathcal{L}\left[\Omega^{<\alpha}\right]$ and $R_\alpha(x) \iff [x]_\alpha \in \mathcal{R}\left[\Omega^{<\alpha}\right]$.

Let $\mathbf{G}$ be the class of all countable linear orders I, labeled with a P_j, L_j and R_j, such that for every triplet P, L, R, that for every pair $i, i' \in I$ such that i' is the successor of i, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.

For every triplet, let $\sigma_j = \sigma$ *be such That* $\sigma(i) \in \{1, \omega, \omega^*, \omega^* + \omega\}$ *and* $L(i) = 1$ *iff* $\sigma(i) \in \{1, \omega\}$ *and* $R(i) = 1$ *iff* $\sigma(i) \in \{1, \omega^*\}$*.*

Then, $C = \bigcup_{I \in \mathbf{G}} \sum_{i \in I} \mathcal{S}_{\alpha_0}^{\sigma_0(i)}$*. Furthermore, its* **MSO** *theory is independent of the choice of* $\alpha_0, \ldots, \alpha_k$*.*

Theorem 6.11. *Let* $\alpha_k > \ldots > \alpha_1 > 0$ *be ordinals.*

Satisfiability of $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha_1}}, \ldots, \mathbf{Int}_{\Omega^{<\alpha_k}}]$ *over all countable linear orders is decidable.*

Proof. First, by corollary 6.5, we can convert any formula in

$$\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha_1}}, \ldots, \mathbf{Int}_{\Omega^{<\alpha_k}}]$$

to an equivalent formula φ in

$$\mathbf{MSO}[[\cdot]_{\alpha_1}, L_{\alpha_1}, R_{\alpha_1}], \ldots, \mathbf{MSO}[[\cdot]_{\alpha_k}, L_{\alpha_k}, R_{\alpha_k}]$$

Now, we shall replace every occurrence of $[\cdot]_{\alpha_i}$ in φ with P_i, every occurrence of L_{α_i} with L_i, and every occurrence of R_{α_i} with R_i, getting a new formula φ'.

Then, satisfiability of φ over all countable linear orders, amounts to satisfiability of φ' over C, which is computable by corollary 6.8. $\square$