

# Orders

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## 1 Properties

**Definition 1.** A property  $\mathbf{P}$  of linear orders is a class of linear orders which is closed under isomorphism.

**Definition 2.** A property  $\mathbf{P}$  of linear orders is monotone if for every linear order  $M$ ,  $M \in \mathbf{P}$  implies that every suborder of  $M$  is in  $\mathbf{P}$ .

**Definition 3.** A property  $\mathbf{P}$  of linear orders is symmetric if for every linear order  $M$ ,  $M \in \mathbf{P}$  iff  $M^R \in \mathbf{P}$ .

**Definition 4.** A property  $\mathbf{P}$  of linear orders is an additive property if for every linear orders  $M_1$  and  $M_2$ ,  $M_1 + M_2 \in \mathbf{P}$  iff  $M_1, M_2 \in \mathbf{P}$ .

**Definition 5.** Let  $\mathbf{P}$  be a property of linear orders.

We define  $\mathbf{P}$ -bounded to be the class of linear orders  $M$  such that for every  $x, y \in M$ , the bounded subinterval  $[x, y]$  is in  $\mathbf{P}$ .

**Definition 6.** A property  $\mathbf{P}$  of linear orders is almost anti-symmetric if for every linear order  $M$ ,  $M \in \mathbf{P}$  and  $M^R \in \mathbf{P}$  imply that  $M$  is finite.

**Definition 7.** A property  $\mathbf{P}$  of linear orders is good if it is monotone, additive and contains at least one infinite linear order.

**Definition 8.** A property  $\mathbf{P}$  of linear orders is a star property if for every linear orders  $M$ , and every family  $\mathcal{F} \subseteq \mathbf{P}$  of subintervals of  $M$  such that  $J_1 \cap J_2 \neq \emptyset$  for every  $J_1, J_2 \in \mathcal{F}$ , we have that  $\bigcup \mathcal{F} \in \mathbf{P}$ .

**Lemma 1** (Star Lemma). Let  $\mathbf{P}$  be an additive property of linear orders.

Then the property  $\mathbf{P}$ -bounded has the star property.

*Proof.* Let  $M$  be a linear order, and let  $\mathcal{F} \subseteq \mathbf{P}$ -bounded be a family of subintervals of  $M$ .

Let  $[x, y] \subseteq \bigcup \mathcal{F}$  be any bounded subinterval. We need to prove it is in  $\mathbf{P}$ .

Suppose  $x \in J_1$  and  $y \in J_2$  for  $J_1, J_2 \in \mathcal{F}$ .

Since  $J_1 \cap J_2 \neq \emptyset$ , we can take  $z \in J_1 \cap J_2$ .

Then  $[x, z] \subseteq J_1$  and  $[z, y] \subseteq J_2$ , and thus by  $\mathbf{P}$ -bounded,  $[x, z], [z, y] \in \mathbf{P}$ .

However,  $\mathbf{P}$  is additive. Since  $[x, y]$  is either the sum or difference of  $[x, z]$  and  $[z, y]$ , we have that  $[x, y] \in \mathbf{P}$ .  $\square$

**Lemma 2.** *Let  $\mathbf{P}$  be a star property.*

*Then for every linear order  $M$ , and every point  $x \in M$ , there exists a largest subinterval  $J \subseteq M$  such that  $J \in \mathbf{P}$ .*

*Thus, we can define an equivalence relation  $\sim_{\mathbf{P}}$  on  $M$  such that  $x \sim_{\mathbf{P}} y$  iff  $x$  and  $y$  are in the same largest  $\mathbf{P}$ -subinterval.*

*Proof.* Let  $J \subseteq M$  be the union of all **P-bounded**-subintervals containing  $x$ . All such subintervals intersect at  $x$ .

Therefore, by the star lemma,  $J$  is in **P-bounded**, and by definition  $J$  is the largest  $\mathbf{P}$ -subinterval containing  $x$ .

