Bipartite Perfect Matching is in Quasi-NC

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Outline

Notation

Notation

Throughout this presentation, *ALL* the graphs are:

- undirected
- bipartite
- balanced
- labeled

Usually we call our graph G and the set of edges E.

The set of edges can be regarded as a relation $E \subseteq [n] \times [n]$.

NC

Definition

- We define $\mathcal{NC}^k$ to be the class of problems that can be solved by a polynomial-time uniform family of circuits of polynomial size and depth $\mathcal{O}(\log^k n)$.

- Equivalently, $\mathcal{NC}^k$ is the class of problems that can be solved by a polynomial number of processors in $\mathcal{O}(\log^k n)$ time.

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$$\mathcal{NC} = \bigcup_{k \geq 0} \mathcal{NC}^k$$

Fact

DET, *the problem of computing the determinant of an $n \times n$ matrix with poly(n) entries, is in* $\mathcal{NC}^2$.

Perfect Matching

Definition

- A *perfect matching* of a graph G is a set of edges such that every vertex is incident to exactly one edge.
- The *perfect matching problem* is to determine whether a graph has a perfect matching, or to find one.
- The *decision version* of the perfect matching problem is denoted PM.
- The *search version* of the perfect matching problem is denoted SearchPM.

State of the Art

It has already been known that perfect matching (whether decision or search) can be solved in randomized $\mathcal{NC}$.

Open Problem

Can perfect matching be solved in $\mathcal{NC}$?

Today

There exist algorithms in:

- Quasi$-\mathcal{NC}^2$ for PM (that is, with $\mathcal{O}(n^{\log n})$ processors and $\mathcal{O}(\log^2 n)$ depth)
- randomized $\mathcal{NC}^2$ for PM with only $\mathcal{O}(\log^2 n)$ random bits.
- (We will not see this) Quasi$-\mathcal{NC}^2$ for PM (that is, with $\mathcal{O}(n^{\log n})$ processors and $\mathcal{O}(\log^2 n)$ depth)
- (We will not see this) $\mathcal{NC}^3$ for SearchPM with only $\mathcal{O}(\log^2 n)$ random bits.

Definitions (Bi-adjacency Matrix)

- The *bi-adjacency matrix*: $\mathbf{A}_G = \mathbf{A}_E = A$ of G (or E) is an $n \times n$ matrix where $A_{ij} = 1_{(i,j) \in E}$
- We write $B \leq C$ iff $\forall i, j. B_{ij} \leq C_{ij}$
- A *permutation matrix* is a matrix $\mathbf{A}_\Gamma$ where $\Gamma = \{(i, \sigma i) : i \in [n]\}$ for some permutation $\sigma \in S_n$

Exercise

For which $E \subseteq [n] \times [n]$ is $\mathbf{A}_E$ a permutation matrix?

Exercise

For which $B \in \mathbb{R}_{\geq 0}^{n \times n}$ and E does $B \leq \mathbf{A}_E$ hold?

Definitions (Weight Functions)

- A *weight function* is a function $w : E \to \mathbb{N}$.
- We extend it naturally to a function $w : 2^E \to \mathbb{N}$ by setting $w(S) = \sum_{e \in S} w(e)$ for all $S \subseteq E$.
- Even more generally (why?), we extend it to a *linear function* $w : \mathbb{R}^{n \times n} \to \mathbb{R}_{\geq 0}$ by setting $w(A) = \sum_{ij \in E} w(ij) A_{ij}$ for all $A \in \mathbb{R}^{n \times n}$.

Definitions (Perfect Matching)

- A *perfect matching* is a set of edges M such that every vertex is incident to exactly one edge in M.
- Equivalently, M is a perfect matching iff $\mathbf{A}_M$ is a permutation matrix.

Birkhoff's Theorem I

Definition (Doubly Stochastic Matrix)

- A matrix $M \in \mathbb{R}_{\geq 0}^{n \times n}$ is *doubly stochastic* if the sum of the entries in each row and column is 1.
- Equivalently, M is doubly stochastic if $M\mathbf{1} = M^T\mathbf{1} = \mathbf{1}$.

