Orders

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Contents

1 Preorders

We begin by studying the classes of preorders. Basically, we define a *class* as a class which is close under isomorphism. We then define the sum operation on preorders. This will be used to create new classes from old ones.

Definitions 1.1 (Preorder)**.** *A* (labeled) preorder *is a a set M together with a binary relation $\leq$ on M such that $\leq$ is reflexive and transitive, possibly endowed with monadic predicates (labels) over some first-order monadic signature.*

Definition 1.2 (class of preorders)**.** *A* class $\mathbf{P}$ *of preorders is a class of preorders which is closed under isomorphism.*

Definition 1.3. *a class $\mathbf{P}$ of preorders is* monotone *if for every preorder M, $M \in \mathbf{P}$ implies that every suborder of M is in $\mathbf{P}$.*

Definition 1.4. *Let M be a preorder.*
Then M^* is the dual/reverse preorder of M.

Definition 1.5 (Sum of preorders)**.** *Let I be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders over a disjoint signature (i.e., for every $i \in I$, I and M_i have disjoint sets of labels).*
The sum $M = \sum_{i\in I} M_i$ is defined as follows:
The domain is $M = \biguplus_{i\in I} M_i$ (a disjoint union).
Let $\leq_i$ be the preorder on M_i.
Let $x \in M_i$ and $y \in M_j$.
Then we define $x \leq y$ iff either $i = j$ and $x \leq_i y$ or $i < j$.
The labels are inherited from either I or the M_i's.
If $I = 2$, we define $M_0 + M_1 := \sum_{i\in 2} M_i$.

Lemma 1.6. *Let I be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders, over a disjoint signature.*
Then $M = \sum_{i\in I} M_i$ is a preorder.

Proof. Reflexivity is clear.
For transitivity, suppose $x \leq y$ and $y \leq z$.
Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.
Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.
Otherwise, $i < k$, and thus $x \leq z$, as required. $\square$

Definition 1.7. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be classes of preorders.*
Then we define

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

The labels are inherited from either $\mathbf{P}_1$ or $\mathbf{P}_2$.

Definition 1.8. *a class $\mathbf{P}$ of preorders is an* additive class *if for every preorders M_1 and M_2, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

Definition 1.9 (Kleene plus). *Let* $\mathbf{P}$ *be a class of preorders.*

We define its Kleene plus as the smallest class of preorders $\mathbf{P}^+$ *which contains* $\mathbf{P}$ *and is closed under finite sums.*

That is, $1^+ = \{1, 2, \ldots\}$*, and* $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$.

Definition 1.10 (Sum of a family of classes over a preorder). *Let* I *be a preorder.*

Let $\{\mathbf{Q}_i\}_{i \in I}$ *be a family of classes of preorders, each with a signature disjoint from that of* I.

Then we define

$$\sum_{i \in I} \mathbf{Q}_i := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q}_i \right\}$$

The labels are inherited from either I *or the* $\mathbf{Q}_i$*'s.*

Definition 1.11 (Sum of a class over a preorder). *Let* I *be a preorder.*

Let $\mathbf{Q}$ *be a single class of preorders, with a signature disjoint from that of* I.

Then we define

$$\sum_{I} \mathbf{Q} := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q} \right\}$$

Note 1.12. *Let* I *be a preorder, and let* $\mathbf{Q}$ *be a class of preorders with a signature disjoint from that of* I.

By the previous two definitions,

$$\sum_{I} \mathbf{Q} = \sum_{i \in I} \mathbf{Q}$$

Definition 1.13 (Sum of a class over a class). *Let* $\mathbf{P}$ *be a class of preorders.*

Let $\mathbf{Q}$ *be a class of preorders over a signature disjoint from that of* $\mathbf{P}$.

Then we define,

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_{I} \mathbf{Q} : I \in \mathbf{P} \right\}$$

2 Linear Orders

In this chapter we focus on linear orders, also known as total orders or chains.

Definitions 2.1 (Linear order)**.** *A* linear order *is a preorder which is symmetric and total.*

Definition 2.2 (class of linear orders)**.** *A* class $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 2.3 (Subintervals)**.** *Let M be a linear order, and let $x, y \in M$, such that $x \leq y$.*

Then we define the bounded subintervals $[x, y]$*,* $(x, y]$*,* $[x, y)$ *and* (x, y) *as usual.*

We also define the semi-bounded subintervals $(-\infty, x]$*,* $[x, \infty)$*,* $(-\infty, x)$ *and* (x, ∞) *as usual.*

We also define the unbounded subinterval $(-\infty, \infty)$ *as the whole linear order M, as usual.*

A subinterval *is either a bounded subinterval, a semi-bounded subinterval or the unbounded subinterval.*

If $x > y$ then we define the intervals as follows:

$$\begin{aligned} [x, y] &:= [y, x] \\ (x, y] &:= (y, x] \\ [x, y) &:= [y, x) \\ (x, y) &:= (y, x) \end{aligned}$$

Definition 2.4. *Let M be a linear order.*

A set $A \subseteq M$ is left cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $y < x$.*

A set $A \subseteq M$ is right cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $x < y$.*

A set $A \subseteq M$ is bi-directionally cofinal *in M if it is both left and right cofinal.*

Lemma 2.5. *Let $\mathbf{P}$ be an additive class of linear orders.*

Let $M \in \mathbf{P}$ be a linear order.

Let $x, y \in M$ be any two points in M.

Then, $[x, y] \in \mathbf{P}$.

Proof. WLOG, suppose $x \leq y$.

Note that,

$$M = (-\infty, \infty) = (-\infty, x) + [x, y] + (y, \infty)$$

when $(-\infty, x)$ and/or (y, ∞) may be empty.

Since $\mathbf{P}$ is an additive class, we conclude that $[x, y] \in \mathbf{P}$. □

Corollary 2.6. *Let* $\mathbf{P}$ *be a nontrivial additive class of linear orders.*

Then $1 \in \mathbf{P}$.