Thus we can define the equivalence relation  $\sim_{\mathbf{P}}$  as above.  $\square$

## 2 Hausdorff Rank

**Definition 9.** *Let  $\mathbf{Q}$  be a good property of linear orders.*

*We define a property  $\mathbf{Q}^{\leq \alpha}$  for every ordinal  $\alpha$  as follows:*

- $\mathbf{Q}^{\leq 0}$  is the class of finite linear orders.
- For  $\alpha > 0$ ,  $\mathbf{Q}^{\leq \alpha}$  is the class of linear orders  $M$  such that  $M = \sum_{i \in I} M_i$  for some  $I \in \mathbf{Q}$  where for all  $i \in I$ ,  $M_i \in \mathbf{Q}^{\beta_i}$  for some  $\beta_i < \alpha$

*We define further  $\mathbf{Q}^{< \alpha} = \bigcup_{\beta < \alpha} \mathbf{Q}^{\leq \beta}$  and  $\mathbf{Q}^{= \alpha} = \mathbf{Q}^{\leq \alpha} - \mathbf{Q}^{< \alpha}$ .*

*We define  $\mathbf{hrank}_{\mathbf{Q}}(M) = \alpha$  iff  $M \in \mathbf{Q}^{= \alpha}$ . This is a partial map from linear orders to ordinals.*

**Observations 1.** *Let  $\mathbf{Q}$  be a good property.*

*We claim the following without proof:*

- $\mathbf{Q}^{\leq 1} = \mathbf{Q}$ .
- For all  $\alpha$ ,  $\mathbf{Q}^{\leq \alpha}$  is a good property.
- $\mathbf{Q}^{\leq \alpha} \subsetneq \mathbf{Q}^{\leq \beta}$  iff  $\alpha < \beta$ .

**Definitions 1.**  $\mathcal{B}_{< \alpha} := \mathbf{Q}^{< \alpha}$ -**bounded** is the class of linear orders of rank  $< \alpha$  on bounded intervals.

$\mathcal{R}_{< \alpha}$  is the class of linear orders  $M$  where  $M + 1 \in \mathcal{B}_{< \alpha}$ .

$\mathcal{L}_{< \alpha}$  is the class of linear orders  $M$  where  $1 + M \in \mathcal{B}_{< \alpha}$ .

**Claim 1.** *The following are equal:*

1.  $\mathbf{Q}^{< \alpha}$
2.  $\{M : 1 + M + 1 \in \mathcal{B}_{< \alpha}\}$ .
3.  $\mathcal{L}_{< \alpha} \cap \mathcal{R}_{< \alpha}$

*Proof.* The only nontrivial direction is 3 implies 2, which follows from the star property of  $\mathcal{B}_{< \alpha}$ .  $\square$

**Lemma 3.** *Let  $M$  be a linear order. Then there exists a largest subinterval  $J \subseteq M$  such that  $x \in J$  and  $J \in \mathcal{B}_{<\alpha}$ .*

**Definition 10.** *Let  $M$  be a linear order. Let  $x \in M$ . We define  $M_\alpha[x]$  to be the largest subinterval  $M \subseteq M$  such that  $x \in M$  and  $M \in \mathcal{B}_{<\alpha}$ .*

*We define  $\sim_\alpha$  to be the equivalence relation on  $M$  such that  $x \sim_\alpha y$  iff  $M_\alpha[x] = M_\alpha[y]$ .*

**Lemma 4.** *Let  $M$  be a linear order. Let  $P, L, R \subseteq M$  be relations, such that:*

- *$P$  represents  $\sim_\alpha$  on  $M$ .*
- *$L$  is such that  $x \in L$  iff  $M_\alpha[x] \in \mathcal{L}_{<\alpha}$ .*
- *$R$  is such that  $x \in R$  iff  $M_\alpha[x] \in \mathcal{R}_{<\alpha}$ .*

*Then for some linear order  $I$  there exists a decomposition  $M = \sum_{i \in I} M_i$  such that  $M_i \in \mathcal{B}_{<\alpha}$  for all  $i \in I$ ,  $M_i$  is monochromatic with respect to  $P$ ,  $L$  and  $R$ .*

*Furthermore, let  $\tau_i$  be the  $n$ -type of  $M_i, p_i, q_i, r_i$  in  $\mathbf{MSO}[p, \ell, r]$ , where  $p_i = 1_{M_i \subseteq P}$ ,  $q_i = 1_{M_i \subseteq L}$  and  $r_i = 1_{M_i \subseteq R}$ . Then the following hold*

- *if  $i$  has a successor,  $p(\tau_i) \neq p(\tau_{i+1})$*
- *if  $i$  has a successor, either  $r(\tau_i) = 0$  or  $\ell(\tau_{i+1}) = 0$*

*Proof.* Take  $I = M / \sim_\alpha$ .

