Part 1. Beginning

1. Properties

Definition 1. *A* (non-trivial) property *of linear orders is a (non-trivial) class of linear orders, closed under isomorphism.*

If the property $\mathbf{P}$ *holds for a linear order* L*, we write* $L \models \mathbf{P}$.

Definition 2. *A non-trivial property* $\mathbf{P}$ *is an* additive *property if for every two linear orders* I *and* J*,* I *and* J *satisfy* $\mathbf{P}$ *if and only if* $I + J$ *satisfies* $\mathbf{P}$.

Observation 1. *Let* $\mathbf{P}$ *be an additive property.*

Then,

(1) $1 \models \mathbf{P}$
(2) *If* $J \subseteq I$ *is a subinterval and* $I \models \mathbf{P}$ *then* $J \models \mathbf{P}$.

Definition 3. *A non-trivial property* $\mathbf{P}$ *is a* monotone *property if for every linear order* I *and* every $J \subseteq I$*, if* I *satisfies* $\mathbf{P}$ *then* J *satisfies* $\mathbf{P}$.

Definition 4. *Let* I *be a linear order.*

I *is called* $\sigma - \mathbf{P}$ *if every bounded subinterval of* I *satisfies the property* $\mathbf{P}$.

Definition 5. *A non-trivial property* $\mathbf{P}$ *is a* star *property if for every linear order* L *and every family* $\mathcal{F}$ *of* $\mathbf{P}$*-intervals of* L *such that* $\bigcap \mathcal{F} \neq \emptyset$*. Then* $\bigcup \mathcal{F}$ *is a* $\mathbf{P}$.

Lemma 1. *Every* $\sigma - \mathbf{P}$ *property is a star property.*

Let $I = \bigcup \mathcal{F}$. Since $\bigcap \mathcal{F} \neq \emptyset$, it follows that I is an interval. Let $w \in \bigcap \mathcal{F}$ be a common point.

Suppose $x_1 \in F_1 \in \mathcal{F}$ and $x_2 \in F_2 \in \mathcal{F}$. Then $[x_1, w] \subseteq F_1$ and $[w, x_2] \subseteq F_2$. Those intervals satisfy $\mathbf{P}$ and thus $[x_1, x_2]$ satisfies $\mathbf{P}$ as it is either their sum or difference.

Corollary 1. *Let* L *be a linear order.*

Every point in L *is contained in a largest (in* L*)* $\sigma - \mathbf{P}$ *interval.*

Proof. The union of all $\sigma - \mathbf{P}$ intervals containing x is again a $\sigma - \mathbf{P}$ interval containing x by the star lemma. □

Corollary 2. *Let* L *be a linear order and* α *be an ordinal.*

If I *and* J *are largest* $\sigma - \mathbf{P}$ *intervals, then* $I \cap J = \emptyset$ *or* $I = J$.

Proof. Suppose $I \cap J \neq \emptyset$. Then they are both the largest $\sigma - \mathbf{P}$ interval containing some $w \in I \cap J$. Thus, $I = J$. □

Definition 6. *Let* L *be a linear order. We define the relation* $\sim_{\mathbf{P}}$ *on* L *as follows:*

We say that $x \sim_{\mathbf{P}} y$ *if and only if they are contained in the same largest* $\sigma - \mathbf{P}$ *interval.*

The equivalence classes are precisely those largest $\sigma - \mathbf{P}$ *intervals.*

Definition 7. *Let* I *be an interval and* α *be an ordinal.* I *is called* right-extensible $\sigma - \mathbf{P}$ *if* I *is* $I + 1$ *is* $\sigma - \mathbf{P}$.

Similarly, I *is called* left-extensible $\sigma - \mathbf{P}$ *if* $1 + I$ *is* $\sigma - \mathbf{P}$.

Lemma 2. *Let* I_1*,* I_2 *be two intervals. Then the following are equivalent:*

(1) $I_1 + I_2$ *is* $\sigma - \mathbf{P}$

(2) $I_1 + 1$ *and* $1 + I_2$ *are* $\sigma - \mathbf{P}$
(3) $I_1 + 1 + I_2$ *is* $\sigma - \mathbf{P}$

Proof. Obviously if $I_1 + I_2$ is $\sigma - \mathbf{P}$, then $I_1 + 1$ and $1 + I_2$ are $\sigma - \mathbf{P}$, as they are subintervals of $I_1 + I_2$. So 1 implies 2.

Since $\sigma - \mathbf{P}$ is a star property, 2 implies 3.

Now, as $I_1 + I_2$ is a union of a finite number of intervals of $I_1 + 1 + I_2$, 3 implies 1. □

Lemma 3. *Let L be any linear order and α be an ordinal.*

Then there exists some linear order γ and a decomposition $L = \sum_{i \in \gamma} L_i$ such that each L_i is $\sigma - \mathbf{P}$, and if i' is the successor of i in γ, then it can't be the case that both L_i is right-extensible $\sigma - \mathbf{P}$ and that $L_{i'}$ is left-extensible $\sigma - \mathbf{P}$.