Proof. Let $M \in \mathbf{P}$ be any linear order and let $x \in M$ be any point in M.

Apply lemma 2.5 to the linear order M, and the points x and x, to conclude that $[x, x] \equiv 1 \in \mathbf{P}$. $\square$

Note 2.7. *Note that corollary 2.6 is false if we do not restrict ourselves to linear orders.*

For example, $(1 \uplus 1)^+$ *is a class of preorders which is additive, but does not contain* 1.

Corollary 2.8. *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Let M *be a linear order.*

Let $x, y \in M$ *be any two points in a linear order* M. *Then the following are equivalent:*

1. $(x, y) \in \mathbf{P}$
2. $(x, y] \in \mathbf{P}$
3. $[x, y) \in \mathbf{P}$
4. $[x, y] \in \mathbf{P}$

Proof. This is just applying the definition of an additive class to the orders $[x, y]$ and 1. $\square$

Corollary 2.9. *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Let M *be a linear order.*

Let $x, y, z \in M$ *be any three points in a linear order* M, *such that* $[x, y] \in \mathbf{P}$ *and* $[y, z] \in \mathbf{P}$.

Then $[x, z] \in \mathbf{P}$.

Proof. If $y \in [x, z]$, then $[x, z] = [x, y] + (y, z]$, and $(y, z] \in \mathbf{P}$ by corollary 2.8.

Otherwise, either $x \in [y, z]$ or $z \in [x, y]$. WLOG, suppose $z \in [x, y]$.

Then $[x, y] = [x, z] + (z, y]$, so $[x, z] \in \mathbf{P}$ by the fact that $\mathbf{P}$ is additive. $\square$

Definitions 2.10. *Let* $\mathbf{P}$ *be a class of linear orders.*

We define the following classes of linear orders:

- $\mathcal{B}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x, y \in M$, *the bounded subinterval* $[x, y]$ *is in* $\mathbf{P}$.
- $\mathcal{L}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x \in M$, *the left-bounded ray* $[x, \infty)$ *is in* $\mathbf{P}$.
- $\mathcal{R}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x \in M$, *the right-bounded ray* $(-\infty, x]$ *is in* $\mathbf{P}$.

Definition 2.11. *a class* $\mathbf{P}$ *of linear orders is a* star class *if for every linear orders* M*, and every family* $\mathcal{F} \subseteq \mathbf{P}$ *of subintervals of* M *such that* $J_1 \cap J_2 \neq \emptyset$ *for every* $J_1, J_2 \in \mathcal{F}$*, we have that* $\bigcup \mathcal{F} \in \mathbf{P}$*.*

Lemma 2.12. *Let* $\mathbf{P}$ *be a star class.*

Then for every linear order M*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$*.*

Equivalently, we can define a convex equivalence relation $\sim_{\mathbf{P}}$ *on* M *such that* $x \sim_{\mathbf{P}} y$ *iff* $[x, y] \in \mathbf{P}$*.*

That is, $x \sim_{\mathbf{P}} y$ *iff* x *and* y *are in the same largest* $\mathbf{P}$*-subinterval.*

Proof. Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in $\mathcal{B}[\mathbf{P}]$, and by definition J is the largest $\mathbf{P}$-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. □

Lemma 2.13 (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Then the class $\mathcal{B}[\mathbf{P}]$ *is a star class.*

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by the definition of $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. Since $\mathbf{P}$ is additive, by corollary 2.9, we have $[x, y] \in \mathbf{P}$. □

Lemma 2.14. *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Then,

1. $\mathcal{L}[\mathbf{P}] = \{M : M + 1 \in \mathcal{B}[\mathbf{P}]\}$
2. $\mathcal{R}[\mathbf{P}] = \{M : 1 + M \in \mathcal{B}[\mathbf{P}]\}$
3. $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}] = \{M : 1 + M + 1 \in \mathcal{B}[\mathbf{P}]\}$

Proof. Let M be a linear order.

1. Suppose $M + \{\infty\} \in \mathcal{B}[\mathbf{P}]$. Then for every $x \in M$, we have $[x, \infty] \in \mathbf{P}$, and thus $[x, \infty) \in \mathbf{P}$. Therefore, $M \in \mathcal{L}[\mathbf{P}]$.

   Conversely, if $M \in \mathcal{L}[\mathbf{P}]$, let $x, y \in M$ be any two points in $M + 1$.

   If $y < \infty$, then $[x, y] \subseteq [x, \infty)$. Since $[x, \infty) \in \mathbf{P}$, we conclude that $[x, y] \in \mathbf{P}$. Otherwise, if $y = \infty$, then $[x, y] = [x, \infty] = [x, \infty) + \{\infty\}$, and thus $[x, y] \in \mathbf{P}$.

2. The second case is dual to the first case.

3. We will show a triple inclusion.

   If $M \in \mathbf{P}$, then by additivity, $1 + M \in \mathbf{P}$ and $M + 1 \in \mathbf{P}$, and thus $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$.

   If $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, then by lemma 2.13, $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$.

   If $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$, then M is a bounded subinterval of $1 + M + 1$, so $M \in \mathcal{B}[\mathbf{P}]$.

□

Lemma 2.15. *Let* $\mathbf{P}$ *be an additive class of linear orders.*
Then,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = \mathbf{P} & \\ & \uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) \\ & \uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}]) \\ & \uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Proof. By lemma 2.14, we conclude that $\mathcal{L}[\mathbf{P}], \mathcal{R}[\mathbf{P}] \subseteq \mathcal{B}[\mathbf{P}]$, since $M + 1 \in \mathbf{P}$ and $1 + M \in \mathbf{P}$ both imply $1 + M + 1 \in \mathbf{P}$.

