Orders

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Definition 1. *Let L be a linear order.*

We define $\mathbf{hrank}(L) \leq 0$ *iff L is finite.*

Let $\alpha > 0$ be an ordinal.

We define $\mathbf{hrank}(L) \leq \alpha$ *iff* $L = \sum_{i \in I} L_i$ *for some linear order I, where* $\mathbf{hrank}(L_i) < \alpha$ *and I is a finite sum of 1, ω and $-\omega$.*

We write $\mathbf{hrank}(L) = \alpha$ *iff α is the least ordinal such that* $\mathbf{hrank}(L) \leq \alpha$.

We write $\mathbf{hrank}(L) = \perp$ *iff there is no ordinal α such that* $\mathbf{hrank}(L) \leq \alpha$.

We will be working with scattered linear orders.

Claim 1. *Let L be a countable linear order.*

Then $\mathbf{hrank}(L)$ *is defined iff L is scattered.*

Proof. To prove $\implies$ is easy, as a scattered sum of scattered linear orders is scattered.

For the other direction... TODO. □

Notations 1. *Let $\mathcal{H}_{<\alpha}$ be the class of linear orders of Hausdorff rank $< \alpha$ and $\mathcal{H}_{=\alpha}$ be the class of linear orders of Hausdorff rank $= \alpha$.*

Let $\mathcal{B}_{<\alpha}$ be the class of linear orders of Hausdorff rank $< \alpha$ on bounded subintervals.

Let $\mathcal{Q}_{<\alpha} = \{L : 1 + L \in \mathcal{B}_{<\alpha}\}$.

Let $\mathcal{R}_{<\alpha} = \{L : L + 1 \in \mathcal{B}_{<\alpha}\}$.

Clearly, $\mathcal{H}_{<\alpha}, \mathcal{Q}_{<\alpha}, \mathcal{R}_{<\alpha} \subseteq \mathcal{B}_{<\alpha}$.

Claim 2. $\mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha} = \{L : 1 + L + 1 \in \mathcal{B}_{<\alpha}\}$.

Proof. The $\supseteq$ direction is obvious. The $\subseteq$ direction follows from the star property of $\mathcal{B}_{<\alpha}$. □

Lemma 1. *Let L be a linear order. Then there exists a largest subinterval $M \subseteq L$ such that $x \in M$ and $M \in \mathcal{B}_{<\alpha}$.*

Definition 2. *Let L be a linear order. Let $x \in L$. We define $M_\alpha[x]$ to be the largest subinterval $M \subseteq L$ such that $x \in M$ and $M \in \mathcal{B}_{<\alpha}$.*

We define $\sim_\alpha$ to be the equivalence relation on L such that $x \sim_\alpha y$ iff $M_\alpha[x] = M_\alpha[y]$.

Lemma 2. *Let L be a linear order. Let $P, Q, R \subseteq L$ be relations, such that:*

- *P represents $\sim_\alpha$ on L.*
- *Q is such that $x \in Q$ iff $M_\alpha[x] \in \mathcal{Q}_{<\alpha}$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_{<\alpha}$.*

Then for some linear order I there exists a decomposition $L = \sum_{i\in I} L_i$ such that $L_i \in \mathcal{B}_{<\alpha}$ for all $i \in I$, L_i is monochromatic with respect to P, Q and R.

Furthermore, let τ_i be the n-type of L_i, p_i, q_i, r_i in $\mathbf{MSO}[p, q, r]$, where $p_i = 1_{L_i \subseteq P}$, $q_i = 1_{L_i \subseteq Q}$ and $r_i = 1_{L_i \subseteq R}$. Then the following hold

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $q(\tau_{i+1}) = 0$*

Proof. Take $I = L/\sim_\alpha$.

Then $L = \sum_{i\in I} L_i$ where L_i is the $\sim_\alpha$-equivalence class of i.

Then L_i is monochromatic with respect to P, Q and R.

The only thing left to prove is the last two conditions. The first follows from the fact that P represents $\sim_\alpha$.

