

# Bipartite Perfect Matching is in Quasi-NC

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# Outline

## Abstract

This is a summary of "Bipartite Perfect Matching is in Quasi-NC" by Fenner, Gurjar, and Thierauf.

- 1 Introduction
- 2 Combinatorics and Probability
- 3 Isolation
- 4 Cycles and Circulation
- 5 The Theorem

# Notation

## Notation

Throughout this presentation, *ALL* the graphs are:

- undirected
- bipartite
- balanced
- labeled

Usually we call our graph  $G$  and the set of edges  $E$ .

The set of edges can be regarded as a relation  $E \subseteq [n] \times [n]$ .

## NC

## Definition

- We define  $\mathcal{NC}^k$  to be the class of problems that can be solved by a polynomial-time uniform family of circuits of polynomial size and depth  $\mathcal{O}(\log^k n)$ .
- Equivalently,  $\mathcal{NC}^k$  is the class of problems that can be solved by a polynomial number of processors in  $\mathcal{O}(\log^k n)$  time.



$$\mathcal{NC} = \bigcup_{k \geq 0} \mathcal{NC}^k$$

## Fact

DET, the problem of computing the determinant of an  $n \times n$  matrix with  $\text{poly}(n)$ -bounded entries, is in  $\mathcal{NC}^2$ .

# Perfect Matching

## Definition

- A *perfect matching* of a graph  $G$  is a set of edges such that every vertex is incident to exactly one edge.
- The *perfect matching problem* is to determine whether a graph has a perfect matching, or to find one.
- The *decision version* of the perfect matching problem is denoted PM.
- The *search version* of the perfect matching problem is denoted SearchPM.

# State of the Art

It has already been known that perfect matching (whether decision or search) can be solved in randomized  $\mathcal{NC}$ .

## Open Problem

Can perfect matching be solved in  $\mathcal{NC}$ ?

## Today

There exist algorithms in:

- Quasi- $\mathcal{NC}^2$  for PM (that is, with  $\mathcal{O}(n^{\log n})$  processors and  $\mathcal{O}(\log^2 n)$  depth)
- randomized  $\mathcal{NC}^2$  for PM with only  $\mathcal{O}(\log^2 n)$  random bits.
- (We will not see this) Quasi- $\mathcal{NC}^2$  for PM (that is, with  $\mathcal{O}(n^{\log n})$  processors and  $\mathcal{O}(\log^2 n)$  depth)
- (We will not see this)  $\mathcal{NC}^3$  for SearchPM with only  $\mathcal{O}(\log^2 n)$  random bits.

## Definitions (Bi-adjacency Matrix)

- The *bi-adjacency matrix*:  $\mathbf{A}_G = \mathbf{A}_E = A$  of  $G$  (or  $E$ ) is an  $n \times n$  matrix where  $A_{ij} = 1_{(i,j) \in E}$
- We write  $B \leq C$  iff  $\forall i, j. B_{ij} \leq C_{ij}$
- A *permutation matrix* is a matrix  $\mathbf{A}_\Gamma$  where  $\Gamma = \{(i, \sigma i) : i \in [n]\}$  for some permutation  $\sigma \in S_n$

## Exercise

For which  $E \subseteq [n] \times [n]$  is  $\mathbf{A}_E$  a permutation matrix?

## Exercise

For which  $B \in \mathbb{R}_{\geq 0}^{n \times n}$  and  $E$  does  $B \leq \mathbf{A}_E$  hold?

## Definitions (Weight Functions)

- A *weight function* is a function  $w : E \rightarrow \mathbb{N}$ .
- We extend it naturally to a function  $w : 2^E \rightarrow \mathbb{N}$  by setting  $w(S) = \sum_{e \in S} w(e)$  for all  $S \subseteq E$ .
- Even more generally (why?), we extend it to a *linear function*  $w : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$  by setting  $w(A) = \sum_{ij \in E} w(ij)A_{ij}$  for all  $A \in \mathbb{R}^{n \times n}$ .



## Definitions (Perfect Matching)

- A *perfect matching* is a set of edges  $M$  such that every vertex is incident to exactly one edge in  $M$ .
- Equivalently,  $M$  is a perfect matching iff  $\mathbf{A}_M$  is a permutation matrix.

# Birkhoff's Theorem I

## Definition (Doubly Stochastic Matrix)

- A matrix  $M \in \mathbb{R}_{\geq 0}^{n \times n}$  is *doubly stochastic* if the sum of the entries in each row and column is 1.
- Equivalently,  $M$  is doubly stochastic if  $M\mathbf{1} = M^T\mathbf{1} = \mathbf{1}$ .

## Definition (Convex Hull)

The *convex hull* of a set  $S$  is the smallest convex set containing  $S$ , denoted  $\mathbf{CH}(S)$ .

