Orders

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Contents

1 Preorders

We begin by studying the classes of preorders. Basically, we define a *class* as a class which is close under isomorphism. We then define the sum operation on preorders. This will be used to create new classes from old ones. Even though we will only use it for linear orders, we can define it for any preorder.

Definitions 1.1 (Preorder)**.** *A* (labeled) preorder *is a a set M together with a binary relation $\leq$ on M such that $\leq$ is reflexive and transitive, possibly endowed with monadic predicates (labels) over some first-order monadic signature.*

Definition 1.2 (class of preorders)**.** *A* class $\mathbf{P}$ *of preorders is a collection of preorders, all defined over one signature, which is closed under isomorphism.*

Definition 1.3. *a class $\mathbf{P}$ of preorders is* monotone *if for every preorder M, $M \in \mathbf{P}$ implies that every suborder of M is in $\mathbf{P}$.*

Definition 1.4. *Let M be a preorder.*
Then M^* is the dual/reverse preorder of M.

Definition 1.5 (Sum of preorders)**.** *Let I be a preorder.*
Let $\{M_i\}_{i\in I}$ be a family of preorders over some signature.
The sum $M = \sum_{i\in I} M_i$ is defined as follows:
The domain is $M = \biguplus_{i\in I} M_i$ (a disjoint union).
Let $\leq_i$ be the preorder on M_i.
Let $x \in M_i$ and $y \in M_j$.
Then we define $x \leq y$ iff either $i = j$ and $x \leq_i y$ or $i < j$.
The labels are inherited from the M_i's.
If $I = 2$, we define $M_0 + M_1 := \sum_{i\in 2} M_i$.

Lemma 1.6. *Let I be a preorder.*
Let $\{M_i\}_{i\in I}$ be a family of preorders over some signature.
Then $M = \sum_{i\in I} M_i$ is a preorder.

Proof. Reflexivity is clear.
For transitivity, suppose $x \leq y$ and $y \leq z$.
Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.
Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.
Otherwise, $i < k$, and thus $x \leq z$, as required. □

Definition 1.7. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be classes of preorders over some signature.*
Then we define

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

The labels are inherited from either $\mathbf{P}_1$ or $\mathbf{P}_2$.

Definition 1.8. *Let $\mathbf{P}$ be a class of preorders.*
$\mathbf{P}$ *is called an* additive class *if for every preorders M_1 and M_2, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

Definition 1.9 (Kleene plus)**.** *Let* $\mathbf{P}$ *be a class of preorders.*

We define its Kleene plus as the smallest class of preorders $\mathbf{P}^+$ *which contains* $\mathbf{P}$ *and is closed under finite sums.*

That is, $1^+ = \{1, 2, \ldots\}$*, and* $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$.

Definition 1.10 (Sum of a family of classes over a preorder)**.** *Let* I *be a preorder.*

Let $\{\mathbf{Q}_i\}_{i \in I}$ *be a family of classes of preorders over some signature.*

Then we define

$$\sum_{i \in I} \mathbf{Q}_i := \left\{ \sum_{i \in I} M_i : \forall i \in I. M_i \in \mathbf{Q}_i \right\}$$

The labels are inherited from $\mathbf{Q}_i$*'s.*

Definition 1.11 (Sum of a class over a preorder)**.** *Let* $\mathbf{Q}$ *be a class of preorders.*

Let I *be a preorder.*

Then we define

$$\sum_{I} \mathbf{Q} := \left\{ \sum_{i \in I} M_i : \forall i \in I. M_i \in \mathbf{Q} \right\}$$

Note 1.12. *Let* $\mathbf{Q}$ *be a class of preorders.*

Let I *be a preorder.*

By the previous two definitions,

$$\sum_{I} \mathbf{Q} = \sum_{i \in I} \mathbf{Q}$$

Definition 1.13 (Sum of a class over a class)**.** *Let* $\mathbf{P}$ *be a class of preorders.*

Let $\mathbf{Q}$ *be a class of preorders.*

Then we define,

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_{I} \mathbf{Q} : I \in \mathbf{P} \right\}$$

Lemma 1.14 (Associativity of sum)**.** *Let* I *be a preorder.*

Let $\{J_i\}_{i \in I}$ *be a family of mutually disjoint preorders over some signature.*

Let $\{K_j\}_{j \in \biguplus_i J_i}$ *be a family of preorders over some signature.*

Then,

$$\sum_{i \in I} \sum_{j \in J_i} K_j \cong \sum_{j \in \sum_{i \in I} J_i} K_j$$

Proof. This follows from the definition of the sum operation. □

Corollary 1.15 (Associativity of sum for classes)**.** *Let* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *be classes.*

Then $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$.

Lemma 1.16 (Sum and union commute)**.** *Let $\mathcal{P}$ be a family of classes.*

Let $\mathbf{Q}$ be a class.

Then $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$.

Proof. This is obvious from the definition of the sum operation. $\square$

2 Linear Orders

In this chapter we focus on linear orders, also known as total orders, intervals and chains.

Definitions 2.1 (Linear order)**.** *A* linear order *is a preorder which is antisymmetric and total.*

Definition 2.2 (class of linear orders)**.** *A* class $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 2.3 (Subintervals)**.** *Let M be a linear order, and let $x, y \in M$, such that $x \leq y$.*

Then we define the bounded subintervals *$[x, y]$, $(x, y]$, $[x, y)$ and (x, y) as usual.*

We also define the semi-bounded subintervals *$(-\infty, x]$, $[x, \infty)$, $(-\infty, x)$ and (x, ∞) as usual.*

We also define the unbounded subinterval *$(-\infty, \infty)$ as the whole linear order M, as usual.*

A subinterval *is either a bounded subinterval, a semi-bounded subinterval or the unbounded subinterval.*

If $x > y$ then we define the intervals as follows:

$$\begin{aligned} [x, y] &:= [y, x] \\ (x, y] &:= (y, x] \\ [x, y) &:= [y, x) \\ (x, y) &:= (y, x) \end{aligned}$$

Definition 2.4. *Let M be a linear order.*

A set $A \subseteq M$ is left cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $y < x$.*

A set $A \subseteq M$ is right cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $x < y$.*

A set $A \subseteq M$ is bi-directionally cofinal *in M if it is both left and right cofinal.*

Lemma 2.5. *Let $\mathbf{P}$ be an additive class of linear orders.*

Let $M \in \mathbf{P}$ be a linear order.

Let $x, y \in M$ be any two points in M.

Then, $[x, y] \in \mathbf{P}$.

Proof. WLOG, suppose $x \leq y$.

Note that,

$$M = (-\infty, \infty) = (-\infty, x) + [x, y] + (y, \infty)$$

when $(-\infty, x)$ and/or (y, ∞) may be empty.

Since $\mathbf{P}$ is an additive class, we conclude that $[x, y] \in \mathbf{P}$. □

Corollary 2.6. *Let* $\mathbf{P}$ *be a nontrivial additive class of linear orders.*

Then $1 \in \mathbf{P}$*.*

Proof. Let $M \in \mathbf{P}$ be any linear order and let $x \in M$ be any point in M.