The second follows because if it were not the case, then L_i and L_{i+1} would be the same $\sim_\alpha$-equivalence class. □

Lemma 3. *Let I be a linear order. Let $n \in \mathbb{N}$. Let p, q, r be boolean variables.*

Let τ_i be an assignment of satisfiable n-types in $\mathbf{MSO}[p, q, r]$ for all $i \in I$. Assume that

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $q(\tau_{i+1}) = 0$*

Then there exists a linear order L and $P, Q, R \subseteq L$ such that:

- *P represents $\sim_\alpha$ on L.*
- *Q is such that $x \in Q$ iff $M_\alpha[x] \in \mathcal{Q}_{<\alpha}$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_{<\alpha}$.*

such that for all $i \in I$, L_i is a $\sim_\alpha$-equivalence class of L, and is thus monochromatic with respect to P, Q and R.

Furthermore, the n-type of L_i, p_i, q_i, r_i in $\mathbf{MSO}[p, q, r]$ is τ_i, where $p_i = 1_{L_i \subseteq P}$, $q_i = 1_{L_i \subseteq Q}$ and $r_i = 1_{L_i \subseteq R}$,

Proof. Since τ_i is satisfiable, we can take L_i to be a linear order of n-type τ_i such that:

- If $q(\tau_i) = r(\tau_i) = 1$, then $L_i \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.
- If $q(\tau_i) = 1$ and $r(\tau_i) = 0$, then $L_i \in \mathcal{Q}_{<\alpha} - \mathcal{R}_{<\alpha}$.

- If $q(\tau_i) = 0$ and $r(\tau_i) = 1$, then $L_i \in \mathcal{R}_{<\alpha} - \mathcal{Q}_{<\alpha}$.
- If $q(\tau_i) = r(\tau_i) = 0$, then $L_i \in \mathcal{B}_{<\alpha} - (\mathcal{Q}_{<\alpha} \cup \mathcal{R}_{<\alpha})$.

Let $L = \sum_{i \in I} L_i$.

By definition each L_i is in $\mathcal{B}_{<\alpha}$. We need to prove that each L_i is a largest $\mathcal{B}_{<\alpha}$-subinterval in L.

On the contrary, suppose that there exist $i' \neq i$ such that $[L_i, L_{i'}] \in \mathcal{B}_{<\alpha}$. WLOG, $L_i < L_{i'}$.

Since I is scattered, take some $i \leq a < b \leq i'$ such that there is no element between a and b in I.

Then $L_a \in \mathcal{R}_{<\alpha}$ and $L_b \in \mathcal{Q}_{<\alpha}$, in contradiction. □

Lemma 4. *Let L be a scattered countable linear order.*

Let $J \subseteq L$ be some subinterval in $\mathcal{B}_{<\alpha}$.

Then $\mathbf{hrank}(J) \leq \alpha$.

Furthermore, $\mathbf{hrank}(J) < \alpha$ *iff* $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Proof. Let $\{x_i\}_{i \in I} \subseteq J$ be a bidirectional, cofinal, weakly monotone I-sequence in J, i.e, $x_i \leq x_j$ if $i \leq j$ for $I \subseteq \mathbb{Z}$.

Write $J = \sum_{i \in I} [x_i, x_{i+1}]$. Then every $[x_i, x_{i+1}]$ is of Hausdorff rank $< \alpha$.

Thus, $\mathbf{hrank}(J) \leq \alpha$.

Suppose $\mathbf{hrank}(J) < \alpha$, then obviously $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Conversely, suppose $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Then $1+J+1 \in \mathcal{B}_{<\alpha}$. But it is a bounded interval, so $\mathbf{hrank}(1 + J + 1) < \alpha$ and thus $\mathbf{hrank}(J) < \alpha$. □

Lemma 5. *Let $J \subseteq L$ be a subinterval.*

Then $\mathbf{hrank}(J) \leq \alpha$ *iff J is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals.*

Proof. From the previous lemma, it is clear that if J is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals, then $\mathbf{hrank}(J) \leq \alpha$, since the rank bound is preserved under finite sums.

Conversely, suppose $\mathbf{hrank}(J) \leq \alpha$.