It is precisely the set of all convex combinations of elements of  $S$ , i.e.,  $\mathbf{CH}(S) = \left\{ \sum_{i=1}^k \lambda_i x_i : x_i \in S, \lambda_i \geq 0, \sum_{i=1}^k \lambda_i = 1 \right\}$ .

# Birkhoff's Theorem II

## Theorem (Birkhoff, 1946)

*The set of doubly stochastic matrices, denoted  $\mathcal{B}_n$ , is the convex hull of the permutation matrices.*

## Hint

Use Hall's marriage theorem.

# Perfect Matching Polytope

## Theorem (Birkhoff, 1946)

*The set of doubly stochastic matrices is the convex hull of the permutation matrices.*

## Definition (Perfect Matching Polytope)

The *perfect matching polytope*  $\mathcal{P}_G$  is the convex hull of the bi-adjacency matrices of all the perfect matchings of  $G$ .

## Corollary

*The perfect matching polytope  $\mathcal{P}_G$  is exactly the matrices  $B$  in the Birkhoff polytope  $\mathcal{B}_n$  such that  $B \leq \mathbf{A}_G$ :*

$$\mathcal{P}_G = \{B : B \in \mathcal{B}_n \wedge B \leq \mathbf{A}_G\}$$

## Definition

Let  $M$  be a perfect matching. We define its *sign*  $\text{sgn} M = \text{sgn} \sigma$  where  $\sigma$  is the permutation given by  $M$ .

## Definition

Let  $\{w_i : E \rightarrow \mathbb{N}\}_{i \in [k]}$  be weight functions.

We define the *weighted Edmonds matrix*  $D^{E,w}$  as follows:

$$\left(D^{E,w}\right)_{ij} = \begin{cases} x_{ij}^{(k)w_k(ij)} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $a \in \mathbb{N}^k$  be a vector of variables. We also denote  $D_a^{E,w} = D^{E,w}(x_{ij}^{(k)} := a_k)$

## Definitions

- A weight function  $w : E \rightarrow \mathbb{N}$  is called *isolating* for  $E$  if the existence of a perfect matching in  $E$  implies that  $E$  has a unique minimum weight perfect matching, which is then called *isolated*.
- Similarly, a sequence of weight functions  $w = w_{i \in [k]}$  is called isolating for  $E$  if the existence of a perfect matching in  $E$  implies that  $E$  has a unique minimum weight perfect matching with respect to  $w = (w_0, \dots, w_k)$ , ordered lexicographically.

## Corollary

- If  $E$  has no perfect matchings, then  $\det D^{E,w} = 0$ .
- If  $w$  is isolating for  $E$ , then  $\det D^{E,w} \neq 0 \iff$  there is a perfect matching.

### Example

Let  $E = \{e_k\}_k$  be an enumeration of the edges. Let  $w : E \rightarrow \mathbb{N}$  be defined by  $w(e_k) = 2^k$ . Every subset of  $E$  has a unique weight. In particular,  $w$  is isolating for  $E$ . Is this enough?

# Isolation Lemma

## Lemma (Mulmuley, Vazirani & Vazirani 1987)

*Let  $B$  be a finite set.*

*Let  $k$  be a positive integer.*

*Let  $\mathcal{F} \subseteq 2^B$  be a nonempty family of subsets of  $B$ .*

*Let  $w : B \rightarrow k$  be a be chosen uniformly at random among all functions from  $B$  to  $[k]$ .*

*Then, with probability at least  $1 - |B|/k$ , there exists a unique set  $S \in \mathcal{F}$  with a minimum weight among all sets in  $\mathcal{F}$ .*



## Corollary

*Let  $w : E \rightarrow [n^3]$  be a random weight function.  
Then with probability at least  $1 - 1/n$ ,  $w$  is isolating for  $G$ .*

## Proof.

Take  $\mathcal{F}$  as the family of perfect matchings of  $G$ .  
As,  $|E| \leq n^2$ , by the isolation lemma, with probability at least

$$(1 - |E|/n^3) \geq 1 - n^2/n^3 = 1 - 1/n$$

$w$  is isolating for  $G$ . □

## Corollary

*There exists an RNC algorithm that can solve PM, which uses  $\text{poly}(n)$  random bits.*

# Circulation

## Definition

Let  $w : E \rightarrow \mathbb{N}$  be a weight function. The  $w$ -circulation of a cycle  $C = e_1, \dots, e_{2k}$  is defined as

$$c_w(C) = |w(e_1) - w(e_2) + \dots + w(e_{2k-1}) - w(e_{2k})|$$

This is well defined because we take the absolute value.

## Remark

Let  $M_1$  and  $M_2$  be two perfect matchings of  $G$ , and suppose  $C \subseteq M_1 \triangle M_2$ .