Definition (Convex Hull)

The *convex hull* of a set S is the smallest convex set containing S, denoted $\mathbf{CH}(S)$.

It is precisely the set of all convex combinations of elements of S, i.e., $\mathbf{CH}(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : x_i \in S, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}$.

Birkhoff's Theorem II

Theorem (Birkhoff, 1946)

The set of doubly stochastic matrices, denoted $\mathcal{B}_n$, is the convex hull of the permutation matrices.

Hint

Use Hall's marriage theorem.

Introduction
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Combinatorics and Probability
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Isolation
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Cycles and Circulation
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The Proof
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Perfect Matching Polytope

Theorem (Birkhoff, 1946)

The set of doubly stochastic matrices is the convex hull of the permutation matrices.

Definition (Perfect Matching Polytope)

The *perfect matching polytope* $\mathcal{P}_G$ is the convex hull of the bi-adjacency matrices of all the perfect matchings of G.

Corollary

The perfect matching polytope P_G is exactly the matrices B in the Birkhoff polytope B_n such that $B \leq \mathbf{A}_G$:

$$\mathcal{P}_G = \{B : B \in \mathcal{B}_n \wedge B \leq \mathbf{A}_G\}$$

Definition

Let M be a perfect matching. We define its *sign* $\mathrm{sgn}M = \mathrm{sgn}\sigma$ where σ is the permutation given by M.

Definition

Let $w : E \to \mathbb{N}$ be a weight function.
Let a be a number or a variable.
We define a matrix $D_a^{E,w}$ as follows:

$$\left(D_a^{E,w}\right)_{ij} = \begin{cases} a^{w(ij)} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Claim

Let a be a number or a variable. Then:

$$\det D_a^{E,w} = \sum_{M \text{ perfect matching}} \mathrm{sgn}M \cdot a^{w(M)}$$

Introduction
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Combinatorics and Probability
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Isolation
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Cycles and Circulation
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The Proof
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Definition

A weight function $w : E \to \mathbb{N}$ is called *isolating* for E if the existence of a perfect matching in E implies that E has a unique minimum weight perfect matching, which is then called *isolated*.

Corollary

Let x be a variable.

- *If E has no perfect matchings, then $\det D_x^{E,w} = 0$.*
- *If w is isolating for E, then $\det D_x^{E,w} \neq 0 \iff$ there is a perfect matching.*

Example

Let $E = \{e_k\}_k$ be an enumeration of the edges. Let $w : E \to \mathbb{N}$ be defined by $w(e_k) = 2^k$. *Every* subset of E has a unique weight. In particular, w is isolating for E.

Isolation Lemma

Lemma (Mulmuley, Vazirani & Vazirani 1987)

Let B be a finite set.

Let k be a positive integer.

Let $\mathcal{F} \subseteq 2^B$ be a nonempty family of subsets of B.

Let $w : B \to k$ be a be chosen uniformly at random among all functions from B to $[k]$.

Then, with probability at least $1 - |B|/k$, there exists a unique set $S \in \mathcal{F}$ with a minimum weight among all sets in $\mathcal{F}$.

Corollary

Let $w : E \to [n^3]$ be a random weight function.
Then with probability at least $1 - 1/n$, w is isolating for G.

Proof.

Take $\mathcal{F}$ as the family of perfect matchings of G.
As, $|E| \leq n^2$, by the isolation lemma, with probability at least

$$(1 - |E|/n^3) \geq 1 - n^2/n^3 = 1 - 1/n$$

w is isolating for G. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary

There exists an RNC algorithm that can solve PM, which uses
poly(n) random bits.

Circulation

Definition

Let $w : E \to \mathbb{N}$ be a weight function. The *w-circulation* of a cycle $C = e_1, \ldots, e_{2k}$ is defined as

$$c_w(C) = |w(e_1) - w(e_2) + \ldots + w(e_{2k-1}) - w(e_{2k})|$$

This is well defined because we take the absolute value.

