Orders

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1 Order Theory

Definition 1. *Let M be a linear order.*

A set $A \subseteq M$ is left cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $y \leq x$.*

A set $A \subseteq M$ is right cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $x \leq y$.*

A set $A \subseteq M$ is bi-directionally cofinal *in M if it is both left and right cofinal.*

2 Properties

Definition 2. *A* property $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 3. *A property $\mathbf{P}$ of linear orders is* monotone *if for every linear order M, $M \in \mathbf{P}$ implies that every suborder of M is in $\mathbf{P}$.*

Definition 4. *A property $\mathbf{P}$ of linear orders is* symmetric *if for every linear order M, $M \in \mathbf{P}$ iff $M^R \in \mathbf{P}$.*

Definition 5. *A property $\mathbf{P}$ of linear orders is an* additive property *if for every linear orders M_1 and M_2, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

Definition 6. *Let $\mathbf{P}$ be a property of linear orders.*

We define **$\mathbf{P}$-bounded** *to be the class of linear orders M such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in $\mathbf{P}$.*

Definition 7. *A property $\mathbf{P}$ of linear orders is* almost anti-symmetric *if for every linear order M, $M \in \mathbf{P}$ and $M^R \in \mathbf{P}$ imply that M is finite.*

Definition 8. *A property $\mathbf{P}$ of linear orders is* good *if it is monotone, additive and contains at least one infinite linear order.*

Definition 9. *A property $\mathbf{P}$ of linear orders is a* star property *if for every linear orders M, and every family $\mathcal{F} \subseteq \mathbf{P}$ of subintervals of M such that $J_1 \cap J_2 \neq \emptyset$ for every $J_1, J_2 \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \in \mathbf{P}$.*

Lemma 1 (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then the property $\mathbf{P}$**-bounded** *has the star property.*

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq$ **P-bounded** be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by **P-bounded**, $[x, z], [z, y] \in \mathbf{P}$. However, $\mathbf{P}$ is additive. Since $[x, y]$ is either the sum or difference of $[x, z]$ and $[z, y]$, we have that $[x, y] \in \mathbf{P}$. □

Lemma 2. *Let* $\mathbf{P}$ *be a star property.*

Then for every linear order M*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$*.*

Thus, we can define an equivalence relation $\sim_{\mathbf{P}}$ *on* M *such that* $x \sim_{\mathbf{P}} y$ *iff* x *and* y *are in the same largest* $\mathbf{P}$*-subinterval.*

Proof. Let $J \subseteq M$ be the union of all **P-bounded**-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in **P-bounded**, and by definition J is the largest $\mathbf{P}$-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. □

3 Hausdorff Rank

Definition 10. *Let* $\mathbf{Q}$ *be a good property of linear orders.*

We define a property $\mathbf{Q}^{\leq \alpha}$ *for every ordinal* α *as follows:*

- $\mathbf{Q}^{\leq 0}$ *is the class of finite linear orders.*
- *For* $\alpha > 0$, $\mathbf{Q}^{\leq \alpha}$ *is the class of linear orders* M *such that* $M = \sum_{i \in I} M_i$ *for some* $I \in \mathbf{Q}$ *where for all* $i \in I$, $M_i \in \mathbf{Q}^{\beta_i}$ *for some* $\beta_i < \alpha$

We define further $\mathbf{Q}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{Q}^{\leq \beta}$ *and* $\mathbf{Q}^{=\alpha} = \mathbf{Q}^{\leq \alpha} - \mathbf{Q}^{<\alpha}$*.*

We define $\mathbf{hrank_Q}(M) = \alpha$ *iff* $M \in \mathbf{Q}^{=\alpha}$*. This is a* partial map *from linear orders to ordinals.*

Observations 1. *Let* $\mathbf{Q}$ *be a good property.*

We claim the following without proof:

- $\mathbf{Q}^{\leq 1} = \mathbf{Q}$.
- *For all* α, $\mathbf{Q}^{\leq \alpha}$ *is a good property.*
- $\mathbf{Q}^{\leq \alpha} \subsetneq \mathbf{Q}^{\leq \beta}$ *iff* $\alpha < \beta$*.*

Definitions 1. $\mathcal{B}_{<\alpha} := \mathbf{Q}^{<\alpha}$**-bounded** *is the class of linear orders of rank $< \alpha$ on bounded intervals.*

$\mathcal{L}_{<\alpha}$ is the class of linear orders M where $1 + M \in \mathcal{B}_{<\alpha}$. Equivalently, M is of rank $< \alpha$ on right-bounded rays.

