Orders

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1 Properties

Definition 1. *A* property $\mathbf{P}$ *of linear orders is a class of linear orders which is closed under isomorphism.*

Definition 2. *A property* $\mathbf{P}$ *of linear orders is* monotone *if for every linear order* M*,* $M \in \mathbf{P}$ *implies that every suborder of* M *is in* $\mathbf{P}$*.*

Definition 3. *A property* $\mathbf{P}$ *of linear orders is* symmetric *if for every linear order* M*,* $M \in \mathbf{P}$ *iff* $M^R \in \mathbf{P}$*.*

Definition 4. *A property* $\mathbf{P}$ *of linear orders is an* additive property *if for every linear orders* M_1 *and* M_2*,* $M_1 + M_2 \in \mathbf{P}$ *iff* $M_1, M_2 \in \mathbf{P}$*.*

Definition 5. *Let* $\mathbf{P}$ *be a property of linear orders.*

We define **P-bounded** *to be the class of linear orders* M *such that for every* $x, y \in M$*, the bounded subinterval* $[x, y]$ *is in* $\mathbf{P}$*.*

Definition 6. *A property* $\mathbf{P}$ *of linear orders is* almost anti-symmetric *if for every linear order* M*,* $M \in \mathbf{P}$ *and* $M^R \in \mathbf{P}$ *imply that* M *is finite.*

Definition 7. *A property* $\mathbf{P}$ *of linear orders is* good *if it is monotone, additive and contains at least one infinite linear order.*

Definition 8. *A property* $\mathbf{P}$ *of linear orders is a* star property *if for every linear orders* M*, and every family* $\mathcal{F} \subseteq \mathbf{P}$ *of subintervals of* M *such that* $J_1 \cap J_2 \neq \emptyset$ *for every* $J_1, J_2 \in \mathcal{F}$*, we have that* $\bigcup \mathcal{F} \in \mathbf{P}$*.*

Lemma 1 (Star Lemma)**.** *Let* $\mathbf{P}$ *be an additive property of linear orders.*

Then the property **P-bounded** *has the star property.*

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq$ **P-bounded** be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by **P-bounded**, $[x, z], [z, y] \in \mathbf{P}$. However, $\mathbf{P}$ is additive. Since $[x, y]$ is either the sum or difference of $[x, z]$ and $[z, y]$, we have that $[x, y] \in \mathbf{P}$. $\square$

Lemma 2. *Let* $\mathbf{P}$ *be a star property.*

Then for every linear order M*, and every point* $x \in M$*, there exists a largest subinterval* $J \subseteq M$ *such that* $J \in \mathbf{P}$*.*

Thus, we can define an equivalence relation $\sim_{\mathbf{P}}$ *on* M *such that* $x \sim_{\mathbf{P}} y$ *iff* x *and* y *are in the same largest* $\mathbf{P}$*-subinterval.*

Proof. Let $J \subseteq M$ be the union of all **P-bounded**-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in **P-bounded**, and by definition J is the largest **P**-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. □

2 Hausdorff Rank

Definition 9. *Let* $\mathbf{Q}$ *be a good property of linear orders.*

We define a property $\mathbf{Q}^{\leq\alpha}$ *for every ordinal* α *as follows:*

- $\mathbf{Q}^{\leq 0}$ *is the class of finite linear orders.*
- *For* $\alpha > 0$*,* $\mathbf{Q}^{\leq\alpha}$ *is the class of linear orders* M *such that* $M = \sum_{i \in I} M_i$ *for some* $I \in \mathbf{Q}$ *where for all* $i \in I$*,* $M_i \in \mathbf{Q}^{\beta_i}$ *for some* $\beta_i < \alpha$

We define further $\mathbf{Q}^{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{Q}^{\leq\beta}$ *and* $\mathbf{Q}^{=\alpha} = \mathbf{Q}^{\leq\alpha} - \mathbf{Q}^{<\alpha}$*.*

We define $\mathbf{hrank}_{\mathbf{Q}}(M) = \alpha$ *iff* $M \in \mathbf{Q}^{=\alpha}$*. This is a* partial map *from linear orders to ordinals.*

