Orders

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February 2, 2025

1 Properties

Lemma 1. *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then the property $\mathbf{bounded} - \mathbf{P}$ *is has the star property.*

2 Hausdorff Rank

Definition 1. *Let* L *be a linear order.*

We define $\mathbf{hrank}(L) \leq 0$ *iff* L *is finite.*

Let $\alpha > 0$ *be an ordinal.*

We define $\mathbf{hrank}(L) \leq \alpha$ *iff* $L = \sum_{i \in I} L_i$ *for some linear order* I*, where* $\mathbf{hrank}(L_i) < \alpha$ *and* I *is a finite sum of* 1*,* ω *and* $-\omega$*.*

We write $\mathbf{hrank}(L) = \alpha$ *iff* α *is the least ordinal such that* $\mathbf{hrank}(L) \leq \alpha$*.*

We will be working with scattered linear orders.

Claim 1. *Let* L *be a countable linear order.*

Then $\mathbf{hrank}(L)$ *is defined iff* L *is scattered.*

Proof. To prove $\implies$ is easy, as a scattered sum of scattered linear orders is scattered.

For the other direction... TODO. □

Notations 1. *Let* $\mathcal{H}_{<\alpha}$ *be the class of linear orders of Hausdorff rank* $< \alpha$ *and* $\mathcal{H}_{=\alpha}$ *be the class of linear orders of Hausdorff rank* $= \alpha$*.*

Let $\mathcal{B}_{<\alpha}$ *be the class of linear orders of Hausdorff rank* $< \alpha$ *on bounded subintervals.*

Let $\mathcal{Q}_{<\alpha} = \{L : 1 + L \in \mathcal{B}_{<\alpha}\}$*.*

Let $\mathcal{R}_{<\alpha} = \{L : L + 1 \in \mathcal{B}_{<\alpha}\}$*.*

Clearly, $\mathcal{H}_{<\alpha}, \mathcal{Q}_{<\alpha}, \mathcal{R}_{<\alpha} \subseteq \mathcal{B}_{<\alpha}$*.*

Clearly, $\mathcal{H}_{<\alpha+1} = \{L : \mathbf{hrank}(L) \leq \alpha\}$*.*

Claim 2. *The following are equal:*

1. $\mathcal{H}_{<\alpha}$

2. $\{L : 1 + L + 1 \in \mathcal{B}_{<\alpha}\}$.

3. $\mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$

Proof. The equivalence of 1 and 2 is clear, and obviously 2 implies 3.

The other direction (3 implies 2) follows from the star property of $\mathcal{B}_{<\alpha}$. $\square$

Lemma 2. *Let L be a linear order. Then there exists a largest subinterval $M \subseteq L$ such that $x \in M$ and $M \in \mathcal{B}_{<\alpha}$.*

Definition 2. *Let L be a linear order. Let $x \in L$. We define $M_\alpha[x]$ to be the largest subinterval $M \subseteq L$ such that $x \in M$ and $M \in \mathcal{B}_{<\alpha}$.*

We define $\sim_\alpha$ to be the equivalence relation on L such that $x \sim_\alpha y$ iff $M_\alpha[x] = M_\alpha[y]$.

Lemma 3. *Let L be a linear order. Let $P, Q, R \subseteq L$ be relations, such that:*

- *P represents $\sim_\alpha$ on L.*
- *Q is such that $x \in Q$ iff $M_\alpha[x] \in \mathcal{Q}_{<\alpha}$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_{<\alpha}$.*

Then for some linear order I there exists a decomposition $L = \sum_{i \in I} L_i$ such that $L_i \in \mathcal{B}_{<\alpha}$ for all $i \in I$, L_i is monochromatic with respect to P, Q and R.

Furthermore, let τ_i be the n-type of L_i, p_i, q_i, r_i in $\mathbf{MSO}[p, q, r]$, where $p_i = 1_{L_i \subseteq P}$, $q_i = 1_{L_i \subseteq Q}$ and $r_i = 1_{L_i \subseteq R}$. Then the following hold

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $q(\tau_{i+1}) = 0$*

Proof. Take $I = L/\sim_\alpha$.