$\mathcal{R}_{<\alpha}$ is the class of linear orders M where $M + 1 \in \mathcal{B}_{<\alpha}$. Equivalently, M is of rank $< \alpha$ on left-bounded rays.

Lemma 3. *The following are equal:*

1. $\mathbf{Q}^{<\alpha}$
2. $\{M : 1 + M + 1 \in \mathcal{B}_{<\alpha}\}$.
3. $\mathcal{L}_{<\alpha} \cap \mathcal{R}_{<\alpha}$

Proof. The only nontrivial direction is 3 implies 2, which follows from the star property of $\mathcal{B}_{<\alpha}$ (Or alternatively, it can be seen clearly from the ray-based formulation). □

Lemma 4. *A countable linear order which has rank $< \alpha$ on bounded subintervals is of rank $\leq \alpha$. That is,*

$$\mathcal{B}_{<\alpha} \subseteq \mathbf{Q}^{\leq\alpha}$$

Proof. Let M be a countable linear order of rank $< \alpha$.

Then $M = \sum_{i \in I} M_i$ where $M_i \in \mathbf{Q}^{<\alpha}$.

Let $\{x_i\}_{i \in I} \subseteq M$ be a bidirectional, cofinal, weakly monotone I-sequence in M, i.e, $x_i \leq x_j$ if $i \leq j$ for $I \subseteq \mathbb{Z}$.

Write $M = \sum_{i \in I} [x_i, x_{i+1}]$. Then every $[x_i, x_{i+1}]$ is of Hausdorff rank $< \alpha$.

Thus, $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$, which completes the proof. □

Lemma 5. *Let M be a countable linear order.*

Suppose $\mathbf{Q} = \{M : \exists n \in \mathbb{N}. M \subseteq \mathbb{Z} \cdot n\}$ (This does not necessarily work for other $\mathbf{Q}$!).

Then $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$ iff M is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals.

Proof. From the previous lemma, it is clear that if M is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals, then $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$, since the rank bound is preserved under finite sums.

Conversely, suppose $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$.

If $M = \sum_{i \in \mathbb{Z}} M_i$ for some M_i of Hausdorff rank $< \alpha$, take $x, y \in M$. Then let $x \in M_{i_1}$ and $y \in M_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} M_i$. But the last sum is of rank $< \alpha$ and thus $[x, y]$ is of rank $< \alpha$. That is, $M \in \mathcal{B}_{<\alpha}$.

Since every subinterval of rank $\leq \alpha$ is a finite sum of $\mathbb{Z}$-sums of intervals of rank $< \alpha$, we are done. □

Lemma 6. *Let $M = \sum_{i \in I} M_i$ where $I \in \mathbf{Q}$ and $\mathbf{hrank}_{\mathbf{Q}}(M_i) < \alpha$ for all $i \in I$.*

Then $\mathbf{hrank}_{\mathbf{Q}}(M) = \alpha$ iff either:

- α *is a successor ordinal, and for all* $i \in I$, $\mathbf{hrank_Q}(M_i) = \alpha - 1$.

Definition 11 (Compatibility)**.** *Let* $I \in \mathbf{Q}$. *Let* M *and* N *be countable linear orders. Let* $\alpha = \mathbf{hrank_Q}(M)$ *and* $\beta = \mathbf{hrank_Q}(N)$.

Suppose $M = \sum_{i \in I} M_i$ *and* $N = \sum_{i \in I} N_i$ *where* $\mathbf{hrank_Q}(M_i) < \alpha$ *and* $\mathbf{hrank_Q}(N_i) < \beta$ *for all* $i \in I$, *and* $M_i \equiv_n N_i$ *for all* $i \in I$.

Then the decompositions $M = \sum_{i \in I} M_i$ *and* $N = \sum_{i \in I} N_i$ *are* compatible *if one of the following holds:*

- α *and* β *are both successor ordinals, and for all* $i \in I$, $\mathbf{hrank_Q}(M_i) = \alpha - 1$ *if and only if* $\mathbf{hrank_Q}(N_i) = \beta - 1$.

- α *and* β *are both limit ordinals, and for every subset* $J \subseteq I$, *the subset* $\mathbf{hrank_Q}(J)$ *is cofinal in* α *iff the subset* $\mathbf{hrank_Q}(J)$ *is cofinal in* β.

M *and* N *are* compatible *if for some* $I \in \mathbf{Q}$, *there exists such a pair of compatible decompositions of* M *and* N.

