On Detection With Partial Information In The Gaussian Setup

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Abstract— **We introduce the problem of communication with partial information, where there is an asymmetry between the transmitter and the receiver codebooks. Practical applications of the proposed setup include the robust signal hashing problem within the context of multimedia security and asymmetric communications with resource-lacking receivers. We study this setup in a binary detection theoretic context for the additive colored Gaussian noise channel. In our proposed setup, the partial information available at the detector consists of dimensionality-reduced versions of the transmitter codewords, where the dimensionality reduction is achieved via a linear transform. We first derive the corresponding MAP-optimal detection rule and the corresponding conditional probability of error (conditioned on the partial information the detector possesses). Then, we constructively quantify an optimal class of linear transforms, where the cost function is the expected Chernoff bound on the conditional probability of error of the MAP-optimal detector.**

I. INTRODUCTION

In this paper, we introduce a communication-theoretic paradigm, which we name as "communication with partial information", and subsequently study it within a detection-theoretic context (therefore the term "detection with partial information") in a particular case of the Gaussian setup. In the proposed paradigm, there is an inherent asymmetry between the information the transmitter and the receiver possess in terms of the utilized codebooks. In particular, in the "detection with partial information" setup, the codebook of the receiver is formed via applying a non-invertible process on the codebook of the transmitter; hence *the codebooks are different*. Thus, the information available at the transmitter forms a "superset" of the information available at the receiver. Note that, a reminiscent asymmetric structure between the transmitter and the receiver also exists in the well-known family of problems, termed as "communication with side information" [1], [2], [3], [4]. However, in the paradigm of "communication with side information" (unlike the proposed "communication with partial information" setup), the utilized codebooks at the receiver and the transmitter are the same; in addition, either the transmitter or the receiver is "favored"

with the presence of "extra" information (which amounts to the "side information").

It appears that, there are at least two significant applications that motivate the formulation of the "communication with partial information" approach:

- The first application can be viewed to fall within the category of "robust signal hashing" in the signal processing & multimedia security literature [5], [6], [7], [8]. In robust signal hashing, a content owner provides "robust hash value"s of the protected content (that is some dimensionality-reduced versions of the protected content) to a third party, which searches the content using its robust hash values as *the partial information* at the receiver end. These robust hash values represent "the content's significant features" and are ideally approximately-invariant under acceptable modifications to the content. In practical applications, the third party that performs the hash-based search is usually *not trusted*; hence, there is a significant issue of privacy. In particular, given a robust hash value, it should ideally be impossible to retrieve the original protected content from a privacy viewpoint. The setup proposed in this paper can be used as a detection-theoretic model to analyze the hash-based detection problem: the protected content is represented by the transmitted signal; the robust hash values used in the search are represented by the partial information available at the receiver; a perceptually-acceptable modification to the protected content is represented by the channel noise.

- The second application includes all instances of point-to-point communications, where there is an inherent asymmetry between the transmitter and the receiver in terms of their storage capabilities and computational resources. In particular, the cases, when the receiver is unable to store the codebook used by the encoder (due to a limit on the memory) or utilize the codebook used by the encoder (due to a limit on the computational resources), can be studied within the framework of "communication with partial information". In such cases, one potential remedy is the receiver's using a "simplified" (i.e., dimensionality-reduced) version of the codebook of the encoder. In practice, such situations may typically arise, for instance, when there is a bi-directional communication between a sensor and the base station (the resource-limited receiver representing the sensor) or when there is a bi-directional communication between a controller and a remote measurement unit. In such applications, the simplified version of

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the encoder codebook is represented by the partial information at the receiver side.