If $J = \sum_{i \in \mathbb{Z}} J_i$ for some J_i of Hausdorff rank $< \alpha$, take $x, y \in J$. Then let $x \in J_{i_1}$ and $y \in J_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} J_i$. But the last sum is of rank $< \alpha$ and thus $[x, y]$ is of rank $< \alpha$. That is, $J \in \mathcal{B}_{<\alpha}$.

Since every subinterval of rank $\leq \alpha$ is a finite sum of $\mathbb{Z}$-sums of intervals of rank $< \alpha$, we are done. □

Corollary 1. *Let $J \subseteq L$ be a subinterval.*

Then $\mathbf{hrank}(J) \leq \alpha$ *iff J is a finite sum of largest $\mathcal{B}_{<\alpha}$-subintervals in L*

Lemma 6. *Let $\mathcal{C} \in \{\mathcal{Q}_{<\alpha}, \mathcal{R}_{<\alpha}\}$.*

There exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, for every ordinal $\alpha \geq f(n)$, and for every linear order L with $\mathbf{hrank}(L) \geq f(n)$, *there exists some linear order $L' \in \mathcal{C}$ such that $L \equiv_n L'$.*

Corollary 2. *Over scattered with interpretations of P, Q and R as above, the properties* $\mathbf{hrank}(\cdot) \leq \alpha$*,* $\mathbf{hrank}(\cdot) < \alpha$ *and* $\mathbf{hrank}(\cdot) = \alpha$ *over subintervals are all expressible in* $\mathbf{MSO}[P, Q, R]$*.*

Proof. For $\mathbf{hrank}(\cdot) \leq \alpha$ and $\mathbf{hrank}(\cdot) < \alpha$, we can use the previous lemmas.

For $\mathbf{hrank}(\cdot) = \alpha$, we can use the previous two. $\square$

Theorem 1. *There is a an algorithm solving satisfiability for* $\mathbf{MSO}[P, Q, R]$ *over scattered linear orders, given an oracle which solves the satisfiability problem for* $\mathbf{MSO}$ *over scattered linear orders.*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[P, Q, R]$ formula φ of quantifier-depth n. outputs an $\mathbf{MSO}[\{X_\tau\}_\tau]$ formula ψ.

Let P_L, Q_L, R_L be the interpretations of P, Q, R on L.

Then

$$L, P := P_L, Q := Q_L, R := R_L \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where $I_\tau = \{i \in I : L_i \models \tau\}$ for every n-type τ.

Let T be the set of n-types in $\mathbf{MSO}[p, q, r]$ which satisfy $q(\tau) = 1 \iff \tau \in \mathcal{Q}_{<\alpha}$ and $r(\tau) = 1 \iff \tau \in \mathcal{R}_{<\alpha}$.

Let $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee q(\tau_2) = 0)\}$.

Then T and S can be calculated using the oracle.

Then ψ is an $\mathbf{MSO}[T, S]$ formula.

Then we define an $\mathbf{MSO}[p, q, r]$ formula ψ' as follows:

ψ' claims that there exists a partition (with possible empty sets) $\{Y_\tau\}_\tau$ of I such that

- Every $i \in I$ is in some Y_τ for $\tau \in T$.
- If $i' = i + 1$ in I, then for some $(\tau_1, \tau_2) \in S$, $i \in Y_{\tau_1}$ and $i' \in Y_{\tau_2}$.

Now we claim that φ is satisfiable in some linear order, iff ψ' is satisfiable in some linear order.

Suppose φ is satisfiable in some linear order L.

Take a decomposition $L = \sum_{i \in I} L_i$ as in lemma 2.

Then ψ holds over the assignment $X_\tau := I_\tau$. But by lemma 2, this assignment satisfies the condition required for ψ' to hold. Then ψ' holds over I.

Conversely, suppose psi' holds in I.

Let $X_\tau := Z_\tau$ be the assignment that is guaranteed by psi'.

Let tau_i be the unique τ such that $i \in Z_\tau$.

Then the conditions for lemma 3 are guaranteed.

Thus, take L as in lemma 3. Then ψ holds over I when we set $X_i := Z_\tau$. But $Z_\tau = I_\tau$ for all τ, so φ holds over L. $\square$