Definition 12. *Let* $\mathbf{Q}$ *be a good property.*

We say that $\mathbf{Q}$ *is* compatibility preserving *if for every* $I \in \mathbf{Q}$ *and every pair of compatible decompositions* $M = \sum_{i \in I} M_i$ *and* $N = \sum_{i \in I} N_i$, *where* $\mathbf{hrank_Q}(M) = \alpha$ *and* $\mathbf{hrank_Q}(N) = \beta$, *and* $\mathbf{hrank_Q}(M_i) < \alpha$ *and* $\mathbf{hrank_Q}(N_i) < \beta$ *for all* $i \in I$,

Lemma 7. *Suppose* $M = \sum_{i \in I} M_i$ *and* $N = \sum_{i \in I} N_i$ *are a pair of compatible decompositions.*

Let $J \subseteq I$ *be a subinterval.*

Let $M_J = \sum_{i \in J} M_i$ *and* $N_J = \sum_{i \in J} N_i$. *Then:*

1. $\mathbf{hrank_Q}(M_J) = \alpha$ *iff* $\mathbf{hrank_Q}(N_J) = \beta$.

2. *If* $\mathbf{hrank_Q}(M_J) = \alpha$ *and* $\mathbf{hrank_Q}(N_J) = \beta$, *then the decompositions* $M_J = \sum_{i \in J} M_i$ *and* $N_J = \sum_{i \in J} N_i$ *are compatible.*

Proof. First let us proof 1. Suppose $\mathbf{hrank_Q}(M_J) = \alpha$. □

4 Decidability of the rank

Lemma 8. *There exists a global computable function* $f : \mathbb{N} \to \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, $\mathbf{type}_n\left[\mathcal{H}_{f(n)+1}\right] = \mathbf{type}_n\left[\mathcal{H}_{f(n)}\right]$.

Equivalently, every linear order of finite rank is n*-equivalent to some linear order of rank* $\leq f(n)$.

Proof. Since there exist only a finite number of n-types, and the ω-sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$ is monotone, it must stabilize at some point.

This point is computable as a function of n, because $\mathbf{type}_n[\mathcal{H}_k]$ is computable for every finite k. □

Lemma 9. *There exist global computable functions $a, b : \mathbb{N} \to \mathbb{N}$ such that for all $n, c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 \geq a(n)$ and $c_1 \equiv c_2 \mod b(n)$,*

$$\mathbf{type}_n [\mathcal{H}_{c_1}] = \mathbf{type}_n [\mathcal{H}_{c_2}]$$

Equivalently, the sequence $\{\mathbf{type}_n [\mathcal{H}_k]\}_{k\in\omega}$ is ultimately periodic for all $n \in \mathbb{N}$. Furthermore, the starting point and the period itself can be computed as a function of n.

Proof. Let $n \in \mathbb{N}$.

Since there exist only a finite number of possible sets of n-types, there exist (and can be computed) some $a(n) > f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n \left[\mathcal{H}_{a(n)}\right] = \mathbf{type}_n \left[\mathcal{H}_{a(n)+b(n)}\right]$$

We shall prove by induction that for all $c \geq a(n) + b(n)$,

$$\mathbf{type}_n [\mathcal{H}_c] = \mathbf{type}_n \left[\mathcal{H}_{c+b(n)}\right]$$

This will complete the proof.

The base case $c = a(n)$ has been proven in the beginning.

Suppose the induction hypothesis holds for c.

Let M be of rank $c + 1$.

Write $M = \sum_{i\in I} M_i$ where $\mathbf{hrank}_{\mathbf{Q}} (M_i) < c + 1$, and $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c$ infinitely many times.

By the induction hypothesis, if $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c$, we can find $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}} (N_i) = c + b(n)$. Setting $N_i := M_i$ for all other i, we conclude that $N := \sum_{i\in I} N_i$ is n-equivalent to M.

However, clearly $\mathbf{hrank}_{\mathbf{Q}} (N) = c + b(n) + 1$. So overall,

$$\mathbf{type}_n [\mathcal{H}_{c+1}] \subseteq \mathbf{type}_n \left[\mathcal{H}_{c+b(n)+1}\right]$$

Conversely, suppose M is of rank $c + b(n) + 1$. Write $M = \sum_{i\in I} M_i$ where $\mathbf{hrank}_{\mathbf{Q}} (M_i) < c+b(n)+1$, and $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c+b(n)$ infinitely many times.

By the induction hypothesis, we can find for all i such that $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c + b(n)$ some $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}} (N_i) = c$. Furthermore, since $c \geq a(n) > f(n)$, we can find $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}} (N_i) \leq f(n) < c$ for all other i.

