Orders

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Contents

1 Preorders

We begin by studying the properties of preorders. Basically, we define a *property* as a class which is close under isomorphism. We then define the sum operation on preorders. This will be used to create new properties from old ones.

Definitions 1.1 ((Labeled) preorder)**.** *A* preorder *is a a set M together with a binary relation $\leq$ on M such that $\leq$ is reflexive and transitive.*

A labeled preorder *is a preorder M together with a labeling function $\gamma : M \to C$, where C is a set of labels (colors).*

Definition 1.2 (Property of preorders)**.** *A* property $\mathbf{P}$ *of preorders is a class of* labeled *preorders which is closed under isomorphism.*

Definition 1.3. *A property $\mathbf{P}$ of preorders is* monotone *if for every preorder M, $M \in \mathbf{P}$ implies that every suborder of M is in $\mathbf{P}$.*

Definition 1.4. *Let M be a (labeled) preorder.*

Then M^* is the dual/reverse (labeled) preorder of M.

Definition 1.5 (Sum of preorders)**.** *Let I be a preorder, and let $\{M_i\}_{i \in I}$ be a family of labeled preorders.*

The sum $M = \sum_{i \in I} M_i$ is defined as follows:

The domain is $M = \biguplus_{i \in I} M_i$ (a disjoint union).

Let $\leq_i$ be the preorder on M_i.

Let $x \in M_i$ and $y \in M_j$.

Then we define $x \leq y$ iff either $i = j$ and $x \leq_i y$ or $i < j$.

The labels are defined naturally.

If $I = 2$, we define $M_1 + M_2 := \sum_{i \in 2} M_i$.

Lemma 1.6. *Let I be a preorder, and let $\{M_i\}_{i \in I}$ be a family of preorders.*

Then $M = \sum_{i \in I} M_i$ is a preorder.

Proof. Reflexivity is clear.

For transitivity, suppose $x \leq y$ and $y \leq z$.

Suppose $x \in M_i$, $y \in M_j$, $z \in M_k$.

Then $i \leq j$ and $j \leq k$, so $i \leq k$. If $i = k$, then necessarily $i = j = k$, and so $x \leq_i y$ and $y \leq_i z$, so $x \leq_i z$, so $x \leq z$, as required.

Otherwise, $i < k$, and thus $x \leq z$, as required. $\square$

Definition 1.7. *Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be properties of preorders.*

Then we define

$$\mathbf{P}_1 + \mathbf{P}_2 := \{M_1 + M_2 : M_1 \in \mathbf{P}_1 \wedge M_2 \in \mathbf{P}_2\}$$

Definition 1.8. *A property $\mathbf{P}$ of preorders is an* additive property *if for every preorders M_1 and M_2, $M_1 + M_2 \in \mathbf{P}$ iff $M_1, M_2 \in \mathbf{P}$.*

Definition 1.9 (Kleene plus). *Let* $\mathbf{P}$ *be a property of preorders.*

We define its Kleene plus as the smallest property of preorders $\mathbf{P}^+$ *which contains* $\mathbf{P}$ *and is closed under finite sums.*

That is, $1^+ = \{1, 2, \ldots\}$*, and* $\mathbf{P}^+ = \sum_{1^+} \mathbf{P}$.

Definition 1.10 (Sum of a property over a preorder). *Let* I *be a preorder.*

Let $\mathbf{Q}$ *be a property of preorders.*

Then we define

$$\sum_I \mathbf{Q} := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q} \right\}$$

Definition 1.11 (Sum of a family of properties over a preorder). *Let* I *be a preorder.*

Let $\{\mathbf{Q}_i\}_{i \in I}$ *be a family of properties of preorders.*

Then we define

$$\sum_{i \in I} \mathbf{Q}_i := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q}_i \right\}$$

Note 1.12. *By the previous two definitions,*

$$\sum_I \mathbf{Q} = \sum_{i \in I} \mathbf{Q}$$

Definition 1.13 (Sum of properties over a labeled preorder). *Let* I *be a labeled preorder, with a labeling function* $\gamma : I \to \vec{C}$*, where* $\vec{C}$ *is a set of colors.*

Let $\vec{\mathbf{Q}} = \{\mathbf{Q}_c\}_{c \in \vec{C}}$ *be a family of properties of preorders, indexed by the colors.*

Then we define

$$\sum_I \left[\vec{C} \leftarrow \vec{\mathbf{Q}} \right] := \left\{ \sum_{i \in I} M_i : \forall i \in I . M_i \in \mathbf{Q}_{\gamma(i)} \right\}$$

Notes 1.14.

1. *We can see a sum over an unlabeled preorder* I *as a sum over a labeled preorder with a constant labeling function* $\gamma : I \to \{1\}$.
2. *We can see* $P_1 + P_2$ *as a sum over* $I = \{1, 2\}$*, colored with* $\gamma(i) = i$.

