Orders

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1 Properties

Definition 1. *A* property $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 2. *A property* $\mathbf{P}$ *of linear orders is* monotone *if for every linear order* L*,* $L \in \mathbf{P}$ *implies that every suborder of* L *is in* $\mathbf{P}$*.*

Definition 3. *A property* $\mathbf{P}$ *of linear orders is* symmetric *if for every linear order* L*,* $L \in \mathbf{P}$ *iff* $L^R \in \mathbf{P}$*.*

Definition 4. *A property* $\mathbf{P}$ *of linear orders is an* additive property *if for every linear orders* L_1 *and* L_2*,* $L_1 + L_2 \in \mathbf{P}$ *iff* $L_1, L_2 \in \mathbf{P}$*.*

Definition 5. *A property* $\mathbf{P}$ *of linear orders is a* star property *if for every family* $\mathcal{F}$ *of linear orders in* $\mathbf{P}$ *such that* $\bigcap \mathcal{F} \neq \emptyset$*,* $\bigcup \mathcal{F} \in \mathbf{P}$*.*

Definition 6. *Let* $\mathbf{P}$ *be a property of linear orders.*

We define $\mathbf{bounded}-\mathbf{P}$ *to be the class of linear orders* L *such that for every* $x, y \in L$*, the bounded subinterval* $[x, y]$ *is in* $\mathbf{P}$*.*

Definition 7. *A property* $\mathbf{P}$ *of linear orders is* almost anti-symmetric *if for every linear order* L*,* $L \in \mathbf{P}$ *and* $L^R \in \mathbf{P}$ *imply that* L *is finite.*

Definition 8. *A property* $\mathbf{P}$ *of linear orders is* good *if it is monotone, additive and contains at least one infinite linear order.*

Lemma 1. *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then the property $\mathbf{bounded} - \mathbf{P}$ *is has the star property.*

2 Hausdorff Rank

Definition 9. *Let* $\mathbf{Q}$ *be a good property of linear orders.*

We define a property $\mathbf{Q}^\alpha$ *for every ordinal* α *as follows:*

- $\mathbf{Q}^0$ *is the class of finite linear orders.*

- *For $\alpha > 0$, $\mathbf{Q}^\alpha$ is the class of linear orders L such that $L = \sum_{i\in I} L_i$ for some $I \in \mathbf{Q}$ where for all $i \in I$, $L_i \in \mathbf{Q}^{\beta_i}$ for some $\beta_i < \alpha$*

We define $\mathbf{hrank_Q}(L) = \alpha$ iff α is the least ordinal such that $L \in \mathbf{Q}^\alpha$. This is a partial *map from linear orders to ordinals.*

Observations 1. *We claim the following without proof:*

- $\mathbf{Q}^1 = \mathbf{Q}$.
- *For all α, $\mathbf{Q}^\alpha$ is a good property.*
- $\mathbf{Q}^\alpha \subsetneq \mathbf{Q}^\beta$ *iff* $\alpha < \beta$.

Notations 1. *Let $\mathcal{H}_\alpha$ be the class of linear orders of Hausdorff rank $< \alpha$ and $\mathcal{H}_{=\alpha}$ be the class of linear orders of Hausdorff rank $= \alpha$.*

Let $\mathcal{B}_\alpha$ be the class of linear orders of Hausdorff rank $< \alpha$ on bounded subintervals.

Let $\mathcal{Q}_\alpha = \{L : 1 + L \in \mathcal{B}_\alpha\}$.

Let $\mathcal{R}_\alpha = \{L : L + 1 \in \mathcal{B}_\alpha\}$.

Clearly, $\mathcal{H}_\alpha, \mathcal{Q}_\alpha, \mathcal{R}_\alpha \subseteq \mathcal{B}_\alpha$.

Clearly, $\mathcal{H}_{\alpha+1} = \{L : \mathbf{hrank_Q}(L) \leq \alpha\}$.

Claim 1. *The following are equal:*

1. $\mathcal{H}_\alpha$
2. $\{L : 1 + L + 1 \in \mathcal{B}_\alpha\}$.
3. $\mathcal{Q}_\alpha \cap \mathcal{R}_\alpha$

Proof. The equivalence of 1 and 2 is clear, and obviously 2 implies 3.

