# Orders

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## Contents

# 1 Preorders

We begin by studying the properties of preorders. Basically, we define a *property* as a class which is close under isomorphism. We then define the sum operation on preorders. This will be used to create new properties from old ones.

**Definitions 1.1** (Preorder)**.** *A* preorder *is a a set $M$ together with a binary relation $\leq$ on $M$ such that $\leq$ is reflexive and transitive, possibly endowed with monadic predicates (labels) over some first-order monadic signature.*

**Definition 1.2** (Property of preorders)**.** *A* property $\mathbf{P}$ *of preorders is a class of preorders which is closed under isomorphism.*

**Definition 1.3.** *A property $\mathbf{P}$ of preorders is* monotone *if for every preorder $M$, $M \in \mathbf{P}$ implies that every suborder of $M$ is in $\mathbf{P}$.*

**Definition 1.4.** *Let $M$ be a preorder.*
*Then $M^*$ is the dual/reverse preorder of $M$.*

**Definition 1.5** (Sum of preorders)**.** *Let $I$ be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders over a disjoint signature (i.e., for every $i \in I$, $I$ and $M_i$ have disjoint sets of labels).*
*The sum $M = \sum_{i\in I} M_i$ is defined as follows:*
*The domain is $M = \biguplus_{i\in I} M_i$ (a disjoint union).*
*Let $\leq_i$ be the preorder on $M_i$.*
*Let $x \in M_i$ and $y \in M_j$.*
*Then we define $x \leq y$ iff either $i = j$ and $x \leq_i y$ or $i < j$.*
*The labels are inherited from either $I$ or the $M_i$'s.*
*If $I = 2$, we define $M_1 + M_2 := \sum_{i\in 2} M_i$.*

**Lemma 1.6.** *Let $I$ be a preorder, and let $\{M_i\}_{i\in I}$ be a family of preorders, over a disjoint signature.*
*Then $M = \sum_{i\in I} M_i$ is a preorder.*

*Proof.* Reflexivity is clear.
For transitivity, suppose $x \leq y$ and $y \leq z$.
Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.
Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.
Otherwise, $i < k$, and thus $x \leq z$, as required. □

**Definition 1.7.** *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be properties of preorders.*
*Then we define*

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

*The labels are inherited from either $\mathbf{P}_1$ or $\mathbf{P}_2$.*

**Definition 1.8.** *A property $\mathbf{P}$ of preorders is an* additive property *if for every preorders $M_1$ and $M_2$, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

**Definition 1.9** (Kleene plus). *Let* $\mathbf{P}$ *be a property of preorders.*

*We define its Kleene plus as the smallest property of preorders* $\mathbf{P}^+$ *which contains* $\mathbf{P}$ *and is closed under finite sums.*

*That is,* $1^+ = \{1, 2, \ldots\}$*, and* $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$.

**Definition 1.10** (Sum of a property over a preorder). *Let* $I$ *be a preorder.*

*Let* $\mathbf{Q}$ *be a property of preorders.*

*Then we define*

$$\sum_I \mathbf{Q} := \left\{ \sum_{i \in I} M_i : \forall i \in I. M_i \in \mathbf{Q} \right\}$$

**Definition 1.11** (Sum of a family of properties over a preorder). *Let* $I$ *be a preorder.*

*Let* $\{\mathbf{Q}_i\}_{i \in I}$ *be a family of properties of preorders over a disjoint signature.*

*Then we define*

$$\sum_{i \in I} \mathbf{Q}_i := \left\{ \sum_{i \in I} M_i : \forall i \in I. M_i \in \mathbf{Q}_i \right\}$$

*The labels are inherited from either* $I$ *or the* $\mathbf{Q}_i$*'s.*

**Note 1.12.** *By the previous two definitions, if* $I$ *has no labels,*

$$\sum_I \mathbf{Q} = \sum_{i \in I} \mathbf{Q}$$

**Definition 1.13** (Sum of a property over a property). *Let* $\mathbf{P}$ *be a property of preorders.*

*Let* $\mathbf{Q}$ *be a property of preorders over a disjoint signature.*

*Then we define,*

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_I \mathbf{Q} : I \in \mathbf{P} \right\}$$

# 2 Linear Orders

**Definitions 2.1** (Linear order)**.** *A* linear order *a preorder which is symmetric and total.*

**Definition 2.2** (Property of linear orders)**.** *A* property $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

**Definition 2.3.** *Subintervals Let $M$ be a linear order, and let $x, y \in M$, such that $x \leq y$.*

*Then we define the* bounded subintervals $[x, y]$, $(x, y]$, $[x, y)$ *and* $(x, y)$ *as usual.*

*We also define the* semi-bounded subintervals $(-\infty, x]$, $[x, \infty)$, $(-\infty, x)$ *and* $(x, \infty)$ *as usual.*

*We also define* the unbounded subinterval $(-\infty, \infty)$ *as the whole linear order* $M$*, as usual.*

*A* subinterval *is either a bounded subinterval, a semi-bounded subinterval or the unbounded subinterval.*