Definition 1.15 (Sum of a property over a property). *Let* $\mathbf{P}$ *be a property of unlabeled preorders.*

Let $\mathbf{Q}$ *be a property of preorders.*

Then we define,

$$\sum_{\mathbf{P}} \mathbf{Q} := \left\{ \sum_I \mathbf{Q} : I \in \mathbf{P} \right\}$$

Definition 1.16 (Sum of a property over a labeled property). *Let* $\mathbf{P}$ *be a property of labeled preorders, over a set of colors* $\vec{C}$*.*

Let $\vec{\mathbf{Q}} = \{\mathbf{Q}_c\}_{c \in \vec{C}}$ *be a family of properties of preorders,*

Then we define,

$$\sum_{\mathbf{P}} \left[\vec{C} \leftarrow \vec{\mathbf{Q}}\right] := \left\{\sum_{I} \left[\vec{C} \leftarrow \vec{M}\right] : I \in \mathbf{P}\right\}$$

2 Linear Orders

Definitions 2.1 ((Labeled) linear order)**.** *A* (labeled) linear order *a (labeled) preorder which is symmetric and total.*

Definition 2.2 (Property of linear orders)**.** *A* property $\mathbf{P}$ *of linear orders is a class of* labeled *linear orders which is closed under isomorphism.*

Definition 2.3. *Subintervals Let M be a linear order, and let $x, y \in M$, such that $x \leq y$.*

Then we define the bounded subintervals $[x, y]$, $(x, y]$, $[x, y)$ *and* (x, y) *as usual.*

We also define the semi-bounded subintervals $(-\infty, x]$, $[x, \infty)$, $(-\infty, x)$ *and* (x, ∞) *as usual.*

We also define the unbounded subinterval $(-\infty, \infty)$ *as the whole linear order M, as usual.*

A subinterval *is either a bounded subinterval, a semi-bounded subinterval or the unbounded subinterval.*

If $x > y$ then we define the intervals as follows:

$$[x, y] := [y, x]$$
$$(x, y] := (y, x]$$
$$[x, y) := [y, x)$$
$$(x, y) := (y, x)$$

Definition 2.4. *Let M be a linear order.*

A set $A \subseteq M$ is left cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $y < x$.*

A set $A \subseteq M$ is right cofinal *in M if for every $x \in M$, there exists $y \in A$ such that $x < y$.*

A set $A \subseteq M$ is bi-directionally cofinal *in M if it is both left and right cofinal.*

Lemma 2.5. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let $M \in \mathbf{P}$ be a linear order.

Let $x, y \in M$ be any two points in a linear order M.

Then, $[x, y] \in \mathbf{P}$.

Proof. WLOG, suppose $x \leq y$.

Note that,

$$M = (-\infty, \infty) = (-\infty, x) + [x, y] + (y, \infty)$$

when $(-\infty, x)$ and/or (y, ∞) may be empty.

Since $\mathbf{P}$ is an additive property, we conclude that $[x, y] \in \mathbf{P}$. $\square$

Corollary 2.6. *Let $\mathbf{P}$ be a nontrivial additive property of linear orders.*

Then $1 \in \mathbf{P}$.

Proof. Let $M \in \mathbf{P}$ be any linear order and let $x \in M$ be any point in M.

Apply lemma 2.5 to the linear order M, and the points x and x, to conclude that $[x, x] \equiv 1 \in \mathbf{P}$. $\square$

Note 2.7. *Note that corollary 2.6 is false if we do not restrict ourselves to linear orders.*

For example, $(1 \uplus 1)^+$ is a property of preorders which is additive, but does not contain 1.

Corollary 2.8. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let M be a linear order.

Let $x, y \in M$ be any two points in a linear order M. Then the following are equivalent:

1. $(x, y) \in \mathbf{P}$
2. $(x, y] \in \mathbf{P}$
3. $[x, y) \in \mathbf{P}$
4. $[x, y] \in \mathbf{P}$

Proof. This is just applying the definition of an additive property to the orders $[x, y]$ and 1. $\square$

Corollary 2.9. *Let $\mathbf{P}$ be an additive property of linear orders.*

Let M be a linear order.

Let $x, y, z \in M$ be any three points in a linear order M, such that $[x, y] \in \mathbf{P}$ and $[y, z] \in \mathbf{P}$.

Then $[x, z] \in \mathbf{P}$.

Proof. If $y \in [x, z]$, then $[x, z] = [x, y] + (y, z]$, and $(y, z] \in \mathbf{P}$ by corollary 2.8.

Otherwise, either $x \in [y, z]$ or $z \in [x, y]$. WLOG, suppose $z \in [x, y]$.