The other direction (3 implies 2) follows from the star property of $\mathcal{B}_\alpha$. □

Lemma 2. *Let L be a linear order. Then there exists a largest subinterval $M \subseteq L$ such that $x \in M$ and $M \in \mathcal{B}_\alpha$.*

Definition 10. *Let L be a linear order. Let $x \in L$. We define $M_\alpha[x]$ to be the largest subinterval $M \subseteq L$ such that $x \in M$ and $M \in \mathcal{B}_\alpha$.*

We define $\sim_\alpha$ to be the equivalence relation on L such that $x \sim_\alpha y$ iff $M_\alpha[x] = M_\alpha[y]$.

Lemma 3. *Let L be a linear order. Let $P, Q, R \subseteq L$ be relations, such that:*

- *P represents $\sim_\alpha$ on L.*
- *Q is such that $x \in Q$ iff $M_\alpha[x] \in \mathcal{Q}_\alpha$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_\alpha$.*

Then for some linear order I there exists a decomposition $L = \sum_{i \in I} L_i$ such that $L_i \in \mathcal{B}_\alpha$ for all $i \in I$, L_i is monochromatic with respect to P, Q and R.

Furthermore, let τ_i be the n-type of L_i, p_i, q_i, r_i in $\mathbf{MSO}[p,q,r]$, where $p_i = 1_{L_i \subseteq P}$, $q_i = 1_{L_i \subseteq Q}$ and $r_i = 1_{L_i \subseteq R}$. Then the following hold

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $q(\tau_{i+1}) = 0$*

Proof. Take $I = L/\sim_\alpha$.

Then $L = \sum_{i \in I} L_i$ where L_i is the $\sim_\alpha$-equivalence class of i.

Then L_i is monochromatic with respect to P, Q and R.

The only thing left to prove is the last two conditions. The first follows from the fact that P represents $\sim_\alpha$.

The second follows because if it were not the case, then L_i and L_{i+1} would be the same $\sim_\alpha$-equivalence class. ☐

Lemma 4. *Let I be a linear order. Let $n \in \mathbb{N}$. Let p, q, r be boolean variables.*

Let τ_i be an assignment of satisfiable n-types in $\mathbf{MSO}[p,q,r]$ for all $i \in I$. Assume that

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $q(\tau_{i+1}) = 0$*

Then there exists a linear order L and $P, Q, R \subseteq L$ such that:

- *P represents $\sim_\alpha$ on L.*
- *Q is such that $x \in Q$ iff $M_\alpha[x] \in \mathcal{Q}_\alpha$.*
- *R is such that $x \in R$ iff $M_\alpha[x] \in \mathcal{R}_\alpha$.*

such that for all $i \in I$, L_i is a $\sim_\alpha$-equivalence class of L, and is thus monochromatic with respect to P, Q and R.

Furthermore, the n-type of L_i, p_i, q_i, r_i in $\mathbf{MSO}[p,q,r]$ is τ_i, where $p_i = 1_{L_i \subseteq P}$, $q_i = 1_{L_i \subseteq Q}$ and $r_i = 1_{L_i \subseteq R}$,

Proof. Since τ_i is satisfiable, we can take L_i to be a linear order of n-type τ_i such that:

- If $q(\tau_i) = r(\tau_i) = 1$, then $L_i \in \mathcal{Q}_\alpha \cap \mathcal{R}_\alpha$.
- If $q(\tau_i) = 1$ and $r(\tau_i) = 0$, then $L_i \in \mathcal{Q}_\alpha - \mathcal{R}_\alpha$.
- If $q(\tau_i) = 0$ and $r(\tau_i) = 1$, then $L_i \in \mathcal{R}_\alpha - \mathcal{Q}_\alpha$.
- If $q(\tau_i) = r(\tau_i) = 0$, then $L_i \in \mathcal{B}_\alpha - (\mathcal{Q}_\alpha \cup \mathcal{R}_\alpha)$.

Let $L = \sum_{i \in I} L_i$.

By definition each L_i is in $\mathcal{B}_\alpha$. We need to prove that each L_i is a largest $\mathcal{B}_\alpha$-subinterval in L.

On the contrary, suppose that there exist $i' \neq i$ such that $[L_i, L_{i'}] \in \mathcal{B}_\alpha$. WLOG, $L_i < L_{i'}$.

Since I is scattered, take some $i \leq a < b \leq i'$ such that there is no element between a and b in I.

Then $L_a \in \mathcal{R}_\alpha$ and $L_b \in \mathcal{Q}_\alpha$, in contradiction. □

Lemma 5. *Let L be a countable linear order.*

Let $J \subseteq L$ be some subinterval in $\mathcal{B}_\alpha$.