Thus,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = (\mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]) & \\ & \uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) \\ & \uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}]) \\ & \uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Since by lemma 2.14 $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, we conclude what we wanted to prove. □

Lemma 2.16. *Let* $\mathbf{P}$ *be an additive class of linear orders.*
Let M, M_1, M_2 *be linear orders such that* $M = M_1 + M_2$.
Then,

1. $M \in \mathcal{B}[\mathbf{P}] \iff M_1 \in \mathcal{L}[\mathbf{P}] \wedge M_2 \in \mathcal{R}[\mathbf{P}]$

Proof. From lemma 2.14, we know that

1.

$$\begin{aligned}M \in \mathcal{B}[\mathbf{P}] &\iff M_1 + M_2 \in \mathcal{B}[\mathbf{P}] \\ &\iff M_1 + 1 \in \mathcal{B}[\mathbf{P}] \wedge 1 + M_2 \in \mathcal{B}[\mathbf{P}] \\ &\iff M_1 \in \mathcal{L}[\mathbf{P}] \wedge M_2 \in \mathcal{R}[\mathbf{P}]\end{aligned}$$

□

Corollary 2.17. *Let* $\mathbf{P}$ *be an additive class of linear orders.*
Then,

$$\mathcal{B}\left[\mathbf{P}\right] \setminus \left(\mathcal{L}\left[\mathbf{P}\right] \cup \mathcal{R}\left[\mathbf{P}\right]\right) = \left(\mathcal{L}\left[\mathbf{P}\right] \setminus \mathcal{R}\left[\mathbf{P}\right]\right) + \left(\mathcal{R}\left[\mathbf{P}\right] \setminus \mathcal{L}\left[\mathbf{P}\right]\right)$$

Lemma 2.18 (Associativity of sum)**.** *Let* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *be classes.*
Then $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$*.*

Proof. It follows directly from the associativity of the sum operation on linear orders. Actually, it generalizes to any algebraic equation which holds on linear orders. □

Lemma 2.19 (Sum and union commute)**.** *Let* $\mathcal{P}$ *be a family of classes.*
Let $\mathbf{Q}$ *be a class.*
Then $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$*.*

Proof. This is obvious from the definition of the sum operation. □

Definition 2.20. *We define* **CNT** *as the class of all countable linear orders.*

3 General Hausdorff Rank

The concept of rank provides a powerful tool for measuring the complexity of linear orders and related structures. In this chapter, we introduce the Hausdorff rank and its generalizations, which allow us to stratify classes of orders according to their structural depth. The results here lay the groundwork for the analysis of types and decidability in subsequent chapters.

Definition 3.1. *Let $\mathbf{Q}$ be a class of linear orders.*
We define a class $\mathbf{H}_{\mathbf{Q}}^{<\alpha}$ for every ordinal α as follows:

- *For $\alpha = 0$, $\mathbf{H}_{\mathbf{Q}}^{<0} = \emptyset$.*
- *For $\alpha = 1$, $\mathbf{H}_{\mathbf{Q}}^{<1} = \{1\}$.*
- *For $\alpha = \gamma + 1$ where $\gamma > 0$,*

$$\mathbf{H}_{\mathbf{Q}}^{<\alpha} = \sum_{\mathbf{Q}} \mathbf{H}_{\mathbf{Q}}^{<\gamma}$$

- *For α a limit ordinal,*

$$\mathbf{H}_{\mathbf{Q}}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{H}_{\mathbf{Q}}^{<\beta}$$

Example 3.2. *Let $\mathbf{Q}$ be a class of linear orders.*
Then $\mathbf{H}_{\mathbf{Q}}^{\leq 1} = \mathbf{Q}$.

Definition 3.3. *Let $\mathbf{Q}$ be a class of linear orders.*
Let α, β be ordinals such that with $0 < \alpha < \beta$.
We define,

1. $\mathbf{H}_{\mathbf{Q}}^{\leq\alpha} := \mathbf{H}_{\mathbf{Q}}^{<\alpha+1}$
2. $\mathbf{H}_{\mathbf{Q}}^{=\alpha} := \mathbf{H}_{\mathbf{Q}}^{\leq\alpha} \setminus \mathbf{H}_{\mathbf{Q}}^{<\alpha}$
3. $\mathbf{H}_{\mathbf{Q}}^{[\alpha,\beta)} := \mathbf{H}_{\mathbf{Q}}^{<\beta} \setminus \mathbf{H}_{\mathbf{Q}}^{<\alpha}$

Definition 3.4. *Let $\mathbf{Q}$ be a class of linear orders.*
We define the $\mathbf{Q}$-Hausdorff rank as a partial *mapping from linear orders to ordinals, such that*

$$\mathbf{hrank}_{\mathbf{Q}}(M) = \min\left\{\alpha : M \in \mathbf{H}_{\mathbf{Q}}^{\leq\alpha}\right\}$$

Equivalently, $\mathbf{hrank}_{\mathbf{Q}}(M)$, is the unique ordinal α such that $M \in \mathbf{H}_{\mathbf{Q}}^{=\alpha}$ (if it exists, otherwise it is undefined).

Definition 3.5. *Let $\gamma \geq \omega$ be a limit ordinal.*
We define $\Gamma_\gamma := \{\beta : \beta \subseteq \gamma^* + \gamma\}^+$.
We define $\omega := \Gamma_\omega$.

Example 3.6.

$$\omega = \{1, \omega, \omega^*\}^+$$

Observation 3.7. *Let $\gamma \geq \omega$ be a limit ordinal.*

Then Γ_γ is a monotone, additive class of linear orders.