*If $x > y$ then we define the intervals as follows:*

$$[x, y] := [y, x]$$
$$(x, y] := (y, x]$$
$$[x, y) := [y, x)$$
$$(x, y) := (y, x)$$

**Definition 2.4.** *Let $M$ be a linear order.*

*A set $A \subseteq M$ is* left cofinal *in $M$ if for every $x \in M$, there exists $y \in A$ such that $y < x$.*

*A set $A \subseteq M$ is* right cofinal *in $M$ if for every $x \in M$, there exists $y \in A$ such that $x < y$.*

*A set $A \subseteq M$ is* bi-directionally cofinal *in $M$ if it is both left and right cofinal.*

**Lemma 2.5.** *Let $\mathbf{P}$ be an additive property of linear orders.*

*Let $M \in \mathbf{P}$ be a linear order.*

*Let $x, y \in M$ be any two points in a linear order $M$.*

*Then, $[x, y] \in \mathbf{P}$.*

*Proof.* WLOG, suppose $x \leq y$.

Note that,

$$M = (-\infty, \infty) = (-\infty, x) + [x, y] + (y, \infty)$$

when $(-\infty, x)$ and/or $(y, \infty)$ may be empty.

Since $\mathbf{P}$ is an additive property, we conclude that $[x, y] \in \mathbf{P}$. □

**Corollary 2.6.** *Let $\mathbf{P}$ be a nontrivial additive property of linear orders.*

*Then $1 \in \mathbf{P}$.*

*Proof.* Let $M \in \mathbf{P}$ be any linear order and let $x \in M$ be any point in $M$.

Apply lemma 2.5 to the linear order $M$, and the points $x$ and $x$, to conclude that $[x, x] \equiv 1 \in \mathbf{P}$. $\square$

**Note 2.7.** *Note that corollary 2.6 is false if we do not restrict ourselves to linear orders.*

*For example, $(1 \uplus 1)^+$ is a property of preorders which is additive, but does not contain* $1$*.*

**Corollary 2.8.** *Let $\mathbf{P}$ be an additive property of linear orders.*

*Let $M$ be a linear order.*

*Let $x, y \in M$ be any two points in a linear order $M$. Then the following are equivalent:*

1. $(x, y) \in \mathbf{P}$
2. $(x, y] \in \mathbf{P}$
3. $[x, y) \in \mathbf{P}$
4. $[x, y] \in \mathbf{P}$

*Proof.* This is just applying the definition of an additive property to the orders $[x, y]$ and $1$. $\square$

**Corollary 2.9.** *Let $\mathbf{P}$ be an additive property of linear orders.*

*Let $M$ be a linear order.*

*Let $x, y, z \in M$ be any three points in a linear order $M$, such that $[x, y] \in \mathbf{P}$ and $[y, z] \in \mathbf{P}$.*

*Then $[x, z] \in \mathbf{P}$.*

*Proof.* If $y \in [x, z]$, then $[x, z] = [x, y] + (y, z]$, and $(y, z] \in \mathbf{P}$ by corollary 2.8.

Otherwise, either $x \in [y, z]$ or $z \in [x, y]$. WLOG, suppose $z \in [x, y]$.

Then $[x, y] = [x, z] + (z, y]$, so $[x, z] \in \mathbf{P}$ by the fact that $\mathbf{P}$ is additive. $\square$

**Definitions 2.10.** *Let $\mathbf{P}$ be a property of linear orders.*

*We define the following properties of linear orders:*

- *$\mathcal{B}[\mathbf{P}]$ is the class of linear orders $M$ such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in $\mathbf{P}$.*
- *$\mathcal{L}[\mathbf{P}]$ is the class of linear orders $M$ such that for every $x \in M$, the left-bounded ray $[x, \infty)$ is in $\mathbf{P}$.*
- *$\mathcal{R}[\mathbf{P}]$ is the class of linear orders $M$ such that for every $x \in M$, the right-bounded ray $(-\infty, x]$ is in $\mathbf{P}$.*

**Definition 2.11.** *A property $\mathbf{P}$ of linear orders is a* star property *if for every linear orders $M$, and every family $\mathcal{F} \subseteq \mathbf{P}$ of subintervals of $M$ such that $J_1 \cap J_2 \neq \emptyset$ for every $J_1, J_2 \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \in \mathbf{P}$.*

**Lemma 2.12.** *Let* $\mathbf{P}$ *be a star property.*

*Then for every linear order* $M$*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$*.*

*Equivalently, we can define a convex equivalence relation* $\sim_{\mathbf{P}}$ *on* $M$ *such that* $x \sim_{\mathbf{P}} y$ *iff* $[x, y] \in \mathbf{P}$*.*

*That is,* $x \sim_{\mathbf{P}} y$ *iff* $x$ *and* $y$ *are in the same largest* $\mathbf{P}$*-subinterval.*

*Proof.* Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$-subintervals containing $x$. All such subintervals intersect at $x$.