Then $[x, y] = [x, z] + (z, y]$, so $[x, z] \in \mathbf{P}$ by the fact that $\mathbf{P}$ is additive. $\square$

Definitions 2.10. *Let $\mathbf{P}$ be a property of linear orders.*

We define the following properties of linear orders:

- *$\mathcal{B}[\mathbf{P}]$ is the class of linear orders M such that for every $x, y \in M$, the bounded subinterval $[x, y]$ is in $\mathbf{P}$.*
- *$\mathcal{L}[\mathbf{P}]$ is the class of linear orders M such that for every $x \in M$, the left-bounded ray $[x, \infty) = \{y \in M : x \leq y\}$ is in $\mathbf{P}$.*
- *$\mathcal{R}[\mathbf{P}]$ is the class of linear orders M such that for every $x \in M$, the right-bounded ray $(-\infty, x] = \{y \in M : y \leq x\}$ is in $\mathbf{P}$.*

Definition 2.11. *A property $\mathbf{P}$ of linear orders is a* star property *if for every linear orders M, and every family $\mathcal{F} \subseteq \mathbf{P}$ of subintervals of M such that $J_1 \cap J_2 \neq \emptyset$ for every $J_1, J_2 \in \mathcal{F}$, we have that $\bigcup \mathcal{F} \in \mathbf{P}$.*

Lemma 2.12. *Let $\mathbf{P}$ be a star property.*

Then for every linear order M, and every point $x \in M$, there exists a largest subinterval $J \subseteq M$ such that $J \in \mathbf{P}$.

Equivalently, we can define a convex equivalence relation $\sim_{\mathbf{P}}$ on M such that $x \sim_{\mathbf{P}} y$ iff $[x, y] \in \mathbf{P}$.

That is, $x \sim_{\mathbf{P}} y$ iff x and y are in the same largest $\mathbf{P}$-subinterval.

Proof. Let $J \subseteq M$ be the union of all $\mathcal{B}[\mathbf{P}]$-subintervals containing x. All such subintervals intersect at x.

Therefore, by the star lemma, J is in $\mathcal{B}[\mathbf{P}]$, and by definition J is the largest $\mathbf{P}$-subinterval containing x.

Thus we can define the equivalence relation $\sim_{\mathbf{P}}$ as above. $\square$

Lemma 2.13 (Star Lemma)**.** *Let $\mathbf{P}$ be an additive property of linear orders.*

Then the property $\mathcal{B}[\mathbf{P}]$ is a star property.

Proof. Let M be a linear order, and let $\mathcal{F} \subseteq \mathcal{B}[\mathbf{P}]$ be a family of subintervals of M.

Let $[x, y] \subseteq \bigcup \mathcal{F}$ be any bounded subinterval. We need to prove it is in $\mathbf{P}$.

Suppose $x \in J_1$ and $y \in J_2$ for $J_1, J_2 \in \mathcal{F}$.

Since $J_1 \cap J_2 \neq \emptyset$, we can take $z \in J_1 \cap J_2$.

Then $[x, z] \subseteq J_1$ and $[z, y] \subseteq J_2$, and thus by the definition of $\mathcal{B}[\mathbf{P}]$, $[x, z], [z, y] \in \mathbf{P}$. Since $\mathbf{P}$ is additive, by corollary 2.9, we have $[x, y] \in \mathbf{P}$. $\square$

Lemma 2.14. *Let $\mathbf{P}$ be an additive property of linear orders.*

Then,

1. $\mathcal{L}[\mathbf{P}] = \{M : M + 1 \in \mathcal{B}[\mathbf{P}]\}$
2. $\mathcal{R}[\mathbf{P}] = \{M : 1 + M \in \mathcal{B}[\mathbf{P}]\}$
3. $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}] = \{M : 1 + M + 1 \in \mathcal{B}[\mathbf{P}]\}$

Proof. Let M be a linear order.

1. Suppose $M + \{\infty\} \in \mathcal{B}[\mathbf{P}]$. Then for every $x \in M$, we have $[x, \infty] \in \mathbf{P}$, and thus $[x, \infty) \in \mathbf{P}$. Therefore, $M \in \mathcal{L}[\mathbf{P}]$.

   Conversely, if $M \in \mathcal{L}[\mathbf{P}]$, let $x, y \in M$ be any two points in $M + 1$.

   If $y < \infty$, then $[x, y] \subseteq [x, \infty)$. Since $[x, \infty) \in \mathbf{P}$, we conclude that $[x, y] \in \mathbf{P}$. Otherwise, if $y = \infty$, then $[x, y] = [x, \infty] = [x, \infty) + \{infty\}$, and thus $[x, y] \in \mathbf{P}$.

2. The second case is dual to the first case.

3. We will show a triple inclusion.

   If $M \in \mathbf{P}$, then by additivity, $1 + M \in \mathbf{P}$ and $M + 1 \in \mathbf{P}$, and thus $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$.

   If $M \in \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, then by lemma 2.13, $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$.