Our contributions in this paper can be listed as follows:

- We introduce the paradigm of "communication with partial information" and study it within the context of binary detection in the Gaussian setup. We believe the main philosophy behind this formulation (i.e., introducing an asymmetry between the transmitter and the receiver in the sense of utilized codebooks) can be used to analyze various problems of interest in communication theory and signal processing.
- Within the binary hypothesis testing setup, we study a case, where the disturbance on the transmitter output consists of additive colored Gaussian noise, and the detector partial information is produced via applying a linear (dimensionality-reducing) transform on the encoder codebook. Consequently, we present the following results:
  - We derive the MAP-optimal detection rule and the corresponding probability of error, both of which are conditioned on the partial information available at the detector.
  - We construct a class of *optimal* linear transforms, which minimize the expected (with respect to the joint distribution of the detector partial information) Chernoff bound on the aforementioned probability of detection error.

In Sec. II, we present the notation that is used throughout the paper and specify the formal problem statement. In Sec. III, we derive the MAP-optimal detection rule conditioned on the partial information available at the receiver. In Sec. IV, we quantify an optimal (in the sense of the expected value of the Chernoff bound on the detection error probability) class of linear transforms that are used to generate the receiver partial information. We present illustrative numerical results in Sec. V, followed by discussions and conclusion in Sec. VI.

II. NOTATION AND PROBLEM STATEMENT

A. Notation

Boldface lowercase and uppercase letters denote vectors and matrices, respectively; the corresponding regular letters with subscripts denote their individual elements. For instance, given a vector $\mathbf{a}$, a_i represents its i-th element; given a matrix $\mathbf{A}$, A_{ij} denotes its (i,j)-th element. Note that, we do not use a separate notation for random vectors; we assume that it is clear from the context.

Given a matrix $\mathbf{A}$, $\mathbf{A}^T$, $r(\mathbf{A})$ and $\det(\mathbf{A})$ denote its transpose, rank and determinant, respectively; further, $\mathbf{I}_n$ denotes the identity matrix of size $n \times n$. Given the vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\langle \mathbf{x}, \mathbf{y} \rangle$ indicates the inner product that induces the Euclidean norm, i.e., $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i$; accordingly the induced Euclidean norm is denoted by $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$.

Definition 2.1: Given $\mathbf{A} \in \mathbb{R}^{m \times n}$, such that $r(\mathbf{A}) = k \leq \min(m, n)$, *Singular Value Decomposition* (SVD) of $\mathbf{A}$ is

unique (up to ordering) and defined as

$$\mathbf{A} \triangleq \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T, \tag{2.1}$$

where $\mathbf{U} \in \mathbb{R}^{m \times k}$, $\mathbf{V} \in \mathbb{R}^{n \times k}$, $\mathbf{\Lambda} \in \mathbb{R}^{k \times k}$ are called the left-singular vector matrix (orthonormal), the right-singular vector matrix (orthonormal) and the singular value matrix of $\mathbf{A}$, respectively. The matrix $\mathbf{\Lambda}$ is positive-definite diagonal; we denote its entries along the diagonal by $\{\sigma_i(\mathbf{A})\}_{i=1}^k$, which are the non-zero singular values of $\mathbf{A}$, and assumed to be in non-increasing order without loss of generality.

For a square matrix $\mathbf{A}$ of size $k \times k$ and of rank $r \leq k$, $\{\lambda_i(\mathbf{A})\}_{i=1}^r$ denote its non-zero eigenvalues; in case $\mathbf{A}$ is a symmetric matrix, $\{\lambda_i\}$ are assumed to be in non-decreasing order. We use $\mathcal{N}(\mu, \mathbf{\Sigma})$ to denote a multivariate Gaussian distribution, with mean vector μ and covariance matrix $\mathbf{\Sigma}$. Furthermore, $\mathbf{Q}(\cdot)$ denotes the standard Q-function: $\mathbf{Q}(\alpha) \triangleq \int_\alpha^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