Apply lemma 2.5 to the linear order M, and the points x and x, to conclude that $[x, x] \equiv 1 \in \mathbf{P}$. □

Note 2.7. *Note that corollary 2.6 is false if we do not restrict ourselves to linear orders.*

For example, $(\mathbf{1} \uplus \mathbf{1})^+$ *is a class of preorders which is additive, but does not contain* $\mathbf{1}$*.*

Corollary 2.8. *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Let M *be a linear order.*

Let $x, y \in M$ *be any two points in a linear order* M*. Then the following are equivalent:*

1. $(x, y) \in \mathbf{P}$
2. $(x, y] \in \mathbf{P}$
3. $[x, y) \in \mathbf{P}$
4. $[x, y] \in \mathbf{P}$

Proof. This is just applying the definition of an additive class to the orders $[x, y]$ and 1. □

Corollary 2.9. *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Let M *be a linear order.*

Let $x, y, z \in M$ *be any three points in a linear order* M*, such that* $[x, y] \in \mathbf{P}$ *and* $[y, z] \in \mathbf{P}$*.*

Then $[x, z] \in \mathbf{P}$*.*

Proof. If $y \in [x, z]$, then $[x, z] = [x, y] + (y, z]$, and $(y, z] \in \mathbf{P}$ by corollary 2.8.

Otherwise, either $x \in [y, z]$ or $z \in [x, y]$. WLOG, suppose $z \in [x, y]$.

Then $[x, y] = [x, z] + (z, y]$, so $[x, z] \in \mathbf{P}$ by the fact that $\mathbf{P}$ is additive. □

Definitions 2.10. *Let* $\mathbf{P}$ *be a class of linear orders.*

We define the following classes of linear orders:

- $\mathcal{B}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x, y \in M$*, the bounded subinterval* $[x, y]$ *is in* $\mathbf{P}$*.*
- $\mathcal{L}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x \in M$*, the left-bounded ray* $[x, \infty)$ *is in* $\mathbf{P}$*.*
- $\mathcal{R}[\mathbf{P}]$ *is the class of linear orders* M *such that for every* $x \in M$*, the right-bounded ray* $(-\infty, x]$ *is in* $\mathbf{P}$*.*

Definition 2.11. *a class* $\mathbf{P}$ *of linear orders is a* star class *if for every linear orders* M*, and every family* $\mathcal{F} \subseteq \mathbf{P}$ *of subintervals of* M *such that* $J_1 \cap J_2 \neq \emptyset$ *for every* $J_1, J_2 \in \mathcal{F}$*, we have that* $\bigcup \mathcal{F} \in \mathbf{P}$*.*

Lemma 2.12. *Let* $\mathbf{P}$ *be a star class.*

Then for every linear order M*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$*.*

Equivalently, we can define a convex equivalence relation $\sim_{\mathbf{P}}$ *on* M *such that* $x \sim_{\mathbf{P}} y$ *iff* $[x, y] \in \mathbf{P}$*.*

That is, $x \sim_{\mathbf{P}} y$ *iff* x *and* y *are in the same largest* $\mathbf{P}$*-subinterval.*

Proof. Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in $\mathcal{B}[\mathbf{P}]$, and by definition J is the largest $\mathbf{P}$-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. $\square$

Lemma 2.13 (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Then the class $\mathcal{B}[\mathbf{P}]$ *is a star class.*

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by the definition of $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. Since $\mathbf{P}$ is additive, by corollary 2.9, we have $[x, y] \in \mathbf{P}$. $\square$

Lemma 2.14. *Let* $\mathbf{P}$ *be an additive class of linear orders.*

Then,

1. $\mathcal{L}[\mathbf{P}] = \{M : M + \mathbf{1} \in \mathcal{B}[\mathbf{P}]\}$
2. $\mathcal{R}[\mathbf{P}] = \{M : \mathbf{1} + M \in \mathcal{B}[\mathbf{P}]\}$
3. $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}] = \{M : \mathbf{1} + M + \mathbf{1} \in \mathcal{B}[\mathbf{P}]\}$

Proof. Let M be a linear order.

1. Suppose $M + \{\infty\} \in \mathcal{B}[\mathbf{P}]$. Then for every $x \in M$, we have $[x, \infty] \in \mathbf{P}$, and thus $[x, \infty) \in \mathbf{P}$. Therefore, $M \in \mathcal{L}[\mathbf{P}]$.

   Conversely, if $M \in \mathcal{L}[\mathbf{P}]$, let $x, y \in M$ be any two points in $M + \mathbf{1}$.

   If $y < \infty$, then $[x, y] \subseteq [x, \infty)$. Since $[x, \infty) \in \mathbf{P}$, we conclude that $[x, y] \in \mathbf{P}$. Otherwise, if $y = \infty$, then $[x, y] = [x, \infty] = [x, \infty) + \{\infty\}$, and thus $[x, y] \in \mathbf{P}$.

2. The second case is dual to the first case.

3. We will show a triple inclusion.

   If $M \in \mathbf{P}$, then by additivity, $\mathbf{1} + M \in \mathbf{P}$ and $M + \mathbf{1} \in \mathbf{P}$, and thus $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$.

   If $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, then by lemma 2.13, $\mathbf{1} + M + \mathbf{1} \in \mathcal{B}[\mathbf{P}]$.

   If $\mathbf{1} + M + \mathbf{1} \in \mathcal{B}[\mathbf{P}]$, then M is a bounded subinterval of $\mathbf{1} + M + \mathbf{1}$, so $M \in \mathcal{B}[\mathbf{P}]$.

$\square$

Lemma 2.15. *Let* $\mathbf{P}$ *be an additive class of linear orders.*
Then,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = \mathbf{P} &\\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}])\\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])\\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Proof. By lemma 2.14, we conclude that $\mathcal{L}[\mathbf{P}], \mathcal{R}[\mathbf{P}] \subseteq \mathcal{B}[\mathbf{P}]$, since $M + 1 \in \mathbf{P}$ and $1 + M \in \mathbf{P}$ both imply $1 + M + 1 \in \mathbf{P}$.