   If $1 + M + 1 \in \mathcal{B}[\mathbf{P}]$, then M is a bounded subinterval of $1 + M + 1$, so $M \in \mathcal{B}[\mathbf{P}]$.

□

Lemma 2.15. *Let* $\mathbf{P}$ *be an additive property of linear orders.*
Then,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = \mathbf{P} & \\ & \uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) \\ & \uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}]) \\ & \uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Proof. By lemma 2.14, we conclude that $\mathcal{L}[\mathbf{P}], \mathcal{R}[\mathbf{P}] \subseteq \mathcal{B}[\mathbf{P}]$, since $M + 1 \in \mathbf{P}$ and $1 + M \in \mathbf{P}$ both imply $1 + M + 1 \in \mathbf{P}$.

Thus,

$$\begin{aligned}\mathcal{B}[\mathbf{P}] = (\mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]) & \\ & \uplus (\mathcal{L}[\mathbf{P}] \setminus \mathcal{R}[\mathbf{P}]) \\ & \uplus (\mathcal{R}[\mathbf{P}] \setminus \mathcal{L}[\mathbf{P}]) \\ & \uplus (\mathcal{B}[\mathbf{P}] \setminus (\mathcal{L}[\mathbf{P}] \cup \mathcal{R}[\mathbf{P}]))\end{aligned}$$

Since by lemma 2.14 $\mathbf{P} = \mathcal{L}[\mathbf{P}] \cap \mathcal{R}[\mathbf{P}]$, we conclude what we wanted to prove. □

Lemma 2.16 (Associativity of sum)**.** *Let* $\mathbf{P}_1$*,* $\mathbf{P}_2$ *and* $\mathbf{P}_3$ *be properties.*
Then $\sum_{\mathbf{P}_1} \sum_{\mathbf{P}_2} \mathbf{P}_3 = \sum_{\sum_{\mathbf{P}_1} \mathbf{P}_2} \mathbf{P}_3$.

Proof. It follows directly from the associativity of the sum operation on linear orders. Actually, it generalizes to any algebraic equation which holds on linear orders. □

Lemma 2.17 (Sum and union commute)**.** *Let* $\mathcal{P}$ *be a family of properties.*
Let $\mathbf{Q}$ *be a property.*
Then $\sum_{\bigcup \mathcal{P}} \mathbf{Q} = \bigcup_{\mathbf{P} \in \mathcal{P}} \sum_{\mathbf{P}} \mathbf{Q}$.

Proof. This is obvious from the definition of the sum operation. □

Definition 2.18. *We define* **CNT** *as the class of all countable linear orders.*

Definition 2.19. *Let* $\beta \geq \omega$ *be a limit ordinal.*
We define $\Gamma_\beta := \{\gamma : \gamma \subseteq \beta^* + \beta\}^+$.

Example 2.20.

$$\Gamma_\omega = \{1, \omega, \omega^*\}^+$$

Observation 2.21. *Let* $\beta \geq \omega$ *be a limit ordinal.*
Then Γ_β *is a monotone, additive property of linear orders.*

3 General Hausdorff Rank

Definition 3.1. *Let $\mathbf{Q}$ be a property of linear orders.*
We define a property $\mathbf{Q}^\alpha$ for every ordinal $\alpha \geq 0$ as follows:

- *For $\alpha = 0$, $\mathbf{Q}^0 = \{1\}$.*
- *For $\alpha = \gamma + 1$,*
$$\mathbf{Q}^\alpha = \sum_{\mathbf{Q}} \mathbf{Q}^\gamma$$
- *For α a limit ordinal,*
$$\mathbf{Q}^\alpha = \bigcup_{\beta<\alpha} \mathbf{Q}^\beta$$

Example 3.2. *Let $\mathbf{Q}$ be a property of linear orders.*
Then $\mathbf{Q}^1 = \mathbf{Q}$.

Lemma 3.3. *Let $\mathbf{Q}$ be a property of linear orders.*
Let $\alpha \geq 0$, $\delta \geq 0$ be ordinals.
Then,
$$\mathbf{Q}^{\alpha+\delta} = \sum_{\mathbf{Q}^\delta} \mathbf{Q}^\alpha$$

Proof. We shall prove this by induction on $\delta \geq 0$.
For $\delta = 0$ we need to prove

$$\mathbf{Q}^\alpha = \sum_{\mathbf{Q}^0} \mathbf{Q}^\alpha.$$

Which is true by definition, since $\mathbf{Q}^0 = \{1\}$.
For $\delta = \gamma + 1$, using the induction hypothesis,

$$\begin{aligned}
\mathbf{Q}^{\alpha+\delta} &= \mathbf{Q}^{\alpha+\gamma+1} \\
&= \sum_{\mathbf{Q}} \mathbf{Q}^{\alpha+\gamma} \\
&= \sum_{\mathbf{Q}} \sum_{\mathbf{Q}^\gamma} \mathbf{Q}^\alpha \\
&= \sum_{\sum_{\mathbf{Q}} \mathbf{Q}^\gamma} \mathbf{Q}^\alpha \\
&= \sum_{\mathbf{Q}^{\gamma+1}} \mathbf{Q}^\alpha \\
&= \sum_{\mathbf{Q}^\delta} \mathbf{Q}^\alpha
\end{aligned}$$