Then $\mathbf{hrank_Q}(J) \leq \alpha$.

Furthermore, $\mathbf{hrank_Q}(J) < \alpha$ *iff* $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Proof. Let $\{x_i\}_{i \in I} \subseteq J$ be a bidirectional, cofinal, weakly monotone I-sequence in J, i.e, $x_i \leq x_j$ if $i \leq j$ for $I \subseteq \mathbb{Z}$.

Write $J = \sum_{i \in I} [x_i, x_{i+1}]$. Then every $[x_i, x_{i+1}]$ is of Hausdorff rank $< \alpha$.

Thus, $\mathbf{hrank_Q}(J) \leq \alpha$.

Suppose $\mathbf{hrank_Q}(J) < \alpha$, then obviously $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Conversely, suppose $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.

Then $1+J+1 \in \mathcal{B}_\alpha$. But it is a bounded interval, so $\mathbf{hrank_Q}(1+J+1) < \alpha$ and thus $\mathbf{hrank_Q}(J) < \alpha$. □

Lemma 6. *Let $J \subseteq L$ be a subinterval.*

Then $\mathbf{hrank_Q}(J) \leq \alpha$ *iff J is a finite sum of $\mathcal{B}_\alpha$-subintervals.*

Note: this lemma does not work if we take a general $\mathbf{Q}$ *property.*

Proof. From the previous lemma, it is clear that if J is a finite sum of $\mathcal{B}_\alpha$-subintervals, then $\mathbf{hrank_Q}(J) \leq \alpha$, since the rank bound is preserved under finite sums.

Conversely, suppose $\mathbf{hrank_Q}(J) \leq \alpha$.

If $J = \sum_{i \in \mathbb{Z}} J_i$ for some J_i of Hausdorff rank $< \alpha$, take $x, y \in J$. Then let $x \in J_{i_1}$ and $y \in J_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} J_i$. But the last sum is of rank $< \alpha$ and thus $[x, y]$ is of rank $< \alpha$. That is, $J \in \mathcal{B}_\alpha$.

Since every subinterval of rank $\leq \alpha$ is a finite sum of $\mathbb{Z}$-sums of intervals of rank $< \alpha$, we are done. □

Corollary 1. *Let $J \subseteq L$ be a subinterval.*

Then $\mathbf{hrank_Q}(J) \leq \alpha$ *iff J is a finite sum of largest $\mathcal{B}_\alpha$-subintervals in L*

Lemma 7. *There exists a global computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,* $\mathbf{type}_n\left[\mathcal{H}_{f(n)+1}\right] = \mathbf{type}_n\left[\mathcal{H}_{f(n)}\right]$.

Equivalently, every linear order of finite rank is n-equivalent to some linear order of rank $\leq f(n)$.

Proof. Since there exist only a finite number of n-types, and the ω-sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k\in\omega}$ is monotone, it must stabilize at some point.

This point is computable as a function of n, because $\mathbf{type}_n[\mathcal{H}_k]$ is computable for every finite k. □

Lemma 8. *There exist global computable functions $a, b : \mathbb{N} \to \mathbb{N}$ such that for all $n, c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 \geq a(n)$ and $c_1 \equiv c_2 \mod b(n)$,*

$$\mathbf{type}_n[\mathcal{H}_{c_1}] = \mathbf{type}_n[\mathcal{H}_{c_2}]$$

Equivalently, the sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k\in\omega}$ is ultimately periodic for all $n \in \mathbb{N}$. Furthermore, the starting point and the period itself can be computed as a function of n.

Proof. Let $n \in \mathbb{N}$.

Since there exist only a finite number of possible sets of n-types, there exist (and can be computed) some $a(n) > f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n\left[\mathcal{H}_{a(n)}\right] = \mathbf{type}_n\left[\mathcal{H}_{a(n)+b(n)}\right]$$

We shall prove by induction that for all $c \geq a(n) + b(n)$,

$$\mathbf{type}_n[\mathcal{H}_c] = \mathbf{type}_n\left[\mathcal{H}_{c+b(n)}\right]$$

This will complete the proof.

The base case $c = a(n)$ has been proven in the beginning.

Suppose the induction hypothesis holds for c.

Let L be of rank $c + 1$.

Write $L = \sum_{i\in I} L_i$ where $\mathbf{hrank}_{\mathbf{Q}}(L_i) < c + 1$, and $\mathbf{hrank}_{\mathbf{Q}}(L_i) = c$ infinitely many times.