Then  $M = \sum_{i \in I} M_i$  where  $M_i$  is the  $\sim_\alpha$ -equivalence class of  $i$ .

Then  $M_i$  is monochromatic with respect to  $P$ ,  $L$  and  $R$ .

The only thing left to prove is the last two conditions. The first follows from the fact that  $P$  represents  $\sim_\alpha$ .

The second follows because if it were not the case, then  $M_i$  and  $M_{i+1}$  would be the same  $\sim_\alpha$ -equivalence class.  $\square$

**Lemma 5.** *Let  $I$  be a linear order. Let  $n \in \mathbb{N}$ . Let  $p, \ell, r$  be boolean variables.*

*Let  $\tau_i$  be an assignment of satisfiable  $n$ -types in  $\mathbf{MSO}[p, \ell, r]$  for all  $i \in I$ . Assume that*

- *if  $i$  has a successor,  $p(\tau_i) \neq p(\tau_{i+1})$*
- *if  $i$  has a successor, either  $r(\tau_i) = 0$  or  $\ell(\tau_{i+1}) = 0$*

*Then there exists a linear order  $M$  and  $P, L, R \subseteq M$  such that:*

- *$P$  represents  $\sim_\alpha$  on  $M$ .*
- *$L$  is such that  $x \in L$  iff  $M_\alpha[x] \in \mathcal{L}_{<\alpha}$ .*
- *$R$  is such that  $x \in R$  iff  $M_\alpha[x] \in \mathcal{R}_{<\alpha}$ .*

such that for all  $i \in I$ ,  $M_i$  is a  $\sim_\alpha$ -equivalence class of  $M$ , and is thus monochromatic with respect to  $P$ ,  $L$  and  $R$ .

Furthermore, the  $n$ -type of  $M_i, p_i, q_i, r_i$  in  $\mathbf{MSO}[p, \ell, r]$  is  $\tau_i$ , where  $p_i = 1_{M_i \subseteq P}$ ,  $q_i = 1_{M_i \subseteq L}$  and  $r_i = 1_{M_i \subseteq R}$ ,

*Proof.* Since  $\tau_i$  is satisfiable, we can take  $M_i$  to be a linear order of  $n$ -type  $\tau_i$  such that:

- If  $\ell(\tau_i) = r(\tau_i) = 1$ , then  $M_i \in \mathcal{L}_{<\alpha} \cap \mathcal{R}_{<\alpha}$ .
- If  $\ell(\tau_i) = 1$  and  $r(\tau_i) = 0$ , then  $M_i \in \mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}$ .
- If  $\ell(\tau_i) = 0$  and  $r(\tau_i) = 1$ , then  $M_i \in \mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}$ .
- If  $\ell(\tau_i) = r(\tau_i) = 0$ , then  $M_i \in \mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})$ .

Let  $M = \sum_{i \in I} M_i$ .

By definition each  $M_i$  is in  $\mathcal{B}_{<\alpha}$ . We need to prove that each  $M_i$  is a largest  $\mathcal{B}_{<\alpha}$ -subinterval in  $M$ .

On the contrary, suppose that there exist  $i' \neq i$  such that  $[M_i, M_{i'}] \in \mathcal{B}_{<\alpha}$ . WLOG,  $M_i < M_{i'}$ .

Since  $I$  is scattered, take some  $i \leq a < b \leq i'$  such that there is no element between  $a$  and  $b$  in  $I$ .

Then  $M_a \in \mathcal{R}_{<\alpha}$  and  $M_b \in \mathcal{L}_{<\alpha}$ , in contradiction.  $\square$

**Lemma 6.** *Let  $M$  be a countable linear order.*

*Let  $J \subseteq M$  be some subinterval in  $\mathcal{B}_{<\alpha}$ .*

*Then  $\mathbf{hrank}_{\mathbf{Q}}(J) \leq \alpha$ .*

*Furthermore,  $\mathbf{hrank}_{\mathbf{Q}}(J) < \alpha$  iff  $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$ .*

*Proof.* Let  $\{x_i\}_{i \in I} \subseteq J$  be a bidirectional, cofinal, weakly monotone  $I$ -sequence in  $J$ , i.e,  $x_i \leq x_j$  if  $i \leq j$  for  $I \subseteq \mathbb{Z}$ .

