

Orders

Alon Gurny

February 19, 2025

1 Properties

Definition 1. A property \mathbf{P} of linear orders is a class of linear orders which is closed under isomorphism.

Definition 2. A property \mathbf{P} of linear orders is monotone if for every linear order M , $M \in \mathbf{P}$ implies that every suborder of M is in \mathbf{P} .

Definition 3. A property \mathbf{P} of linear orders is symmetric if for every linear order M , $M \in \mathbf{P}$ iff $M^R \in \mathbf{P}$.

Definition 4. A property \mathbf{P} of linear orders is an additive property if for every linear orders M_1 and M_2 , $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.

Definition 5. Let \mathbf{P} be a property of linear orders.

We define \mathbf{P} -bounded to be the class of linear orders M such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in \mathbf{P} .

Definition 6. A property \mathbf{P} of linear orders is almost anti-symmetric if for every linear order M , $M \in \mathbf{P}$ and $M^R \in \mathbf{P}$ imply that M is finite.

Definition 7. A property \mathbf{P} of linear orders is good if it is monotone, additive and contains at least one infinite linear order.

Definition 8. A property \mathbf{P} of linear orders is a star property if for every linear orders M , and every family $\mathcal{F} \subseteq \mathbf{P}$ of subintervals of M such that $J_1 \cap J_2 \neq \emptyset$ for every $J_1, J_2 \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \in \mathbf{P}$.

Lemma 1 (Star Lemma). Let \mathbf{P} be an additive property of linear orders.

Then the property \mathbf{P} -bounded has the star property.

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathbf{P}$ -bounded be a family of subintervals of M .

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in \mathbf{P} .

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by \mathbf{P} -bounded, $[x, z], [z, y] \in \mathbf{P}$.

However, \mathbf{P} is additive. Since $[x, y]$ is either the sum or difference of $[x, z]$ and $[z, y]$, we have that $[x, y] \in \mathbf{P}$. \square

Lemma 2. *Let \mathbf{P} be a star property.*

Then for every linear order M , and every point $x \in M$, there exists a largest subinterval $J \subseteq M$ such that $J \in \mathbf{P}$.

Thus, we can define an equivalence relation $\sim_{\mathbf{P}}$ on M such that $x \sim_{\mathbf{P}} y$ iff x and y are in the same largest \mathbf{P} -subinterval.

Proof. Let $J \subseteq M$ be the union of all **P-bounded**-subintervals containing x . All such subintervals intersect at x .

Therefore, by the star lemma, J is in **P-bounded**, and by definition J is the largest \mathbf{P} -subinterval containing x .

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. \square

2 Hausdorff Rank

Definition 9. *Let \mathbf{Q} be a good property of linear orders.*

We define a property $\mathbf{Q}^{\leq \alpha}$ for every ordinal α as follows:

- $\mathbf{Q}^{\leq 0}$ is the class of finite linear orders.
- For $\alpha > 0$, $\mathbf{Q}^{\leq \alpha}$ is the class of linear orders M such that $M = \sum_{i \in I} M_i$ for some $I \in \mathbf{Q}$ where for all $i \in I$, $M_i \in \mathbf{Q}^{\beta_i}$ for some $\beta_i < \alpha$

We define further $\mathbf{Q}^{< \alpha} = \bigcup_{\beta < \alpha} \mathbf{Q}^{\leq \beta}$ and $\mathbf{Q}^{= \alpha} = \mathbf{Q}^{\leq \alpha} - \mathbf{Q}^{< \alpha}$.

We define $\mathbf{hrank}_{\mathbf{Q}}(M) = \alpha$ iff $M \in \mathbf{Q}^{= \alpha}$. This is a partial map from linear orders to ordinals.

Observations 1. *Let \mathbf{Q} be a good property.*

We claim the following without proof:

- $\mathbf{Q}^{\leq 1} = \mathbf{Q}$.
- For all α , $\mathbf{Q}^{\leq \alpha}$ is a good property.
- $\mathbf{Q}^{\leq \alpha} \subsetneq \mathbf{Q}^{\leq \beta}$ iff $\alpha < \beta$.

Definitions 1. $\mathcal{B}_{< \alpha} := \mathbf{Q}^{< \alpha}$ -**bounded** is the class of linear orders of rank $< \alpha$ on bounded intervals.

$\mathcal{R}_{< \alpha}$ is the class of linear orders M where $M + 1 \in \mathcal{B}_{< \alpha}$.

$\mathcal{L}_{< \alpha}$ is the class of linear orders M where $1 + M \in \mathcal{B}_{< \alpha}$.