Then  $c_w(C) = |w(C \cap M_1) - w(C \cap M_2)|$ .

## Claim

*The symmetric difference of two perfect matchings is a union of disjoint cycles.*

*Each cycle consists of interleaved edges from the two matchings.*

## Proof.

The degree of each vertex is 1 in a perfect matching.

Thus, in the symmetric difference of two perfect matchings, each vertex has degree 0 or 2.

Therefore it is a union of disjoint cycles. The rest is easy to see. □

## Lemma

*Let  $w : E \rightarrow \mathbb{N}$  be a weight function.*

*If every cycle in  $G$  has a nonzero circulation, then  $w$  is isolating for  $G$ .*

## Proof.

On the contrary, suppose that  $M_1$  and  $M_2$  are two minimum perfect matchings of  $G$ . Choose some cycle  $C \subseteq M_1 \triangle M_2$ . Since  $c_w(C) > 0$ , we have  $w(C \cap M_1) \neq w(C \cap M_2)$ . WLOG, assume that  $w(C \cap M_1) < w(C \cap M_2)$ . Then  $M_2 \triangle C$  is a perfect matching with weight less than  $w(M_2) = w(M_1)$ , contradicting minimality. □

## Definition

Suppose  $G$  has a perfect matching. Let  $w : E \rightarrow \mathbb{N}$  be a weight function.

Then  $G_w \subseteq G$  is the union of all minimum weight perfect matchings of  $G$ , relatively to  $w$ .

## Theorem

Let  $G^0 = G$  be and  $G^{i+1} = (G^i)_{w_i}$ .

Let  $k \geq \log n - 2$  and let  $w_0, \dots, w_k : E \rightarrow \mathbb{N}$  be weight functions such that  $w_i$  gives nonzero circulation to all the cycles of size at most  $4 \cdot 2^i$  in  $G_i$ .

Then the joint weight function  $w = (w_0, \dots, w_k)$  is isolating.

## Lemma

*Let  $r \geq 2$  be even. Suppose  $G$  has no cycles of size at most  $2r$ . Then  $G$  has at most  $n^4$  cycles of size at most  $4r$ .*

## Proof.

- Equivalently, there is at most one path of length  $\leq r$  between any two vertices.
- Let  $v_0, v_1, v_2, v_3$  be vertices in  $V_1$ , such that the distance between  $v_i$  and  $v_{i+1 \bmod 4}$  is  $\leq r$ .
- There exists a *partial function* from  $V_1^4$  to cycles in  $G$  that maps  $(v_0, v_1, v_2, v_3)$  to the cycle composed of the unique paths of length  $\leq r$  between  $v_i$  and  $v_{i+1 \bmod 4}$ .
- Let  $C$  be a cycle of size  $\leq 4r$ .  
We can choose arbitrarily  $v_0, v_1, v_2, v_3 \in C$  such that the distance between  $v_i$  and  $v_{i+1 \bmod 4}$  is  $\leq r$ .
- This function is onto the set of cycles of size  $\leq 4r$ . Thus, there are at most  $|V_1|^4 = n^4$  cycles of size  $\leq 4r$ .



## Lemma

*Suppose  $G$  has a perfect matching and let  $w : E \rightarrow \mathbb{N}$  be a weight function.*

*Then every perfect matching in  $G_w$  has the same weight as every minimum weight matching in  $G$ .*



## Proof.

If  $w$  is isolating,  $G_w$  is just a single perfect matching.

Otherwise, suppose the minimum weight for perfect matchings in  $G$  is  $m$ . Suppose there exist  $t$  such matchings and let  $X$  be their average. Then,  $w(X) = m$ .

Let  $N$  be a perfect matching in  $G_w$ . Every edge is contained in some minimum weight perfect matching, so  $X - 1/tN \geq 0$ .

It can easily be seen that the sum of every row and column is  $(t-1)/t$  and thus the matrix  $\frac{t}{t-1}X - \frac{1}{t-1}N$  is doubly stochastic. Therefore, it lies in  $\mathcal{P}_G$  and thus has weight at least  $m$ .

That is,

$$m \leq w\left(\frac{t}{t-1}X - \frac{1}{t-1}N\right) = m \cdot \frac{t}{t-1} - w(N) \cdot \frac{1}{t-1}$$

From that it follows that  $w(N) \leq m$ . □

### Lemma

*Suppose  $G$  has a perfect matching and let  $w : E \rightarrow \mathbb{N}$  be a weight function.*

*Then the  $w$ -circulation of every cycle in  $G_w$  is zero.*

## Proof.

Let  $X$  be the average of all perfect matchings in  $G_w$ .

Let  $t$  be the number of perfect matchings,  $\varepsilon = 1/t$ , and let  $M_1, \dots, M_t$  be the perfect matchings. Then  $X = \varepsilon \sum_{i=1}^t M_i$ .