Write  $J = \sum_{i \in I} [x_i, x_{i+1}]$ . Then every  $[x_i, x_{i+1}]$  is of Hausdorff rank  $< \alpha$ .

Thus,  $\mathbf{hrank}_{\mathbf{Q}}(J) \leq \alpha$ .

Suppose  $\mathbf{hrank}_{\mathbf{Q}}(J) < \alpha$ , then obviously  $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$ .

Conversely, suppose  $J \in \mathcal{Q}_{<\alpha} \cap \mathcal{R}_{<\alpha}$ .

Then  $1 + J + 1 \in \mathcal{B}_{<\alpha}$ . But it is a bounded interval, so  $\mathbf{hrank}_{\mathbf{Q}}(1 + J + 1) < \alpha$  and thus  $\mathbf{hrank}_{\mathbf{Q}}(J) < \alpha$ .  $\square$

**Lemma 7.** *Let  $J \subseteq M$  be a subinterval.*

*Then  $\mathbf{hrank}_{\mathbf{Q}}(J) \leq \alpha$  iff  $J$  is a finite sum of  $\mathcal{B}_{<\alpha}$ -subintervals.*

*Note: this lemma does not work if we take a general  $\mathbf{Q}$  property.*

*Proof.* From the previous lemma, it is clear that if  $J$  is a finite sum of  $\mathcal{B}_{<\alpha}$ -subintervals, then  $\mathbf{hrank}_{\mathbf{Q}}(J) \leq \alpha$ , since the rank bound is preserved under finite sums.

Conversely, suppose  $\mathbf{hrank}_{\mathbf{Q}}(J) \leq \alpha$ .

If  $J = \sum_{i \in \mathbb{Z}} J_i$  for some  $J_i$  of Hausdorff rank  $< \alpha$ , take  $x, y \in J$ . Then let  $x \in J_{i_1}$  and  $y \in J_{i_2}$ .

Then  $[x, y] \subseteq \sum_{i \in [i_1, i_2]} J_i$ . But the last sum is of rank  $< \alpha$  and thus  $[x, y]$  is of rank  $< \alpha$ . That is,  $J \in \mathcal{B}_{<\alpha}$ .

Since every subinterval of rank  $\leq \alpha$  is a finite sum of  $\mathbb{Z}$ -sums of intervals of rank  $< \alpha$ , we are done.  $\square$

**Corollary 1.** *Let  $J \subseteq M$  be a subinterval.*

*Then  $\mathbf{hrank}_{\mathbf{Q}}(J) \leq \alpha$  iff  $J$  is a finite sum of largest  $\mathcal{B}_{<\alpha}$ -subintervals in  $M$*

**Lemma 8.** *There exists a global computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n \in \mathbb{N}$ ,  $\mathbf{type}_n[\mathcal{H}_{f(n)+1}] = \mathbf{type}_n[\mathcal{H}_{f(n)}]$ .*

*Equivalently, every linear order of finite rank is  $n$ -equivalent to some linear order of rank  $\leq f(n)$ .*

*Proof.* Since there exist only a finite number of  $n$ -types, and the  $\omega$ -sequence  $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$  is monotone, it must stabilize at some point.

This point is computable as a function of  $n$ , because  $\mathbf{type}_n[\mathcal{H}_k]$  is computable for every finite  $k$ .  $\square$

**Lemma 9.** *There exist global computable functions  $a, b : \mathbb{N} \rightarrow \mathbb{N}$  such that for all  $n, c_1, c_2 \in \mathbb{N}$  such that  $c_1, c_2 \geq a(n)$  and  $c_1 \equiv c_2 \pmod{b(n)}$ ,*

$$\mathbf{type}_n[\mathcal{H}_{c_1}] = \mathbf{type}_n[\mathcal{H}_{c_2}]$$

*Equivalently, the sequence  $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$  is ultimately periodic for all  $n \in \mathbb{N}$ . Furthermore, the starting point and the period itself can be computed as a function of  $n$ .*

*Proof.* Let  $n \in \mathbb{N}$ .