Remark

Let M_1 and M_2 be two perfect matchings of G, and suppose $C \subseteq M_1 \triangle M_2$.
Then $c_w(C) = |w(C \cap M_1) - w(C \cap M_2)|$.

Claim

The symmetric difference of two perfect matchings is a union of disjoint cycles.

Each cycle consists of interleaved edges from the two matchings.

Proof.

The degree of each vertex is 1 in a perfect matching.

Thus, in the symmetric difference of two perfect matchings, each vertex has degree 0 or 2.

Therefore it is a union of disjoint cycles. The rest is easy to see. □

Lemma

Let $w : E \to \mathbb{N}$ be a weight function.
If every cycle in G has a nonzero circulation, then w is isolating for G.

Proof.

On the contrary, suppose that M_1 and M_2 are two minimum perfect matchings of G. Choose some cycle $C \subseteq M_1 \triangle M_2$. Since $c_w(C) > 0$, we have $w(C \cap M_1) \neq w(C \cap M_2)$. WLOG, assume that $w(C \cap M_1) < w(C \cap M_2)$. Then $M_2 \triangle C$ is a perfect matching with weight less than $w(M_2) = w(M_1)$, contradicting minimality. $\qquad\square$

Definition

Suppose G has a perfect matching. Let $w : E \to \mathbb{N}$ be a weight function.

Then $G_w \subseteq G$ is the union of all minimum weight perfect matchings of G, relatively to w.

Theorem

Let $G^0 = G$ be and $G^{i+1} = \left(G^i \right)_{w_i}$.

Let $k \geq \log n - 2$ and let $w_0, \ldots, w_k : E \to \mathbb{N}$ be weight functions such that w_i gives nonzero circulation to all the cycles of size at most $4 \cdot 2^i$ in G_i.

Then the joint weight function $w = (w_0, \ldots, w_k)$ is isolating.

Lemma

Let $r \geq 2$ be even. Suppose G has no cycles of size at most $2r$.
Then G has at most n^4 cycles of size at most $4r$.

Proof.

- Equivalently, there is at most one path of length $\leq r$ between any two vertices.

- Let v_0, v_1, v_2, v_3 be vertices in V_1, such that the distance between v_i and $v_{i+1 \mod 4}$ is $\leq r$.

- There exists a *partial function* from V_1^4 to cycles in G that maps (v_0, v_1, v_2, v_3) to the cycle composed of the unique paths of length $\leq r$ between v_i and $v_{i+1 \mod 4}$.

- Let C be a cycle of size $\leq 4r$.
  We can choose arbitrarily $v_0, v_1, v_2, v_3 \in C$ such that the distance between v_i and $v_{i+1 \mod 4}$ is $\leq r$.

- This function is onto the set of cycles of size $\leq 4r$. Thus, there are at most $|V_1|^4 = n^4$ cycles of size $\leq 4r$.

$\square$

Lemma

Suppose G has a perfect matching and let $w : E \to \mathbb{N}$ be a weight function.

Then every perfect matching in G_w has the same weight as every minimum weight matching in G.

Proof.

If w is isolating, G_w is just a single perfect matching.

Otherwise, suppose the minimum weight for perfect matchings in G is m. Suppose there exist t such matchings and let X be their average. Then, $w(X) = m$.

Let N be a perfect matching in G_w. Every edge is contained in some minimum weight perfect matching, so $X - 1/tN \geq 0$.

It can easily be seen that the sum of every row and column is $(t-1)/t$ and thus the matrix $\frac{t}{t-1}X - \frac{1}{t-1}N$ is doubly stochastic. Therefore, it lies in $\mathcal{P}_G$ and thus has weight at least m. That is,

$$m \leq w\left(\frac{t}{t-1}X - \frac{1}{t-1}N\right) = m \cdot \frac{t}{t-1} - w(N) \cdot \frac{1}{t-1}$$

From that it follows that $w(N) \leq m$. □

Lemma

Suppose G has a perfect matching and let $w : E \to \mathbb{N}$ be a weight function.
Then the w-circulation of every cycle in G_w is zero.