Proof. Let $\gamma = L/\sim_{\mathbf{P}}$. Then $L = \sum_{i \in \gamma} L_i$, and each L_i is largest $\sigma - \mathbf{P}$. In particular, $L_i + L_{i'}$ is not $\sigma - \mathbf{P}$ for i and i' such that i' is the successor of i in γ. That means that it can't be the case that both L_i is right-extensible $\sigma - \mathbf{P}$ and that $L_{i'}$ is left-extensible $\sigma - \mathbf{P}$. □

2. Hausdorff rank

Definition 8. *Let L be a linear order. Let $\mathbf{Q}$ be an additive property.*

We define the $\mathbf{Q}$-Hausdorff rank $\mathbf{rank}(L)$ *as follows:*

We define $\mathbf{rank}(L) \leq 0$ iff L is finite.

We define $\mathbf{rank}(L) \leq \alpha$ for $\alpha > 0$ iff
$L = \sum_{i \in \gamma} L_i$ where γ satisfies $\mathbf{Q}$, and for all $i \in \gamma$, $\mathbf{rank}(L_i) < \alpha$.

This is called a γ-decomposition of L or a $\mathbf{Q}$-decomposition of L.

$\mathbf{rank}(L) = \alpha$ *if α is the least ordinal such that* $\mathbf{rank}(L) \leq \alpha$.

If $\mathbf{rank}(L) \leq \alpha$ *does not hold for any ordinal α, then we write* $\mathbf{rank}(L) = \bot$.

Claim 1. *For every ordinal α, the property $\mathbf{rank}(\cdot) \leq \alpha$ is additive.*

Equivalently, $\mathbf{rank}(L_1 + L_2) = \max(\mathbf{rank}(L_1), \mathbf{rank}(L_2))$.

Furthermore, $\mathbf{rank}(\cdot) < \alpha$ *is additive.*

Proof. Since $\mathbf{Q}$ is additive, it is obvious that $\mathbf{rank}(\cdot) \leq \alpha$ is additive. The rest follows easily. □

Claim 2. *Let $\mathbf{Q}$ be an* **MSO**-*definable additive property, monotone property.*

Let α be an ordinal.

Then the property $\mathbf{rank}(\cdot) \leq \alpha$ is monotone.

Proof. If $\alpha = 0$, it is obvious.

Now, suppose L is of rank $\leq \alpha$. Write $L = \sum_{i \in \gamma} L_i$ where γ satisfies $\mathbf{Q}$.

Let $M \subseteq L$. Let $M_i = M \cap L_i$.

Let $\gamma' = \{i \in \gamma : M_i \neq \emptyset\}$.

Then $M = \sum_{i \in \gamma'} M_i$, where $\mathbf{rank}(M_i) < \alpha$ by induction, and γ' satisfies $\mathbf{Q}$ by monotonicity. □

Claim 3. *Let $\mathbf{Q}$ be an* **MSO**-*definable additive property.*

Then the property $\mathbf{rank}(\cdot) \leq m$ is **MSO**-*definable for all $m \in \mathbb{N}$.*

Proof. For $m = 0$, $\mathbf{rank}(X) \leq 0$ is **MSO**-definable by saying that X is finite.

By induction on m, $\mathbf{rank}(\cdot) \leq m - 1$ is **MSO**-definable.

Then $\mathbf{rank}\,(X) \leq m$ is **MSO**-definable as follows: There exists a partition P of X such that some γ which has a unique representative of every P-equivalence set, satisfies $\mathbf{Q}$. And such that every P-equivalence set has rank $\leq m-1$. $\square$

Claim 4. *Let $\mathbf{Q}$ be an **MSO**-definable additive property.*

Let $n \in \mathbb{N}$. Let γ be a linear order satisfying $\mathbf{Q}$. Let $P_1, \ldots, P_m$ be a partition/coloring of γ.

Let $\tau(X_1, \ldots, X_m)$ be the n-type of $\gamma, \vec{P}$.

Lemma 4. *Let L be a linear order.*

Let α be an ordinal.

Then $\mathbf{rank}\,(L) \leq \alpha$ if and only if there exists a γ-decomposition of L for some $\gamma \models \mathbf{Q}$, such that:

1. $\mathbf{rank}\,(L_i) = \alpha - 1$ *for infinitely many $i \in \gamma$ if α is a successor ordinal*
2. $\sup_{i\in\gamma} \mathbf{rank}\,(L_i) = \alpha$ *if α is a limit ordinal*

Theorem 1. *Let $\mathbf{Q}$ be an **MSO**-definable additive property.*

There exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

Let $\alpha \geq f(n)$ be an ordinal. Then every linear order of rank $\geq f(n)$ is $\equiv_n$-equivalent to some linear order of rank α.

Equivalently, the n-types of linear orders of rank $= \alpha$ are the same for all $\alpha \geq f(n)$.