Since there exist only a finite number of possible sets of  $n$ -types, there exist (and can be computed) some  $a(n) > f(n)$ ,  $a(n) + b(n)$  such that

$$\mathbf{type}_n[\mathcal{H}_{a(n)}] = \mathbf{type}_n[\mathcal{H}_{a(n)+b(n)}]$$

We shall prove by induction that for all  $c \geq a(n) + b(n)$ ,

$$\mathbf{type}_n[\mathcal{H}_c] = \mathbf{type}_n[\mathcal{H}_{c+b(n)}]$$

This will complete the proof.

The base case  $c = a(n)$  has been proven in the beginning.

Suppose the induction hypothesis holds for  $c$ .

Let  $M$  be of rank  $c + 1$ .

Write  $M = \sum_{i \in I} M_i$  where  $\mathbf{hrank}_{\mathbf{Q}}(M_i) < c + 1$ , and  $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c$  infinitely many times.

By the induction hypothesis, if  $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c$ , we can find  $N_i \equiv_n M_i$  with  $\mathbf{hrank}_{\mathbf{Q}}(N_i) = c + b(n)$ . Setting  $N_i := M_i$  for all other  $i$ , we conclude that  $N := \sum_{i \in I} N_i$  is  $n$ -equivalent to  $M$ .

However, clearly  $\mathbf{hrank}_{\mathbf{Q}}(N) = c + b(n) + 1$ . So overall,

$$\mathbf{type}_n[\mathcal{H}_{c+1}] \subseteq \mathbf{type}_n[\mathcal{H}_{c+b(n)+1}]$$

Conversely, suppose  $M$  is of rank  $c + b(n) + 1$ . Write  $M = \sum_{i \in I} M_i$  where  $\mathbf{hrank}_{\mathbf{Q}}(M_i) < c + b(n) + 1$ , and  $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c + b(n)$  infinitely many times.

By the induction hypothesis, we can find for all  $i$  such that  $\mathbf{hrank}_{\mathbf{Q}}(M_i) = c + b(n)$  some  $N_i \equiv_n M_i$  with  $\mathbf{hrank}_{\mathbf{Q}}(N_i) = c$ . Furthermore, since  $c \geq a(n) > f(n)$ , we can find  $N_i \equiv_n M_i$  with  $\mathbf{hrank}_{\mathbf{Q}}(N_i) \leq f(n) < c$  for all other  $i$ .

We conclude that  $N := \sum_{i \in I} N_i$  is  $n$ -equivalent to  $M$ . However, clearly  $\mathbf{hrank}_{\mathbf{Q}}(N) = c + 1$ . So overall,

$$\mathbf{type}_n[\mathcal{H}_{c+b(n)+1}] \subseteq \mathbf{type}_n[\mathcal{H}_{c+1}]$$

So we have proven the induction step, and the lemma follows.  $\square$

**Lemma 10.** *Let  $n \in \mathbb{N}$ , and let  $\alpha \geq \omega$  be an ordinal.*

*Then,*

$$\mathbf{type}_n[\mathbf{Q}^{=\alpha}] = \bigcup_{c < b(n)} \mathbf{type}_n[\mathcal{H}_{c+b(n)}]$$

*In particular,  $\mathbf{type}_n[\mathbf{Q}^{=\alpha}]$  can be computed, and is independent of the choice  $\alpha \geq \omega$ .*

*Proof.* TBC.  $\square$

*Proof.* By induction on  $\alpha \geq f(n)$  suppose that for all  $f(n) \leq \beta < \alpha$ ,

$$\mathbf{type}_n[\mathcal{H}_{\beta+1}] = \mathbf{type}_n[\mathcal{H}_{f(n)}]$$

Let  $M$  be a scattered linear order of rank  $\alpha$ .

Then  $M = \sum_{i \in I} M_i$  where  $\mathbf{hrank}_{\mathbf{Q}}(M_i) < \alpha$ . By the induction hypothesis, we can find  $N_i \equiv_n M_i$  with  $\mathbf{hrank}_{\mathbf{Q}}(N_i) < f(n)$ .

Let  $N = \sum_{i \in I} N_i$ . Then  $M \equiv_n N$ .