We conclude that $N := \sum_{i\in I} N_i$ is n-equivalent to M. However, clearly $\mathbf{hrank}_{\mathbf{Q}} (N) = c + 1$. So overall,

$$\mathbf{type}_n \left[\mathcal{H}_{c+b(n)+1}\right] \subseteq \mathbf{type}_n [\mathcal{H}_{c+1}]$$

So we have proven the induction step, and the lemma follows. □

Lemma 10. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Then,

$$\mathbf{type}_n [\mathbf{Q}^{=\alpha}] = \bigcup_{c<b(n)} \mathbf{type}_n \left[\mathcal{H}_{c+b(n)}\right]$$

In particular, $\mathbf{type}_n [\mathbf{Q}^{=\alpha}]$ can be computed, and is independent of the choice $\alpha \geq \omega$.

Proof. TBC. ☐

Lemma 11. *Let $\alpha, \beta \geq \omega$ be ordinals.*
Let M, N be countable linear orders.
Suppose $\mathbf{hrank}_{\mathbf{Q}}(M) \in \mathcal{B}_{<\alpha}$ and $\mathbf{hrank}_{\mathbf{Q}}(N) \in \mathcal{B}_{<\beta}$.
Let $M = \sum_{i\in I} M_i$ and $N = \sum_{i\in I} N_i$ be decompositions where $I \in \mathbf{Q}$ and for all $i \in I$, $\mathbf{hrank}_{\mathbf{Q}}(M_i) < \alpha$ and $\mathbf{hrank}_{\mathbf{Q}}(N_i) < \beta$.
Then, $M \in \mathcal{L}_{<\alpha}$ iff $N \in \mathcal{L}_{<\beta}$, and $M \in \mathcal{R}_{<\alpha}$ iff $N \in \mathcal{R}_{<\beta}$.

Proof. We will prove for the case of $\mathcal{L}$, the other case is dual.
We will show the first direction, as the other direction is symmetric.
Suppose $M \in \mathcal{L}_{<\alpha}$. That is, for every $x \in M$, the ray $\{\leq x\}$ is of rank $< \alpha$.
Let $y \in N$. Suppose $y \in M_j$. Let us look at $N' = \sum_{i\leq j} N_i$.
Suppose for the contrary that it is of rank $< \beta$.
TBC. ☐

Corollary 1. *The following sequences stabilize at $f(n)$:*

- $\mathbf{type}_n[\mathcal{H}_\alpha]$
- $\mathbf{type}_n[\mathcal{B}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{L}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{R}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})]$

Proof. The corollary is false and should be fixed. ☐

Definition 13. *Let $\alpha \geq \omega$ be an ordinal.*
Let M be a linear order and $x \in M$.
We define the convex equivalence relation $\sim_\alpha := \sim_{\mathcal{B}_{<\alpha}}$, and $[x]_\alpha := [x]_{\mathcal{B}_{<\alpha}}$ (that is, $[x]_\alpha$ is the largest $\mathcal{B}_{<\alpha}$-subinterval containing x in M).

Lemma 12. *Let M be a linear order. Let $P, L, R \subseteq M$ be relations, such that:*

- *P represents $\sim_\alpha$ on M.*
- *L is such that $x \in L$ iff $[x]_\alpha \in \mathcal{L}_{<\alpha}$.*
- *R is such that $x \in R$ iff $[x]_\alpha \in \mathcal{R}_{<\alpha}$.*

Then for some linear order I there exists a decomposition $M = \sum_{i\in I} M_i$ such that $M_i \in \mathcal{B}_{<\alpha}$ for all $i \in I$, M_i is monochromatic with respect to P, L and R.
Furthermore, the following hold:

- *if i has a successor, $M_i \subseteq P$ iff $M_{i+1} \subseteq P$.*
- *if i has a successor, either $M_i \not\subseteq R$ or $M_{i+1} \not\subseteq L$.*

Proof. Take $I = M/\sim_\alpha$.

Then $M = \sum_{i \in I} M_i$ where M_i is the $\sim_\alpha$-equivalence class of i.

Then M_i is monochromatic with respect to P, L and R.

The only thing left to prove is the last two conditions. The first follows from the fact that P represents $\sim_\alpha$.

The second follows because if it were not the case, then M_i and M_{i+1} would be the same $\sim_\alpha$-equivalence class. $\square$

Lemma 13. *Let I be a linear order. Let $n \in \mathbb{N}$.*

Let τ_i be an assignment of satisfiable n-types in $\mathbf{MSO}[p, \ell, r]$ *for all $i \in I$. Assume that*

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $\ell(\tau_{i+1}) = 0$*

Then there exists a linear order M and $P, L, R \subseteq M$ such that:

- *P represents $\sim_\alpha$ on M.*
- *L is such that $x \in L$ iff $[x]_\alpha \in \mathcal{L}_{<\alpha}$.*
- *R is such that $x \in R$ iff $[x]_\alpha \in \mathcal{R}_{<\alpha}$.*

such that for all $i \in I$, M_i is a $\sim_\alpha$-equivalence class of M, and is thus monochromatic with respect to P, L and R.