B. Problem Statement

We analyze a binary communication system, where the encoder selects one of the two codewords, $\mathbf{x}_0$ and $\mathbf{x}_1$, representing the message bit $i \in \{0, 1\}$, where $\Pr(i = 0) = \Pr(i = 1) = 1/2$; the selected codeword, $\mathbf{x} = \mathbf{x}_i$, is sent through a channel. The encoder output $\mathbf{x}$ is corrupted by an additive, signal-independent, (not necessarily white) Gaussian noise, denoted by $\mathbf{e}$, thereby yielding the overall channel output $\mathbf{y}$. Observing $\mathbf{y}$, the receiver acts as a detector and makes a binary decision, as to the origins of received signal. We pursue a detection-theoretic approach to solve this problem and assume uniform costs. We assume that $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{e}$, and $\mathbf{y}$ are all length-n real-valued vectors, where $\mathbf{x}_0$ and $\mathbf{x}_1$ are independent of each other and $\mathbf{x}_0, \mathbf{x}_1 \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_x)$, $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma}_e)$ is independent of both $\mathbf{x}_0$ and $\mathbf{x}_1$. Here, we also assume that the covariance matrix of the original signals $\mathbf{\Sigma}_x$ and the covariance matrix of the noise $\mathbf{\Sigma}_e$ are *positive definite* (they are also symmetric by construction). See Fig. 1 for a schematic illustration of the proposed problem.
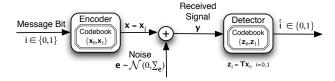

Fig. 1. Block diagram representation of the problem of "binary detection with partial information".

In the considered setup, *the detector does not know the original codewords* $\{\mathbf{x}_0, \mathbf{x}_1\}$, *but only their distributions and their dimensionality-reduced versions,* $\{\mathbf{z}_0, \mathbf{z}_1\}$, where $\mathbf{z}_i = \mathbf{T} \cdot \mathbf{x}_i$, $i = 0, 1$, and $\mathbf{T}$ is a deterministic real matrix of size $m \times n$, $m < n$, $r(\mathbf{T}) = m$. Note that, this implies, $\mathbf{z}_0$ and $\mathbf{z}_1$ are both length-m real-valued vectors. As such, the proposed problem is radically different from the conventional binary detection scenario due to the *mismatch between the codebooks of the encoder and the detector*. Consequently,

we term the problem at hand as "detection with partial information" for the Gaussian case.

An important point here is that, since the receiver fully knows the statistical characterization of the whole system, it is able to apply the MAP decoding rule. In particular, in Sec. III, we derive the MAP detection rule, which is given as a function of the partial information $(\mathbf{z}_0, \mathbf{z}_1)$, and the corresponding conditional probability of error (conditioned on $\mathbf{z}_0$ and $\mathbf{z}_1$). Subsequently, in Sec. IV, we derive the optimal linear transform, $\mathbf{T}$, in the sense of the expected Chernoff bound on the conditional probability of error of the MAP detector.

Remark 2.1: In [9], the authors study a closely-related problem, which can be viewed as the "deterministic variant" of the aforementioned setup. In particular, in [9] the authors assume that the encoder codewords $\{\mathbf{x}_i\}$ are deterministic, unknown and the subsequent analysis is based on the probability of error induced by the GLRT (generalized likelihood ratio test) rule. On the other hand, in this paper, we assume that the encoder codewords $\{\mathbf{x}_i\}$ are random (in particular Gaussian) and perform a MAP-based analysis.

Remark 2.2: Although the problem imposed in this paper is the binary detection case, the analysis can be extended to apply a "union bound based approach" for the L-ary case with little or no difficulty[1]. A similar approach and discussion was provided in [9] for the case of deterministic $\{\mathbf{x}_i\}$.

III. OPTIMAL DETECTION CONDITIONED ON THE PARTIAL INFORMATION

At the detector side, we are given $\{\mathbf{z}_0, \mathbf{z}_1\}$, which yield partial information about the true codewords $\{\mathbf{x}_0, \mathbf{x}_1\}$. The *binary hypothesis testing* approach on the detector side utilizes the MAP detection rule [10]: It operates on the observed data $\mathbf{y}$ (generated by the process explained in Sec. II-B), and makes a binary decision regarding the message bit given $\{\mathbf{z}_0, \mathbf{z}_1\}$. Thus, we aim to solve the following binary hypothesis testing problem:

$$
\begin{aligned}
H_0 &: \quad \mathbf{y} = \mathbf{x}_0 + \mathbf{e} \quad ; \qquad \text{given } \{\mathbf{z}_0, \mathbf{z}_1\}, \\
H_1 &: \quad \mathbf{y} = \mathbf{x}_1 + \mathbf{e} \quad ; \qquad \text{given } \{\mathbf{z}_0, \mathbf{z}_1\}.
\end{aligned}
$$