Additionally,  $\mathbf{hrank}_{\mathbf{Q}}(N) < f(n) + 1$  by the rank definition. However, that means that we can find  $N' \equiv_n N$  with  $\mathbf{hrank}_{\mathbf{Q}}(N') \leq f(n)$  by the definition of  $f(n)$ .  $\square$

**Corollary 2.** *The following sequences stabilize at  $f(n)$ :*

- $\mathbf{type}_n[\mathcal{H}_{\alpha}]$
- $\mathbf{type}_n[\mathcal{B}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{L}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{R}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}]$
- $\mathbf{type}_n[\mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})]$

*Proof.* The corollary is false and should be fixed.  $\square$

**Corollary 3.** *Over countable linear orders with interpretations of  $P$ ,  $L$  and  $R$  as above, the properties  $\mathbf{hrank}_{\mathbf{Q}}(\cdot) \leq \alpha$ ,  $\mathbf{hrank}_{\mathbf{Q}}(\cdot) < \alpha$  and  $\mathbf{hrank}_{\mathbf{Q}}(\cdot) = \alpha$  over subintervals are all expressible in  $\mathbf{MSO}[P, L, R]$ .*

*Proof.* For  $\mathbf{hrank}_{\mathbf{Q}}(\cdot) \leq \alpha$  and  $\mathbf{hrank}_{\mathbf{Q}}(\cdot) < \alpha$ , we can use the previous lemmas.

For  $\mathbf{hrank}_{\mathbf{Q}}(\cdot) = \alpha$ , we can use the previous two.  $\square$

**Theorem 1.** *There is an algorithm solving satisfiability for  $\mathbf{MSO}[P, L, R]$  over countable linear orders, given an oracle which solves the satisfiability problem for  $\mathbf{MSO}$  over countable linear orders.*

*Proof.* By the decomposition theorem, there exists a translation, that given an  $\mathbf{MSO}[P, L, R]$  formula  $\varphi$  of quantifier-depth  $n$ , outputs an  $\mathbf{MSO}[\{X_\tau\}_\tau]$  formula  $\psi$ .

Let  $P_L, Q_L, R_M$  be the interpretations of  $P, L, R$  on  $M$ .

Then

$$M, P := P_L, L := Q_L, R := R_M \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where  $I_\tau = \{i \in I : M_i \models \tau\}$  for every  $n$ -type  $\tau$ .

Let  $T$  be the set of  $n$ -types in  $\mathbf{MSO}[p, \ell, r]$  which satisfy  $\ell(\tau) = 1 \iff \tau \in \mathcal{L}_{<\alpha}$  and  $r(\tau) = 1 \iff \tau \in \mathcal{R}_{<\alpha}$ .

Let  $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee \ell(\tau_2) = 0)\}$ .

Then  $T$  and  $S$  can be calculated using the oracle.

Then  $\psi$  is an  $\mathbf{MSO}[T, S]$  formula.

Then we define an  $\mathbf{MSO}[p, \ell, r]$  formula  $\psi'$  as follows:

$\psi'$  claims that there exists a partition (with possible empty sets)  $\{Y_\tau\}_\tau$  of  $I$  such that

- Every  $i \in I$  is in some  $Y_\tau$  for  $\tau \in T$ .
- If  $i' = i + 1$  in  $I$ , then for some  $(\tau_1, \tau_2) \in S$ ,  $i \in Y_{\tau_1}$  and  $i' \in Y_{\tau_2}$ .

Now we claim that  $\varphi$  is satisfiable in some linear order, iff  $\psi'$  is satisfiable in some linear order.

Suppose  $\varphi$  is satisfiable in some linear order  $M$ .

Take a decomposition  $M = \sum_{i \in I} M_i$  as in lemma 2.

Then  $\psi$  holds over the assignment  $X_\tau := I_\tau$ . But by lemma 2, this assignment satisfies the condition required for  $\psi'$  to hold. Then  $\psi'$  holds over  $I$ .

Conversely, suppose  $\psi'$  holds in  $I$ .

Let  $X_\tau := Z_\tau$  be the assignment that is guaranteed by  $\psi'$ .

Let  $\tau_i$  be the unique  $\tau$  such that  $i \in Z_\tau$ .

Then the conditions for lemma 3 are guaranteed.

Thus, take  $M$  as in lemma 3. Then  $\psi$  holds over  $I$  when we set  $X_i := Z_{\tau_i}$ .

But  $Z_\tau = I_\tau$  for all  $\tau$ , so  $\varphi$  holds over  $M$ .  $\square$