Proof. Let A_k be the set of all satisfiable n-types of rank $\leq k$.

Then A_{k+1} is the closure of A_k under $\mathbf{Q}$-sums.

Thus, the sequence $A_0 \subseteq A_1 \subseteq \ldots$ stabilizes. Let $f(n)$ be defined such that the fixed point is $A_{f(n)-1} = A_{f(n)}$.

Suppose L has rank $\beta \geq f(n)$. Write $L = \sum_{i\in\gamma} L_i$ where $\gamma \models \mathbf{Q}$ and for all $i \in \gamma$, $\mathbf{rank}\,(L_i) < \beta$.

Then for infinitely many $i \in \gamma$, $\mathbf{rank}\,(L_i) = \beta - 1$, otherwise $\mathbf{rank}\,(L) \leq \beta - 1$ would hold.

Therefore, we do the same: we make a sequence α_k such that $\alpha_k = \alpha - 1$ if α is a successor ordinal and $\alpha_k \to \alpha$ if α is a limit ordinal.

Then we replace every L_i in some subsequence of γ whose rank is at least $f(n)$ by some L_i' of rank α_k with the same n-type. $\square$

Corollary 3.

3. Conversion to MSO

This is general.

Definition 9. **MSO**$[P]$ *is the logic of monadic second order logic with a unary second order predicate symbol P, such that $L, X := A \models P(X)$ is true if and only if A is a bounded interval which satisfies $\mathbf{P}$.*

Theorem 2. *There exists a translation which translates any formula $\varphi(\vec{X})$ in **MSO**$[P]$, into a formula $\psi(Q, \vec{Y})$ in **MSO**, such that for every linear order L, there exists a subset $C \subseteq L$ such that,*

$$L, \vec{X} := \vec{A} \models \varphi(\vec{X}) \iff L, Q := C, \vec{Y} := \vec{B} \models \psi(Q, \vec{Y})$$

More

Definition 10. *Let $\varphi(\vec{X})$ be a formula in* **MSO**.

Theorem 3. *Let P be the unary second order predicate symbol, where $L, A \models P(X)$ is true if and only if A is a bounded interval which satisfies* **P**.

There exists a syntactical translation which translates any formula $\varphi(\vec{X})$ in **MSO**$[P]$,

into a sentence ψ in **MSO** *such that φ is true over all properly labeled linear orders, if and only if ψ is true over all linear orders*

Proof. There exists a syntactical translation which translates any formula $\varphi(\vec{X})$ in **MSO**$[P]$, to a formula $\psi'(\vec{Y})$ in **MSO**,

such that for every linear order L and any decomposition $L = \sum_{i\in\gamma} L_i$,

$$L, \vec{A} \models \varphi(\vec{X}) \iff \gamma, \vec{B} \models \psi'(\vec{Y})$$

where $B_\tau = \{i : L_i \models \tau\}$.

Let us add two symbols Z_{left} and Z_{right} to the language of **MSO**$[\vec{Y}]$. We define the sentence $\psi(\vec{Y}, Z_{left}, Z_{right})$ as follows:

$$\begin{aligned}\psi :=& \forall\vec{Y}\forall Z_{right}\forall Z_{left}\\ &\quad \mathbf{partition}(\vec{Y}) \wedge \neg\exists i, i'.i' = i+1 \wedge Z_{right}(i) \wedge Z_{left}(i') \implies \psi'(\vec{Y})\end{aligned}$$

Now we claim that

$$L, \vec{A} \models \varphi(\vec{X})$$

if and only if

$$\gamma, \vec{B}, C_{left}, C_{right} \models \psi(\vec{B}, Z_{left}, Z_{right})$$

where

$$C_{left} = \{i : L_i \text{ is left-extensible } \sigma - \mathbf{P}\}$$
$$C_{right} = \{i : L_i \text{ is right-extensible } \sigma - \mathbf{P}\}$$

and this is obvious.

Now, we claim even further that $\varphi(\vec{X})$ is true over all properly labeled linear orders, if and only if ψ is true over all linear orders.

Suppose ψ is true over all linear orders,

Then in particular $\psi'(\vec{B}, C_{left}, C_{right})$ is true over all the index sets of decompositions of linear orders. But as all the linear orders L can be decomposed, this implies that $\varphi(\vec{X})$ is true over all properly labeled linear orders.

For the other direction: suppose that $\varphi(\vec{X})$ is true over all properly labeled linear orders. Then in particular $\psi'(\vec{B}, C_{left}, C_{right})$ is true over all the index sets of decompositions of linear orders.

Let γ be some order. If the precondition does not hold for some choice, then we are done. Otherwise, $\square$

4. Compositional relation

Definition 11. *Let $L = \sum_{i\in\gamma} L_i$ be a decomposition of L, where $\gamma \in$* **Q** *and for all $i \in \gamma$, $L_i \in$* **P**.

Then for all

5. Homework

5.1. **January 26.** Generalize to trees.