The corresponding MAP detection rule is given by

$$
p\left(\mathbf{y}|H_0\right) \underset{H_1}{\overset{H_0}{\gtrless}} p\left(\mathbf{y}|H_1\right). \tag{3.1}
$$

since we have equal priors and uniform costs. Note that, (3.1) is also known as the maximum-likelihood detection rule [10]. Note that, for all $i \in \{0, 1\}$, we have

$$
p\left(\mathbf{y} \mid H_i\right) = p\left(\mathbf{x}_i + \mathbf{e} \mid \mathbf{z}_i\right)\Big|_{\mathbf{x}_i + \mathbf{e} = \mathbf{y}},
$$

which implies that (3.1) can be rewritten as

$$
p\left(\mathbf{x}_0 + \mathbf{e}|\mathbf{z}_0\right)\big|_{\mathbf{x}_0 + \mathbf{e} = \mathbf{y}} \underset{H_1}{\overset{H_0}{\gtrless}} p\left(\mathbf{x}_1 + \mathbf{e}|\mathbf{z}_1\right)\big|_{\mathbf{x}_1 + \mathbf{e} = \mathbf{y}}. \tag{3.2}
$$

[1]In the L-ary case, the message is $\log L$ bits long; the encoder and receiver codebooks are $\{\mathbf{x}_i\}_{i=0}^{L-1}$ and $\{\mathbf{z}_i\}_{i=0}^{L-1}$, respectively.

Theorem 3.1: The maximum likelihood detection rule (3.2) is given by

$$
\|\boldsymbol{\Sigma}_{y|z}^{-1/2}\left(\mathbf{y} - \mu_{y_0|z_0}\right)\| \underset{H_0}{\overset{H_1}{\gtrless}} \|\boldsymbol{\Sigma}_{y|z}^{-1/2}\left(\mathbf{y} - \mu_{y_1|z_1}\right)\| \tag{3.3}
$$

The corresponding (conditional) probability of error (conditioned on $\mathbf{z}_0$ and $\mathbf{z}_1$) is given by

$$
P_{e|\mathbf{z}_0, \mathbf{z}_1} = Q\left(\frac{\|\boldsymbol{\Sigma}_{y|z}^{-1/2}\left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right)\|}{2}\right) \tag{3.4}
$$

where, for $i \in \{0, 1\}$, $\mu_{y_i|z_i} = \mathrm{E}\left(\mathbf{y}_i \mid \mathbf{z}_i\right)\big|_{\mathbf{y}_i = \mathbf{x}_i + \mathbf{e}} = \boldsymbol{\Sigma}_x \mathbf{T}^T \left(\mathbf{T}\boldsymbol{\Sigma}_x\mathbf{T}^T\right)^{-1} \mathbf{z}_i$; $\boldsymbol{\Sigma}_{y|z}$ is positive definite and given by $\boldsymbol{\Sigma}_{y|z} = \mathrm{Cov}\left(\mathbf{y}_i \mid \mathbf{z}_i\right)\big|_{\mathbf{y}_i = \mathbf{x}_i + \mathbf{e}, i=0,1} = \boldsymbol{\Sigma}_x + \boldsymbol{\Sigma}_e - \boldsymbol{\Sigma}_x\mathbf{T}^T \left(\mathbf{T}\boldsymbol{\Sigma}_x\mathbf{T}^T\right)^{-1} \mathbf{T}\boldsymbol{\Sigma}_x$.