Notation 3.8. *When we omit the subscript in $\mathbf{H}^{<\alpha}$, we mean $\mathbf{H}_\omega^{<\alpha}$, and similarly for $\mathbf{H}^{\leq\alpha}$, $\mathbf{H}^{=\alpha}$, $\mathbf{H}^{[\alpha,\beta)}$, and $\mathbf{hrank}\,(M)$.*

4 ω-Hausdorff rank

In this chapter, we focus on the special case of the Hausdorff rank associated with the ordinal ω. This case is of particular interest due to its connections with countable structures and its role in the classification of infinite linear orders. We introduce new families of classes and analyze their relationships, providing tools that will be essential for the study of types and decidability.

Definition 4.1. *Let $\alpha > 0$ be an ordinal.*
We define:

1. *(Right α-Major)* $\mathbf{RM}_\alpha := \mathcal{R}\left[\mathbf{H}^{<\alpha}\right] \setminus \mathcal{L}\left[\mathbf{H}^{<\alpha}\right]$
2. *(Left α-Major)* $\mathbf{LM}_\alpha := \mathcal{L}\left[\mathbf{H}^{<\alpha}\right] \setminus \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]$
3. *(Bounded α-Major)* $\mathbf{BM}_\alpha := \mathcal{B}\left[\mathbf{H}^{<\alpha}\right] \setminus \left(\mathcal{L}\left[\mathbf{H}^{<\alpha}\right] \cup \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]\right)$

Note 4.2. *Obviously $\mathbf{LM}_\alpha = \mathbf{RM}_\alpha^*$ by symmetry.*
By corollary 2.17, $\mathbf{BM}_\alpha = \mathbf{LM}_\alpha + \mathbf{RM}_\alpha$.
Also, by the definition:

$$\mathcal{B}\left[\mathbf{H}^{<\alpha}\right] = \mathbf{H}^{<\alpha} \uplus \mathbf{LM}_\alpha \uplus \mathbf{RM}_\alpha \uplus \mathbf{BM}_\alpha$$

Lemma 4.3. *Let $\alpha > 0$ be an ordinal.*
Then $\mathcal{R}\left[\mathbf{H}^{<\alpha}\right] = \sum_\omega \mathbf{H}^{<\alpha}$.

Proof. ($\supseteq$) Let $M \in \sum_\omega \mathbf{H}^{<\alpha}$ be a linear order.
Let $M = \sum_{i\in\omega} M_i$ be the decomposition of M, where $M_i \in \mathbf{H}^{<\alpha}$.
Let $x, y \in M$ be any two points in M. WLOG $x \leq y$.
Suppose $x \in M_i$ and $y \in M_j$ for $i, j \in \omega$.
Since i and j have a finite distance in ω, we conclude $[x, y] \subseteq M_i + \ldots + M_j$, and thus $[x, y] \subseteq \left(\mathbf{H}^{<\alpha}\right)^+ = \mathbf{H}^\alpha$.
($\subseteq$) Let $M \in \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]$ be a linear order.
Since M is countable, let $\{x_i\}_{i\in\omega} M$ be a right cofinal ω-sequence in M.
Let $M_0 = (-\infty, x_0]$ and and $M_i = (x_{i-1}, x_i]$ for $i > 0$.
Then $M = \sum_{i\in\omega} M_i$.
But M_i is a right-bounded interval and thus $M_i \in \mathbf{H}^{<\alpha}$, so $M \in \sum_\omega \mathbf{H}^{<\alpha}$. ∎

An immediate corollary of lemma 4.3 is that "major" is a good name.

Corollary 4.4. *Let $\alpha > 0$ be an ordinal.*
Then,

1. $\mathbf{H}^{\leq\alpha} = \left(\mathcal{B}\left[\mathbf{H}^{<\alpha}\right]\right)^+$
2. $\mathbf{H}^{=\alpha} = \left(\mathbf{LM}_\alpha \uplus \mathbf{RM}_\alpha\right)^+$

Lemma 4.5. *Let α be an ordinal.*

Then, we have the following:

$$\mathbf{RM}_{\alpha+1} = \sum_{\omega} \mathbf{H}^{=\alpha}$$

Proof. ($\subseteq$) Let $M \in \mathbf{RM}_{\alpha+1}$.

By lemma 4.3 and corollary 4.4

$$\mathcal{R}\left[\mathbf{H}^{<\alpha+1}\right] = \sum_{\omega} \mathbf{H}^{<\alpha+1} = \sum_{\omega} \mathbf{H}^{\leq\alpha} = \sum_{\omega} \left(\mathcal{B}\left[\mathbf{H}^{<\alpha}\right]\right)^{+} = \sum_{\omega} \mathcal{B}\left[\mathbf{H}^{<\alpha}\right]$$

since by definition 4.1, $M \in \mathcal{R}\left[\mathbf{H}^{<\alpha+1}\right]$, we conclude that $M = \sum_{i\in\omega} M_i$ for a sequence $\{M_i\}_{i\in\omega} \subseteq \mathcal{B}\left[\mathbf{H}^{<\alpha}\right]$.

If $M_i \in \mathbf{H}^{=\alpha}$ held for only finitely many $i \in \omega$, we would have $M \in \mathbf{H}^{\leq\alpha}$, which is a contradiction to $M \notin \mathcal{L}\left[\mathbf{H}^{<\alpha}\right]$.

Thus, $M_i \in \mathbf{H}^{=\alpha}$ holds for infinitely many $i \in \omega$, and thus (by adjoining $M_i \in \mathbf{H}^{<\alpha}$ to the next $\mathbf{H}^{=\alpha}$ one) we conclude $M \in \sum_{\omega} \mathbf{H}^{=\alpha}$.

($\supseteq$) Let $M \in \sum_{\omega} \mathbf{H}^{=\alpha}$.

Since $M \in \sum_{\omega} \mathbf{H}^{<\alpha+1}$, by lemma 4.3, $M \in \mathcal{R}\left[\mathbf{H}^{<\alpha+1}\right]$.

