1 Ordinals

In this section the logic is always MSO, and the structures are ordinals.

Notations 1. *Let $\iota_\alpha(X)$ denote that the set X is a closed interval isomorphic to α.*

Let $\pi_\alpha(X)$ denote that the set X is a finite union of intervals and is isomorphic to α.

Let $\kappa_\alpha(x)$ denote that x is a multiple of α.

Proposition 1. *Let $\alpha = \omega^\beta$ be an ordinals and $x, y \in \alpha$ some elements.*

Then, $[x, y]$ is isomorphic to α iff there exists some z which is a multiple of α, such that y is the least greater multiple of α and $z \leq x < y$.

Proof. Divide $x = \gamma \cdot \alpha + \delta$ where $\delta < \alpha$, and as $x + \alpha = y$, $y = \gamma \cdot \alpha + \delta + \alpha$.

Now, as α is additively indecomposable, $\delta + \alpha = \alpha$, so $y = \gamma \cdot \alpha + \alpha = (\gamma + 1) \cdot \alpha$. Choosing $z = \gamma + 1$, indeed y is the least greter multiple of α, and $z \leq x < y$, as we need. □

Corollary 1. *Let $\alpha = \omega^\beta$. Over ordinals, ι_α is expressible via κ_α.*

Proof. By the previous lemma,

$$\iota_\alpha(X) \iff \exists x, y, z. X = [x, y] \wedge \kappa_z \wedge y = \min_a \{a : z < a \wedge \kappa_\alpha(a)\}$$

□

Proposition 2. *Let $\alpha = \omega^\beta$. Then, a set X satisfies π_α iff it is a finite sum $X = \sum_{i=1}^m X_i$ where for $i < m$, X_i is a strict prefix of some interval satisfying ι_α, and for $i = m$, X_i satisfies ι_α.*

Proof. Suppose X_i is indeed an interval which is less then α for $i < m$ and equals α for $i = m$. Since α is additively indecomposable, the sum $\sum_{i=1}^m X_i$ is indeed α. Therefore, X is a disjoint sum of intervals isomorphic to α.

For the other direction, suppose X is a disjoint sum of intervals isomorphic to α. Let the intervals be X_i for $i = 1, ..., m$. Since α is additively indecomposable, nessessarily $X_i < \alpha$ for $i < m$ and $X_m = \alpha$.

Also, note that for $i < m$, the interval from the left end of X_i to the right end of X_m is at least α, since it contains $X_i \cup X_m$ which is isomorphic to α, and thus there is an interval in which X_i is contained which is isomorphic to α. □

Corollary 2. *Let $\alpha = \omega^\beta$. Then π_α is expressible via ι_α.*

Proof. By the previous corollary, it is enough to formulate that there exists a finite set of left endpoints and right endpoints, such that except for the last ones, All the intervals are strictly the prefix of an interval isomorphic to α, and such that the last interval is isomorphic to α.

$$\pi_\alpha(X) \iff \exists L \exists R. finite(L) \wedge finite(R) \wedge \min(L \cup R) \in L \wedge \max(L \cup R) \in R \wedge \forall x, y \in L. x < y \implies \exists z \in$$

□

Proposition 3. *Let α be any ordinal. Let the cantor decomposition be*

$$\alpha = \sum_{i=1}^{k} \alpha_i$$

where $\alpha_i = \omega^{\beta_i}$.

Then, a set X satisfies π_α iff it is an ordered sum $X = \sum_{i=1}^{k} X_i$ where X_i satisfies π_{α_i}.

Proof. Obviously a disjoint sum of such intervals satisfies π_α.

For the other direction, suppose X is a union of closed intervals and is isomorphic to α. Let $\varphi : X \to \alpha$ be the isomorphism map. Recall that $\alpha = \sum_{i=1}^{k} Y_i$ where $Y_i \cong \alpha_i$.

Let $X_i := \varphi^{-1}(Y_i)$ for $i = 1, ..., k$. Then it can easily be seen that X_i is a disjoint sum of closed intervals. □

Corollary 3. *Let α be any ordinal. Then π_α is expressible via $\pi_{\alpha_1}, ..., \pi_{\alpha_k}$ for some ordinals $\alpha_1 = \omega^{\beta_1}, ..., \alpha_k = \omega^{\beta_k}$.*

Proof. This is pretty much the same trick as the previous corrollary, but is very tedious to write down. □

Theorem 1. *Let α be any ordinal.*

Over ordinals, π_α is expressible via $\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}$ where $\alpha_1 = \omega^{\beta_1}, ..., \alpha_k = \omega^{\beta_k}$.

Proof. Combine the three previous corollaries. □

Definition 1. *Let $\alpha > 0$ be a nonzero ordinal.*

Let ξ be any ordinal. Let γ and δ be the unique ordinals such that $\xi = \gamma\alpha + \delta$.

Then, define $\xi/\alpha := \gamma$.

That is, ξ/α is the quotient of ξ by α, "rounded down".

Now, suppose $\mathcal{C}$ be a class of ordinals.

Define $\mathcal{C}_{/\alpha} := \{\xi/\alpha : \xi \in \mathcal{C}\}$

Define $\mathcal{C}_{\alpha|} := \left\{\delta_{\alpha\nmid\xi} : \xi \in \mathcal{C}\right\} = \{0 : \xi \in \mathcal{C}, \alpha \mid \xi\} \cup \{1 : \xi \in \mathcal{C}, \alpha \nmid \xi\}$.

Theorem 2. *Let $\mathcal{C}$ be a class of ordinals, and let $\alpha_1 = \omega^{\beta_1}, ..., \alpha_k = \omega^{\beta_k}$ be ordinals, where $\beta_1 \geq \cdots \geq \beta_k$, that is, $\alpha_k \mid \cdots \mid \alpha_1$.*

There is a computable function from a fully-quantified formula φ in

$$MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}\right]$$

to fully-quantified formulae ψ, χ in

$$MSO\left[\kappa_{\frac{\alpha_1}{\alpha_k}}, ..., \kappa_{\frac{\alpha_{k-1}}{\alpha_k}}\right]$$

such that φ is true over $\mathcal{C}$ iff ψ is true over $\mathcal{C}_{/\alpha_k} + 1$.

Proof. Denote $\alpha := \alpha_k$.

By the composition theorem, given φ of quantifier rank ρ, we can compute ψ', with free variables $\{X_\tau\}_\tau$, where τ is a ρ-type.

such that for any $\xi = \sum_{i \in \xi/\alpha} \xi_i + \delta$, for $\delta < \alpha$,

(denoting $\xi_{\xi/\alpha} := \delta$ we get $\xi = \sum_{i \in \xi/\alpha+1} \xi_i$),

$$\xi \vDash \varphi \iff \xi/\alpha + 1, Q_1, ..., Q_r \vDash \psi'$$

where $Q_\tau = \{\xi_i \vDash \tau : i \in \xi/\alpha + 1\}$

Now, let $\upsilon(X)$ be the formula $X \subseteq \{max\} \vee X^{\complement} \subseteq \{max\}$ (that is, up to the maximal element, X is all or nothing).

Then, set $\psi :=$ "$\forall \tau$"$\forall X_t \upsilon(X_t) \implies \psi'$.

TBC.

$\square$