Proof: See Appendix I. ∎

Remark 3.1: Using Theorem 3.1, we see that, if $\mathbf{z}_0 = \mathbf{z}_1$, conditional probability of error is $1/2$, which is meaningful. Then, there is nothing to discriminate from the detector's perspective thereby converting the detection to a fair coin toss.

Remark 3.2: The argument of the Q-function in (3.4) is always non-negative. This allows us to set a tight bound on the expected probability of error, and analyze it in Sec. IV.

IV. OPTIMAL LINEAR OPERATORS IN THE EXPECTATION SENSE

In this section, our performance criterion is based on the *expected* (unconditional) probability of error of the MAP detector, denoted by P_e, given by

$$
\begin{aligned}
P_e &= \mathrm{E}_{\{\mathbf{z}_0, \mathbf{z}_1\}}\left[P_{e|\mathbf{z}_0, \mathbf{z}_1}\right], \\
&= \mathrm{E}_{\{\mathbf{z}_0, \mathbf{z}_1\}}\left(Q\left(\frac{\|\boldsymbol{\Sigma}_{y|z}^{-1/2}\left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right)\|}{2}\right)\right) \tag{4.1}
\end{aligned}
$$

where $\mathrm{E}_{\{\mathbf{z}_0, \mathbf{z}_1\}}(.)$ denotes expectation with respect to the joint distribution of $\mathbf{z}_0$ and $\mathbf{z}_1$, and the right hand side follows from (3.4).

Remark 4.1: It appears to be manageable to find a linear transform that minimizes the conditional probability of error, $P_{e|\mathbf{z}_0, \mathbf{z}_1}$ (see, for instance, [9]) as a function of the transmitted signals, $\mathbf{x}_0$ and $\mathbf{x}_1$, which would yield an "input-adaptive optimal transform". On the other hand, the expected probability of error given by (4.1) is not tractable for an analogous analysis, carried out to characterize the optimal linear transform $\mathbf{T}$ that minimizes it. This stems from the fact that, such an optimal $\mathbf{T}$ would be a function of the overall statistics of the system (corresponding to applying the operator of $\mathrm{E}_{\{\mathbf{z}_0, \mathbf{z}_1\}}(.)$ in (4.1)) rather than individual realizations, which yields a "complicated" cost function to minimize; the result of the expectation operation, i.e., the $m \times m$-fold integration in (4.1) is not given in terms of standard analytical functions. Therefore, we continue our analysis by characterizing linear operator(s) that minimize *a tight upper bound* on the expected probability of error defined by (4.1)

Hence, we proceed with the following approach: We first bound $P_{e|\mathbf{z}_0,\mathbf{z}_1}$ for any given pair of $\{\mathbf{z}_0,\mathbf{z}_1\}$ from above and make use of the fact that expected value of this upper bound is an upper bound on P_e (since, by definition, $P_{e|\mathbf{z}_0,\mathbf{z}_1} \geq 0$). Also, note that the use of an *upper* bound clearly makes sense since we aim to *minimize P_e*. The upper bound on $P_{e|\mathbf{z}_0,\mathbf{z}_1}$ that we use is the *Chernoff bound* on the Q-function (see *Basic Inequality* in [12]), which is an exponentially decaying and a sufficiently tight bound. The expected Chernoff bound, which replaces the primary objective function P_e in the design of optimal linear transform $\mathbf{T}$ due to its analytical tractability and sufficient tightness, is derived in the following proposition.