Therefore, by the star lemma, $J$ is in $\mathcal{B}[\mathbf{P}]$, and by definition $J$ is the largest $\mathbf{P}$-subinterval containing $x$.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. □

**Lemma 2.13** (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

*Then the property* $\mathcal{B}[\mathbf{P}]$ *is a star property.*

*Proof.* Let $M$ be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of $M$.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by the definition of $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. Since $\mathbf{P}$ is additive, by corollary 2.9, we have $[x, y] \in \mathbf{P}$. □

**Lemma 2.14.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

*Then,*

1. $\mathcal{L}[\mathbf{P}] = \{M : M + 1 \in \mathcal{B}[\mathbf{P}]\}$
2. $\mathcal{R}[\mathbf{P}] = \{M : 1 + M \in \mathcal{B}[\mathbf{P}]\}$
3. $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}] = \{M : 1 + M + 1 \in \mathcal{B}[\mathbf{P}]\}$

*Proof.* Let $M$ be a linear order.

1. Suppose $M + \{\infty\} \in \mathcal{B}[\mathbf{P}]$. Then for every $x \in M$, we have $[x, \infty] \in \mathbf{P}$, and thus $[x, \infty) \in \mathbf{P}$. Therefore, $M \in \mathcal{L}[\mathbf{P}]$.

   Conversely, if $M \in \mathcal{L}[\mathbf{P}]$, let $x, y \in M$ be any two points in $M + 1$.

   If $y < \infty$, then $[x, y] \subseteq [x, \infty)$. Since $[x, \infty) \in \mathbf{P}$, we conclude that $[x, y] \in \mathbf{P}$. Otherwise, if $y = \infty$, then $[x, y] = [x, \infty] = [x, \infty) + \{\infty\}$, and thus $[x, y] \in \mathbf{P}$.

2. The second case is dual to the first case.

3. We will show a triple inclusion.

   If $M \in \mathbf{P}$, then by additivity, $1 + M \in \mathbf{P}$ and $M + 1 \in \mathbf{P}$, and thus $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$.

   If $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, then by lemma 2.13, $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$.

   If $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$, then $M$ is a bounded subinterval of $1 + M + 1$, so $M \in \mathcal{B}[\mathbf{P}]$.

$\square$

**Lemma 2.15.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

*Then,*

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = \mathbf{P}&\\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}])\\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])\\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

*Proof.* By lemma 2.14, we conclude that $\mathcal{L}[\mathbf{P}], \mathcal{R}[\mathbf{P}] \subseteq \mathcal{B}[\mathbf{P}]$, since $M + 1 \in \mathbf{P}$ and $1 + M \in \mathbf{P}$ both imply $1 + M + 1 \in \mathbf{P}$.

Thus,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = (\mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}])&\\ &\uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}])\\ &\uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])\\ &\uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Since by lemma 2.14 $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, we conclude what we wanted to prove. $\square$

**Lemma 2.16.** *Let* $\mathbf{P}$ *be a property of linear orders.*

*Let* $M \in \mathcal{B}[\mathbf{P}]$ *be a linear order.*

*Let* $x \in M$ *be a non-extreme point in* $M$*.*

*Then* $[x, \infty) \in \mathcal{L}[\mathbf{P}]$ *and* $(-\infty, x] \in \mathcal{R}[\mathbf{P}]$*.*

*Furthermore,* $[x, \infty) \in \mathcal{R}[\mathbf{P}]$ *iff* $M \in \mathcal{R}[\mathbf{P}]$*, and* $(-\infty, x] \in \mathcal{L}[\mathbf{P}]$ *iff* $M \in \mathcal{L}[\mathbf{P}]$*.*

*Proof.* This follows immediately from definitions 2.10. $\square$

**Corollary 2.17.**

$$\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]) = (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) + (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}])$$

**Lemma 2.18** (Associativity of sum)**.** *Let* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *be properties.*

*Then* $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$*.*

*Proof.* It follows directly from the associativity of the sum operation on linear orders. Actually, it generalizes to any algebraic equation which holds on linear orders. $\square$

**Lemma 2.19** (Sum and union commute)**.** *Let* $\mathcal{P}$ *be a family of properties.*

*Let* $\mathbf{Q}$ *be a property.*

*Then* $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$*.*

*Proof.* This is obvious from the definition of the sum operation. $\square$

**Definition 2.20.** *We define* $\mathbf{CNT}$ *as the class of all countable linear orders.*

**Definition 2.21.** *Let $\gamma \geq \omega$ be a limit ordinal.*

*We define $\Gamma_\gamma := \{\beta : \beta \subseteq \gamma^* + \gamma\}^+$.*

*We define $\Omega := \Gamma_\omega$.*

**Example 2.22.**

$$\Omega = \{1, \omega, \omega^*\}^+$$

**Observation 2.23.** *Let $\gamma \geq \omega$ be a limit ordinal.*

*Then $\Gamma_\gamma$ is a monotone, additive property of linear orders.*

# 3 General Hausdorff Rank

**Definition 3.1.** *Let* $\mathbf{Q}$ *be a property of linear orders.*
*We define a property* $\mathbf{Q}^{<\alpha}$ *for every ordinal* $\alpha$ *as follows:*