For δ a limit ordinal, using the induction hypothesis,

$$\begin{aligned}
\mathbf{Q}^{\alpha+\delta} &= \bigcup_{\gamma<\delta} \mathbf{Q}^{\alpha+\gamma} \\
&= \bigcup_{\gamma<\delta} \sum_{\mathbf{Q}^{\gamma}} \mathbf{Q}^{\alpha} \\
&= \sum_{\bigcup_{\gamma<\delta} \mathbf{Q}^{\gamma}} \mathbf{Q}^{\alpha} \\
&= \sum_{\mathbf{Q}^{\delta}} \mathbf{Q}^{\alpha}
\end{aligned}$$

$\square$

Definition 3.4. *Let* $\mathbf{Q}$ *be a property of linear orders.*

Let M *be a linear order, such that* $M \in (\mathbf{Q}^{\alpha})^{+}$ *for some ordinal* α.

We define the $\mathbf{Q}$-Hausdorff rank *of* M *as*

$$\mathbf{hrank}_{\mathbf{Q}}(M) = \sup\left\{\beta : M \notin \left(\mathbf{Q}^{\beta}\right)^{+}\right\}$$

where the supremum is taken over all ordinals β. *(Recall that the supremum of the empty set is defined to be* 0.*)*

Example 3.5. *Let* $\mathbf{Q}$ *be a property of linear orders.*

Let M *be a linear order.*

Then $\mathbf{hrank}_{\mathbf{Q}}(M) = 0$ *if and only* M *is finite.*

4 ω-Hausdorff rank

Definitions 4.1. *Let $\alpha \geq 0$*

We define:

1. $\mathcal{S}_\alpha^1 := \mathcal{B}\left[\Gamma_\omega^\alpha\right]$
2. $\mathcal{S}_\alpha^\omega := \mathcal{L}\left[\Gamma_\omega^\alpha\right] \setminus \mathcal{R}\left[\Gamma_\omega^\alpha\right]$
3. $\mathcal{S}_\alpha^{\omega^*} := \mathcal{R}\left[\Gamma_\omega^\alpha\right] \setminus \mathcal{L}\left[\Gamma_\omega^\alpha\right]$
4. $\mathcal{S}_\alpha^{\omega^*+\omega} := \mathcal{B}\left[\Gamma_\omega^\alpha\right] \setminus \left(\mathcal{L}\left[\Gamma_\omega^\alpha\right] \cup \mathcal{R}\left[\Gamma_\omega^\alpha\right]\right)$

The names will soon be justified.

Lemma 4.2. *Let $\alpha \geq 0$ be an ordinal. Let $s \in \{\omega, \omega^*, \omega^* + \omega\}$.*

Suppose that $\alpha = \sup_{i \in s}(\alpha_i + 1)$ for ordinals $\alpha_i > 0$ for all $i \in s$.

Then, we have the following:

$$\mathcal{S}_\alpha^s = \sum_{i \in s} \Gamma_\omega^{\alpha_i}$$

Proof. TBC. □

Corollary 4.3. *Let $\alpha \geq 0$, $\delta \geq 0$ be ordinals.*

Let $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$

Then,

$$\mathcal{S}_{\alpha+\delta}^s = \sum_{\mathcal{S}_\delta^s} \Gamma_\omega^\alpha$$

Proof. For $s = 1$, it follows from lemma 3.3.

Otherwise, suppose that $\delta = \sup_{i \in s}(\delta_i + 1)$.

Then $\alpha + \delta = \sup_{i \in s}(\alpha_i + \delta_i + 1)$.

$$\mathcal{S}_{\alpha+\delta}^s = \sum_{i \in s} \mathcal{S}_{\alpha+\delta_i+1}^s = \sum_{i \in s} \sum_{\Gamma_\omega^{\delta_i+1}} \Gamma_\omega^\alpha = \sum_{\sum_{i \in s} \Gamma_\omega^{\delta_i+1}} \Gamma_\omega^\alpha = \sum_{\mathcal{S}_\delta^s} \Gamma_\omega^\alpha$$

□

5 Type Theory

Definition 5.1. *Let* $\mathbf{P}$ *be a property of preorders.*

Let $n \in \mathbb{N}$.

We define $\mathbf{type}_n[\mathbf{P}]$ *as the set of all* n*-types satisfiable in* $\mathbf{P}$.