Since each edge is contained in some perfect matching, we have  $X_{ij} \geq \varepsilon$  for all  $i, j$  such that  $(i, j) \in E$ .

Let  $C = e_1, \dots, e_{2p}$  be a cycle in  $H$ .

Define  $Y$  by

$$Y_{ij} = \begin{cases} X_{ij} + (-1)^k \varepsilon & \text{if } (i, j) = e_k \in C, \\ X_{ij} & \text{otherwise.} \end{cases}$$

Then clearly  $Y \geq 0$  and  $Y\mathbf{1} = Y^T\mathbf{1} = \mathbf{1}$ . Therefore,  $Y$  lies in the perfect matching polytope.

Since all the perfect matchings have the same weight,

$w(Y) = w(X)$  and thus  $w(Y - X) = 0$ . But

$c_w(C) = \varepsilon w(Y - X)$ , and thus  $c_w(C) = 0$ . □

## Lemma

*Using  $\mathcal{O}(\log n)$  random bits we can generate a weight assignment  $w : E \rightarrow \mathbb{N}$  with  $w \leq \text{poly}(n)$  such that for every set of  $n^4$  cycles,  $w$  gives nonzero circulation to all of them with probability at least  $1 - 1/n$ .*

## Proof.

Let  $w(e_k) = 2^k$ . Let the cycles be  $C_1, \dots, C_s$ . Then,  $c_w(C_i) \neq 0$  for all  $i$  is equivalent to  $\prod_{i=1}^{n^4} c_w(C_i) \neq 0$ . This product is bounded by  $\text{poly}(n)^{n^4} = 2^{\text{poly}(n)}$ .

Thus, it has at most  $\text{poly}(n)$  prime factors, say  $k$  prime factors. Choose  $t = kn$ .

Then if we choose a random prime amongst the first  $t$  primes  $[p_1, \dots, p_t]$ , then if the product is nonzero, with probability at least  $1 - 1/n$  it is still true modulo the chosen prime.  $\square$

## Theorem

*Let  $G^0 = G$  be and  $G^{i+1} = (G^i)_{w_i}$ .*

*Let  $k \geq \log n - 2$  and let  $w_0, \dots, w_k : E \rightarrow \mathbb{N}$  be weight functions such that for every set of  $w_i$  gives nonzero circulation to all the cycles of size at most  $4 \cdot 2^i$  in  $G_i$ .*

*Then the joint weight function  $w = (w_0, \dots, w_k)$  is isolating.*

## Proof.

Since  $w_i$  gives nonzero circulation to all the cycles in  $G^i$ , all those cycles do not appear in  $G^{i+1}$ .

In particular,  $G^k$  has no cycles of size at most  $4 \cdot 2^k \geq n$ .

That is,  $G^k$  has a unique perfect matching. We want to show that it is isolated by  $w$ . This follows from the next lemma. □

## Lemma

*Suppose  $M_1$  is a matching that appears in  $G^i$  but not in  $G^{i+1}$  and  $M_2$  is a matching that appears in  $G_{i+1}$ .  
Then  $w_i(M_2) < w_i(M_1)$ .*

## Proof.

Since  $M_2$  appears in  $G_{i+1}$ , it has the same weight as any  $w_i$ -minimum weight matching in  $G_i$ . That is  $w_i(M_2) \leq w_i(M_1)$ . But if  $w_i(M_1) = w_i(M_2)$  held, then  $M_1$  would have been in  $G^{i+1}$  by its definition. □

# Deciding PM in randomized $\mathcal{NC}^2$

The following algorithm solves PM in randomized  $\mathcal{NC}^2$  with  $\mathcal{O}(\log^2 n)$  random bits.

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**Algorithm 1** Decide PM (randomized)

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Let  $k = \log n - 2$ .

Generate random weight functions  $w_0, \dots, w_k$ .

Generate random numbers  $r_0, \dots, r_k$ .

Create  $w_0, \dots, w_k$ .

Combine  $w = (w_0, \dots, w_k)$ .

Calculate  $\det D_r^{G,w}$ .

**return**  $\det D_r^{G,w} \neq 0$

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# Deciding PM in randomized Quasi- $\mathcal{NC}^2$

The following algorithm solves PM in Quasi- $\mathcal{NC}^2$ .

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**Algorithm 2** Decide PM (deterministic)

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Let  $k = \log n - 2$ .

**for all** Possible weight functions  $w_0, \dots, w_k$  and  $r$ , simultaneously  
**do**

    Calculate  $\det D_r^{G,w}$ .

**if**  $\det D_r^{G,w} \neq 0$  **then**

**return** True

**end if**

**return** False

**end for**

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## Questions?