- *For* $\alpha = 0$, $\mathbf{Q}^0 = \{1\}$.
- *For* $\alpha = \gamma + 1$,
$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}$$
- *For* $\alpha$ *a limit ordinal,*
$$\mathbf{Q}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{Q}^{<\beta}$$

**Example 3.2.** *Let* $\mathbf{Q}$ *be a property of linear orders.*
*Then* $\mathbf{Q}^1 = \mathbf{Q}$.

**Lemma 3.3.** *Let* $\mathbf{Q}$ *be a property of linear orders.*
*Let* $\alpha, \delta$ *be ordinals.*
*Then,*
$$\mathbf{Q}^{<\alpha+\delta} = \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}$$

*Proof.* We shall prove this by induction on $\delta \geq 0$.
For $\delta = 0$ we need to prove
$$\mathbf{Q}^{<\alpha} = \sum_{\mathbf{Q}^0} \mathbf{Q}^{<\alpha}.$$

Which is true by definition, since $\mathbf{Q}^0 = \{1\}$.
For $\delta = \gamma + 1$, using the induction hypothesis,
$$\begin{aligned}
\mathbf{Q}^{<\alpha+\delta} &= \mathbf{Q}^{<\alpha+\gamma+1} \\
&= \sum_{\mathbf{Q}} \mathbf{Q}^{<\alpha+\gamma} \\
&= \sum_{\mathbf{Q}} \sum_{\mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\sum_{\mathbf{Q}} \mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\gamma+1}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}
\end{aligned}$$

For $\delta$ a limit ordinal, using the induction hypothesis,

$$\begin{aligned}
\mathbf{Q}^{<\alpha+\delta} &= \bigcup_{\gamma<\delta} \mathbf{Q}^{<\alpha+\gamma} \\
&= \bigcup_{\gamma<\delta} \sum_{\mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\bigcup_{\gamma<\delta} \mathbf{Q}^{<\gamma}} \mathbf{Q}^{<\alpha} \\
&= \sum_{\mathbf{Q}^{<\delta}} \mathbf{Q}^{<\alpha}
\end{aligned}$$

□

**Definition 3.4.** *Let* $\mathbf{Q}$ *be a property of linear orders.*
*Let* $\alpha$ *be an ordinal.*
*We define* $\mathbf{Q}^{=\alpha} := \mathbf{Q}^{<\alpha+1} \setminus \mathbf{Q}^{<\alpha}$.

**Definition 3.5.** *Let* $\mathbf{Q}$ *be a property of linear orders.*
*Let* $M$ *be a linear order, such that* $M \in (\mathbf{Q}^{<\alpha})^{+}$ *for some ordinal* $\alpha$.
*We define the* $\mathbf{Q}$-Hausdorff rank *of* $M$ *as*

$$\mathbf{hrank}_{\mathbf{Q}}(M) = \sup\left\{\beta : M \notin \left(\mathbf{Q}^{<\beta}\right)^{+}\right\}$$

*where the supremum is taken over all ordinals* $\beta$*. (Recall that the supremum of the empty set is defined to be* $0$*.)*

**Example 3.6.** *Let* $\mathbf{Q}$ *be a property of linear orders.*
*Let* $M$ *be a linear order.*
*Then* $\mathbf{hrank}_{\mathbf{Q}}(M) = 0$ *if and only* $M$ *is finite.*

## 4 $\omega$-Hausdorff rank

**Definitions 4.1.** *Let $\alpha > 0$ be an ordinal.*

*We define:*

1. $\mathcal{S}_\alpha^1 := \Omega^\alpha$
2. $\mathcal{S}_\alpha^\omega := \mathcal{L}\left[\Omega^\alpha\right] \setminus \mathcal{R}\left[\Omega^\alpha\right]$
3. $\mathcal{S}_\alpha^{\omega^*} := \mathcal{R}\left[\Omega^\alpha\right] \setminus \mathcal{L}\left[\Omega^\alpha\right]$
4. $\mathcal{S}_\alpha^{\omega^*+\omega} := \mathcal{B}\left[\Omega^\alpha\right] \setminus \left(\mathcal{L}\left[\Omega^\alpha\right] \cup \mathcal{R}\left[\Omega^\alpha\right]\right)$

*The names will soon be justified.*

**Lemma 4.2.** *Let $\alpha > 0$ be an ordinal.*

*Then,*

1. $\mathcal{L}\left[\Omega^\alpha\right] = \sum_\omega \Omega^\alpha$.
2. $\mathcal{R}\left[\Omega^\alpha\right] = \sum_{\omega^*} \Omega^\alpha$.
3. $\mathcal{B}\left[\Omega^\alpha\right] = \sum_{\omega^*+\omega} \Omega^\alpha$.

*Proof.* Let us prove the first part. ($\supseteq$) Let $M \in \sum_\omega \Omega^\alpha$ be a linear order.

Let $M = \sum_{i\in\omega} M_i$ be the decomposition of $M$, where $M_i \in \Omega^\alpha$.

Let $x, y \in M$ be any two points in $M$. WLOG $x \leq y$.

Suppose $x \in M_i$ and $y \in M_j$ for $i, j \in \omega$.