Lemma 5.2. *Let* $\mathbf{Q}$ *be a property of preorders.*

There exists a computable function $f_{\mathbf{Q}} = f : \mathbb{N} \to \mathbb{N}$ *such that for all* $n \in \mathbb{N}$, *and all* $a \in \mathbb{N}$ *such that* $a \geq f(n)$, $\mathbf{type}_n[\mathbf{Q}^a] = \mathbf{type}_n\left[\mathbf{Q}^{f(n)}\right]$.

Equivalently, every preorders of finite rank is n*-equivalent to some preorder of rank* $\leq f(n)$.

Proof. Since there are only finitely many n-types, and the ω-sequence

$$\left\{\mathbf{type}_n\left[\mathbf{Q}^k\right]\right\}_{k\in\omega}$$

is monotone, there must be some k where the sequence stabilizes.

This point k is computable as a function of n, because $\mathbf{type}_n\left[\mathbf{Q}^k\right]$ is computable for every finite k. □

Lemma 5.3. *There exist global computable functions* $a, b : \mathbb{N} \to \mathbb{N}$ *such that for all* $n, c_1, c_2 \in \mathbb{N}$ *such that* $c_1, c_2 \geq a(n)$ *and* $c_1 \equiv c_2 \mod b(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c_1}] = \mathbf{type}_n[\mathbf{Q}^{=c_2}]$$

Equivalently, the sequence $\left\{\mathbf{type}_n\left[\mathbf{Q}^k\right]\right\}_{k\in\omega}$ *is ultimately periodic for all* $n \in \mathbb{N}$. *Furthermore, the starting point and the period itself can be computed as a function of* n.

Proof. Let $n \in \mathbb{N}$.

Since there are only finitely many sets of n-types, there exist (and can be computed) some $a(n) > f(n)$, $a(n) + b(n)$ such that

$$\mathbf{type}_n\left[\mathbf{Q}^{=a(n)}\right] = \mathbf{type}_n\left[\mathbf{Q}^{=a(n)+b(n)}\right]$$

holds for every s.

We shall prove by induction that for all $c \geq a(n)$,

$$\mathbf{type}_n[\mathbf{Q}^{=c}] = \mathbf{type}_n\left[\mathbf{Q}^{=c+b(n)}\right]$$

This will complete the proof.

The base case $c = a(n)$ has been proven in the beginning.

Suppose the induction hypothesis holds for c.

Let M be of rank $c + 1$.

Write $M = \sum_{i\in I} M_i$ where $M_i \in \Gamma_\omega^{<c+1}$, and $M_i \in \Gamma_\omega^{=c}$ infinitely many times.

By the induction hypothesis, if $M_i \in \Gamma_\omega^{=c}$, we can find $N_i \equiv_n M_i$ with $N_i \in \Gamma_\omega^{=c+b(n)}$. Setting $N_i := M_i$ for all other i, we conclude that $N := \sum_{i\in I} N_i$ is n-equivalent to M.

However, clearly $N \in \Gamma_\omega^{=c+b(n)+1}$. So overall,

$$\mathbf{type}_n \left[\mathbf{Q}^{=c+1}\right] \subseteq \mathbf{type}_n \left[\mathbf{Q}^{=c+b(n)+1}\right]$$

Conversely, suppose M is of rank $c+b(n)+1$. Write $M = \sum_{i \in I} M_i$ where $M_i \in \Gamma_\omega^{<c+b(n)+1}$, and $M_i \in \Gamma_\omega^{=c+b(n)}$ infinitely many times.

By the induction hypothesis, we can find for all i such that $M_i \in \Gamma_\omega^{=c+b(n)}$ some $N_i \equiv_n M_i$ with $N_i \in \Gamma_\omega^{=c}$. Furthermore, since $c \geq a(n) > f(n)$, we can find $N_i \equiv_n M_i$ with $N_i \in \Gamma_\omega^{f(n)<c}$ for all other i.

We conclude that $N := \sum_{i \in I} N_i$ is n-equivalent to M. However, clearly $N \in \Gamma_\omega^{=c+1}$. So overall,

$$\mathbf{type}_n \left[\mathbf{Q}^{=c+b(n)+1}\right] \subseteq \mathbf{type}_n \left[\mathbf{Q}^{=c+1}\right]$$

So we have proven the induction step, and the lemma follows. $\square$

Corollary 5.4. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Let $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$ be a shape.

Then there exists a computable function $b(n)$ such that for all $c_1, c_2 \in \mathbb{N}$ such that $c_1, c_2 > a(n)$ and $c_1 \equiv c_2 \mod b(n)$, we have

$$\mathbf{type}_n \left[\mathcal{S}^s_{c_1}\right] = \mathbf{type}_n \left[\mathcal{S}^s_{c_2}\right]$$

Proof. For $s = 1$, it follows from lemma 5.2, since $\mathcal{S}^1_c = \mathbf{Q}^{<c} = \mathbf{Q}^{c-1}$. and $c > a(n) \geq f(n)$ so $c - 1 \geq f(n)$ for $c \in \{c_1, c_2\}$.