By corollary 4.4,

$$M \in \sum_{\omega} \mathbf{H}^{=\alpha} = \sum_{\omega} \left(\mathbf{LM}_{\alpha} \uplus \mathbf{RM}_{\alpha}\right)^{+} = \sum_{\omega} \left(\mathbf{LM}_{\alpha} \uplus \mathbf{RM}_{\alpha}\right)$$

Suppose $M = \sum_{i\in\omega} M_i$ where $M_i \in \{\mathbf{LM}_{\alpha}, \mathbf{RM}_{\alpha}\}$. By the pigeonhole principle, there are either infinitely many $M_i \in \mathbf{LM}_{\alpha}$ or infinitely many $M_i \in \mathbf{RM}_{\alpha}$. WLOG, suppose $M_i \in \mathbf{RM}_{\alpha}$ for infinitely many $i \in \omega$.

Then, since $M_i \in \mathbf{H}^{<\alpha+1}$, we have $M_i \in \mathbf{H}^{=\alpha}$.

Suppose by contradiction $M = \mathbf{H}^{\leq\alpha} = \left(\mathcal{B}\left[\mathbf{H}^{<\alpha}\right]\right)^{+}$. In particular, by the pigeonhole principle, there exists some $N \in \omega$ such that $\sum_{N\leq i<\omega} M_i \in \mathcal{B}\left[\mathbf{H}^{<\alpha}\right]$, which is a contradiction because it follows that $M_{N+1} \in \omega< \alpha$ as it is bounded between M_N and M_{N+2}. $\square$

Lemma 4.6. *Let $\{\alpha_i\}_{i\in\omega}$ be a strictly increasing ordinal sequence, and let $\alpha = \sup_{\alpha_i}$.*

Then,

$$\mathbf{RM}_{\alpha} = \sum_{i\in\omega} \mathbf{H}^{[\alpha_i,\alpha)}$$

Proof. ($\subseteq$) Let $M \in \mathbf{RM}_{\alpha}$. Let $y_{i i<\omega}$ be a right cofinal ω-sequence in M.

Thus we can choose some x_0 far enough such that $(-\infty, x_0] \in \mathbf{H}^{[\alpha_0,\alpha)}$, and $x_0 > y_0$. Now by induction we choose x_1 such that $(x_0, x_1] \in \mathbf{H}^{[\alpha_1,\alpha)}$, and $x_1 > y_1$.

By iterating ω times we get an ω-sequence $\{M_i\}_{i\in\omega}$ such that $M = \sum_{i\in\omega} M_i$ and $M_i \in \mathbf{H}^{[\alpha_i,\alpha)}$, where $M_i = (x_{i-1}, x_i]$ (where $x_{-1} := -\infty$).

($\supseteq$) Let $M \in \sum_{i\in\omega} \mathbf{H}^{[\alpha_i,\alpha)}$. It is obvious that $M \in \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]$ since every right-bounded ray is in $\mathbf{H}^{\leq\alpha_i}$ for some $i \in \omega$.

However, $M \notin \mathbf{H}^{<\alpha_i}$ for any $i \in \omega$, so $M \notin \mathbf{H}^{<\alpha}$.

$\square$

Lemma 4.7. *Let $\{\alpha_i\}_{i\in\omega}$ be a non-decreasing ordinal sequence, and let $\alpha = \sup_{\alpha_i+1}$.*

Then,

$$\mathbf{RM}_\alpha = \sum_{i\in\omega} \mathbf{H}^{[\alpha_i,\alpha)}$$

Proof. It is just a way to write lemma 4.5 and lemma 4.6 together more succinctly. $\square$

Note 4.8. *In the proof of lemma 4.7, we actually use the fact that we work over $\omega = \Gamma_\omega$. This proof would not have worked over Γ_β for $\beta > \omega$.*

5 Type Theory

Type theory provides a framework for analyzing the expressive power of logical languages over classes of structures. In this chapter, we introduce the notion of types for classes of preorders and study their computability. The results here connect the structural classes of orders with logical definability, setting the stage for the study of decidability in the next chapter.

Definition 5.1. *Let* $\mathbf{P}$ *be a class of preorders.*

Let $n \in \mathbb{N}$.

We define $\mathbf{type}_n[\mathbf{P}]$ *as the set of all* n*-types satisfiable in* $\mathbf{P}$.

Definition 5.2. *a class* $\mathbf{P}$ *of preorders is* computable *if* $n \mapsto \mathbf{type}_n[\mathbf{P}]$ *is a computable function.*

Equivalently, satisfiability of $\mathbf{MSO}$ *over* $\mathbf{P}$ *is decidable.*

Lemma 5.3. *Let* $\mathbf{Q}$ *be a class of preorders.*

There exists a computable function $f_{\mathbf{Q}} = f : \mathbb{N} \to \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and every ordinal* $\alpha \geq f(n)$, $\mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{<\alpha}\right] = \mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{<f(n)}\right]$.

Proof. Since there are only finitely many n-types, and the ordinal sequence

$$\left\{\mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{<\kappa}\right]\right\}_{\kappa}$$

is monotone, there must be some minimal $\kappa_0 \in \omega$ where the sequence stabilizes.

This κ_0 is computable as a function of n, because $\mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{<\kappa}\right]$ is computable for every finite κ. □

Lemma 5.4. *There exist global computable functions* $a, b : \mathbb{N} \to \mathbb{N}$ *such that for all* $n, c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$,

$$\mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{=c_1}\right] = \mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{=c_2}\right]$$

Proof. Let $n \in \mathbb{N}$.