Since $i$ and $j$ have a finite distance in $\omega$, we conclude $[x, y] \subseteq M_i + \ldots + M_j$, and thus $[x, y] \subseteq \left(\Omega^\alpha\right)^+ = \Omega^\alpha$.

($\subseteq$) Let $M \in \mathcal{B}\left[\Omega^\alpha\right]$ be a linear order.

Since $M$ is countable, let $\{x_i\}_{i\in\omega} M$ be a bidirectionally cofinal $\omega$-sequence in $M$.

Then $M = \sum_{i\in\omega} M_i$ where $M_i = [x_i, x_{i'})$ for $i'$ the successor of $i$ in $I$.

But $M_i$ is a bounded interval and thus $M_i \in \Omega^\alpha$, so $M \in \sum_\omega \Omega^\alpha$.

The second part is symmetric.

The third part follows from corollary 2.17:

$$\begin{aligned}
\mathcal{B}\left[\Omega^\alpha\right] &= \mathcal{R}\left[\Omega^\alpha\right] + \mathcal{L}\left[\Omega^\alpha\right] \\
&= \sum_{\omega^*} \Omega^\alpha + \sum_{\omega} \Omega^\alpha \\
&= \sum_{\omega^*+\omega} \Omega^\alpha
\end{aligned}$$

$\square$

**Lemma 4.3.** *Let $\alpha > 0$ be an ordinal.*

*Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$.*

*Suppose that $\alpha = \sup_{i\in s}\left(\alpha_i + 1\right)$ for ordinals $\{\alpha_i\}_{i\in s}$.*

*Then, we have the following:*

$$\mathcal{S}^s_\alpha = \sum_{i\in s} \Omega^{\alpha_i}$$

**Note 4.4.** *For the proof of this lemma, we actually use the fact that we work over $\Omega$. This proof would not have worked over $\Gamma_\beta$ for $\beta > \omega$.*

*Proof.* TBC. □

**Corollary 4.5.** *Let $\alpha, \delta > 0$ be ordinals.*

*Let $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$*

*Then,*

$$\mathcal{S}^s_{\alpha+\delta} = \sum_{\mathcal{S}^s_\delta} \Omega^\alpha$$

*Proof.* For $s = 1$, it follows from lemma 3.3.

Otherwise, suppose that $\delta = \sup_{i\in s}(\delta_i + 1)$.

Then $\alpha + \delta = \sup_{i\in s}(\alpha_i + \delta_i + 1)$.

$$\mathcal{S}^s_{\alpha+\delta} = \sum_{i\in s} \mathcal{S}^s_{\alpha+\delta_i+1} = \sum_{i\in s} \sum_{\Omega^{\delta_i+1}} \Omega^\alpha = \sum_{\sum_{i\in s}\Omega^{\delta_i+1}} \Omega^\alpha = \sum_{\mathcal{S}^s_\delta} \Omega^\alpha$$

□

## 5 Type Theory

**Definition 5.1.** *Let* $\mathbf{P}$ *be a property of preorders.*

*Let* $n \in \mathbb{N}$.

*We define* $\mathbf{type}_n[\mathbf{P}]$ *as the set of all* $n$*-types satisfiable in* $\mathbf{P}$.

**Definition 5.2.** *A property* $\mathbf{P}$ *of preorders is* computable *if* $n \mapsto \mathbf{type}_n[\mathbf{P}]$ *is a computable function.*

**Lemma 5.3.** *Let* $\mathbf{Q}$ *be a property of preorders.*

*There exists a computable function* $f_{\mathbf{Q}} = f : \mathbb{N} \to \mathbb{N}$ *such that for every* $n \in \mathbb{N}$ *and every ordinal* $\alpha \geq f(n)$, $\mathbf{type}_n[\mathbf{Q}^{<\alpha}] = \mathbf{type}_n\left[\mathbf{Q}^{f(n)}\right]$.

*Proof.* Since there are only finitely many $n$-types, and the ordinal sequence

$$\left\{\mathbf{type}_n\left[\mathbf{Q}^{<\kappa}\right]\right\}_\kappa$$

is monotone, there must be some minimal $\kappa_0 \in \omega$ where the sequence stabilizes.

This $\kappa_0$ is computable as a function of $n$, because $\mathbf{type}_n[\mathbf{Q}^{<\kappa}]$ is computable for every finite $\kappa$. □

**Lemma 5.4.** *There exist global computable functions* $a, b : \mathbb{N} \to \mathbb{N}$ *such that for all* $n, c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c_1}] = \mathbf{type}_n[\mathbf{Q}^{=c_2}]$$

*Proof.* Let $n \in \mathbb{N}$.

