1 Ordinals

In this section the logic is always MSO, and the structures are ordinals.

Notations 1. *Let $\iota_\alpha(X)$ denote that the set X is a closed interval isomorphic to α.*

Let $\pi_\alpha(X)$ denote that the set X is a finite union of intervals and is isomorphic to α.

Let $\kappa_\alpha(x)$ denote that x is a multiple of α.

Proposition 1. *Let $\alpha = \omega^\beta$ be an ordinals and $x, y \in \alpha$ some elements.*

Then, $[x, y]$ is isomorphic to α iff there exists some z which is a multiple of α, such that y is the least greater multiple of α and $z \leq x < y$.

Proof. Divide $x = \gamma \cdot \alpha + \delta$ where $\delta < \alpha$, and as $x + \alpha = y$, $y = \gamma \cdot \alpha + \delta + \alpha$.

Now, as α is additively indecomposable, $\delta + \alpha = \alpha$, so $y = \gamma \cdot \alpha + \alpha = (\gamma + 1) \cdot \alpha$. Choosing $z = \gamma + 1$, indeed y is the least greter multiple of α, and $z \leq x < y$, as we need. □

Corollary 1. *Let $\alpha = \omega^\beta$. Over ordinals, ι_α is expressible via κ_α.*

Proof. By the previous lemma,

$$\iota_\alpha(X) \iff \exists x, y, z. X = [x, y] \land \kappa_z \land y = \min_a \{a : z < a \land \kappa_\alpha(a)\}$$

□

Proposition 2. *Let $\alpha = \omega^\beta$. Then, a set X satisfies π_α iff it is a finite sum $X = \sum_{i=1}^m X_i$ where for $i < m$, X_i is a strict prefix of some interval satisfying ι_α, and for $i = m$, X_i satisfies ι_α.*

Proof. Suppose X_i is indeed an interval which is less then α for $i < m$ and equals α for $i = m$. Since α is additively indecomposable, the sum $\sum_{i=1}^m X_i$ is indeed α. Therefore, X is a disjoint sum of intervals isomorphic to α.

For the other direction, suppose X is a disjoint sum of intervals isomorphic to α. Let the intervals be X_i for $i = 1, ..., m$. Since α is additively indecomposable, nessessarily $X_i < \alpha$ for $i < m$ and $X_m = \alpha$.

Also, note that for $i < m$, the interval from the left end of X_i to the right end of X_m is at least α, since it contains $X_i \cup X_m$ which is isomorphic to α, and thus there is an interval in which X_i is contained which is isomorphic to α. □

Corollary 2. *Let $\alpha = \omega^\beta$. Then π_α is expressible via ι_α.*

Proof. By the previous corollary, it is enough to formulate that there exists a finite set of left endpoints and right endpoints, such that except for the last ones, All the intervals are strictly the prefix of an interval isomorphic to α, and such that the last interval is isomorphic to α.

$$\pi_\alpha(X) \iff \exists L \exists R. finite(L) \land finite(R) \land \min(L \cup R) \in L \land \max(L \cup R) \in R \land \forall x, y \in L. x < y \implies \exists z \in$$

□

Proposition 3. *Let α be any ordinal. Let the cantor decomposition be*

$$\alpha = \sum_{i=1}^{k} \alpha_i$$

where $\alpha_i = \omega^{\beta_i}$.

Then, a set X satisfies π_α iff it is an ordered sum $X = \sum_{i=1}^{k} X_i$ where X_i satisfies π_{α_i}.

Proof. Obviously a disjoint sum of such intervals satisfies π_α.

For the other direction, suppose X is a union of closed intervals and is isomorphic to α. Let $\varphi : X \to \alpha$ be the isomorphism map. Recall that $\alpha = \sum_{i=1}^{k} Y_i$ where $Y_i \cong \alpha_i$.

Let $X_i := \varphi^{-1}(Y_i)$ for $i = 1, ..., k$. Then it can easily be seen that X_i is a disjoint sum of closed intervals. □

Corollary 3. *Let α be any ordinal. Then π_α is expressible via $\pi_{\alpha_1}, ..., \pi_{\alpha_k}$ for some ordinals $\alpha_1 = \omega^{\beta_1}, ..., \alpha_k = \omega^{\beta_k}$.*

Proof. This is pretty much the same trick as the previous corrollary, but is very tedious to write down. □

Theorem 1. *Let α be any ordinal.*

Over ordinals, π_α is expressible via $\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}$ where $\alpha_1 = \omega^{\beta_1}, ..., \alpha_k = \omega^{\beta_k}$.

Proof. Combine the three previous corollaries. □

Definition 1. *Let $\alpha > 0$ be a nonzero ordinal.*

Let ξ be any ordinal. Let γ and δ be the unique ordinals such that $\xi = \gamma\alpha + \delta$.

Then, define $\xi/\alpha := \gamma$ and $\xi\%\alpha := \delta$.

That is, ξ/α is the quotient of ξ by α, "rounded down".