Claim 1. *The following are equal:*

1. $\mathbf{Q}^{< \alpha}$
2. $\{M : 1 + M + 1 \in \mathcal{B}_{< \alpha}\}$.
3. $\mathcal{L}_{< \alpha} \cap \mathcal{R}_{< \alpha}$

Proof. The only nontrivial direction is 3 implies 2, which follows from the star property of $\mathcal{B}_{< \alpha}$. \square

Lemma 3. *Let M be a linear order. Then there exists a largest subinterval $J \subseteq M$ such that $x \in J$ and $J \in \mathcal{B}_{<\alpha}$.*

Definition 10. *Let M be a linear order. Let $x \in M$. We define $M_\alpha[x]$ to be the largest subinterval $M \subseteq M$ such that $x \in M$ and $M \in \mathcal{B}_{<\alpha}$.*

We define \sim_α to be the equivalence relation on M such that $x \sim_\alpha y$ iff $M_\alpha[x] = M_\alpha[y]$.

Lemma 4. *Let M be a countable linear order.*

Let $J \subseteq M$ be some subinterval in $\mathcal{B}_{<\alpha}$.

Then $\mathbf{hrank}_Q(J) \leq \alpha$.

Furthermore, $\mathbf{hrank}_Q(J) < \alpha$ iff $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Proof. Let $\{x_i\}_{i \in I} \subseteq J$ be a bidirectional, cofinal, weakly monotone I -sequence in J , i.e. $x_i \leq x_j$ if $i \leq j$ for $I \subseteq \mathbb{Z}$.

Write $J = \sum_{i \in I} [x_i, x_{i+1}]$. Then every $[x_i, x_{i+1}]$ is of Hausdorff rank $< \alpha$.

Thus, $\mathbf{hrank}_Q(J) \leq \alpha$.

Suppose $\mathbf{hrank}_Q(J) < \alpha$, then obviously $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Conversely, suppose $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Then $1 + J + 1 \in \mathcal{B}_{<\alpha}$. But it is a bounded interval, so $\mathbf{hrank}_Q(1 + J + 1) < \alpha$ and thus $\mathbf{hrank}_Q(J) < \alpha$. \square

Lemma 5. *Let $J \subseteq M$ be a subinterval.*

Then $\mathbf{hrank}_Q(J) \leq \alpha$ iff J is a finite sum of $\mathcal{B}_{<\alpha}$ -subintervals.

Note: this lemma does not work if we take a general Q property.

Proof. From the previous lemma, it is clear that if J is a finite sum of $\mathcal{B}_{<\alpha}$ -subintervals, then $\mathbf{hrank}_Q(J) \leq \alpha$, since the rank bound is preserved under finite sums.

Conversely, suppose $\mathbf{hrank}_Q(J) \leq \alpha$.

If $J = \sum_{i \in \mathbb{Z}} J_i$ for some J_i of Hausdorff rank $< \alpha$, take $x, y \in J$. Then let $x \in J_{i_1}$ and $y \in J_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} J_i$. But the last sum is of rank $< \alpha$ and thus $[x, y]$ is of rank $< \alpha$. That is, $J \in \mathcal{B}_{<\alpha}$.

Since every subinterval of rank $\leq \alpha$ is a finite sum of \mathbb{Z} -sums of intervals of rank $< \alpha$, we are done. \square

Corollary 1. *Let $J \subseteq M$ be a subinterval.*

Then $\mathbf{hrank}_Q(J) \leq \alpha$ iff J is a finite sum of largest $\mathcal{B}_{<\alpha}$ -subintervals in M

Corollary 2. *Over countable linear orders with interpretations of P , L and R as above, the properties $\mathbf{hrank}_Q(\cdot) \leq \alpha$, $\mathbf{hrank}_Q(\cdot) < \alpha$ and $\mathbf{hrank}_Q(\cdot) = \alpha$ over subintervals are all expressible in $\mathbf{MSO}[P, L, R]$.*

Proof. For $\mathbf{hrank}_Q(\cdot) \leq \alpha$ and $\mathbf{hrank}_Q(\cdot) < \alpha$, we can use the previous lemmas.

For $\mathbf{hrank}_Q(\cdot) = \alpha$, we can use the previous two. \square

3 Decidability of the rank

Lemma 6. *Let M be a linear order. Let $P, L, R \subseteq M$ be relations, such that:*

- *P represents \sim_α on M .*
- *L is such that $x \in L$ iff $M_\alpha[x] \in \mathcal{L}_{<\alpha}$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_{<\alpha}$.*

Then for some linear order I there exists a decomposition $M = \sum_{i \in I} M_i$ such that $M_i \in \mathcal{B}_{<\alpha}$ for all $i \in I$, M_i is monochromatic with respect to P , L and R .