Since there are only finitely many sets of $n$-types, there exist (and can be computed) some $a(n) \geq f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$$

By induction if follows that for all $c \geq a(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c}] = \mathbf{type}_n\left[\mathbf{Q}^{=c+b(n)}\right]$$

since $\mathbf{Q}^{=c+1} = \sum_{\mathbf{Q}} \mathbf{Q}^{=c}$. □

**Corollary 5.5.** *Let* $n \in \mathbb{N}$*, and let* $\alpha \geq \omega$ *be an ordinal.*

*Let* $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$ *be a shape.*

*Then there exists a computable function* $b(n)$ *such that for all* $c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$*, we have*

$$\mathbf{type}_n\left[\mathcal{S}^s_{c_1}\right] = \mathbf{type}_n\left[\mathcal{S}^s_{c_2}\right]$$

*Proof.* For $s = 1$, it follows from lemma 5.3, since $\mathcal{S}^1_c = \mathbf{Q}^{<c}$ and $c \geq a(n) \geq f(n)$ for $c \in \{c_1, c_2\}$.

For $s \in \{\omega, \omega^*, \omega^* + \omega\}$, it follows easily from lemma 4.3 and lemma 5.4. □

**Lemma 5.6.** *For every $n \in \mathbb{N}$ and for every ordinal $\alpha \geq \omega$,*

$$\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right] = \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]$$

*In particular, $\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.*

*Proof.* By induction on $\alpha \geq \omega$.

Let $\{\alpha_i\}_{i\in\omega}$ be an $\omega$-sequence of ordinals such that $a(n) \leq \alpha_i$ for all $i \in \omega$, and $\sup_{i\in\omega}(\alpha_i + 1) = \alpha$.

Then $\mathbf{Q}^{=\alpha} = \sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{Q}^{=\alpha_i}$ and thus,

$$\begin{aligned}
\mathbf{type}_n\left[\mathbf{Q}^{=\alpha}\right] &= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \mathbf{Q}^{=\alpha_i}\right] \\
&= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{i\in\omega} \bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\sum_{\mathbf{Q}} \bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \sum_{\mathbf{Q}} \mathbf{Q}^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c+1}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)} \mathbf{Q}^{=a(n)+c}\right]
\end{aligned}$$

where the last transition is because $\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$. □

**Corollary 5.7.** *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

*Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$ be a shape.*

$$\mathbf{type}_n\left[\mathcal{S}^s_\alpha\right] = \mathbf{type}_n\left[\sum_{s} \bigcup_{c<b(n)} \Omega^{=a(n)+c}\right]$$

*In particular, $\mathbf{type}_n\left[\mathcal{S}^s_\alpha\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.*

*Proof.* There exists a sequence $\{\alpha_i\}_{i\in s}$ such that $a(n) \leq \alpha_i$ for all $i \in s$, and $\sup_{i\in s}(\alpha_i + 1) = \alpha$.

Then $\mathcal{S}^s_\alpha = \sum_{i\in s} \Omega^{=\alpha_i}$, and thus,

$$\begin{aligned}
\mathbf{type}_n\left[\mathcal{S}^s_\alpha\right] &= \mathbf{type}_n\left[\sum_{i\in s}\Omega^{=\alpha_i}\right] \\
&= \mathbf{type}_n\left[\sum_{s}\bigcup_{c<b(n)}\Omega^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\sum_{s}\Omega^{=a(n)+c}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\mathcal{S}^s_{a(n)+c+1}\right] \\
&= \mathbf{type}_n\left[\bigcup_{c<b(n)}\mathcal{S}^s_{a(n)+c}\right]
\end{aligned}$$

where the last transition is by corollary 5.5.

□

## 6 Decidability of the rank

**Definition 6.1.** *Let* $\mathbf{Q}$ *be a property of linear orders.*

*Let* $M$ *be a linear order.*

*We define the predicate* $\mathbf{Int_Q}(J)$ *as true in* $M$ *iff* $J$ *is a* $\mathbf{Q}$*-subinterval of* $M$*.*

**Lemma 6.2.** *Let* $\alpha > 0$ *be an ordinal.*

*Then predicates* $\mathbf{Int}_{\Omega^{\leq\alpha}}$, $\mathbf{Int}_{\Omega^{=\alpha}}$ *are expressible in* $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$*.*

*Proof.* Obviously,

$$\mathbf{Int}_{\Omega^{=\alpha}} \iff \mathbf{Int}_{\Omega^{\leq\alpha}} \wedge \neg\mathbf{Int}_{\Omega^{<\alpha}}$$

So it is enough to express $\mathbf{Int}_{\Omega^{\leq\alpha}}$.

Now, $J$ is a $\Omega^{\leq\alpha}$-subinterval of $M$ iff $J \in \sum_{\Omega} \Omega^{<\alpha}$.

But this can be expressed in **MSO** since it is expressible to check whether an arbitrary subset is in $\Omega$. □

**Definition 6.3.** *Let* $\alpha > 0$ *be an ordinal.*

*Let* $M$ *be a linear order and* $x \in M$*.*

*We define the convex equivalence relation:*

$$\sim_\alpha := \sim_{\mathcal{B}[\Omega^\alpha]}$$

*and* $[x]_\alpha := [x]_{\mathcal{B}[\Omega^\alpha]}$*.*

*That is,* $[x]_\alpha$ *is the largest* $\mathcal{B}[\Omega^\alpha]$*-subinterval containing* $x$ *in* $M$*.*