Now, suppose $\mathcal{C}$ be a class of ordinals.

Define $\mathcal{C}_{/\alpha} := \{\xi/\alpha : \xi \in \mathcal{C}\}$ Define $\mathcal{C}_{\%\alpha} := \{\xi\%\alpha : \xi \in \mathcal{C}\}$

Theorem 2. *Let $\mathcal{C}$ be a class of ordinals, and let $\alpha_1 = \omega^{\beta_1}, ..., \alpha_k = \omega^{\beta_k}$ be ordinals, where $\beta_1 \geq \cdots \geq \beta_k$, that is, $\alpha_k \mid \cdots \mid \alpha_1$.*

Then the logic $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}\right]$ is decidable over $\mathcal{C}$,

using oracles to decide $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$ over α and over $\mathcal{C}_{\%\alpha_k}$ and decide MSO over $\mathcal{C}_{/\alpha_k}$.

Proof. Denote $\alpha := \alpha_k$.

By the composition theorem, given φ of quantifier rank ρ, we can compute ψ', with free variables $\{X_\tau\}_\tau$, where τ is a ρ-type.

such that for any $\xi = \sum_{i \in \xi/\alpha} \xi_i + \delta$, for $\delta < \alpha$,

(denoting $\xi_{\xi/\alpha} := \delta$ we get $\xi = \sum_{i \in \xi/\alpha+1} \xi_i$),

$$\xi \vDash \varphi \iff \xi/\alpha + 1, Q_1, ..., Q_r \vDash \psi'$$

where $Q_\tau = \{\xi_i \vDash \tau : i \in \xi/\alpha + 1\}$

Now, since the Q_τ are all (except maybe the last index) or nothing (except including maybe last the index), we can actually describe them with two propositional/boolean variables. That is, we can replace each time X_τ appears in ψ', by $\{i < \max : y_\tau\} \cup \{\max : z_\tau\}$ where y_τ and z_τ are two new propositional variables, and then:

$$\xi \vDash \varphi \iff \xi/\alpha + 1, Y_1, Z_1, ..., Y_r, Z_r \vDash \psi''$$

where $Y_\tau := \alpha \vDash \tau$, and $Z_\tau := \delta \vDash \tau$, and ψ'' is ψ' after the replacement.

Therefore, given a class $\mathcal{C}$ of ordinals, we first determine for each τ what Y_τ is, using the fact that α is decidable over $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$, and that over α we can replace κ_{α_k} with $= 0$.

Now, we calculate all the possible values of Z_τ: for that, we check if τ holds in all of $\mathcal{C}_{\%\alpha}$, and if $\neg\tau$ holds in all of $\mathcal{C}_{\%\alpha}$.

This is possible since $\mathcal{C}_{\%\alpha}$ is decidable over $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$, and again, we can replace κ_{α_k} with $= 0$.

Finally, for each possible combination of values $Y_1, Z_1, ..., Y_r, Z_r$, we calculate whether ψ'' holds over $\mathcal{C}_{/\alpha}$. This is possible since MSO is decidable over $\mathcal{C}_{/\alpha}$.

Then, suppose that φ holds over all $\xi \in \mathcal{C}$.

Then, it must be the case that ψ'' holds for any such combination.

And also for the contrary, every ordinal $\xi \in \mathcal{C}$ gives rise to some combination. □

Corollary 4. *Suppose in the previous theorem that $\mathcal{C}$ is either:*

1. *a **single countable** ordinal: $\mathcal{C} = \{\xi\}$*
2. *the class of **all countable** ordinals*

Then $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_k}\right]$ is decidable.

Proof. For a single countable ordinal ξ: α over $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$ is decidable by induction on k (for the induction step), since MSO is decidable over a single countable ordinal (for the base case)

$\mathcal{C}_{/\alpha} = \{\xi/\alpha\}$ again is decidable for the same reason.

and $\mathcal{C}_{\%\alpha} = \{\xi\%\alpha\}$ is a single countable ordinal and so MSO is decidable over it.

Now, suppose $\mathcal{C}$ is the class of all countable ordinals, then $CC_{/\alpha} = \mathcal{C}$, so $MSO\left[\kappa_{\alpha_1}, ..., \kappa_{\alpha_{k-1}}\right]$ is decidable over it by induction on k, or since MSO is decidable over the class of all countable ordinals (in the base case),

and $\mathcal{C}_{\%\alpha} = \{\delta : \delta < \alpha\}$. MSO is decidable over it, since a formula φ is true over $\{\delta : \delta < \alpha\}$ iff $\varphi \vee \exists X.\kappa_\alpha$ is true over the class of all countable ordinals, and we have already established that this $MSO\left[\kappa_\alpha\right]$ is decidable over a single countable ordinal. □

Theorem 3. *Under the same conditions, $MSO\left[\pi_{\alpha_1}, ..., \pi_{\alpha_k}\right]$ is decidable.*

Proof. Combine all the results we have had so far. □