Then $L = \sum_{i \in I} L_i$ where L_i is the $\sim_\alpha$-equivalence class of i.

Then L_i is monochromatic with respect to P, Q and R.

The only thing left to prove is the last two conditions. The first follows from the fact that P represents $\sim_\alpha$.

The second follows because if it were not the case, then L_i and L_{i+1} would be the same $\sim_\alpha$-equivalence class. $\square$

Lemma 4. *Let I be a linear order. Let $n \in \mathbb{N}$. Let p, q, r be boolean variables.*

Let τ_i be an assignment of satisfiable n-types in $\mathbf{MSO}[p, q, r]$ for all $i \in I$. Assume that

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $q(\tau_{i+1}) = 0$*

Then there exists a linear order L and $P, Q, R \subseteq L$ such that:

- *P represents $\sim_\alpha$ on L.*

- *Q is such that $x \in Q$ iff $M_\alpha[x] \in \mathcal{Q}_{<\alpha}$.*

- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_{<\alpha}$.*

such that for all $i \in I$, L_i is a $\sim_\alpha$-equivalence class of L, and is thus monochromatic with respect to P, Q and R.

Furthermore, the n-type of L_i, p_i, q_i, r_i in $\mathbf{MSO}[p, q, r]$ is τ_i, where $p_i = 1_{L_i \subseteq P}$, $q_i = 1_{L_i \subseteq Q}$ and $r_i = 1_{L_i \subseteq R}$,

Proof. Since τ_i is satisfiable, we can take L_i to be a linear order of n-type τ_i such that:

- If $q(\tau_i) = r(\tau_i) = 1$, then $L_i \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

- If $q(\tau_i) = 1$ and $r(\tau_i) = 0$, then $L_i \in \mathcal{Q}_{<\alpha} - \mathcal{R}_{<\alpha}$.

- If $q(\tau_i) = 0$ and $r(\tau_i) = 1$, then $L_i \in \mathcal{R}_{<\alpha} - \mathcal{Q}_{<\alpha}$.

- If $q(\tau_i) = r(\tau_i) = 0$, then $L_i \in \mathcal{B}_{<\alpha} - (\mathcal{Q}_{<\alpha} \cup \mathcal{R}_{<\alpha})$.

Let $L = \sum_{i \in I} L_i$.

By definition each L_i is in $\mathcal{B}_{<\alpha}$. We need to prove that each L_i is a largest $\mathcal{B}_{<\alpha}$-subinterval in L.

On the contrary, suppose that there exist $i' \neq i$ such that $[L_i, L_{i'}] \in \mathcal{B}_{<\alpha}$. WLOG, $L_i < L_{i'}$.

Since I is scattered, take some $i \leq a < b \leq i'$ such that there is no element between a and b in I.

Then $L_a \in \mathcal{R}_{<\alpha}$ and $L_b \in \mathcal{Q}_{<\alpha}$, in contradiction. □

Lemma 5. *Let L be a scattered countable linear order.*

Let $J \subseteq L$ be some subinterval in $\mathcal{B}_{<\alpha}$.

Then $\mathbf{hrank}\,(J) \leq \alpha$.

Furthermore, $\mathbf{hrank}\,(J) < \alpha$ *iff $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.*

Proof. Let $\{x_i\}_{i \in I} \subseteq J$ be a bidirectional, cofinal, weakly monotone I-sequence in J, i.e, $x_i \leq x_j$ if $i \leq j$ for $I \subseteq \mathbb{Z}$.

Write $J = \sum_{i \in I} [x_i, x_{i+1}]$. Then every $[x_i, x_{i+1}]$ is of Hausdorff rank $< \alpha$.

Thus, $\mathbf{hrank}\,(J) \leq \alpha$.