*We define* $\sigma_\alpha(x)$ *as the* $\alpha$*-shape of* $[x]_\alpha$*.*

*We define* $L_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{L}[\Omega^{<\alpha}]}$ *and* $R_\alpha(x) = \mathbf{1}_{[x]_\alpha \in \mathcal{R}[\Omega^{<\alpha}]}$*.*

**Lemma 6.4.** *Let* $M$ *be a linear order and* $\alpha > 0$ *an ordinal.*

*Let* $J \subseteq M$ *be an interval.*

*Then* $J \in \Omega^{<\alpha}$ *iff it is contained in a single* $\sim_\alpha$*-equivalence class* $K$*, such that:*

- *Either* $K \in \mathcal{L}[\Omega^{<\alpha}]$ *or there exists some* $x \in K$ *such that* $x < J$*.*
- *Either* $K \in \mathcal{R}[\Omega^{<\alpha}]$ *or there exists some* $x \in K$ *such that* $x > J$*.*

*Proof.* TBC. □

**Corollary 6.5.** *Let* $\alpha > 0$ *be an ordinal.*

*The predicate* $\mathbf{Int}_{\Omega^{<\alpha}}$ *is* **MSO***-expressible over* $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$*.*

**Theorem 6.6.** *Let* $\mathbf{P}$ *be a computable property of linear orders of some finite signature.*

*Let* $\{\mathbf{Q}_i\}_{i\in I}$ *be a* finite *family of computable properties of linear orders over some finite signature which is disjoint from the signature of* $\mathbf{P}$*.*

*Then* $\bigcup_{I\in\mathbf{P}} \sum_{i\in I} \mathbf{Q}_i$ *is a computable property of linear orders.*

*Proof.* We will use the decomposition theorem. Let $\tau(X_1, \ldots, X_m)$ be an $n$-type.

Then we can compute a formula $\psi(\xi)$ (where $\xi$ has the type of a coloring whose range is the set of $n$-types) such that for any linear order $M = \sum_{i \in I} M_i$,

and any given $A_1, \ldots, A_m \subseteq M$,

$$M \models \tau(A_1, \ldots, A_m) \iff I \models \psi(\Xi)$$

where $\Xi$ is the coloring assigning $i \in I$ the $n$-type of $M_i$.

TBC. $\square$

**Lemma 6.7.** *Let $\alpha$ be an ordinal.*

*Let $P$, $L$ and $R$ be first-order unary predicates.*

*Let $C$ be the class of all countable linear orders labeled with $P$, $L$ and $R$, such that $P$ represents $\sim_\alpha$, $L_\alpha(x) \iff [x]_\alpha \in \mathcal{L}[\Omega^{<\alpha}]$ and $R_\alpha(x) \iff [x]_\alpha \in \mathcal{R}[\Omega^{<\alpha}]$.*

*Let $\mathbf{G}$ be the class of all countable linear orders $I$, labeled with a $P$, $L$ and $R$, such that for every pair $i, i' \in I$ such that $i'$ is the successor of $i$, $P(i) \neq P(i')$, and either $R(i) = 0$ or $L(i') = 0$.*

*Let $\sigma(i) \in \{1, \omega, \omega^*, \omega^* + \omega\}$ be such that $L(i) = 1$ iff $\sigma(i) \in \{1, \omega\}$ and $R(i) = 1$ iff $\sigma(i) \in \{1, \omega^*\}$.*

*Then, $C = \bigcup_{I \in \mathbf{G}} \sum_{i \in I} \mathcal{S}_\alpha^{\sigma(i)}$.*

*Proof.* ($\subseteq$) Let $M$ be a countable linear order labeled with $P$, $L$ and $R$ as above.

Let $I = M/\sim_\alpha$ be the quotient of $M$ by the equivalence relation $\sim_\alpha$.

Then $M = \sum_{i \in I} M_i$, where $\{M_i\}_{i \in I}$ are the $\sim_\alpha$-equivalence class of $I$.

Then for each $i \in I$, $M_i \in \mathcal{B}[\Omega^{<\alpha}]$, and by definition $\sigma(i) = \sigma_\alpha(M_i)$.

Let $i'$ be the successor of $i$ in $I$.

Then $P(i) \neq P(i')$ since $P$ represents $\sim_\alpha$.

Furthermore, suppose $R(i) = L(i') = 1$ holds. Then $M_i \in \mathcal{R}[\Omega^{<\alpha}]$ and $M_{i'} \in \mathcal{L}[\Omega^{<\alpha}]$. so $M_i$ and $M_{i'}$ are the same $\sim_\alpha$-equivalence class of $M$, which is a contradiction.

Thus either $R(i) = 0$ or $L(i') = 0$.

($\supseteq$) Let $M = \sum_{i \in I} M_i$ be a linear order such that $I \in \mathbf{G}$ and $M_i \in \mathcal{S}_\alpha^{\sigma(i)}$ for each $i \in I$.

In particular $M_i \in \mathcal{B}[\Omega^{<\alpha}]$ for each $i \in I$, so it is contained in a single $\sim_\alpha$-equivalence class of $M$.

Suppose that there exist distinct $j, k \in I$ such that $j < k$, and $M_j, M_k$ are in the same $\sim_\alpha$-equivalence class.