Thus,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = (\mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]) &\\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}])\\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])\\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Since by lemma 2.14 $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, we conclude what we wanted to prove. $\square$

Lemma 2.16. *Let* $\mathbf{P}$ *be an additive class of linear orders.*
Let M, M_1, M_2 *be linear orders such that* $M = M_1 + M_2$.
Then,

1. $M \in \mathcal{B}[\mathbf{P}] \iff M_1 \in \mathcal{L}[\mathbf{P}] \wedge M_2 \in \mathcal{R}[\mathbf{P}]$

Proof. From lemma 2.14, we know that

1.

$$\begin{aligned}M \in \mathcal{B}[\mathbf{P}] &\iff M_1 + M_2 \in \mathcal{B}[\mathbf{P}]\\ &\iff M_1 + 1 \in \mathcal{B}[\mathbf{P}] \wedge 1 + M_2 \in \mathcal{B}[\mathbf{P}]\\ &\iff M_1 \in \mathcal{L}[\mathbf{P}] \wedge M_2 \in \mathcal{R}[\mathbf{P}]\end{aligned}$$

$\square$

Corollary 2.17. *Let* $\mathbf{P}$ *be an additive class of linear orders.*
Then,

$$\mathcal{B}\left[\mathbf{P}\right] \setminus \left(\mathcal{L}\left[\mathbf{P}\right] \cup \mathcal{R}\left[\mathbf{P}\right]\right) = \left(\mathcal{L}\left[\mathbf{P}\right] \setminus \mathcal{R}\left[\mathbf{P}\right]\right) + \left(\mathcal{R}\left[\mathbf{P}\right] \setminus \mathcal{L}\left[\mathbf{P}\right]\right)$$

Definition 2.18. *We define* $\mathbf{CNT}$ *as the class of all countable linear orders.*

3 The Hausdorff Rank

Definition 3.1.

$$\Omega = \{1, \omega, \omega^*\}^+$$

Observation 3.2. *Ω is a monotone, additive class of linear orders.*

Definition 3.3. *We define a class $\mathbf{H}^{<\alpha}$ for every ordinal α as follows:*

- *For $\alpha = 0$, $\mathbf{H}^{<0} = \emptyset$.*
- *For $\alpha = 1$, $\mathbf{H}^{<1} = \{1\}$.*
- *For $\alpha = \gamma + 1$ where $\gamma > 0$,*

$$\mathbf{H}^{<\alpha} = \sum_{\Omega} \mathbf{H}^{<\gamma}$$

- *For α a limit ordinal,*

$$\mathbf{H}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{H}^{<\beta}$$

Definition 3.4. *Let α, β be ordinals such that with $0 < \alpha < \beta$.*
We define,

1. $\mathbf{H}^{\leq\alpha} := \mathbf{H}^{<\alpha+1}$
2. $\mathbf{H}^{=\alpha} := \mathbf{H}^{\leq\alpha} \setminus \mathbf{H}^{<\alpha}$
3. $\mathbf{H}^{[\alpha,\beta)} := \mathbf{H}^{<\beta} \setminus \mathbf{H}^{<\alpha}$

Definition 3.5. *We define the Hausdorff rank as a* partial *mapping from linear orders to ordinals, such that*

$$\mathbf{hrank}(M) = \min\left\{\alpha : M \in \mathbf{H}^{\leq\alpha}\right\}$$

Equivalently,

$$\mathbf{hrank}(M) = \alpha \iff M \in \mathbf{H}^{=\alpha}$$

Definition 3.6. *Let $\alpha > 0$ be an ordinal.*
We define:

1. *(Right α-Major)* $\mathbf{RM}_\alpha := \mathcal{R}[\mathbf{H}^{<\alpha}] \setminus \mathcal{L}[\mathbf{H}^{<\alpha}]$
2. *(Left α-Major)* $\mathbf{LM}_\alpha := \mathcal{L}[\mathbf{H}^{<\alpha}] \setminus \mathcal{R}[\mathbf{H}^{<\alpha}]$
3. *(Bounded α-Major)* $\mathbf{BM}_\alpha := \mathcal{B}[\mathbf{H}^{<\alpha}] \setminus (\mathcal{L}[\mathbf{H}^{<\alpha}] \cup \mathcal{R}[\mathbf{H}^{<\alpha}])$

Note 3.7. *Obviously $\mathbf{LM}_\alpha = \mathbf{RM}_\alpha^*$ by symmetry.*
By corollary 2.17, $\mathbf{BM}_\alpha = \mathbf{LM}_\alpha + \mathbf{RM}_\alpha$.
Also, by the definition and lemma 2.14,

$$\mathcal{B}\left[\mathbf{H}^{<\alpha}\right] = \mathbf{H}^{<\alpha} \uplus \mathbf{LM}_\alpha \uplus \mathbf{RM}_\alpha \uplus \mathbf{BM}_\alpha$$

$$\mathbf{LM}_\alpha \uplus \mathbf{RM}_\alpha \uplus \mathbf{BM}_\alpha \subseteq \mathbf{H}^{=\alpha}$$

Lemma 3.8. *Let $\alpha > 0$ be an ordinal.*
Then $\mathcal{R}\left[\mathbf{H}^{<\alpha}\right] = \sum_\omega \mathbf{H}^{<\alpha}$.

Proof. ($\supseteq$) Let $M \in \sum_\omega \mathbf{H}^{<\alpha}$ be a linear order.
Let $M = \sum_{i\in\omega} M_i$ be the decomposition of M, where $M_i \in \mathbf{H}^{<\alpha}$.
Let $x, y \in M$ be any two points in M. WLOG $x \leq y$.
Suppose $x \in M_i$ and $y \in M_j$ for $i, j \in \omega$.
Since i and j have a finite distance in ω, we conclude $[x, y] \subseteq M_i + \ldots + M_j$, and thus $[x, y] \subseteq \left(\mathbf{H}^{<\alpha}\right)^+ = \mathbf{H}^{<\alpha}$.
($\subseteq$) Let $M \in \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]$ be a linear order.
Since M is countable, let $\{x_i\}_{i\in\omega} M$ be a right cofinal ω-sequence in M.
Let $M_0 = (-\infty, x_0]$ and and $M_i = (x_{i-1}, x_i]$ for $i > 0$.
Then $M = \sum_{i\in\omega} M_i$.
But M_i is a right-bounded interval and thus $M_i \in \mathbf{H}^{<\alpha}$, so $M \in \sum_\omega \mathbf{H}^{<\alpha}$.
□

An immediate corollary of lemma 3.8 is that "major" is a good name, in the sense that every interval of rank α is a finite sum of α-major and "α-minor" (i.e, of rank $< \alpha$) intervals.

Corollary 3.9. *Let $\alpha > 0$ be an ordinal.*
Then,

$$\mathbf{H}^{\leq\alpha} = \left(\mathcal{B}\left[\mathbf{H}^{<\alpha}\right]\right)^+$$

Proof. By the definition,

$$\begin{aligned}
\mathbf{H}^{\leq\alpha} &= \sum_{\Omega} \mathbf{H}^{<\alpha} \\
&= \sum_{\mathbf{FNT}} \left(\mathbf{H}^{<\alpha} \uplus \sum_{\omega} \mathbf{H}^{<\alpha} \uplus \sum_{\omega^*} \mathbf{H}^{<\alpha} \uplus \sum_{\omega^*+\omega} \mathbf{H}^{<\alpha}\right) \\
&= \sum_{\mathbf{FNT}} \left(\mathbf{H}^{<\alpha} \uplus \mathbf{LM}_\alpha \uplus \mathbf{RM}_\alpha \uplus \mathbf{BM}_\alpha\right) \\
&= \left(\mathcal{B}\left[\mathbf{H}^{<\alpha}\right]\right)^+
\end{aligned}$$

□

Corollary 3.10. *Let $\alpha > 0$ be an ordinal.*
Then,

$$\begin{aligned}
\mathbf{H}^{=\alpha} = \mathbf{H}^{\leq\alpha} \\
&+ \left(\mathbf{LM}_\alpha \uplus \mathbf{RM}_\alpha \uplus \mathbf{BM}_\alpha\right) \\
&+ \mathbf{H}^{\leq\alpha}
\end{aligned}$$

Proof. ($\supseteq$) is immediate.
($\subseteq$) Let $M \in \mathbf{H}^{=\alpha}$.
Then M is a finite sum of $\mathcal{B}\left[\mathbf{H}^{<\alpha}\right]$-intervals by the previous lemma, but not all of them are $\mathbf{H}^{<\alpha}$-intervals, otherwise we would have $M \in \mathbf{H}^{<\alpha}$, which is a contradiction. □

Lemma 3.11. *Let $\alpha > 0$ be an ordinal.*

Let $\{\alpha_i\}_{i\in\omega}$ be a non-decreasing ω-sequence of ordinals.