Observations 1. *Let* $\mathbf{Q}$ *be a good property.*

We claim the following without proof:

- $\mathbf{Q}^{\leq 1} = \mathbf{Q}$*.*
- *For all* α*,* $\mathbf{Q}^{\leq\alpha}$ *is a good property.*
- $\mathbf{Q}^{\leq\alpha} \subsetneq \mathbf{Q}^{\leq\beta}$ *iff* $\alpha < \beta$*.*

Definitions 1. $\mathcal{B}_{<\alpha} := \mathbf{Q}^{<\alpha}$**-bounded** *is the class of linear orders of rank* $< \alpha$ *on bounded intervals.*

$\mathcal{R}_{<\alpha}$ *is the class of linear orders* M *where* $M + 1 \in \mathcal{B}_{<\alpha}$*.*

$\mathcal{L}_{<\alpha}$ *is the class of linear orders* M *where* $1 + M \in \mathcal{B}_{<\alpha}$*.*

Lemma 3. *The following are equal:*

1. $\mathbf{Q}^{<\alpha}$
2. $\{M : 1 + M + 1 \in \mathcal{B}_{<\alpha}\}$*.*
3. $\mathcal{L}_{<\alpha} \cap \mathcal{R}_{<\alpha}$

Proof. The only nontrivial direction is 3 implies 2, which follows from the star property of $\mathcal{B}_{<\alpha}$. □

Lemma 4. *A countable linear order which has rank $< \alpha$ on bounded subintervals is of rank $\leq \alpha$. That is,*

$$\mathcal{B}_{<\alpha} \subseteq \mathbf{Q}^{\leq \alpha}$$

Proof. Let M be a countable linear order of rank $< \alpha$.

Then $M = \sum_{i \in I} M_i$ where $M_i \in \mathbf{Q}^{<\alpha}$.

Let $\{x_i\}_{i \in I} \subseteq M$ be a bidirectional, cofinal, weakly monotone I-sequence in M, i.e, $x_i \leq x_j$ if $i \leq j$ for $I \subseteq \mathbb{Z}$.

Write $M = \sum_{i \in I} [x_i, x_{i+1}]$. Then every $[x_i, x_{i+1}]$ is of Hausdorff rank $< \alpha$.

Thus, $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$, which completes the proof. □

Lemma 5. *Let M be a countable linear order.*

Suppose $\mathbf{Q} = \{M : \exists n \in \mathbb{N}. M \subseteq \mathbb{Z} \cdot n\}$ *(This does not necessarily work for other* $\mathbf{Q}$*!).*

Then $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$ *iff* M *is a finite sum of* $\mathcal{B}_{<\alpha}$*-subintervals.*

Proof. From the previous lemma, it is clear that if M is a finite sum of $\mathcal{B}_{<\alpha}$-subintervals, then $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$, since the rank bound is preserved under finite sums.

Conversely, suppose $\mathbf{hrank}_{\mathbf{Q}}(M) \leq \alpha$.

If $M = \sum_{i \in \mathbb{Z}} M_i$ for some M_i of Hausdorff rank $< \alpha$, take $x, y \in M$. Then let $x \in M_{i_1}$ and $y \in M_{i_2}$.

Then $[x, y] \subseteq \sum_{i \in [i_1, i_2]} M_i$. But the last sum is of rank $< \alpha$ and thus $[x, y]$ is of rank $< \alpha$. That is, $M \in \mathcal{B}_{<\alpha}$.

Since every subinterval of rank $\leq \alpha$ is a finite sum of $\mathbb{Z}$-sums of intervals of rank $< \alpha$, we are done. □

3 Decidability of the rank

Lemma 6. *There exists a global computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,* $\mathbf{type}_n\left[\mathcal{H}_{f(n)+1}\right] = \mathbf{type}_n\left[\mathcal{H}_{f(n)}\right]$.

Equivalently, every linear order of finite rank is n-equivalent to some linear order of rank $\leq f(n)$.