By the induction hypothesis, if $\mathbf{hrank}_{\mathbf{Q}}(L_i) = c$, we can find $N_i \equiv_n L_i$ with $\mathbf{hrank}_{\mathbf{Q}}(N_i) = c + b(n)$. Setting $N_i := L_i$ for all other i, we conclude that $N := \sum_{i\in I} N_i$ is n-equivalent to L.

However, clearly $\mathbf{hrank}_{\mathbf{Q}}(N) = c + b(n) + 1$. So overall,

$$\mathbf{type}_n[\mathcal{H}_{c+1}] \subseteq \mathbf{type}_n\left[\mathcal{H}_{c+b(n)+1}\right]$$

Conversely, suppose L is of rank $c + b(n) + 1$. Write $L = \sum_{i\in I} L_i$ where $\mathbf{hrank}_{\mathbf{Q}}(L_i) < c + b(n) + 1$, and $\mathbf{hrank}_{\mathbf{Q}}(L_i) = c + b(n)$ infinitely many times.

By the induction hypothesis, we can find for all i such that $\mathbf{hrank}_{\mathbf{Q}}(L_i) = c + b(n)$ some $N_i \equiv_n L_i$ with $\mathbf{hrank}_{\mathbf{Q}}(N_i) = c$. Furthermore, since $c \geq a(n) > f(n)$, we can find $N_i \equiv_n L_i$ with $\mathbf{hrank}_{\mathbf{Q}}(N_i) \leq f(n) < c$ for all other i.

We conclude that $N := \sum_{i\in I} N_i$ is n-equivalent to L. However, clearly $\mathbf{hrank}_{\mathbf{Q}}(N) = c + 1$. So overall,

$$\mathbf{type}_n\left[\mathcal{H}_{c+b(n)+1}\right] \subseteq \mathbf{type}_n[\mathcal{H}_{c+1}]$$

So we have proven the induction step, and the lemma follows. □

Lemma 9. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Then,

$$\mathbf{type}_n \left[\mathcal{H}_\alpha\right] = \bigcup_{c<b(n)} \mathbf{type}_n \left[\mathcal{H}_{c+b(n)}\right]$$

In particular, $\mathbf{type}_n \left[\mathcal{H}_\alpha\right]$ can be computed, and is independent of the choice $\alpha \geq \omega$.

Proof. TBC. $\square$

Proof. By induction on $\alpha \geq f(n)$ suppose that for all $f(n) \leq \beta < \alpha$,

$$\mathbf{type}_n \left[\mathcal{H}_{\beta+1}\right] = \mathbf{type}_n \left[\mathcal{H}_{f(n)}\right]$$

Let L be a scattered linear order of rank α.

Then $L = \sum_{i \in I} L_i$ where $\mathbf{hrank}_{\mathbf{Q}}\left(L_i\right) < \alpha$. By the induction hypothesis, we can find $N_i \equiv_n L_i$ with $\mathbf{hrank}_{\mathbf{Q}}\left(N_i\right) < f(n)$.

Let $N = \sum_{i \in I} N_i$. Then $L \equiv_n N$.

Additionally, $\mathbf{hrank}_{\mathbf{Q}}\left(N\right) < f(n) + 1$ by the rank definition. However, that means that we can find $N' \equiv_n N$ with $\mathbf{hrank}_{\mathbf{Q}}\left(N'\right) \leq f(n)$ by the definition of $f(n)$. $\square$

Corollary 2. *The following sequences stabilize at $f(n)$:*

- $\mathbf{type}_n \left[\mathcal{H}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{B}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{Q}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{R}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{Q}_\alpha - \mathcal{R}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{R}_\alpha - \mathcal{Q}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{B}_\alpha - \left(\mathcal{Q}_\alpha \cup \mathcal{R}_\alpha\right)\right]$

Proof. Let A_k be the set of all satisfiable n-types of rank $< k$. Then A_{k+1} is the closure of A_k under finite sums of $\subseteq \mathbb{Z}$-sums.

The sequence $A_0 \subseteq A_1 \subseteq \ldots$ stabilizes at some point. Suppose $A_{f(n)} = A_{f(n)+1}$.

Suppose L has rank $\beta \geq f(n)$.

Write $L = \sum_{i \in I} L_i$ where $\mathbf{hrank}_{\mathbf{Q}}\left(L_i\right) < \beta$, and I is a finite sum of $\subseteq \mathbb{Z}$.