Suppose $\mathbf{hrank}\,(J) < \alpha$, then obviously $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Conversely, suppose $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Then $1 + J + 1 \in \mathcal{B}_{<\alpha}$. But it is a bounded interval, so $\mathbf{hrank}\,(1 + J + 1) < \alpha$ and thus $\mathbf{hrank}\,(J) < \alpha$. □

Lemma 6. *Let $J \subseteq L$ be a subinterval.*

Then $\mathbf{hrank}\,(J) \leq \alpha$ *iff J is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals.*

Note: this lemma does not work if we take a general $\mathbf{Q}$ *property.*

Proof. From the previous lemma, it is clear that if J is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals, then $\mathbf{hrank}\,(J) \leq \alpha$, since the rank bound is preserved under finite sums.

Conversely, suppose $\mathbf{hrank}\,(J) \leq \alpha$.

If $J = \sum_{i \in \mathbb{Z}} J_i$ for some J_i of Hausdorff rank $< \alpha$, take $x, y \in J$. Then let $x \in J_{i_1}$ and $y \in J_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} J_i$. But the last sum is of rank $< \alpha$ and thus $[x, y]$ is of rank $< \alpha$. That is, $J \in \mathcal{B}_{<\alpha}$.

Since every subinterval of rank $\leq \alpha$ is a finite sum of $\mathbb{Z}$-sums of intervals of rank $< \alpha$, we are done. ☐

Corollary 1. *Let $J \subseteq L$ be a subinterval.*

Then $\mathbf{hrank}\,(J) \leq \alpha$ *iff J is a finite sum of largest $\mathcal{B}_{<\alpha}$-subintervals in L*

Lemma 7. *There exists a global computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, The ordinal-sequence A_β of all n-types satisfiable in $\mathcal{H}_{<\beta}$ stabilizes at $f(n)$, i.e, $A_\beta = A_{f(n)}$ for all $\beta \geq f(n)$.*

Proof. Let τ be an n-type. We prove by induction on $\beta \geq f(n)$ that $\tau \in A_{f(n)}$.

The base clear is clear as $\beta = f(n)$ implies $\tau \in A_{f(n)+1}$, that is $\tau \in A_{f(n)}$.

The induction step is also clear, because if $\beta > f(n)$, then we can write $\tau = \sum_{i \in I} \tau_i$ where τ_i of rank $< \beta$, and thus by induction $\tau_i \in A_{f(n)}$, thus $\tau \in A_{f(n)+1} = A_{f(n)}$. ☐

Lemma 8. *For every ordinal $\alpha \geq f(n)$, and for every linear order L with* $\mathbf{hrank}\,(L) \geq f(n)$, *and for every class $\mathcal{C}$ which is one of:*

1. $\mathcal{H}_{<\alpha}$,
2. $\mathcal{Q}_{<\alpha} - \mathcal{R}_{<\alpha}$,
3. $\mathcal{R}_{<\alpha} - \mathcal{Q}_{<\alpha}$,
4. $\mathcal{B}_{<\alpha} - (\mathcal{Q}_{<\alpha} \cup \mathcal{R}_{<\alpha})$.

there exists some linear order $L' \in \mathcal{C}$ such that $L \equiv_n L'$.

Proof. Let A_k be the set of all satisfiable n-types of rank $< k$. Then A_{k+1} is the closure of A_k under finite sums of $\subseteq \mathbb{Z}$-sums.

The sequence $A_0 \subseteq A_1 \subseteq \ldots$ stabilizes at some point. Suppose $A_{f(n)} = A_{f(n)+1}$.

Suppose L has rank $\beta \geq f(n)$.

Write $L = \sum_{i \in I} L_i$ where $\mathbf{hrank}\,(L_i) < \beta$, and I is a finite sum of $\subseteq \mathbb{Z}$.

If β is a limit ordinal, then there must be a bi-cofinal sequence i_k such that $\mathbf{hrank}\,(L_{i_k}) \to \beta$.

If β is a successor ordinal, then $\mathbf{hrank}\,(L_i) = \beta - 1$ must hold infinitely many times.

Now we proceed by induction on $\alpha \geq f(n)$.