For $s \in \{\omega, \omega^*, \omega^* + \omega\}$, it follows easily from lemma 4.2 and lemma 5.3. $\square$

Lemma 5.5. *Let $n \in \mathbb{N}$, and let $\alpha \geq \omega$ be an ordinal.*

Let $s \in \{1, \omega, \omega^*, \omega^* + \omega\}$ be a shape.

$$\mathbf{type}_n \left[\mathcal{S}^s_\alpha\right] = \bigcup_{c<b(n)} \mathcal{S}^s_{a(n)+c}$$

In particular, $\mathbf{type}_n \left[\mathcal{S}^s_\alpha\right]$ can be computed, and is independent of the choice of $\alpha \geq \omega$.

Proof. TBC.

$\square$

6 Decidability of the rank

Definition 6.1. *Let $\alpha \geq \omega$ be an ordinal.*

Let M be a linear order and $x \in M$.

We define the convex equivalence relation:

$$\sim_\alpha := \sim_{\mathcal{B}[\Gamma_\omega^\alpha]}$$

and $[x]_\alpha := [x]_{\mathcal{B}[\Gamma_\omega^\alpha]}$ (that is, $[x]_\alpha$ is the largest $\mathcal{B}[\Gamma_\omega^\alpha]$-subinterval containing x in M).

We define $\sigma_\alpha(M)$ as the α-shape of M.

Lemma 6.2. *The property $\Gamma_\omega^{=\alpha}$ is expressible over intervals in* $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$.

That is, there exists a formula $\varphi_\alpha(\Pi, \Xi)$ such that for every linear order M and every $\mathcal{B}[\Gamma_\omega^\alpha]$-subinterval I of M, we have

$$M, \Pi, \Xi \models \varphi_\alpha(\Pi, \Xi) \iff I = \sum_{i \in I} M_i \text{ where } M_i \in \Gamma_\omega^{=\alpha} \text{ for all } i$$

Proof. It is equivalent to being a sum of $\sim_\alpha$-subintervals, of which at least one has $\sigma_\alpha \neq 1$. □

Theorem 6.3. *There is an oracle reduction from SAT for* $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$*, to SAT for* $\mathbf{MSO}$.

Proof. By the decomposition theorem, there exists a translation, that given an $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ formula φ of quantifier-depth n. outputs an $\mathbf{MSO}$ formula $\psi(\Pi)$ such that...

Let φ be an $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ formula, and let n be the quantifier-depth of φ.

WLOG, assume that φ is a sentence.

First, let us calculate the sets:

$$T_s := \mathbf{type}_n [\mathcal{S}_\alpha^s]$$

for every shape s.

Now we create the formulae:

$$\theta_s(\Pi, \Xi) := \left\{ i : \bigvee_{\tau \in S_s} \Xi(\Pi(i)) = s \right\}$$

$$L(\Pi, \Xi) := \theta_\omega(\Pi, \Xi) \vee \theta_{\omega^* + \omega}(\Pi, \Xi)$$

$$R(\Pi, \Xi) := \theta_{\omega^*}(\Pi, \Xi) \vee \theta_{\omega^* + \omega}(\Pi, \Xi)$$

We create the formula $\chi(\Pi, \Xi)$ as follows:

$$\chi := \Pi = domain(\Xi) \wedge \forall i, i'.i' = i + 1 \implies i \in R(\Pi, \Xi) \vee i' \in L(\Pi, \Xi)$$

Now we claim that φ is satisfiable in $\mathbf{MSO}[\sim_\alpha, \sigma_\alpha]$ iff $\psi \wedge \chi$ is satisfiable in $\mathbf{MSO}$.

If φ is satisfiable, then there exists a model M of φ.

Let $M = \sum_{i \in I} M_i$ be the decomposition of M where $I = \sim_\alpha$ and M_i are the $\sim_\alpha$-equivalence classes.

By the decomposition theorem, Ψ holds in $I, \Pi := \mathbf{type}_n[\cdot]$.

We claim that χ holds in $I, \Pi := \mathbf{type}_n[\cdot]$.

It follows from the star property of $\sim_\alpha$ that the constraint holds.

Conversely, suppose $\psi \wedge \chi$ is satisfiable in $\mathbf{MSO}$.

Let $I, \Pi := T$ be a model of $\psi \wedge \chi$.

Let us take a model M_i with the appropriate type. Now define $M := \sum_{i \in I} M_i$.

We claim that each M_i is a *maximum* $\mathcal{B}[\Gamma_\omega^\alpha]$-subinterval of M.

Suppose $[M_i, M_j]$ is a $\mathcal{B}[\Gamma_\omega^\alpha]$-subinterval of M.