Furthermore, let τ_i be the n -type of M_i, p_i, q_i, r_i in $\mathbf{MSO}[p, \ell, r]$, where $p_i = 1_{M_i \subseteq P}$, $q_i = 1_{M_i \subseteq L}$ and $r_i = 1_{M_i \subseteq R}$. Then the following hold

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $\ell(\tau_{i+1}) = 0$*

Proof. Take $I = M / \sim_\alpha$.

Then $M = \sum_{i \in I} M_i$ where M_i is the \sim_α -equivalence class of i .

Then M_i is monochromatic with respect to P , L and R .

The only thing left to prove is the last two conditions. The first follows from the fact that P represents \sim_α .

The second follows because if it were not the case, then M_i and M_{i+1} would be the same \sim_α -equivalence class. \square

Lemma 7. *Let I be a linear order. Let $n \in \mathbb{N}$. Let p, ℓ, r be boolean variables.*

Let τ_i be an assignment of satisfiable n -types in $\mathbf{MSO}[p, \ell, r]$ for all $i \in I$. Assume that

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $\ell(\tau_{i+1}) = 0$*

Then there exists a linear order M and $P, L, R \subseteq M$ such that:

- *P represents \sim_α on M .*
- *L is such that $x \in L$ iff $M_\alpha[x] \in \mathcal{L}_{<\alpha}$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_{<\alpha}$.*

such that for all $i \in I$, M_i is a \sim_α -equivalence class of M , and is thus monochromatic with respect to P , L and R .

Furthermore, the n -type of M_i, p_i, q_i, r_i in $\mathbf{MSO}[p, \ell, r]$ is τ_i , where $p_i = 1_{M_i \subseteq P}$, $q_i = 1_{M_i \subseteq L}$ and $r_i = 1_{M_i \subseteq R}$,

Proof. Since τ_i is satisfiable, we can take M_i to be a linear order of n -type τ_i such that:

- If $\ell(\tau_i) = r(\tau_i) = 1$, then $M_i \in \mathcal{L}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.
- If $\ell(\tau_i) = 1$ and $r(\tau_i) = 0$, then $M_i \in \mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}$.
- If $\ell(\tau_i) = 0$ and $r(\tau_i) = 1$, then $M_i \in \mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}$.
- If $\ell(\tau_i) = r(\tau_i) = 0$, then $M_i \in \mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})$.

Let $M = \sum_{i \in I} M_i$.

By definition each M_i is in $\mathcal{B}_{<\alpha}$. We need to prove that each M_i is a largest $\mathcal{B}_{<\alpha}$ -subinterval in M .

On the contrary, suppose that there exist $i' \neq i$ such that $[M_i, M_{i'}] \in \mathcal{B}_{<\alpha}$. WLOG, $M_i < M_{i'}$.

Since I is scattered, take some $i \leq a < b \leq i'$ such that there is no element between a and b in I .

Then $M_a \in \mathcal{R}_{<\alpha}$ and $M_b \in \mathcal{L}_{<\alpha}$, in contradiction. \square

Lemma 8. *There exists a global computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, $\mathbf{type}_n[\mathcal{H}_{f(n)+1}] = \mathbf{type}_n[\mathcal{H}_{f(n)}]$.*

Equivalently, every linear order of finite rank is n -equivalent to some linear order of rank $\leq f(n)$.

Proof. Since there exist only a finite number of n -types, and the ω -sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$ is monotone, it must stabilize at some point.

This point is computable as a function of n , because $\mathbf{type}_n[\mathcal{H}_k]$ is computable for every finite k . \square

Lemma 9. *There exist global computable functions $a, b : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $n, c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 \geq a(n)$ and $c_1 \equiv c_2 \pmod{b(n)}$,*

$$\mathbf{type}_n[\mathcal{H}_{c_1}] = \mathbf{type}_n[\mathcal{H}_{c_2}]$$

Equivalently, the sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$ is ultimately periodic for all $n \in \mathbb{N}$. Furthermore, the starting point and the period itself can be computed as a function of n .

Proof. Let $n \in \mathbb{N}$.

Since there exist only a finite number of possible sets of n -types, there exist (and can be computed) some $a(n) > f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n[\mathcal{H}_{a(n)}] = \mathbf{type}_n[\mathcal{H}_{a(n)+b(n)}]$$

We shall prove by induction that for all $c \geq a(n) + b(n)$,

$$\mathbf{type}_n[\mathcal{H}_c] = \mathbf{type}_n[\mathcal{H}_{c+b(n)}]$$

This will complete the proof.

The base case $c = a(n)$ has been proven in the beginning.

Suppose the induction hypothesis holds for c .

Let M be of rank $c + 1$.