1. If $\mathcal{C} = \mathcal{H}_{<\alpha}$, we take $L' \in A_{f(n)}$, which necessarily has rank $< f(n) \leq \alpha$.

2. If $\mathcal{C} = \mathcal{Q}_{<\alpha} - \mathcal{R}_{<\alpha}$, we take an ω-sequence α_k such that $\alpha_k \to \alpha$ (if α is a limit ordinal) or $\alpha_k = \alpha - 1$ (if α is a successor ordinal).

   Then we take $L' = \sum_{i\in\omega} L'_i$ where $\mathbf{hrank}\left(L'_{i_k}\right) = \alpha_k$ (and $\mathbf{hrank}\left(L'_i\right) = \mathbf{hrank}\left(L_i\right)$ for every other i). Then $L' \in \mathcal{Q}_{<\alpha} - \mathcal{R}_{<\alpha}$, but also $L' \equiv_n L$.

3. This is just the same with $-\omega$ instead of ω.

4. This is just the same with $\mathbb{Z}$ instead of $-\omega$.

$\square$

Corollary 2. *Over scattered with interpretations of P, Q and R as above, the properties* $\mathbf{hrank}(\cdot) \leq \alpha$*,* $\mathbf{hrank}(\cdot) < \alpha$ *and* $\mathbf{hrank}(\cdot) = \alpha$ *over subintervals are all expressible in* $\mathbf{MSO}[P, Q, R]$*.*

Proof. For $\mathbf{hrank}(\cdot) \leq \alpha$ and $\mathbf{hrank}(\cdot) < \alpha$, we can use the previous lemmas.
For $\mathbf{hrank}(\cdot) = \alpha$, we can use the previous two. $\square$

Theorem 1. *There is a an algorithm solving satisfiability for* $\mathbf{MSO}[P, Q, R]$ *over scattered linear orders, given an oracle which solves the satisfiability problem for* $\mathbf{MSO}$ *over scattered linear orders.*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[P, Q, R]$ formula φ of quantifier-depth n. outputs an $\mathbf{MSO}[\{X_\tau\}_\tau]$ formula ψ.

Let P_L, Q_L, R_L be the interpretations of P, Q, R on L.

Then

$$L, P := P_L, Q := Q_L, R := R_L \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where $I_\tau = \{i \in I : L_i \models \tau\}$ for every n-type τ.

Let T be the set of n-types in $\mathbf{MSO}[p, q, r]$ which satisfy $q(\tau) = 1 \iff \tau \in \mathcal{Q}_{<\alpha}$ and $r(\tau) = 1 \iff \tau \in \mathcal{R}_{<\alpha}$.

Let $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee q(\tau_2) = 0)\}$.

Then T and S can be calculated using the oracle.

Then ψ is an $\mathbf{MSO}[T, S]$ formula.

Then we define an $\mathbf{MSO}[p, q, r]$ formula ψ' as follows:

ψ' claims that there exists a partition (with possible empty sets) $\{Y_\tau\}_\tau$ of I such that

- Every $i \in I$ is in some Y_τ for $\tau \in T$.

- If $i' = i + 1$ in I, then for some $(\tau_1, \tau_2) \in S$, $i \in Y_{\tau_1}$ and $i' \in Y_{\tau_2}$.

Now we claim that φ is satisfiable in some linear order, iff ψ' is satisfiable in some linear order.

Suppose φ is satisfiable in some linear order L.

Take a decomposition $L = \sum_{i\in I} L_i$ as in lemma 2.

Then ψ holds over the assignment $X_\tau := I_\tau$. But by lemma 2, this assignment satisfies the condition required for ψ' to hold. Then ψ' holds over I.

Conversely, suppose psi' holds in I.

Let $X_\tau := Z_\tau$ be the assignment that is guaranteed by psi'.

Let tau_i be the unique τ such that $i \in Z_\tau$.

Then the conditions for lemma 3 are guaranteed.

Thus, take L as in lemma 3. Then ψ holds over I when we set $X_i := Z_\tau$. But $Z_\tau = I_\tau$ for all τ, so φ holds over L. $\square$