Proof. Since there exist only a finite number of n-types, and the ω-sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$ is monotone, it must stabilize at some point.

This point is computable as a function of n, because $\mathbf{type}_n[\mathcal{H}_k]$ is computable for every finite k. □

Lemma 7. *There exist global computable functions $a, b : \mathbb{N} \to \mathbb{N}$ such that for all $n, c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 \geq a(n)$ and $c_1 \equiv c_2 \mod b(n)$,*

$$\mathbf{type}_n[\mathcal{H}_{c_1}] = \mathbf{type}_n[\mathcal{H}_{c_2}]$$

Equivalently, the sequence $\{\mathbf{type}_n[\mathcal{H}_k]\}_{k \in \omega}$ *is ultimately periodic for all* $n \in \mathbb{N}$*. Furthermore, the starting point and the period itself can be computed as a function of* n*.*

Proof. Let $n \in \mathbb{N}$.

Since there exist only a finite number of possible sets of n-types, there exist (and can be computed) some $a(n) > f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n \left[\mathcal{H}_{a(n)}\right] = \mathbf{type}_n \left[\mathcal{H}_{a(n)+b(n)}\right]$$

We shall prove by induction that for all $c \geq a(n) + b(n)$,

$$\mathbf{type}_n \left[\mathcal{H}_c\right] = \mathbf{type}_n \left[\mathcal{H}_{c+b(n)}\right]$$

This will complete the proof.

The base case $c = a(n)$ has been proven in the beginning.

Suppose the induction hypothesis holds for c.

Let M be of rank $c + 1$.

Write $M = \sum_{i \in I} M_i$ where $\mathbf{hrank}_{\mathbf{Q}} (M_i) < c + 1$, and $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c$ infinitely many times.

By the induction hypothesis, if $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c$, we can find $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}} (N_i) = c + b(n)$. Setting $N_i := M_i$ for all other i, we conclude that $N := \sum_{i \in I} N_i$ is n-equivalent to M.

However, clearly $\mathbf{hrank}_{\mathbf{Q}} (N) = c + b(n) + 1$. So overall,

$$\mathbf{type}_n \left[\mathcal{H}_{c+1}\right] \subseteq \mathbf{type}_n \left[\mathcal{H}_{c+b(n)+1}\right]$$

Conversely, suppose M is of rank $c + b(n) + 1$. Write $M = \sum_{i \in I} M_i$ where $\mathbf{hrank}_{\mathbf{Q}} (M_i) < c + b(n) + 1$, and $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c + b(n)$ infinitely many times.

By the induction hypothesis, we can find for all i such that $\mathbf{hrank}_{\mathbf{Q}} (M_i) = c + b(n)$ some $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}} (N_i) = c$. Furthermore, since $c \geq a(n) > f(n)$, we can find $N_i \equiv_n M_i$ with $\mathbf{hrank}_{\mathbf{Q}} (N_i) \leq f(n) < c$ for all other i.

We conclude that $N := \sum_{i \in I} N_i$ is n-equivalent to M. However, clearly $\mathbf{hrank}_{\mathbf{Q}} (N) = c + 1$. So overall,

$$\mathbf{type}_n \left[\mathcal{H}_{c+b(n)+1}\right] \subseteq \mathbf{type}_n \left[\mathcal{H}_{c+1}\right]$$

So we have proven the induction step, and the lemma follows. □

Lemma 8. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Then,

$$\mathbf{type}_n \left[\mathbf{Q}^{=\alpha}\right] = \bigcup_{c < b(n)} \mathbf{type}_n \left[\mathcal{H}_{c+b(n)}\right]$$

In particular, $\mathbf{type}_n \left[\mathbf{Q}^{=\alpha}\right]$ can be computed, and is independent of the choice $\alpha \geq \omega$.