Proof.

Let X be the average of all perfect matchings in G_w.

Let t be the number of perfect matchings, $\varepsilon = 1/t$, and let $M_1, \ldots, M_t$ be the perfect matchings. Then $X = \varepsilon \sum_{i=1}^{t} M_i$.

Since each edge is contained in some perfect matching, we have $X_{ij} \geq \varepsilon$ for all i, j such that $(i, j) \in E$.

Let $C = e_1, \ldots, e_{2p}$ be a cycle in H.

Define Y by

$$Y_{ij} = \begin{cases} X_{ij} + (-1)^k \varepsilon & \text{if } (i,j) = e_k \in C, \\ X_{ij} & \text{otherwise.} \end{cases}$$

Then clearly $Y \geq 0$ and $Y\mathbf{1} = Y^T\mathbf{1} = \mathbf{1}$. Therefore, Y lies in the perfect matching polytope.

Since all the perfect matchings have the same weight, $w(Y) = w(X)$ and thus $w(Y - X) = 0$. But $c_w(C) = \varepsilon w(Y - X)$, and thus $c_w(C) = 0$. $\qquad \square$

Lemma

Let $s = poly(n)$. Using $\mathcal{O}(\log n)$ random bits we can generate a weight assignment $w : E \to \mathbb{N}$ with $|w| = poly(n)$ such that for every set of s cycles, w gives nonzero circulation to all of them with probability at least $1 - 1/n$.

Proof.

Let the cycles be $C_1, \ldots, C_s$. Then, $c_w(C_i) \neq 0$ for all i is equivalent to $\prod_{i=1}^{s} c_w(C_i) \neq 0$. This product is bounded by $poly(n)^s = 2^{poly(n)}$.

Thus, it has at most $poly(n)$ prime factors, say k prime factors. Choose $t = kn$.

Then if we choose a random prime amongst the first t primes $[p_1, \ldots, p_t]$, then if the product is nonzero, with probability at least $1 - 1/n$ it is still true modulo the chosen prime.

Choose $w(e_k) = 2^k$ and we are done. $\qquad\square$

Theorem

Let $G^0 = G$ be and $G^{i+1} = \left(G^i\right)_{w_i}$.
*Let $k \geq \log n - 2$ and let $w_0, \ldots, w_k : E \to \mathbb{N}$ be weight functions
such that for every set of w_i gives nonzero circulation to all the
cycles of size at most $4 \cdot 2^i$ in G_i.*
Then the joint weight function $w = (w_0, \ldots, w_k)$ is isolating.

Proof.

Since w_i gives nonzero circulation to all the cycles in G^i, all those
cycles do not appear in G^{i+1}.
In particular, G^k has no cycles of size at most $4 \cdot 2^k \geq n$.
That is, G^k has a unique perfect matching. We want to show that
it is isolated by w. This follows from the next lemma. $\qquad\square$

Lemma

Suppose M_1 is a matching that appears in G^i but not in G^{i+1} and M_2 is a matching that appears in G_{i+1}.
Then $w_i(M_2) < w_i(M_1)$.

Proof.

Since M_2 appears in G_{i+1}, it has the same weight as any w_i-minimum weight matching in G_i. That is $w_i(M_2) \le w_i(M_1)$.
But if $w_{i+1}(M_1) = w_{i+1}(M_2)$ held, then M_1 would have been in G^{i+1} by its definition. □

Deciding PM

The following algorithm solves PM in Quasi$-\mathcal{NC}^2$.

Algorithm 1 Decide PM

Let $k = \log n - 2$.
for all possible choices of random values, simulataneously **do**
   Create $w_0, \ldots, w_k$.
   Combine $w = (w_0, \ldots, w_k)$.
   Calculate $\det D_2^{G,w}$.
   if $\det D_2^{G,w} \neq 0$ **then**
     return True
   end if
end for
return False