Suppose $\sup_{i\in\omega}(\alpha_i + 1) = \alpha$.

Then,

$$\mathbf{RM}_\alpha = \sum_{i\in\omega} \mathbf{H}^{[\alpha_i,\alpha)}$$

Proof. ($\subseteq$) Let $M \in \mathbf{RM}_\alpha$. Let $\{y_i\}_{i<\omega}$ be a right cofinal ω-sequence in M.

Thus we can choose some x_0 far enough such that $(-\infty, x_0] \in \mathbf{H}^{[\alpha_0,\alpha)}$, and $x_0 > y_0$. Now by induction we choose x_1 such that $(x_0, x_1] \in \mathbf{H}^{[\alpha_1,\alpha)}$, and $x_1 > y_1$.

By iterating ω times we get an ω-sequence $\{M_i\}_{i\in\omega}$ such that $M = \sum_{i\in\omega} M_i$ and $M_i \in \mathbf{H}^{[\alpha_i,\alpha)}$, where $M_i = (x_{i-1}, x_i]$ (where $x_{-1} := -\infty$).

($\supseteq$) Let $M \in \sum_{i\in\omega} \mathbf{H}^{[\alpha_i,\alpha)}$. It is obvious that $M \in \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]$ since every right-bounded ray is a finite sum of $\mathbf{H}^{<\alpha}$-intervals.

However, $M \notin \mathbf{H}^{<\alpha_i}$ for any $i \in \omega$. If α is a limit ordinal, it implies $M \notin \mathbf{H}^{<\alpha}$, and thus $M \notin \mathcal{L}\left[\mathbf{H}^{<\alpha}\right]$, so we are done.

Otherwise, α is a successor ordinal. Say $\alpha = \gamma + 1$, and WLOG $\alpha_i = \gamma$ for all $i \in \omega$.

If $\gamma = 0$, then indeed $M = \sum_\omega 1 = \omega \in \mathbf{RM}_1$.

Otherwise, suppose for the sake of contradiction that $M \in \mathbf{H}^{<\alpha}$. That is, $M \in \mathbf{H}^{\leq\gamma} = \left(\mathcal{B}\left[\mathbf{H}^{<\gamma}\right]\right)^+$.

Then M can be decomposed into a finite sum of $\mathcal{B}\left[\mathbf{H}^{<\gamma}\right]$-intervals, the last of which contains almost all of the M_i, so WLOG $M \in \mathcal{B}\left[\mathbf{H}^{<\gamma}\right]$.

In particular $M_1 \in \mathbf{H}^{<\gamma}$, since M_1 is bounded between M_0 and M_2, but $M_1 \in \mathbf{H}^{=\gamma}$ which is a contradiction. $\square$

Obviously, we can use the decomposition of $\mathbf{RM}_\alpha$ to decompose $\mathbf{LM}_\alpha$ and $\mathbf{BM}_\alpha$ as well, as $\mathbf{LM}_\alpha = \mathbf{RM}^*_\alpha$ and $\mathbf{BM}_\alpha = \mathbf{LM}_\alpha + \mathbf{RM}_\alpha$.

4 Decidability of the Hausdorff Rank

In this chapter, we will try to establish the decidability of the **MSO**-satisfiability over the minor and major classes: $\mathbf{H}^{<\alpha}$, $\mathbf{RM}_{=\alpha}$, $\mathbf{LM}_{=\alpha}$, and $\mathbf{BM}_{=\alpha}$, for every ordinal $\alpha \geq \omega$.

Definition 4.1. *Let $\mathbf{P}$ be a class of preorders.*

Let $n \in \mathbb{N}$.

We define $\mathbf{type}_n[\mathbf{P}]$ as the set of all n-types satisfiable in $\mathbf{P}$.

Definition 4.2. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be classes of preorders.*

Let $n \in \mathbb{N}$.

*Then we say that $\mathbf{P}_1$ and $\mathbf{P}_2$ are n-*equivalent*, denoted $\mathbf{P}_1 \equiv_n \mathbf{P}_2$, if $\mathbf{type}_n[\mathbf{P}_1] = \mathbf{type}_n[\mathbf{P}_2]$.*

Lemma 4.3. *There exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every ordinal $\alpha \geq f(n)$, $\mathbf{H}^{<\alpha} \equiv_n \mathbf{H}^{<f(n)}$.*

Proof. Since there are only finitely many n-types, and the ordinal sequence

$$\left\{\mathbf{type}_n\left[\mathbf{H}^{<\kappa}\right]\right\}_\kappa$$

is monotone, there must be some minimal $\kappa_0 \in \omega$ where the sequence stabilizes.

This κ_0 is computable as a function of n by exhaustive search. □

Lemma 4.4. *There exist global computable functions $a, b : \mathbb{N} \to \mathbb{N}$ such that for all $n, c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 \geq a(n)$ and $c_1 \equiv c_2 \mod b(n)$, $\mathbf{H}^{=c_1} \equiv_n \mathbf{H}^{=c_2}$.*

Proof. Let $n \in \mathbb{N}$.

Since there are only finitely many sets of n-types, there exist (and can be computed by exhaustive search) some $a(n) \geq f(n)$, $a(n) + b(n)$ such that

$$\mathbf{H}^{=a(n)} \equiv_n \mathbf{H}^{=a(n)+b(n)}$$

By induction if follows that for all $c \geq a(n)$,

$$\mathbf{H}^{=c} \equiv_n \mathbf{H}^{=c+b(n)}$$

since $\mathbf{H}^{=c+1} = \sum_\Omega \mathbf{H}^{=c}$. □

Note 4.5. *Actually we can give explicit formulas for some pair of $a(n)$ and $b(n)$ (not necessarily minimal), but there is no need to do so in this thesis.*

Lemma 4.6. *For every $n \in \mathbb{N}$ and for every ordinal $\alpha \geq \omega$,*

$$\mathbf{RM}_\alpha \equiv_n \sum_\omega \mathbf{H}^{[a(n),a(n)+b(n)]}$$

$$\mathbf{LM}_\alpha \equiv_n \sum_{\omega^*} \mathbf{H}^{[a(n),a(n)+b(n)]}$$

$$\mathbf{BM}_\alpha \equiv_n \sum_{\omega^*+\omega} \mathbf{H}^{[a(n),a(n)+b(n)]}$$

$$\mathbf{H}^{=\alpha} \equiv_n \mathbf{H}^{[a(n),a(n)+b(n)]}$$

In particular, $\mathbf{type}_n\,[\mathbf{RM}_{=\alpha}]$ *can be computed, and is independent of the choice of* $\alpha \geq \omega$*.*

Proof. We proceed by induction on $\alpha \geq \omega$.

