1 Ordinals

In this section the logic is always MSO, and the structures are ordinals.

Notations 1. *Let $\iota_\alpha(X)$ denote that the set X is a left-closed, right-open interval isomorphic to ω^α.*

Let $\pi_\alpha(X)$ denote that the set X is a finite union of intervals and is isomorphic to ω^α.

Let $\kappa_\alpha(x)$ denote that x is a multiple of ω^α.

Proposition 1. *Let $\beta = \omega^\alpha$ be an ordinals and x, y some ordinals.*

Then, $[x, y)$ is isomorphic to ω^α iff there exists some z which is a multiple of ω^α, such that y is the least greater multiple of ω^α s.t. $z \leq x < y$.

Proof. Divide $x = \gamma \cdot \beta + \delta$ where $\delta < \beta$, and as $x + \beta = y$, $y = \gamma \cdot \beta + \delta + \beta$.

Now, as ω^α is additively indecomposable, $\delta + \beta = \beta$, so $y = \gamma \cdot \beta + \beta = (\gamma + 1) \cdot \beta$. Choosing $z = \gamma + 1$, indeed y is the least greter multiple of ω^α, and $z \leq x < y$, as we need. □

Corollary 1. *Over ordinals, ι_α is expressible via κ_α.*

Proof. By the previous lemma,

$$\iota_\alpha(X) \iff \exists x, y, z. X = [x, y] \wedge \kappa_\alpha(z) \wedge y = \min_b \{a : z < b \wedge \kappa_\alpha(b)\}$$

□

Proposition 2. *Let $\beta = \omega^\alpha$. Then, a set X satisfies π_α iff it is a finite sum $X = \sum_{i=1}^m X_i$ where for $i < m$, X_i is a strict prefix of some interval satisfying ι_α, and for $i = m$, X_i satisfies ι_α.*

Proof. Suppose X_i is indeed an interval which is less then ω^α for $i < m$ and equals ω^α for $i = m$. Since ω^α is additively indecomposable, the sum $\sum_{i=1}^m X_i$ is indeed ω^α. Therefore, X is a disjoint sum of intervals isomorphic to ω^α.

For the other direction, suppose X is a disjoint sum of intervals isomorphic to ω^α. Let the intervals be X_i for $i = 1, ..., m$. Since ω^α is additively indecomposable, nessessarily $X_i < \beta$ for $i < m$ and $X_m = \beta$.

Also, note that for $i < m$, the interval from the left end of X_i to the right end of X_m is at least ω^α, since it contains $X_i \cup X_m$ which is isomorphic to ω^α, and thus there is an interval in which X_i is contained which is isomorphic to ω^α. □

Corollary 2. *Over ordinals, π_α is expressible via ι_α.*

Proof. By the previous corollary, it is enough to formulate that there exists a finite set of left endpoints and right endpoints, such that except for the last ones, All the intervals are strictly the prefix of an interval isomorphic to ω^α, and such that the last interval is isomorphic to ω^α.

$$
\begin{aligned}
\pi_\alpha(X) \iff \exists L \exists R.& \\
& finite(L) \\
& \wedge finite(R) \\
& \wedge \min(L \cup R) \in L \\
& \wedge \max(L \cup R) \in R \\
& \wedge \forall x, y \in L. x < y \implies \exists z \in R. x < z < y \\
& \wedge \forall x, y \in R. x < y \implies \exists z \in L. x < z < y \\
& \wedge \forall x \in L - \{\max L\} \, \forall y \in R - \{\max R\} \, \exists z. y < z \wedge \iota_{[x,z]} \\
& \wedge \iota_{\{\max L, \max R\}}
\end{aligned}
\tag{1}
$$

□

Proposition 3. *Let ω^α be any ordinal. Let the Cantor normal form be*

$$\beta = \sum_{i=1}^{k} \beta_i$$

where $\beta_i = \omega^{\alpha_i}$.

Then, a set X satisfies π_α iff it is an ordered sum $X = \sum_{i=1}^{k} X_i$ where X_i satisfies π_{α_i}.

Proof. Obviously a disjoint sum of such intervals satisfies π_α.

For the other direction, suppose X is a union of closed intervals and is isomorphic to ω^α. Let $\varphi : X \to \beta$ be the isomorphism map. Recall that $\beta = \sum_{i=1}^{k} Y_i$ where $Y_i \cong \beta_i$.

Let $X_i := \varphi^{-1}(Y_i)$ for $i = 1, ..., k$. Then it can easily be seen that X_i is a disjoint sum of closed intervals. □

Corollary 3. *Let ω^α be any ordinal. Then π_α is expressible via $\pi_{\alpha_1}, ..., \pi_{\alpha_k}$ for some ordinals $\beta_1 = \omega^{\alpha_1}, ..., \beta_k = \omega^{\alpha_k}$.*

Proof. This is pretty much the same trick as the previous corrollary, but is very tedious to write down. □

Theorem 1. *Let ω^α be any ordinal.*

Over ordinals, π_α is expressible via $\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}$ where $\beta_1 = \omega^{\alpha_1}, ..., \beta_k = \omega^{\alpha_k}$.