Since there are only finitely many sets of n-types, there exist (and can be computed) some $a(n) \geq f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{=a(n)+b(n)}\right]$$

By induction if follows that for all $c \geq a(n)$,

$$\mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{=c}\right] = \mathbf{type}_n\left[\mathbf{H}_{\mathbf{Q}}^{=c+b(n)}\right]$$

since $\mathbf{H}_{\mathbf{Q}}^{=c+1} = \sum_{\mathbf{Q}} \mathbf{H}_{\mathbf{Q}}^{=c}$. □

Corollary 5.5. *Let* $n \in \mathbb{N}$, *and let* $\alpha \geq \omega$ *be an ordinal.*

Then there exists a computable function $b(n)$ *such that for all* $c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$, *we have*

$$\mathbf{type}_n[\mathbf{RM}_{c_1}] = \mathbf{type}_n[\mathbf{RM}_{c_2}]$$

Proof. It follows easily from lemma 4.7 and lemma 5.4. □

Lemma 5.6. *For every $n \in \mathbb{N}$ and for every pair of ordinals $\alpha \geq \omega$, $\beta > \alpha$,*

$$\mathbf{type}_n \left[\mathbf{H}_{\mathbf{Q}}^{[\alpha,\beta)} \right] = \mathbf{type}_n \left[\bigcup_{c<b(n)} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c} \right]$$

In particular, $\mathbf{type}_n \left[\mathbf{H}_{\mathbf{Q}}^{=\alpha} \right]$ *can be computed, and is independent of the choice of $\alpha \geq \omega$.*

Proof. It is enough to prove that

$$\mathbf{type}_n \left[\mathbf{H}_{\mathbf{Q}}^{=\alpha} \right] = \mathbf{type}_n \left[\bigcup_{c<b(n)} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c} \right]$$

We thus proceed by induction on $\alpha \geq \omega$.

Let $\{\alpha_i\}_{i\in\omega}$ be an increasing ω-sequence of ordinals such that $a(n) \leq \alpha_i$ for all $i \in \omega$, and $\sup_{i\in\omega} (\alpha_i + 1) = \alpha$.

Then $\mathbf{H}_{\mathbf{Q}}^{=\alpha} = \sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{H}_{\mathbf{Q}}^{[\alpha_i,\alpha)}$ and thus,

$$\begin{aligned}
\mathbf{type}_n \left[\mathbf{H}_{\mathbf{Q}}^{=\alpha} \right] &= \mathbf{type}_n \left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{H}_{\mathbf{Q}}^{[\alpha_i,\alpha)} \right] \\
&= \mathbf{type}_n \left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \bigcup_{c<b(n)} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c} \right] \\
&= \mathbf{type}_n \left[\sum_{\mathbf{Q}} \bigcup_{c<b(n)} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c} \right] \\
&= \mathbf{type}_n \left[\bigcup_{c<b(n)} \sum_{\mathbf{Q}} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c} \right] \\
&= \mathbf{type}_n \left[\bigcup_{c<b(n)} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c+1} \right] \\
&= \mathbf{type}_n \left[\bigcup_{c<b(n)} \mathbf{H}_{\mathbf{Q}}^{=a(n)+c} \right]
\end{aligned}$$

where the last transition is because $\mathbf{type}_n \left[\mathbf{H}_{\mathbf{Q}}^{=a(n)} \right] = \mathbf{type}_n \left[\mathbf{H}_{\mathbf{Q}}^{=a(n)+b(n)} \right]$. □

Corollary 5.7. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

$$\mathbf{type}_n\left[\mathbf{RM}_\alpha\right] = \mathbf{type}_n\left[\sum_{\omega}\bigcup_{c<b(n)}\mathbf{H}^{=a(n)+c}\right]$$

In particular, $\mathbf{type}_n\left[\mathbf{RM}_\alpha\right]$ *can be computed, and is independent of the choice of* $\alpha \geq \omega$.

Proof. There exists an increasing ω-sequence $\{\alpha_i\}_{i\in\omega}$ such that $a(n) \leq \alpha_i$ for all $i \in \omega$, and $\sup_{i\in\omega}(\alpha_i + 1) = \alpha$.

Then $\mathbf{RM}_\alpha = \sum_{i\in\omega}\mathbf{H}^{=\alpha_i}$, and thus,

$$\begin{aligned}
\mathbf{type}_n\left[\mathbf{RM}_\alpha\right] &= \mathbf{type}_n\left[\sum_{i\in\omega}\mathbf{H}^{=\alpha_i}\right] \\
&= \mathbf{type}_n\left[\sum_{\omega}\bigcup_{c<b(n)}\mathbf{H}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\sum_{\omega}\mathbf{H}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\mathbf{RM}_{a(n)+c+1}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\mathbf{RM}_{a(n)+c}\right]
\end{aligned}$$

where the last transition is by corollary 5.5.

□

6 Decidability of the rank

Decidability questions lie at the heart of mathematical logic and theoretical computer science. In this chapter, we investigate the decidability of rank-related classes for linear orders, connecting the structural results of previous chapters with algorithmic considerations. We introduce key predicates and equivalence relations, and show how they can be expressed and manipulated in logical frameworks.

Definition 6.1. *Let $\mathbf{Q}$ be a class of linear orders.*

Let M be a linear order.

We define the predicate $\mathbf{Int}_{\mathbf{Q}}(J)$ as true in M iff J is a $\mathbf{Q}$-subinterval of M.

Lemma 6.2. *Let $\alpha > 0$ be an ordinal.*

Then predicates $\mathbf{Int}_{\mathbf{H}^{\leq\alpha}}$, $\mathbf{Int}_{\mathbf{H}^{=\alpha}}$ are expressible in $\mathbf{MSO}[\mathbf{Int}_{\mathbf{H}^{<\alpha}}]$.

Proof. Obviously,

$$\mathbf{Int}_{\mathbf{H}^{=\alpha}} \iff \mathbf{Int}_{\mathbf{H}^{\leq\alpha}} \wedge \neg\mathbf{Int}_{\mathbf{H}^{<\alpha}}$$

So it is enough to express $\mathbf{Int}_{\mathbf{H}^{\leq\alpha}}$.

Now, J is a $\mathbf{H}^{\leq\alpha}$-subinterval of M iff $J \in \sum_{\omega} \mathbf{H}^{<\alpha}$.