Let $x \in M_j$ and $y \in M_k$. Then $[x, y] \in \Omega^{<\alpha}$, and thus $[j, k] \in \Omega^{<\alpha}$, and in particular it is sparse.

Then there exist some $j', k' \in I$ such that $j < j' < k' < k$, and $k'$ is the successor of $j'$ in $I$.

Then $M_{j'}$ and $M_{k'}$ are in the same $\sim_\alpha$-equivalence class. Thus it must be the case that $M_{j'} \in \mathcal{R}[\Omega^{<\alpha}]$ and $M_{k'} \in \mathcal{L}[\Omega^{<\alpha}]$, which implies $R(j') = L(k') = 1$, which is a contradiction.

Thus $\{M_i\}_{i \in I}$ are pairwise distinct $\sim_\alpha$-equivalence classes, and obviously the conditions holds, so $M \in C$ and we are done. $\square$

**Corollary 6.8.** *Let $C$ be defined as in lemma 6.7.*

*Then $C$ is a computable property.*

**Theorem 6.9.** *Let $\alpha > 0$ be an ordinal.*

*Satisfiability of $\mathbf{MSO}[\mathbf{Int}_{\Omega^{<\alpha}}]$ over all countable linear orders is decidable.*

*Proof.* First, by corollary 6.5, we can convert any formula in $\mathbf{MSO}[\mathbf{int}_{\Omega^{<\alpha}}]$ to an equivalent formula $\varphi$ in $\mathbf{MSO}[[\cdot]_\alpha, L_\alpha, R_\alpha]$.

Now, we shall replace every occurrence of $[\cdot]_\alpha$ in $\varphi$ with $P$, every occurrence of $L_\alpha$ with $L$, and every occurrence of $R_\alpha$ with $R$, getting a new formula $\varphi'$.

Then, satisfiability of $\varphi$ over all countable linear orders, amounts to satisfiability of $\varphi'$ over $C$, which is computable by corollary 6.8.

Thus we can compute $\mathbf{type}_n[C]$ and $\mathbf{type}_n[\varphi']$, and thus we can compute whether $\varphi$ is satisfiable over all countable linear orders, by seeing if these sets intersect. $\square$

**Theorem 6.10.** *Let $\alpha, \delta_1, \ldots, \delta_k$ be ordinals.*

*Let $\alpha_i = \alpha + \delta_i$ for $i = 1, \ldots, k$.*

*Let $C$ be the class of all countable linear orders labeled with $\pi_\alpha$ and $\sigma_\alpha$, and $\pi_{\alpha_i}$ and $\sigma_{\alpha_i}$ for $i = 1, \ldots, k$.*

*Let $\mathbf{G}$ be the class of all countable linear orders $I$, labeled with a coloring function $\gamma$ whose range is $\{1, \omega, \omega^*, \omega^* + \omega\}$, such that for pair $i, j \in I$ such that $j$ is the successor of $i$, either $\gamma(i) \in \{\omega, \omega^* + \omega\}$ or $\gamma(j) \in \{\omega^*, \omega^* + \omega\}$.*

*Proof.* TBC. $\square$

# 7 Everything Better

**Theorem 7.1.** *Let $\mathcal{C}$ be a computable property of linear orders, such that $\mathcal{C}$ is closed under taking subintervals, projections and inverse-projections (i.e, of one of the colors), and all finite-sums and $\mathcal{C}$-sums.*

*Let $\mathbf{P}_1, \ldots, \mathbf{P}_k \subseteq \mathcal{C}$ be computable properties of linear orders.*

*Let $\mathbf{MSO}[P_1, \ldots, P_k]$ be monadic second order logic of order over $\mathcal{C}$, with $P_1, \ldots, P_k$ as monadic predicates whose semantics are: $P_i(X)$ holds iff $X$ is a subinterval which satisfies $\mathbf{P}_i$.*

*Given $\phi$ a formula of $\mathbf{MSO}[P_1, \ldots, P_k]$ (possibly with free variables) we define*

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

*Then $\mathcal{C}_\phi$ is a computable property of linear orders.*

*Proof.* By structural induction on $\phi$.

Suppose $\phi$ is an atomic formula. If $\phi$ is of the form $X \subseteq Y$ or $X \leq Y$,

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

and thus,

$$\mathbf{type}_n[\mathcal{C}_\phi] = \{\tau \in \mathbf{type}_n[\mathcal{C}] : \tau \models \phi\}$$

which is computable since $\mathbf{type}_n[\mathcal{C}]$ is computable, and we can then compute whether $\tau \models \phi$ for each $\tau \in \mathbf{type}_n[\mathcal{C}]$.

If $\phi$ is of the form $P_i(X)$, then

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models P_i(X)\}$$

and thus,

$$\mathbf{type}_n[\mathcal{C}_\phi] = \mathbf{type}_n[\mathbf{P}_i]$$

which is computable since $\mathbf{P}_i$ is computable.

If $\phi = \neg\phi_1$, then

$$\mathcal{C}_\phi = \mathcal{C} \setminus \mathcal{C}_{\phi_1}$$

□