1 Orders

In this section the logic is always MSO, and the structures are orders. When we mention a linear order $\mathcal{L}$, we assume implicitly that it is countable. When we mention an ordinal α, we assume implicitly that it is a limit ordinal. We shall write this as CLO (countable linear order).

Definition 1. *Let $\mathcal{L}$ be a CLO. We define two relation $\mathbf{rank}^{-}\mathcal{L} \leq \alpha$ and $\mathbf{rank}\mathcal{L} \leq \alpha$ by transfinite induction on α as follows:*

$\mathbf{rank}^{-}\mathcal{L} \leq 0$ iff $\mathcal{L}$ is 1.

$\mathbf{rank}\mathcal{L} \leq \alpha$ iff $\mathcal{L}$ is a finite sum of CLOs $\mathcal{L}_i$ such that $\mathbf{rank}^{-}\mathcal{L}_i < \alpha$ for all i.

For $\alpha > 0$, $\mathbf{rank}^{-}\mathcal{L} \leq \alpha$ iff $\mathcal{L} = \sum_{i \in I} \mathcal{L}_i$ where I is either finite or $\pm\omega$ or η. and $\mathbf{rank}\mathcal{L}_i < \alpha$ for all $i \in I$.

Now, we define $\mathbf{rank}^{-}\mathcal{L}$, if it exists, as the minimal α such that $\mathbf{rank}^{-}\mathcal{L} \leq \alpha$, and similarly we (partially) define $\mathbf{rank}\mathcal{L}$.

Lemma 1. *For every α,*

1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be CLOs such that $\mathbf{rank}\mathcal{L}_i = \alpha_i$.*

   Then $\mathbf{rank}\left(\mathcal{L}_1 + \mathcal{L}_2\right) = \max\left\{\alpha_1, \alpha_2\right\}$.

2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be CLOs such that $\mathbf{rank}^{-}\mathcal{L}_1 = \mathbf{rank}^{-}\mathcal{L}_2 = \alpha$.*

   Then $\mathbf{rank}^{-}\left(\mathcal{L}_1 + \mathcal{L}_2\right) = \alpha + 1$.

Proof. This is obvious from the definition. □

Definition 2 (Good 2-coloring)**.** *Let $\mathcal{L}$ be a CLO. A good 2-coloring of $\mathcal{L}$ is a coloring of $\mathcal{L}$ with two colors, such that: * If there is only a single endpoint, it is red. * If there are two endpoints, the leftmost one is red.*

Lemma 2 (Existence and uniqueness)**.** *Every CLO has a good 2-coloring. All such 2-colorings are isomorphic.*

Proof. Every rank-1 equivalence class is obviously 2-colorable this way, since it either has an endpoint or is isomorphic to $\eta := -\omega + \omega = \mathbf{otp}\left(\mathbb{Z}\right)$

The coloring is obviously unique, except for the case of η, where the two possible good colorings are isomorphic.

Since the rank-1 equivalence classes have no edges between them, the proof is done.

□ □

Lemma 3. *The fact that R is a good 2-coloring is expressible in MSO over CLOs.*

Proof. Obvious.

□ □

Definition 3. *Let $\mathcal{L}$ be a CLO and let α be an ordinal. We define the relation $\sim_\alpha$ on $\mathcal{L}$ as follows: $x \sim_\alpha y$ iff $\mathbf{rank}(x, y) < \alpha$*

Lemma 4. *Let α be a limit ordinal. Then, the following are equivalent:*

1. $x \sim_\alpha y$
2. *the interval* (x, y) *is of* **rank** $< \alpha$
3. *the interval* $[x, y)$ *is of* **rank** $< \alpha$
4. *the interval* $(x, y]$ *is of* **rank** $< \alpha$
5. *the interval* $[x, y]$ *is of* **rank** $< \alpha$

Proof. Follows easily from the definition of **rank**. □

Lemma 5. *The relation $\sim_\alpha$ is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious. Trnasitivity follows easily from the fact that α is a limit ordinal and the previous lemma. □ □

Definition 4. *Let $\mathcal{L}$ be a CLO together with a good coloring C of $\mathcal{L}/\sim_\alpha$, from which there is an induced coloring of $\mathcal{L}$ (which is valid with regard to α-classes).*

We denote by $R_C(x)$ the formula expressing the fact that the α-class of x is red in C.

Lemma 6. *An interval X is of* **rank** $< \alpha$ *iff it is contained in an equivalence class of $\sim_\alpha$.*

Proof. Both directions are obvious.

□ □

Lemma 7. *X is an equivalence class of $\sim_\alpha$ iff X is a maximal subset such that it is connected and every subinterval of X is of* **rank** $< \alpha$.

Proof. It is enough to show that every set with this property is contained in an equivalence class of $\sim_\alpha$, and that the equivalence classes possess this property.

Clearly, every nonempty set X with this property is contained in a single equivalence class: Let $x \in X$ be a specific choice, and let $y \in X$ be another point. Then $[x, y]$ is contained in X since X is connected, and therefore is of **rank** $< \alpha$. Therefore, $x \sim_\alpha y$.

That is, $X = [x]_{\sim_\alpha}$.

Now, let $X = [x]_{\sim_\alpha}$. Since α is a limit ordinal, every subinterval of X is of **rank** $< \alpha$. And also clearly it is connected.

□ □

Corollary 1. *Let $\mathcal{L}$ be a CLO and let C be a good coloring of $\mathcal{L}/\sim_\alpha$. The fact that X is an interval of* **rank** $< \alpha$ *is expressible in MSO over CLOs using R_C.*

Proof. Since an α-class of $\mathcal{L}$ is just a maximal homochromatic connected subset this is clear from the previous lemma.

□ □

Theorem 1. *MSO extended by R_C is decidable over all CLOs with a good coloring.*

Proof. Let $\mathcal{L}$ be a CLO of rank ν and let α be a limit ordinal.

Let $\gamma := \mathcal{L}/\sim_\alpha$. Then γ is of rank $\nu - \alpha$.

Let C be a good coloring of γ.

Then we can write $\mathcal{L} = \sum_{i\in\gamma} L_i$ where each L_i is an α-class.

Moreover, over each L_i, the coloring is monochromatic.

Let φ be a formula of qd rank r over R_C.

Then, by the composition theorem, there exists a formula ψ such that

$$\mathcal{L} \vDash \varphi \iff \gamma, \{Q_\tau\}_\tau \vDash \psi$$

where $Q_\tau := \{L_i : L_i \vDash \tau\}$, where τ is an r-rank.

To decide φ over all $\mathcal{L}$, we need to decide ψ over all the possible combinations of γ and $\{Q_\tau\}_\tau$ that arise from $\mathcal{L}$.

Now clearly, the possible values of γ are all the CLOs.

Now, we need to calculate all the values of τ for which there exists some equivalence class of some $\mathcal{L}$ that satisfies τ. $\square$