For $\alpha = \omega$, let $\{\alpha_i\}_{i\in\omega}$ be an increasing ω-sequence of (finite) ordinals such that $a(n) \leq \alpha_i$ for all $i \in \omega$, and $\sup_{i\in\omega} \alpha_i = \omega$.

By lemma 4.4, we conclude for all $i \in \omega$,

$$\mathbf{H}^{[\alpha_i,\omega)} \equiv_n \mathbf{H}^{[a(n),a(n)+b(n))}$$

Then $\mathbf{RM}_\omega = \sum_\omega \mathbf{H}^{[\alpha_i,\omega)}$ by lemma 3.11, so $\mathbf{RM}_\omega \equiv_n \sum_\omega \mathbf{H}^{[a(n),a(n)+b(n))}$.

$$\mathbf{LM}_\omega \equiv_n \sum_{\omega^*} \mathbf{H}^{[a(n),a(n)+b(n))}$$

and

$$\mathbf{BM}_\omega \equiv_n \sum_{\omega^*+\omega} \mathbf{H}^{[a(n),a(n)+b(n))}$$

and $\mathbf{H}^{\leq\omega} \equiv_n \mathbf{H}^{<f(n)}$.

Thus, by corollary 3.10,

$$\mathbf{H}^{=\omega} \equiv_n \mathbf{H}^{<f(n)} + \left(\sum_{\{\omega,\omega^*,\omega^*+\omega\}} \mathbf{H}^{[a(n),a(n)+b(n))} \right) + \mathbf{H}^{<f(n)}$$

But also,

$$\mathbf{H}^{[a(n),a(n)+b(n))} \equiv_n \mathbf{H}^{<f(n)} + \left(\sum_{\{\omega,\omega^*,\omega^*+\omega\}} \mathbf{H}^{[a(n),a(n)+b(n))} \right) + \mathbf{H}^{<f(n)}$$

Thus $\mathbf{H}^{=\omega} \equiv_n \mathbf{H}^{[a(n),a(n)+b(n))}$ as required.

Now, for $\alpha > \omega$, we can take a set $\{\alpha_i\}_{i\in\omega}$ of ordinals such that $\omega \leq \alpha_i$ for all $i \in \omega$, and $\sup_{i\in\omega} (\alpha_i + 1) = \alpha$.

By the induction hypothesis,

$$\mathbf{H}^{[\alpha_i,\alpha)} \equiv_n \mathbf{H}^{[a(n),a(n)+b(n))}$$

Then $\mathbf{RM}_{=\alpha} = \sum_{i\in\omega} \mathbf{H}^{[\alpha_i,\alpha)}$, and thus

$$\mathbf{RM}_{=\alpha} \equiv_n \sum_{\omega} \mathbf{H}^{[a(n),a(n)+b(n))}$$

as required.

Again, the corollary for $\mathbf{LM}_{=\alpha}$ and $\mathbf{BM}_{=\alpha}$ is immediate.

Similarly, we have by corollary 3.10,

$$\mathbf{H}^{=\alpha} \equiv_n \mathbf{H}^{<f(n)} + \left(\sum_{\{\omega,\omega^*,\omega^*+\omega\}} \mathbf{H}^{[a(n),a(n)+b(n))} \right) + \mathbf{H}^{<f(n)}$$

and thus $\mathbf{H}^{=\alpha} \equiv_n \mathbf{H}^{=\omega}$, so we are done. $\square$

5 Decidability of the the MSO Theory

Definition 5.1. *Let $\mathbf{Q}$ be a class of linear orders.*

Let M be a linear order, and let $J \subseteq M$ be a subset of M.

We define the predicate $\mathrm{Int}_{\mathbf{Q}}(J)$ as true in M iff J is a subinterval of M and $J \in \mathbf{Q}$.

Lemma 5.2. *Let $\alpha > 0$ be an ordinal.*

Then predicates $\mathrm{Int}_{\mathbf{H}^{\leq\alpha}}$, $\mathrm{Int}_{\mathbf{H}^{=\alpha}}$ are expressible in $\mathbf{MSO}[\mathrm{Int}_{\mathbf{H}^{<\alpha}}]$.

Proof. Obviously,

$$\mathrm{Int}_{\mathbf{H}^{=\alpha}} \iff \mathrm{Int}_{\mathbf{H}^{\leq\alpha}} \wedge \neg\mathrm{Int}_{\mathbf{H}^{<\alpha}}$$

So it is enough to express $\mathrm{Int}_{\mathbf{H}^{\leq\alpha}}$.

Now, J is a $\mathbf{H}^{\leq\alpha}$-subinterval of M iff J is a subinterval of M and $J \in \sum_{\Omega} \mathbf{H}^{<\alpha}$.

But this can be expressed in **MSO** since it is expressible to check whether an arbitrary subset is in Ω. $\square$

Definition 5.3. *Let $\alpha > 0$ be an ordinal.*

Let M be a linear order and $x \in M$.

We define the convex equivalence relation:

$$\sim_{\alpha} := \sim_{\mathcal{B}[\mathbf{H}^{<\alpha}]}$$

and $[x]_{\alpha} := [x]_{\mathcal{B}[\mathbf{H}^{<\alpha}]}$.

That is, $[x]_{\alpha}$ is the largest $\mathcal{B}[\mathbf{H}^{<\alpha}]$-subinterval containing x in M.

We define the predicates $\mathbf{L}_{\alpha}(x)$ and $\mathbf{R}_{\alpha}(x)$ on M as follows:

$$x \in \mathbf{L}_{\alpha} \iff [x]_{\alpha} \in \mathcal{L}\left[\mathbf{H}^{<\alpha}\right]$$

$$x \in \mathbf{R}_{\alpha} \iff [x]_{\alpha} \in \mathcal{R}\left[\mathbf{H}^{<\alpha}\right]$$

We define the the α-shape, *$\sigma_{\alpha}(x)$ as follows:*

$$\sigma_{\alpha}(x) := \begin{cases} \mathbf{H}^{<\alpha} & \text{if } x \in \mathbf{L}_{\alpha} \cap \mathbf{R}_{\alpha} \\ \mathbf{RM}_{\alpha} & \text{if } x \in \mathbf{R}_{\alpha} \setminus \mathbf{L}_{\alpha} \\ \mathbf{LM}_{\alpha} & \text{if } x \in \mathbf{L}_{\alpha} \setminus \mathbf{R}_{\alpha} \\ \mathbf{BM}_{\alpha} & \text{if } x \notin \mathbf{L}_{\alpha} \cup \mathbf{R}_{\alpha} \end{cases}$$

Note that σ_{α} is a function from M to classes ("shapes").