Write $M = \sum_{i \in I} M_i$ where $\mathbf{hrank}_{\mathbf{Q}}(M_i) < c + 1$, and $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c$ infinitely many times.

By the induction hypothesis, if $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c$, we can find $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}}(N_i) = c + b(n)$. Setting $N_i := M_i$ for all other i , we conclude that $N := \sum_{i \in I} N_i$ is n -equivalent to M .

However, clearly $\mathbf{hrank}_{\mathbf{Q}}(N) = c + b(n) + 1$. So overall,

$$\mathbf{type}_n[\mathcal{H}_{c+1}] \subseteq \mathbf{type}_n[\mathcal{H}_{c+b(n)+1}]$$

Conversely, suppose M is of rank $c + b(n) + 1$. Write $M = \sum_{i \in I} M_i$ where $\mathbf{hrank}_{\mathbf{Q}}(M_i) < c + b(n) + 1$, and $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c + b(n)$ infinitely many times.

By the induction hypothesis, we can find for all i such that $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c + b(n)$ some $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}}(N_i) = c$. Furthermore, since $c \geq a(n) > f(n)$, we can find $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}}(N_i) \leq f(n) < c$ for all other i .

We conclude that $N := \sum_{i \in I} N_i$ is n -equivalent to M . However, clearly $\mathbf{hrank}_{\mathbf{Q}}(N) = c + 1$. So overall,

$$\mathbf{type}_n[\mathcal{H}_{c+b(n)+1}] \subseteq \mathbf{type}_n[\mathcal{H}_{c+1}]$$

So we have proven the induction step, and the lemma follows. \square

Lemma 10. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Then,

$$\mathbf{type}_n[\mathbf{Q}^{=\alpha}] = \bigcup_{c < b(n)} \mathbf{type}_n[\mathcal{H}_{c+b(n)}]$$

In particular, $\mathbf{type}_n[\mathbf{Q}^{=\alpha}]$ can be computed, and is independent of the choice $\alpha \geq \omega$.

Proof. TBC. \square

Corollary 3. *The following sequences stabilize at $f(n)$:*

- $\mathbf{type}_n[\mathcal{H}_\alpha]$
- $\mathbf{type}_n[\mathcal{B}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{L}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{R}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})]$

Proof. The corollary is false and should be fixed. \square

Theorem 1. *There is an algorithm solving satisfiability for $\mathbf{MSO}[P, L, R]$ over countable linear orders, given an oracle which solves the satisfiability problem for \mathbf{MSO} over countable linear orders.*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[P, L, R]$ formula φ of quantifier-depth n . outputs an $\mathbf{MSO}[\{X_\tau\}_\tau]$ formula ψ .

Let P_L, Q_L, R_M be the interpretations of P, L, R on M .

Then

$$M, P := P_L, L := Q_L, R := R_L \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where $I_\tau = \{i \in I : M_i \models \tau\}$ for every n -type τ .

Let T be the set of n -types in $\mathbf{MSO}[p, \ell, r]$ which satisfy $\ell(\tau) = 1 \iff \tau \in \mathcal{L}_{<\alpha}$ and $r(\tau) = 1 \iff \tau \in \mathcal{R}_{<\alpha}$.

Let $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee \ell(\tau_2) = 0)\}$.

Then T and S can be calculated using the oracle.

Then ψ is an $\mathbf{MSO}[T, S]$ formula.

Then we define an $\mathbf{MSO}[p, \ell, r]$ formula ψ' as follows:

ψ' claims that there exists a partition (with possible empty sets) $\{Y_\tau\}_\tau$ of I such that

- Every $i \in I$ is in some Y_τ for $\tau \in T$.
- If $i' = i + 1$ in I , then for some $(\tau_1, \tau_2) \in S$, $i \in Y_{\tau_1}$ and $i' \in Y_{\tau_2}$.

Now we claim that φ is satisfiable in some linear order, iff ψ' is satisfiable in some linear order.

Suppose φ is satisfiable in some linear order M .

Take a decomposition $M = \sum_{i \in I} M_i$ as in lemma 2.

Then ψ holds over the assignment $X_\tau := I_\tau$. But by lemma 2, this assignment satisfies the condition required for ψ' to hold. Then ψ' holds over I .

Conversely, suppose ψ' holds in I .

Let $X_\tau := Z_\tau$ be the assignment that is guaranteed by ψ' .

Let τ_{i_i} be the unique τ such that $i \in Z_\tau$.

Then the conditions for lemma 3 are guaranteed.

Thus, take M as in lemma 3. Then ψ holds over I when we set $X_i := Z_{\tau_i}$.

But $Z_\tau = I_\tau$ for all τ , so φ holds over M . \square