If β is a limit ordinal, then there must be a bi-cofinal sequence i_k such that $\mathbf{hrank}_{\mathbf{Q}}\left(L_{i_k}\right) \to \beta$.

If β is a successor ordinal, then $\mathbf{hrank}_{\mathbf{Q}}\left(L_i\right) = \beta - 1$ must hold infinitely many times.

Now we proceed by induction on $\alpha \geq f(n)$.

1. If $\mathcal{C} = \mathcal{H}_\alpha$, we take $L' \in A_{f(n)}$, which necessarily has rank $< f(n) \leq \alpha$.

2. If $\mathcal{C} = \mathcal{Q}_\alpha - \mathcal{R}_\alpha$, we take an ω-sequence α_k such that $\alpha_k \to \alpha$ (if α is a limit ordinal) or $\alpha_k = \alpha - 1$ (if α is a successor ordinal).

   Then we take $L' = \sum_{i\in\omega} L'_i$ where $\mathbf{hrank_Q}\left(L'_{i_k}\right) = \alpha_k$ (and $\mathbf{hrank_Q}\left(L'_i\right) = \mathbf{hrank_Q}\left(L_i\right)$ for every other i). Then $L' \in \mathcal{Q}_\alpha - \mathcal{R}_\alpha$, but also $L' \equiv_n L$.

3. This is just the same with $-\omega$ instead of ω.

4. This is just the same with $\mathbb{Z}$ instead of $-\omega$.

$\square$

Corollary 3. *Over countable linear orders with interpretations of P, Q and R as above, the properties $\mathbf{hrank_Q}(\cdot) \leq \alpha$, $\mathbf{hrank_Q}(\cdot) < \alpha$ and $\mathbf{hrank_Q}(\cdot) = \alpha$ over subintervals are all expressible in $\mathbf{MSO}[P,Q,R]$.*

Proof. For $\mathbf{hrank_Q}(\cdot) \leq \alpha$ and $\mathbf{hrank_Q}(\cdot) < \alpha$, we can use the previous lemmas.

For $\mathbf{hrank_Q}(\cdot) = \alpha$, we can use the previous two. $\square$

Theorem 1. *There is a an algorithm solving satisfiability for $\mathbf{MSO}[P,Q,R]$ over countable linear orders, given an oracle which solves the satisfiability problem for $\mathbf{MSO}$ over countable linear orders.*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[P,Q,R]$ formula φ of quantifier-depth n. outputs an $\mathbf{MSO}[\{X_\tau\}_\tau]$ formula ψ.

Let P_L, Q_L, R_L be the interpretations of P, Q, R on L.

Then

$$L, P := P_L, Q := Q_L, R := R_L \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where $I_\tau = \{i \in I : L_i \models \tau\}$ for every n-type τ.

Let T be the set of n-types in $\mathbf{MSO}[p,q,r]$ which satisfy $q(\tau) = 1 \iff \tau \in \mathcal{Q}_\alpha$ and $r(\tau) = 1 \iff \tau \in \mathcal{R}_\alpha$.

Let $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee q(\tau_2) = 0)\}$.

Then T and S can be calculated using the oracle.

Then ψ is an $\mathbf{MSO}[T,S]$ formula.

Then we define an $\mathbf{MSO}[p,q,r]$ formula ψ' as follows:

ψ' claims that there exists a partition (with possible empty sets) $\{Y_\tau\}_\tau$ of I such that

- Every $i \in I$ is in some Y_τ for $\tau \in T$.

- If $i' = i + 1$ in I, then for some $(\tau_1, \tau_2) \in S$, $i \in Y_{\tau_1}$ and $i' \in Y_{\tau_2}$.

Now we claim that φ is satisfiable in some linear order, iff ψ' is satisfiable in some linear order.

Suppose φ is satisfiable in some linear order L.

Take a decomposition $L = \sum_{i\in I} L_i$ as in lemma 2.

Then ψ holds over the assignment $X_\tau := I_\tau$. But by lemma 2, this assignment satisfies the condition required for ψ' to hold. Then ψ' holds over I.

Conversely, suppose ψ' holds in I.

Let $X_\tau := Z_\tau$ be the assignment that is guaranteed by ψ'.

Let tau_i be the unique τ such that $i \in Z_\tau$.

Then the conditions for lemma 3 are guaranteed.

Thus, take L as in lemma 3. Then ψ holds over I when we set $X_i := Z_\tau$. But $Z_\tau = I_\tau$ for all τ, so φ holds over L. $\square$