Lemma 5.4. *Let M be a linear order and $\alpha > 0$ an ordinal.*

Let $J \subseteq M$ be a subinterval.

Then $J \in \mathbf{H}^{<\alpha}$ iff it is contained in a single $\sim_{\alpha}$-equivalence class K, such that:

- *Either $K \in \mathcal{L}[\mathbf{H}^{<\alpha}]$ or there exists some $x \in K$ such that $x < J$.*
- *Either $K \in \mathcal{R}[\mathbf{H}^{<\alpha}]$ or there exists some $x \in K$ such that $x > J$.*

Proof. Suppose $J \in \mathbf{H}^{<\alpha}$. Then obviously J is contained in a single $\sim_\alpha$-equivalence class K.

We will show the first condition, the second is symmetric.

Suppose that for all $x \in K$, $J \leq x$. Then we can write $K = J + J'$. Since $J \in \mathbf{H}^{<\alpha}$, it follows that $K \in \mathcal{L}[\mathbf{H}^{<\alpha}]$. □

Corollary 5.5. *Let $\alpha > 0$ be an ordinal.*

Let P_α be any predicate representing $\sim_\alpha$, let $L_\alpha = \mathbf{L}_\alpha$ and $R_\alpha = \mathbf{R}_\alpha$.

Then $\mathrm{Int}_{\mathbf{H}^{<\alpha}}$ is $\mathbf{MSO}$-expressible over $\mathbf{MSO}[P_\alpha, L_\alpha, R_\alpha]$.

Theorem 5.6. *Let $\mathbf{P}$ be a class of linear orders of some finite signature, including $C_1, \ldots, C_k$.*

Let $\mathcal{Q}$ be a finite *set of classes of linear orders over some finite signature which is disjoint from the signature of $\mathbf{P}$.*

Suppose that the $\mathbf{MSO}$-theories of $\mathbf{P}$ and each $\mathbf{Q} \in \mathcal{Q}$ are computable.

Let $F : 2^k \to \mathcal{Q}$ be any function.

Then the $\mathbf{MSO}$-theory of the class $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} F(C_1(i), \ldots, C_k(i))$ is computable.

Proof. We will use the decomposition theorem. Let φ be a formula of quantifier depth n. WLOG, φ is a sentence.

Then we can compute a formula $\psi(\xi)$ (where ξ has the type of a coloring whose range is the set of n-types) such that for any linear order $M = \sum_{i\in I} M_i$,

$$M \models \varphi \iff I \models \psi(\Xi)$$

where Ξ is the coloring assigning $i \in I$ the n-type of M_i.

Thus, there is some $M \in \bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$, such that $M \models \varphi$ iff there exists some $I \in \mathbf{P}$, and assignment Ξ of n-types, such that $\Xi(i)$ is satisfiable in $\mathbf{Q}_i$ for all $i \in I$, and $I \models \psi(\Xi)$.

Equivalently, φ is satisfiable over $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$ iff

$$\begin{aligned} &\exists\xi.\psi(\xi) \wedge \xi \text{ is a coloring with } n\text{-types} \\ &\qquad \wedge \forall i.\xi(i) \in \mathbf{type}_n\left[F(C_1(i), \ldots, C_k(i))\right] \end{aligned}$$

is satisfiable over $\mathbf{P}$.

The elements of $\mathcal{Q}$ have a computable $\mathbf{MSO}$-theory.

Thus, we can pre-compute $\mathbf{type}_n[F(\vec{c})]$ for any value $\vec{c} \in 2^k$ so we can actually write the formula above in $\mathbf{MSO}$. Furthermore, since $\mathbf{P}$ is computable, we can check whether it is satisfiable over $\mathbf{P}$. So we are done. □

Definition 5.7. *Let α be an ordinal.*

We define the $\mathbf{MSO}[P_\alpha, L_\alpha, R_\alpha]$ formula good_α *as follows:* good_α *is true in a linear order I iff for every pair $i, i' \in M$ such that i' is the successor of i, the following conditions hold:*

- $P_\alpha(i) \neq P_\alpha(i')$
- $R_\alpha(i) = 0$ *or* $L_\alpha(i') = 0$

We further define the class

$$\mathbf{Good}_\alpha := \{I \in \mathbf{CNT}[P_\alpha, L_\alpha, R_\alpha] : I \models \mathrm{good}_\alpha\}$$

as the class of all good_α *linear orders.*

Definition 5.8. *Let $\alpha > 0$ be an ordinal.*

Let $\mathbf{Q}$ be a class of linear orders.

We define the class of linear orders $\mathbf{Q}[\alpha]$ as the class of all linear orders in $\mathbf{Q}$, labeled with P_α, L_α and R_α such that P_α represents the equivalence relation $\sim_\alpha$, $L_\alpha = \mathbf{L}_\alpha$, and $R_\alpha = \mathbf{R}_\alpha$.

Lemma 5.9. *Let α be an ordinal.*

Then,

$$\mathbf{CNT}[\alpha] = \bigcup_{I \in \mathbf{Good}_\alpha} \sum_{i \in I} \sigma_\alpha(i)$$

Proof. ($\subseteq$) Let M be a countable linear order labeled with P_α, L_α and R_α as above.

Let $I = M/\sim_\alpha$ be the quotient of M by the equivalence relation $\sim_\alpha$.

Then $M = \sum_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ are the $\sim_\alpha$-equivalence class of I.

Then for each $i \in I$, $M_i \in \mathcal{B}[\mathbf{H}^{<\alpha}]$, and by definition $\sigma_\alpha(i) = M_i$.

Let i' be the successor of i in I.

Then $P_\alpha(i) \neq P_\alpha(i')$ since P_α represents $\sim_\alpha$.

Furthermore, suppose $R_\alpha(i) = L_\alpha(i') = 1$ holds. Then $M_i \in \mathcal{R}[\mathbf{H}^{<\alpha}]$ and $M_{i'} \in \mathcal{L}[\mathbf{H}^{<\alpha}]$. so M_i and $M_{i'}$ are the same $\sim_\alpha$-equivalence class of M, which is a contradiction.

Thus either $R_\alpha(i) = 0$ or $L_\alpha(i') = 0$.

($\supseteq$) Let $M = \sum_{i \in I} M_i$ be a linear order such that $I \in \mathbf{Good}_\alpha$ and $M_i \in \sigma_\alpha(i)$ for each $i \in I$.

In particular $M_i \in \mathcal{B}[\mathbf{H}^{<\alpha}]$ for each $i \in I$, so it is contained in a single $\sim_\alpha$-equivalence class of M.

Suppose that there exist distinct $j, k \in I$ such that $j < k$, and M_j, M_k are in the same $\sim_\alpha$-equivalence class.

Let $x \in M_j$ and $y \in M_k$. Then $[x, y] \in \mathbf{H}^{<\alpha}$, and thus $[j, k] \in \mathbf{H}^{<\alpha}$, and in particular it is sparse.