Furthermore, the n-type of M_i, p_i, q_i, r_i in $\mathbf{MSO}[p, \ell, r]$ *is τ_i, where $p_i = 1_{M_i \subseteq P}$, $q_i = 1_{M_i \subseteq L}$ and $r_i = 1_{M_i \subseteq R}$,*

Proof. Since τ_i is satisfiable, we can take M_i to be a linear order of n-type τ_i such that:

- If $\ell(\tau_i) = r(\tau_i) = 1$, then $M_i \in \mathcal{L}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.
- If $\ell(\tau_i) = 1$ and $r(\tau_i) = 0$, then $M_i \in \mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}$.
- If $\ell(\tau_i) = 0$ and $r(\tau_i) = 1$, then $M_i \in \mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}$.
- If $\ell(\tau_i) = r(\tau_i) = 0$, then $M_i \in \mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})$.

Let $M = \sum_{i \in I} M_i$.

By definition each M_i is in $\mathcal{B}_{<\alpha}$. We need to prove that each M_i is a largest $\mathcal{B}_{<\alpha}$-subinterval in M.

On the contrary, suppose that there exist $i' \neq i$ such that $[M_i, M_{i'}] \in \mathcal{B}_{<\alpha}$. WLOG, $M_i < M_{i'}$.

Since I is scattered, take some $i \leq a < b \leq i'$ such that there is no element between a and b in I.

Then $M_a \in \mathcal{R}_{<\alpha}$ and $M_b \in \mathcal{L}_{<\alpha}$, in contradiction. $\square$

Lemma 14. *Over countable linear orders with interpretations of P, L and R as above, the property $\mathbf{Q}^{<\alpha}$ is expressible in $\mathbf{MSO}[P, L, R]$.*

Proof. From lemma 3 we can express $\mathbf{Q}^{<\alpha}$ as the intersection of L and R. $\square$

Theorem 1. *There is a an algorithm solving satisfiability for $\mathbf{MSO}[P, L, R]$ over countable linear orders, given an oracle which solves the satisfiability problem for $\mathbf{MSO}$ over countable linear orders.*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[P, L, R]$ formula φ of quantifier-depth n. outputs an $\mathbf{MSO}[\{X_\tau\}_\tau]$ formula ψ.

Let P_L, Q_L, R_M be the interpretations of P, L, R on M.

Then

$$M, P := P_L, L := Q_L, R := R_L \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where $I_\tau = \{i \in I : M_i \models \tau\}$ for every n-type τ.

Let T be the set of n-types in $\mathbf{MSO}[p, \ell, r]$ which satisfy $\ell(\tau) = 1 \iff \tau \in \mathcal{L}_{<\alpha}$ and $r(\tau) = 1 \iff \tau \in \mathcal{R}_{<\alpha}$.

Let $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee \ell(\tau_2) = 0)\}$.

Then T and S can be calculated using the oracle.

Then ψ is an $\mathbf{MSO}[T, S]$ formula.

Then we define an $\mathbf{MSO}[p, \ell, r]$ formula ψ' as follows:

ψ' claims that there exists a partition (with possible empty sets) $\{Y_\tau\}_\tau$ of I such that

- Every $i \in I$ is in some Y_τ for $\tau \in T$.
- If $i' = i + 1$ in I, then for some $(\tau_1, \tau_2) \in S$, $i \in Y_{\tau_1}$ and $i' \in Y_{\tau_2}$.

Now we claim that φ is satisfiable in some linear order, iff ψ' is satisfiable in some linear order.

Suppose φ is satisfiable in some linear order M.

Take a decomposition $M = \sum_{i \in I} M_i$ as in lemma 2.

Then ψ holds over the assignment $X_\tau := I_\tau$. But by lemma 2, this assignment satisfies the condition required for ψ' to hold. Then ψ' holds over I.

Conversely, suppose ψ' holds in I.

Let $X_\tau := Z_\tau$ be the assignment that is guaranteed by ψ'.

Let tau_i be the unique τ such that $i \in Z_\tau$.

Then the conditions for lemma 3 are guaranteed.

Thus, take M as in lemma 3. Then ψ holds over I when we set $X_i := Z_\tau$. But $Z_\tau = I_\tau$ for all τ, so φ holds over M. $\square$