Proof. Combine the three previous corollaries. □

Definition 1. *Let $\beta > 0$ be a nonzero ordinal.*

Let ξ be any ordinal. Let γ and δ be the unique ordinals such that $\xi = \gamma\beta + \delta$.

Then, define $\xi/\beta := \gamma$ and $\xi\%\beta := \delta$.

That is, ξ/β is the quotient of ξ by ω^α, "rounded down".

Now, suppose $\mathcal{C}$ be a class of ordinals.

Define $\mathcal{C}_{/\beta} := \{\xi/\beta : \xi \in \mathcal{C}\}$ Define $\mathcal{C}_{\%\beta} := \{\xi\%\beta : \xi \in \mathcal{C}\}$

Theorem 2. *Let $\mathcal{C}$ be a class of ordinals, and let $\beta_1 = \omega^{\alpha_1}, ..., \beta_k = \omega^{\alpha_k}$ be ordinals, where $\alpha_1 \geq \cdots \geq \alpha_k$, that is, $\beta_k \mid \cdots \mid \beta_1$.*

Then the logic $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}\right]$ is decidable over $\mathcal{C}$,

using oracles to decide $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$ over ω^α and over $\mathcal{C}_{\%\beta_k}$ and decide MSO over $\mathcal{C}_{/\beta_k}$.

Proof. Denote $\beta := \beta_k$.

By the composition theorem, given φ of quantifier rank ρ, we can compute ψ', with free variables $\{X_\tau\}_\tau$, where τ is a ρ-type.

such that for any $\xi = \sum_{i \in \xi/\beta} \xi_i + \delta$, for $\delta < \beta$,

(denoting $\xi_{\xi/\beta} := \delta$ we get $\xi = \sum_{i \in \xi/\beta+1} \xi_i$),

$$\xi \vDash \varphi \iff \xi/\beta + 1, Q_1, ..., Q_r \vDash \psi'$$

where $Q_\tau = \{\xi_i \vDash \tau : i \in \xi/\beta + 1\}$

Now, since the Q_τ are all (except maybe the last index) or nothing (except including maybe last the index), we can actually describe them with two propositional/boolean variables. That is, we can replace each time X_τ appears in ψ', by $\{i < \max : y_\tau\} \cup \{\max : z_\tau\}$ where y_τ and z_τ are two new propositional variables, and then:

$$\xi \vDash \varphi \iff \xi/\beta + 1, Y_1, Z_1, ..., Y_r, Z_r \vDash \psi''$$

where $Y_\tau := \beta \vDash \tau$, and $Z_\tau := \delta \vDash \tau$, and ψ'' is ψ' after the replacement.

Therefore, given a class $\mathcal{C}$ of ordinals, we first determine for each τ what Y_τ is, using the fact that ω^α is decidable over $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$, and that over ω^α we can replace κ_{α_k} with $= 0$.

Now, we calculate all the possible values of Z_τ: for that, we check if τ holds in all of $\mathcal{C}_{\%\beta}$, and if $\neg\tau$ holds in all of $\mathcal{C}_{\%\beta}$.

This is possible since $\mathcal{C}_{\%\beta}$ is decidable over $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$, and again, we can replace κ_{α_k} with $= 0$.

Finally, for each possible combination of values $Y_1, Z_1, ..., Y_r, Z_r$, we calculate whether ψ'' holds over $\mathcal{C}_{/\beta}$. This is possible since MSO is decidable over $\mathcal{C}_{/\beta}$.

Then, suppose that φ holds over all $\xi \in \mathcal{C}$.

Then, it must be the case that ψ'' holds for any such combination.

And also for the contrary, every ordinal $\xi \in \mathcal{C}$ gives rise to some combination. □

Corollary 4. *Suppose in the previous theorem that $\mathcal{C}$ is either:*

1. *a **single countable** ordinal: $\mathcal{C} = \{\xi\}$*
2. *the class of **all countable** ordinals*

Then $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}\right]$ is decidable.

Proof. For a single countable ordinal ξ: ω^α over $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$ is decidable by induction on k (for the induction step), since MSO is decidable over a single countable ordinal (for the base case)

$\mathcal{C}_{/\beta} = \{\xi/\beta\}$ again is decidable for the same reason.

and $\mathcal{C}_{\%\beta} = \{\xi\%\beta\}$ is a single countable ordinal and so MSO is decidable over it.

Now, suppose $\mathcal{C}$ is the class of all countable ordinals, then $\mathcal{C}_{/\beta} = \mathcal{C}$, so $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$ is decidable over it by induction on k, or since MSO is decidable over the class of all countable ordinals (in the base case),

and $\mathcal{C}_{\%\beta} = \{\delta : \delta < \beta\}$. MSO is decidable over it, since a formula φ is true over $\{\delta : \delta < \beta\}$ iff $\varphi \vee \exists X.\kappa_\alpha$ is true over the class of all countable ordinals, and we have already established that this $MSO\left[\kappa_\alpha\right]$ is decidable over a single countable ordinal. □

Theorem 3. *Under the same conditions,* $MSO\left[\pi_{\alpha_1}, ..., \pi_{\alpha_k}\right]$ *is decidable.*

Proof. Combine all the results we have had so far. □