Proof. TBC. □

Corollary 1. *The following sequences stabilize at $f(n)$:*

- $\mathbf{type}_n \left[\mathcal{H}_\alpha\right]$
- $\mathbf{type}_n \left[\mathcal{B}_{<\alpha}\right]$

- $\mathbf{type}_n\left[\mathcal{L}_{<\alpha}\right]$
- $\mathbf{type}_n\left[\mathcal{R}_{<\alpha}\right]$
- $\mathbf{type}_n\left[\mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}\right]$
- $\mathbf{type}_n\left[\mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}\right]$
- $\mathbf{type}_n\left[\mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})\right]$

Proof. The corollary is false and should be fixed. □

Definition 10. *Let $\alpha \geq \omega$ be an ordinal.*

Let M be a linear order and $x \in M$.

We define the convex equivalence relation $\sim_\alpha := \sim_{\mathcal{B}_{<\alpha}}$, and $[x]_\alpha := [x]_{\mathcal{B}_{<\alpha}}$ (that is, $[x]_\alpha$ is the largest $\mathcal{B}_{<\alpha}$-subinterval containing x in M).

Lemma 9. *Let M be a linear order. Let $P, L, R \subseteq M$ be relations, such that:*

- *P represents $\sim_\alpha$ on M.*
- *L is such that $x \in L$ iff $[x]_\alpha \in \mathcal{L}_{<\alpha}$.*
- *R is such that $x \in R$ iff $[x]_\alpha \in \mathcal{R}_{<\alpha}$.*

Then for some linear order I there exists a decomposition $M = \sum_{i \in I} M_i$ such that $M_i \in \mathcal{B}_{<\alpha}$ for all $i \in I$, M_i is monochromatic with respect to P, L and R.

Furthermore, let τ_i be the n-type of M_i, p_i, q_i, r_i in $\mathbf{MSO}[p, \ell, r]$, where $p_i = 1_{M_i \subseteq P}$, $q_i = 1_{M_i \subseteq L}$ and $r_i = 1_{M_i \subseteq R}$. Then the following hold

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $\ell(\tau_{i+1}) = 0$*

Proof. Take $I = M / \sim_\alpha$.

Then $M = \sum_{i \in I} M_i$ where M_i is the $\sim_\alpha$-equivalence class of i.

Then M_i is monochromatic with respect to P, L and R.

The only thing left to prove is the last two conditions. The first follows from the fact that P represents $\sim_\alpha$.

The second follows because if it were not the case, then M_i and M_{i+1} would be the same $\sim_\alpha$-equivalence class. □

Lemma 10. *Let I be a linear order. Let $n \in \mathbb{N}$. Let p, ℓ, r be boolean variables.*

Let τ_i be an assignment of satisfiable n-types in $\mathbf{MSO}[p, \ell, r]$ for all $i \in I$. Assume that

- *if i has a successor, $p(\tau_i) \neq p(\tau_{i+1})$*
- *if i has a successor, either $r(\tau_i) = 0$ or $\ell(\tau_{i+1}) = 0$*

Then there exists a linear order M and $P, L, R \subseteq M$ such that:

- *P represents $\sim_\alpha$ on M.*
- *L is such that $x \in L$ iff $[x]_\alpha \in \mathcal{L}_{<\alpha}$.*
- *R is such that $x \in R$ iff $[x]_\alpha \in \mathcal{R}_{<\alpha}$.*

such that for all $i \in I$, M_i is a $\sim_\alpha$-equivalence class of M, and is thus monochromatic with respect to P, L and R.

Furthermore, the n-type of M_i, p_i, q_i, r_i in $\mathbf{MSO}[p, \ell, r]$ is τ_i, where $p_i = 1_{M_i \subseteq P}$, $q_i = 1_{M_i \subseteq L}$ and $r_i = 1_{M_i \subseteq R}$,

Proof. Since τ_i is satisfiable, we can take M_i to be a linear order of n-type τ_i such that:

- If $\ell(\tau_i) = r(\tau_i) = 1$, then $M_i \in \mathcal{L}_{<\alpha} \cap \mathcal{R}_{<\alpha}$.
- If $\ell(\tau_i) = 1$ and $r(\tau_i) = 0$, then $M_i \in \mathcal{L}_{<\alpha} - \mathcal{R}_{<\alpha}$.
- If $\ell(\tau_i) = 0$ and $r(\tau_i) = 1$, then $M_i \in \mathcal{R}_{<\alpha} - \mathcal{L}_{<\alpha}$.
- If $\ell(\tau_i) = r(\tau_i) = 0$, then $M_i \in \mathcal{B}_{<\alpha} - (\mathcal{L}_{<\alpha} \cup \mathcal{R}_{<\alpha})$.