But this can be expressed in $\mathbf{MSO}$ since it is expressible to check whether an arbitrary subset is in ω. □

Definition 6.3. *Let $\alpha > 0$ be an ordinal.*

Let M be a linear order and $x \in M$.

We define the convex equivalence relation:

$$\sim_\alpha := \sim_{\mathcal{B}[\mathbf{H}^{<\alpha}]}$$

and $[x]_\alpha := [x]_{\mathcal{B}[\mathbf{H}^{<\alpha}]}$.

That is, $[x]_\alpha$ is the largest $\mathcal{B}[\mathbf{H}^{<\alpha}]$-subinterval containing x in M.

We define $\sigma_\alpha(x)$ as the α-shape of $[x]_\alpha$.

We define $L_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{L}[\mathbf{H}^{<\alpha}]}$ and $R_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{R}[\mathbf{H}^{<\alpha}]}$.

Lemma 6.4. *Let M be a linear order and $\alpha > 0$ an ordinal.*

Let $J \subseteq M$ be an interval.

Then $J \in \mathbf{H}^{<\alpha}$ iff it is contained in a single $\sim_\alpha$-equivalence class K, such that:

- *Either $K \in \mathcal{L}[\mathbf{H}^{<\alpha}]$ or there exists some $x \in K$ such that $x < J$.*
- *Either $K \in \mathcal{R}[\mathbf{H}^{<\alpha}]$ or there exists some $x \in K$ such that $x > J$.*

Proof. Suppose $J \in \mathbf{H}^{<\alpha}$. Then obviously J is contained in a single $\sim_\alpha$-equivalence class K.

We will show the first condition, the second is symmetric.

Suppose that for all $x \in K$, $J \leq x$. Then we can write $K = J + J'$. Since $J \in \mathbf{H}^{<\alpha}$, it follows that $K \in \mathcal{L}[\mathbf{H}^{<\alpha}]$. □

Corollary 6.5. *Let $\alpha > 0$ be an ordinal.*

The predicate $\mathbf{Int}_{\mathbf{H}^{<\alpha}}$ is $\mathbf{MSO}$-expressible over $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$.

Theorem 6.6. *Let $\mathbf{P}$ be a computable class of linear orders of some finite signature, including $C_1, \ldots, C_k$.*

Let $\mathcal{Q}$ be a finite *set of computable classes of linear orders over some finite signature which is disjoint from the signature of $\mathbf{P}$.*

Let $F : 2^k \to \mathcal{Q}$ be any function.

Then $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} F(C_1(i), \ldots, C_k(i))$ is a computable class of linear orders.

Proof. We will use the decomposition theorem. Let φ be a formula of quantifier depth n. WLOG, φ is a sentence.

Then we can compute a formula $\psi(\xi)$ (where ξ has the type of a coloring whose range is the set of n-types) such that for any linear order $M = \sum_{i\in I} M_i$,

$$M \models \varphi \iff I \models \psi(\Xi)$$

where Ξ is the coloring assigning $i \in I$ the n-type of M_i.

Thus, there is some $M \in \bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$, such that $M \models \varphi$ iff there exists some $I \in \mathbf{P}$, and assignment Ξ of n-types, such that $\Xi(i)$ is satisfiable in $\mathbf{Q}_i$ for all $i \in I$, and $I \models \psi(\Xi)$.

Equivalently, φ is satisfiable over $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$ iff

$$\begin{aligned}&\exists\xi.\psi(\xi) \wedge \xi \text{ is a coloring with } n\text{-types}\\ &\qquad \wedge \forall i.\xi(i) \in \mathbf{type}_n\left[F(C_1(i), \ldots, C_k(i))\right]\end{aligned}$$

is satisfiable over $\mathbf{P}$.

Since $\mathcal{Q}$ has only computable classes, We can pre-compute $\mathbf{type}_n\left[F(\vec{c})\right]$ for any value $\vec{c} \in 2^k$ so we can actually write the formula above in $\mathbf{MSO}$. Furthermore, since $\mathbf{P}$ is computable, we can check whether it is satisfiable over $\mathbf{P}$. So we are done. ☐

Lemma 6.7. *Let α be an ordinal.*

Let P, L and R be first-order unary predicates.

Let C be the class of all countable linear orders labeled with P, L and R, such that P represents $\sim_\alpha$, $L_\alpha(x) \iff [x]_\alpha \in \mathcal{L}[\mathbf{H}^{<\alpha}]$ and $R_\alpha(x) \iff [x]_\alpha \in \mathcal{R}[\mathbf{H}^{<\alpha}]$.

Let $\mathbf{G}$ be the class of all countable linear orders I, labeled with a P, L and R, such that for every pair $i, i' \in I$ such that i' is the successor of i, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.

Let $\sigma(i)$ be defined as follows:

$$\sigma(i) := \begin{cases}\mathbf{H}^{<\alpha} & \text{if } L(i) = R(i) = 0\\ \mathbf{RM}_{\mathbf{H}^{<\alpha}} & \text{if } L(i) = 0 \wedge R(i) = 1\\ \mathbf{LM}_{\mathbf{H}^{<\alpha}} & \text{if } L(i) = 1 \wedge R(i) = 0\\ \mathbf{BM}_{\mathbf{H}^{<\alpha}} & \text{if } L(i) = R(i) = 1\end{cases}$$

Proof. ($\subseteq$) Let M be a countable linear order labeled with P, L and R as above.

Let $I = M/\sim_\alpha$ be the quotient of M by the equivalence relation $\sim_\alpha$.

Then $M = \sum_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ are the $\sim_\alpha$-equivalence class of I.

Then for each $i \in I$, $M_i \in \mathcal{B}[\mathbf{H}^{<\alpha}]$, and by definition $\sigma(i) = \sigma_\alpha(M_i)$.