In particular, it has a rank, so it is scattered. So in particular, $[i, j] \subseteq I$ is a scattered interval.

If $i = j$ we are done. Otherwise, let i', j' be such that $i \leq i' < j' \leq j$, and $j' = i' + 1$. But it cannot be the case by the constraint.

$\square$

Definition 6.4. *Let $\alpha_1, \ldots, \alpha_k$ be ordinals.*

We define $C[\alpha_1, \ldots, \alpha_k]$ as the class of countable linear orders, labeled with π_{α_i} and σ_{α_i} for $1 \leq i \leq k$.

Theorem 6.5. *Let $\alpha_1, \ldots, \alpha_k$ be ordinals.*

Let α be an ordinal such that $\alpha < \alpha_i$ for all $1 \leq i \leq k$.

Let $\delta_i > 0$ for $1 \leq i \leq k$ be such that $\alpha_i = \alpha + \delta_i$.

Let $\mathbf{P}$ be the class of countable linear orders,

Then $C_0 = \sum_{\mathbf{P}} C_1$.

Definition 6.6. *A property $\mathbf{P}$ of preorders is called a* computable *property if $\mathbf{type}_n[\mathbf{P}]$ is computable as a function of n.*

Theorem 6.7 (Decomposition theorem)**.** *There exists a computable translation $\mathcal{T}$ from MSO formulae to MSO formulae,*

such that for any $M = \sum_{i \in I} M_i$, formula $\varphi\left(\vec{X}\right)$, vector $\vec{A}$ of the same length as $\vec{X}$, if n is the quantifier-depth of φ, then

$$M, \vec{X} := \vec{A} \models \varphi \iff I, \Pi \models \mathcal{T}\varphi$$

where $\Pi(i) = \mathbf{type}_n[M_i]$.

Lemma 6.8. *There exists a global computable function $h : \mathbb{N} \to \mathbb{N}$ such that the following holds.*

Let $\{C_i\}_{i=1}^k$ be a finite set of colors.

Let $\mathbf{P}$ be a property of linear orders, labeled by the colors $\{C_i\}_{i=1}^k$.

Let $\{\mathbf{Q}_i\}_{i=1}^k$ be a finite set of properties of linear orders.

Then $\mathbf{type}_n\left[\sum_{\mathbf{P}}\left[\vec{C} \leftarrow \vec{\mathbf{Q}}\right]\right]$ *is a computable function of* $\mathbf{type}_{h(n)}\left[\mathbf{P}\right]$ *and* $\mathbf{type}_n\left[\vec{\mathbf{Q}}\right] = \{\mathbf{type}_n\left[\mathbf{Q}_i\right]\}_{i=1}^{k}$.

Proof. TBC. $\square$

7 Everything Better

Theorem 7.1. *Let $\mathcal{C}$ be a computable property of linear orders, such that $\mathcal{C}$ is closed under taking subintervals, projections and inverse-projections (i.e, of one of the colors), and all finite-sums and $\mathcal{C}$-sums.*

Let $\mathbf{P}_1, \ldots, \mathbf{P}_k \subseteq \mathcal{C}$ be computable properties of linear orders.

Let $\mathbf{MSO}[P_1, \ldots, P_k]$ be monadic second order logic of order over $\mathcal{C}$, with $P_1, \ldots, P_k$ as monadic predicates whose semantics are: $P_i(X)$ holds iff X is a subinterval which satisfies $\mathbf{P}_i$.

Given ϕ a formula of $\mathbf{MSO}[P_1, \ldots, P_k]$ (possibly with free variables) we define

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

(Note that M above may be a labeled *linear order.)*

Then $\mathcal{C}_\phi$ is a computable property of linear orders.

Proof. By structural induction on ϕ.

Suppose ϕ is an atomic formula. If ϕ is of the form $X \subseteq Y$ or $X \leq Y$,

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models \phi\}$$

and thus,

$$\mathbf{type}_n\left[\mathcal{C}_\phi\right] = \{\tau \in \mathbf{type}_n\left[\mathcal{C}\right] : \tau \models \phi\}$$

which is computable since $\mathbf{type}_n\left[\mathcal{C}\right]$ is computable, and we can then compute whether $\tau \models \phi$ for each $\tau \in \mathbf{type}_n\left[\mathcal{C}\right]$.

If ϕ is of the form $P_i(X)$, then

$$\mathcal{C}_\phi = \{M \in \mathcal{C} : M \models P_i(X)\}$$

and thus,

$$\mathbf{type}_n\left[\mathcal{C}_\phi\right] = \mathbf{type}_n\left[\mathbf{P}_i\right]$$

which is computable since $\mathbf{P}_i$ is computable.

If $\phi = \neg\phi_1$, then

$$\mathcal{C}_\phi = \mathcal{C} \setminus \mathcal{C}_{\phi_1}$$

□