Then there exist some $j', k' \in I$ such that $j < j' < k' < k$, and k' is the successor of j' in I.

Then $M_{j'}$ and $M_{k'}$ are in the same $\sim_\alpha$-equivalence class. Thus it must be the case that $M_{j'} \in \mathcal{R}[\mathbf{H}^{<\alpha}]$ and $M_{k'} \in \mathcal{L}[\mathbf{H}^{<\alpha}]$, which implies $R_\alpha(j') = L_\alpha(k') = 1$, which is a contradiction.

Thus $\{M_i\}_{i \in I}$ are pairwise distinct $\sim_\alpha$-equivalence classes, and obviously the conditions holds, so $M \in C$ and we are done. ∎

Lemma 5.10. *Let $\alpha > 0$ be an ordinal.*

Then the **MSO**-*theory of* $\mathbf{CNT}[\alpha]$ *is computable.*

Proof. Since $\mathbf{Good}_\alpha$ is clearly computable, it follows from combining theorem 5.6 with lemma 5.9 and the computability of $\mathbf{H}^{<\alpha}$, $\mathbf{RM}_\alpha$, $\mathbf{LM}_\alpha$ and $\mathbf{BM}_\alpha$. $\square$

Theorem 5.11. *Let $\alpha > 0$ be an ordinal.*
Satisfiability of $\mathbf{MSO}[\mathrm{Int}_{\mathbf{H}^{<\alpha}}]$ over $\mathbf{CNT}$ is decidable.

Proof. First, by corollary 5.5, we can convert any formula φ in $\mathbf{MSO}[\mathrm{Int}_{\mathbf{H}^{<\alpha}}]$ to a formula φ' in $\mathbf{MSO}\left[P_\alpha, L_\alpha, R_\alpha\right]$ such that φ is satisfiable over $\mathbf{CNT}$ iff φ' is satisfiable over $\mathbf{CNT}\left[\alpha\right]$.

This is decidable by lemma 5.10. $\square$

6 Multiple Ordinals

We can extend the results from the previous section to multiple ordinals.

Lemma 6.1. *Let $\alpha > 0$ be an ordinal and let $\delta \geq \omega$ be a limit ordinal.*
Then,

$$\mathbf{H}^{<\alpha+\delta}[\alpha] = \bigcup_{I \in \mathbf{Good}_\alpha \wedge \mathbf{H}^{<\delta}} \sum_{i \in I} \sigma_\alpha(i)$$

Proof. ($\subseteq$) Let $M \in \mathbf{H}^{<\alpha+\delta}[\alpha]$. By definition, M is a linear order labeled with $P_\alpha, L_\alpha, R_\alpha$ such that the underlying order is in $\mathbf{H}^{<\alpha+\delta}$, P_α represents $\sim_\alpha$, and L_α, R_α are as defined.

Let $I = M/\sim_\alpha$ be the quotient of M by the equivalence relation $\sim_\alpha$. Then $M = \sum_{i \in I} M_i$, where M_i are the $\sim_\alpha$-equivalence classes. By the definition of $\sim_\alpha$, each $M_i \in \mathcal{B}[\mathbf{H}^{<\alpha}]$, and by the definition of $\sigma_\alpha(i)$, $M_i \in \sigma_\alpha(i)$.

Since $M \in \mathbf{H}^{<\alpha+\delta}$, the quotient I is in $\mathbf{H}^{<\delta}$. For each pair i, i' of consecutive elements in I, the labeling ensures that $P_\alpha(i) \neq P_\alpha(i')$ and either $R_\alpha(i) = 0$ or $L_\alpha(i') = 0$, so $I \in \mathbf{Good}_\alpha$. Thus, $M \in \sum_{i \in I} \sigma_\alpha(i)$ for some $I \in \mathbf{Good}_\alpha \wedge \mathbf{H}^{<\delta}$.

($\supseteq$) Let $M = \sum_{i \in I} M_i$ where $I \in \mathbf{Good}_\alpha \wedge \mathbf{H}^{<\delta}$ and $M_i \in \sigma_\alpha(i)$ for each $i \in I$. The labeling $P_\alpha, L_\alpha, R_\alpha$ on M is as required by the definition of $\mathbf{Good}_\alpha$, and each $M_i \in \mathcal{B}[\mathbf{H}^{<\alpha}]$. Since $I \in \mathbf{H}^{<\delta}$, $M \in \mathbf{H}^{<\alpha+\delta}$. Thus, $M \in \mathbf{H}^{<\alpha+\delta}[\alpha]$.

Therefore,

$$\mathbf{H}^{<\alpha+\delta}[\alpha] = \bigcup_{I \in \mathbf{Good}_\alpha \wedge \mathbf{H}^{<\delta}} \sum_{i \in I} \sigma_\alpha(i)$$

□

Corollary 6.2. *Let $\alpha > 0$ be an ordinal and let $\delta \geq \omega$ be a limit ordinal.*
Then,

$$\mathbf{RM}_{\alpha+\delta}[\alpha] = \bigcup_{I \in \mathbf{Good}_\alpha \wedge \mathbf{RM}_\delta} \sum_{i \in I} \sigma_\alpha(i)$$

Proof. It follows from lemma 6.1 together with lemma 3.11. □

The first lemma we shall use is the "extension lemma".

Lemma 6.3. *Let α be an ordinal.*

Let I be a linear order and let $\{M_i\}_{i \in I}$ be a family of linear orders, such that for each pair $i, i' \in I$ such that i' is the successor of i in I, either $\mathbf{R}_\alpha(M_i) = 0$ or $\mathbf{L}_\alpha(M_{i'}) = 0$.

Then,

$$\left(\sum_{i \in I} M_i\right)[\alpha] = \sum_{i \in I} (M_i[\alpha])$$

Proof. It is obvious, but TBC. □

Notation 6.4. *Let $\alpha_1 < \ldots < \alpha_k$ be ordinals.*

Let $\mathbf{P}$ be a class of linear orders.

Then,

$$\mathbf{P}[\alpha_1, \ldots, \alpha_k] := \mathbf{P}[\alpha_1] \cdots [\alpha_k]$$

Corollary 6.5. *Let $\alpha_1 < \ldots < \alpha_k < \alpha$ be ordinals.*

Let I be a linear order and let $\{M_i\}_{i \in I}$ be a family of linear orders, such that for each pair $i, i' \in I$ such that i' is the successor of i in I, either $\mathbf{R}_\alpha(M_i) = 0$ or $\mathbf{L}_\alpha(M_{i'}) = 0$.

Then,

$$\left(\sum_{i \in I} M_i\right)[\alpha_1, \ldots, \alpha_k, \alpha] = \sum_{i \in I} (M_i[\alpha_1, \ldots, \alpha_k, \alpha])$$

Proof. If $\mathbf{R}_\alpha(i) = 0$, then in particular $\mathbf{R}_{\alpha_j}(i) = 0$ for all $j \in [k]$, and similarly for $\mathbf{L}_\alpha(i')$.