Let i' be the successor of i in I.

Then $P(i) \neq P(i')$ since P represents $\sim_\alpha$.

Furthermore, suppose $R(i) = L(i') = 1$ holds. Then $M_i \in \mathcal{R}[\mathbf{H}^{<\alpha}]$ and $M_{i'} \in \mathcal{L}[\mathbf{H}^{<\alpha}]$. so M_i and $M_{i'}$ are the same $\sim_\alpha$-equivalence class of M, which is a contradiction.

Thus either $R(i) = 0$ or $L(i') = 0$.

($\supseteq$) Let $M = \sum_{i \in I} M_i$ be a linear order such that $I \in \mathbf{G}$ and $M_i \in \mathcal{S}_\alpha^{\sigma(i)}$ for each $i \in I$.

In particular $M_i \in \mathcal{B}[\mathbf{H}^{<\alpha}]$ for each $i \in I$, so it is contained in a single $\sim_\alpha$-equivalence class of M.

Suppose that there exist distinct $j, k \in I$ such that $j < k$, and M_j, M_k are in the same $\sim_\alpha$-equivalence class.

Let $x \in M_j$ and $y \in M_k$. Then $[x, y] \in \mathbf{H}^{<\alpha}$, and thus $[j, k] \in \mathbf{H}^{<\alpha}$, and in particular it is sparse.

Then there exist some $j', k' \in I$ such that $j < j' < k' < k$, and k' is the successor of j' in I.

Then $M_{j'}$ and $M_{k'}$ are in the same $\sim_\alpha$-equivalence class. Thus it must be the case that $M_{j'} \in \mathcal{R}[\mathbf{H}^{<\alpha}]$ and $M_{k'} \in \mathcal{L}[\mathbf{H}^{<\alpha}]$, which implies $R(j') = L(k') = 1$, which is a contradiction.

Thus $\{M_i\}_{i \in I}$ are pairwise distinct $\sim_\alpha$-equivalence classes, and obviously the conditions holds, so $M \in C$ and we are done. □

Corollary 6.8. *Let $\alpha > 0$ be an ordinal.*

Let C be defined as in lemma 6.7.

Then C is a computable class of linear orders.

Proof. Since $\gg$ is clearly computable, it follows from combining theorem 6.6 and lemma 6.7. □

Theorem 6.9. *Let $\alpha > 0$ be an ordinal.*

Satisfiability of $\mathbf{MSO}[\mathbf{Int}_{\mathbf{H}^{<\alpha}}]$ *over all countable linear orders is decidable.*

Proof. First, by corollary 6.5, we can convert any formula in $\mathbf{MSO}[\mathbf{Int}_{\mathbf{H}^{<\alpha}}]$ to an equivalent formula φ in $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$.

Now, we shall replace every occurrence of $[\cdot]_\alpha$ in φ with P, every occurrence of L_α with L, and every occurrence of R_α with R, getting a new formula φ'.

Then, satisfiability of φ over all countable linear orders, amounts to satisfiability of φ' over C, which is computable by corollary 6.8. □

7 Decidability - continued

Lemma 7.1. *Let $\alpha_0, \ldots, \alpha_k$, such that $\alpha_i \geq \alpha_{i-1} + \omega$ for $i > 0$.*

Let $P_1, L_1, R_1, \ldots, P_k, L_k, R_k$. first-order unary predicates.

Let C be the class of all countable linear orders labeled with P_j, L_j and R_j, such that for every quadruple α, P, L, R, P represents $\sim_\alpha$, $L_\alpha(x) \iff [x]_\alpha \in \mathcal{L}[\mathbf{H}^{<\alpha}]$ and $R_\alpha(x) \iff [x]_\alpha \in \mathcal{R}[\mathbf{H}^{<\alpha}]$.

Let $\mathbf{G}$ be the class of all countable linear orders I, labeled with a P_j, L_j and R_j, such that for every triplet P, L, R, that for every pair $i, i' \in I$ such that i' is the successor of i, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.

For every triplet, let $\sigma_j = \sigma$ be such That $\sigma(i) \in \{1, \omega, \omega^*, \omega^* + \omega\}$ and $L(i) = 1$ iff $\sigma(i) \in \{1, \omega\}$ and $R(i) = 1$ iff $\sigma(i) \in \{1, \omega^*\}$.

Then, $C = \bigcup_{I \in \mathbf{G}} \sum_{i \in I} \mathcal{S}_{\alpha_0}^{\sigma_0(i)}$. Furthermore, its **MSO** *theory is independent of the choice of $\alpha_0, \ldots, \alpha_k$.*

Theorem 7.2. *Let $\alpha_k > \ldots > \alpha_1 > 0$ be ordinals.*

Satisfiability of $\mathbf{MSO}[\mathbf{Int}_{\mathbf{H}^{<\alpha_1}}, \ldots, \mathbf{Int}_{\mathbf{H}^{<\alpha_k}}]$ *over all countable linear orders is decidable.*

Proof. First, by corollary 6.5, we can convert any formula in

$$\mathbf{MSO}[\mathbf{Int}_{\mathbf{H}^{<\alpha_1}}, \ldots, \mathbf{Int}_{\mathbf{H}^{<\alpha_k}}]$$

to an equivalent formula φ in

$$\mathbf{MSO}[[\cdot]_{\alpha_1}, L_{\alpha_1}, R_{\alpha_1}], \ldots, \mathbf{MSO}[[\cdot]_{\alpha_k}, L_{\alpha_k}, R_{\alpha_k}]$$

Now, we shall replace every occurrence of $[\cdot]_{\alpha_i}$ in φ with P_i, every occurrence of L_{α_i} with L_i, and every occurrence of R_{α_i} with R_i, getting a new formula φ'.

Then, satisfiability of φ over all countable linear orders, amounts to satisfiability of φ' over C, which is computable by corollary 6.8. $\square$