Let $M = \sum_{i \in I} M_i$.

By definition each M_i is in $\mathcal{B}_{<\alpha}$. We need to prove that each M_i is a largest $\mathcal{B}_{<\alpha}$-subinterval in M.

On the contrary, suppose that there exist $i' \neq i$ such that $[M_i, M_{i'}] \in \mathcal{B}_{<\alpha}$. WLOG, $M_i < M_{i'}$.

Since I is scattered, take some $i \leq a < b \leq i'$ such that there is no element between a and b in I.

Then $M_a \in \mathcal{R}_{<\alpha}$ and $M_b \in \mathcal{L}_{<\alpha}$, in contradiction. □

Lemma 11. *Over countable linear orders with interpretations of P, L and R as above, the property $\mathbf{Q}^{<\alpha}$ is expressible in $\mathbf{MSO}[P, L, R]$.*

Proof. From lemma 3 we can express $\mathbf{Q}^{<\alpha}$ as the intersection of L and R. □

Theorem 1. *There is a an algorithm solving satisfiability for $\mathbf{MSO}[P, L, R]$ over countable linear orders, given an oracle which solves the satisfiability problem for $\mathbf{MSO}$ over countable linear orders.*

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[P, L, R]$ formula φ of quantifier-depth n. outputs an $\mathbf{MSO}[\{X_\tau\}_\tau]$ formula ψ.

Let P_L, Q_L, R_M be the interpretations of P, L, R on M.

Then

$$M, P := P_L, L := Q_L, R := R_L \models \varphi \iff I, \{X_\tau := I_\tau\}_\tau \models \psi$$

Where $I_\tau = \{i \in I : M_i \models \tau\}$ for every n-type τ.

Let T be the set of n-types in $\mathbf{MSO}[p, \ell, r]$ which satisfy $\ell(\tau) = 1 \iff \tau \in \mathcal{L}_{<\alpha}$ and $r(\tau) = 1 \iff \tau \in \mathcal{R}_{<\alpha}$.

Let $S = \{(\tau_1, \tau_2) : p(\tau_1) \neq p(\tau_2) \wedge (r(\tau_1) = 0 \vee \ell(\tau_2) = 0)\}$.

Then T and S can be calculated using the oracle.

Then ψ is an $\mathbf{MSO}[T, S]$ formula.

Then we define an $\mathbf{MSO}[p, \ell, r]$ formula ψ' as follows:

ψ' claims that there exists a partition (with possible empty sets) $\{Y_\tau\}_\tau$ of I such that

- Every $i \in I$ is in some Y_τ for $\tau \in T$.
- If $i' = i + 1$ in I, then for some $(\tau_1, \tau_2) \in S$, $i \in Y_{\tau_1}$ and $i' \in Y_{\tau_2}$.

Now we claim that φ is satisfiable in some linear order, iff ψ' is satisfiable in some linear order.

Suppose φ is satisfiable in some linear order M.

Take a decomposition $M = \sum_{i \in I} M_i$ as in lemma 2.

Then ψ holds over the assignment $X_\tau := I_\tau$. But by lemma 2, this assignment satisfies the condition required for ψ' to hold. Then ψ' holds over I.

Conversely, suppose ψ' holds in I.

Let $X_\tau := Z_\tau$ be the assignment that is guaranteed by ψ'.

Let tau_i be the unique τ such that $i \in Z_\tau$.

Then the conditions for lemma 3 are guaranteed.

Thus, take M as in lemma 3. Then ψ holds over I when we set $X_i := Z_\tau$. But $Z_\tau = I_\tau$ for all τ, so φ holds over M. □