So the condition for α implies the similar conditions for $\alpha_1, \ldots, \alpha_k$.

Now, we can apply lemma 6.3 inductively to obtain the result. □

Lemma 6.6. *Let $\alpha_1 < \ldots < \alpha_k < \alpha$ be ordinals. Then,*

$$\mathbf{CNT}[\alpha_1, \ldots, \alpha_k, \alpha] = \bigcup_{I \in \mathbf{Good}_\alpha} \sum_{i \in I} \sigma_\alpha(i)[\alpha_1, \ldots, \alpha_k]$$

Proof. This is a consequence of lemma 5.9 and corollary 6.5. □

Lemma 6.7. *Let $\alpha_1 < \ldots < \alpha_k < \alpha$ and $\delta > 1$ be ordinals. Then,*

$$\mathbf{H}^{<\alpha+\delta}[\alpha_1, \ldots, \alpha_k, \alpha] = \bigcup_{I \in \mathbf{Good}_\alpha \wedge \mathbf{H}^{<\delta}} \sum_{i \in I} \sigma_\alpha(i)[\alpha_1, \ldots, \alpha_k]$$

Proof. This is a consequence of lemma 6.1 and corollary 6.5. □

Lemma 6.8. *Let $\alpha_1 < \ldots < \alpha_k < \alpha$ and δ be ordinals. Then,*

$$\mathbf{RM}_{\alpha+\delta}[\alpha_1, \ldots, \alpha_k, \alpha] = \bigcup_{I \in \mathbf{Good}_\alpha \wedge \mathbf{RM}_\delta} \sum_{i \in I} \sigma_\alpha(i)[\alpha_1, \ldots, \alpha_k]$$

Proof. This is a consequence of corollary 6.2 and corollary 6.5. □

Lemma 6.9. *Let $\alpha_1 < \ldots < \alpha_k < \alpha$ be ordinals.*

Then the **MSO***-theory of* $\mathbf{CNT}[\alpha_1, \ldots, \alpha_k, \alpha]$ *is decidable.*

Proof. Since $\mathbf{Good}_\alpha$ is computable, it follows from combining theorem 5.6 with lemma 6.6 and the computability of $\mathbf{H}^{<\alpha}[\alpha_1, \ldots, \alpha_k]$, $\mathbf{RM}_\alpha[\alpha_1, \ldots, \alpha_k]$, $\mathbf{LM}_\alpha[\alpha_1, \ldots, \alpha_k]$ and $\mathbf{BM}_\alpha[\alpha_1, \ldots, \alpha_k]$. □

Theorem 6.10. *Let $\alpha_1 < \ldots < \alpha_k$ be ordinals.*

Satisfiability of $\mathbf{MSO}[\mathrm{Int}_{\mathbf{H}^{<\alpha_1}}, \ldots, \mathrm{Int}_{\mathbf{H}^{<\alpha_k}}]$ *over* $\mathbf{CNT}$ *is decidable.*

Proof. First, by corollary 5.5, we can convert any formula φ in

$$\mathbf{MSO}[\mathrm{Int}_{\mathbf{H}^{<\alpha_1}}, \ldots, \mathrm{Int}_{\mathbf{H}^{<\alpha_k}}]$$

to a formula φ' in

$$\mathbf{MSO}\left[P_{\alpha_1}, L_{\alpha_1}, R_{\alpha_1}, \ldots, P_{\alpha_k}, L_{\alpha_k}, R_{\alpha_k}\right]$$

such that φ is satisfiable over **CNT** iff φ' is satisfiable over $\mathbf{CNT}\left[\alpha_1, \ldots, \alpha_k\right]$.

This is decidable by lemma 6.9 and lemma 6.6. ☐

7 Shuffle

Lemma 7.1. *Let $M_1, \ldots, M_k$ be linear orders.*
Then,

$$\mathbf{shuffle}\left(M_1, \ldots, M_k\right) = \mathbf{shuffle}\left(\mathbf{shuffle}\left(M_1, \ldots, M_{k-1}\right), M_k\right)$$

8 Decidability of Definable Intervals

Definition 8.1. *Let* **DFN** *be the class of all linear orders defined by an* **MSO**-*formula*

Definition 8.2. *Let κ be a cardinal.*

Let φ be an **MSO**-*formula. We say that φ is* $\leq \kappa$-ambiguous *if it has at most κ many models.*

Definition 8.3. *Let κ be a cardinal.*

Let M be a linear order. We say that M is $\leq \kappa$-ambiguous *if it is a model of some $\leq \kappa$-ambiguous* **MSO**-*formula.*

Lemma 8.4. *Let I be ω labeled over $P_1, \ldots, P_k$, and suppose I is $\leq \kappa$-ambiguous for some cardinal κ. Let F be a function assigning each truth vector in 2^k a definable linear order.*

Then $M = \sum_{i \in I} F(P_1(i), \ldots, P_k(i))$ is $\leq \kappa$-ambiguous.

Proof. It is immediate. $\square$

Lemma 8.5. *Let M_1 be a $\leq \kappa_1$-ambiguous linear order for some cardinal κ_1, and let M_2 be a $\leq \kappa_2$-ambiguous linear order for some cardinal κ_2.*

Then $M_1 + M_2$ is $\leq \kappa_1\kappa_2$-ambiguous.

Proof. It is immediate. $\square$

Lemma 8.6. *Then φ be an $\leq \aleph_0$-ambiguous* **MSO**-*formula. Then every scattered model of φ is in $\mathbf{H}^{<\omega}$.*

Proof. Let $n = \mathbf{qd}(\varphi)$.

Let M be a scattered model of φ.

If $M \in \mathbf{H}^{\geq\omega}$, then for every ordinal $\omega \leq \alpha < \omega_1$, there is some $N \in \mathbf{H}^{=\alpha}$ such that $N \equiv_n M$.

This means that φ has at least one model of every rank between ω and ω_1, which in particular means uncountably many models. $\square$

Lemma 8.7. *Let φ be an* **MSO**-*formula.*

Then φ is $\leq \aleph_0$-ambiguous iff it has only definable models.

Proof. ($\Longleftarrow$) Since there are only countably many definable models, if φ has only definable models, then it has at most countably many models.

($\Longrightarrow$) Suppose φ is $\leq \aleph_0$-ambiguous. By the previous lemma, every scattered model has finite rank, so we will prove first, by ω-induction on the rank, that every scattered model of φ is definable.

Let M be a scattered model of φ. If $\mathbf{hrank}(M) = 0$, then $M = 1$, so it is in particular definable.

Otherwise, let $\mathbf{hrank}(M) = \alpha$, and let $I = M/\sim_\alpha$, and $\{M_i\}_{i \in I}$ be the equivalence classes of M under $\sim_\alpha$.

Also, $\mathbf{hrank}(I) = \alpha - 1$.

By the composition theorem, there is a formula $\psi(\vec{Y})$ such that $M \models \varphi$ iff $I \models \psi(\vec{Q})$, where $Q_\tau = \{i \in I : M_i \models \tau\}$ for every n-type τ.

$\square$