Proposition 4.1: The Chernoff bound on $P_{e|\mathbf{z}_0,\mathbf{z}_1}$ is

$$P_{e|\mathbf{z}_0,\mathbf{z}_1} \leq \frac{1}{2}\exp\left(-\frac{\|\mathbf{\Sigma}_{y|z}^{-1/2}\left(\mu_{y_0|z_0}-\mu_{y_1|z_1}\right)\|^2}{8}\right), \quad (4.2)$$

yielding the following corresponding "expected Chernoff bound" on P_e

$$P_e \leq \frac{1}{2}\left\{\det\left(\mathbf{I}_m+\frac{1}{2}\mathbf{W}\right)\right\}^{-1/2} \quad (4.3)$$

where $\mathbf{W} \overset{\triangle}{=} \mathbf{T}\mathbf{\Sigma}_x\mathbf{\Sigma}_{y|z}^{-1}\mathbf{\Sigma}_x\mathbf{T}^T\left(\mathbf{T}\mathbf{\Sigma}_x\mathbf{T}^T\right)^{-1}$

Proof: See Appendix II. ∎

Remark 4.2: The bound on expected (unconditional) probability of error of the MAP detector, given by (4.3) is the objective function we aim to minimize in this section. The minimization (over $\mathbf{T}$) is carried out over a class of linear transformations that posses certain properties imposed by the physical structure of the analyzed system. The obvious one of these properties is the dimension of the transformation (i.e., the fact that $\mathbf{T}$ is a $m\times n$ matrix); the other one is the constraint on its rank (i.e., the fact that $r(\mathbf{T}) = m$). The rank constraint is set to ensure that the dimensionality of the subspace (which is equal to $r(\mathbf{T})$), to which the partial information shared by the two sides of the communication belongs, is at a certain desired level; this is because of the following fact: the performance of a system, which utilizes a rank-deficient transformation, is analogous to the performance of another system, the transformation of which is full-rank and has the same rank as the previous rank-deficient transformation.

Definition 4.1: The "expected probability of error bound minimizing transform $\mathbf{T}_{opt}$" is given by

$$\mathbf{T}_{opt} = \underset{\substack{\mathbf{T}\in\mathbb{R}^{m\times n}\\r(\mathbf{T})=m}}{\operatorname{argmax}} \quad \det\left(\mathbf{I}_m+\frac{1}{2}\mathbf{W}\right) \quad (4.4)$$

Proposition 4.2: Let $\mathcal{S}_{\mathbf{T}} \overset{\triangle}{=} \left\{\mathbf{T} \mid \mathbf{T}\in\mathbb{R}^{m\times n}, r(\mathbf{T})=m\right\}$, $\mathcal{S}_{\mathbf{M}} \overset{\triangle}{=} \left\{\mathbf{M} \mid \mathbf{M}\in\mathbb{R}^{n\times m}, \mathbf{M}^T\mathbf{M}=\mathbf{I}_m\right\}$, $\mathbf{P} \overset{\triangle}{=} \mathbf{\Lambda}^{-1}\mathbf{F}^T\left(\mathbf{\Sigma}_x+\mathbf{\Sigma}_e\right)\mathbf{F}\mathbf{\Lambda}^{-1}$. Let the SVD of $\mathbf{\Sigma}_x$ and $\mathbf{P}$ be given by $\mathbf{\Sigma}_x = \mathbf{F}\mathbf{\Lambda}^2\mathbf{F}^T$ and $\mathbf{P} = \mathbf{U}_p\mathbf{\Lambda}_p\mathbf{U}_p^T$, respectively, and

$\hat{\mathbf{\Lambda}}_p \overset{\triangle}{=} \mathbf{I}_n-\mathbf{\Lambda}_p^{-1}$. Also define

$$G(\mathbf{M}) \overset{\triangle}{=} \left(\frac{1}{2}\right)^m \prod_{i=1}^m \left[1+\frac{1}{\lambda_i\left(\mathbf{M}^T\hat{\mathbf{\Lambda}}_p\mathbf{M}\right)}\right]$$

$$J(\mathbf{T}) \overset{\triangle}{=} \det\left[\mathbf{I}_m+\frac{1}{2}\mathbf{W}\right].$$

Suppose there exists

$$\mathbf{M}^* = \underset{\mathbf{M}\in\mathcal{S}_{\mathbf{M}}}{\operatorname{argmax}} G(\mathbf{M}). \quad (4.5)$$

Then, letting $\mathbf{T}^* \overset{\triangle}{=} \mathbf{E}\mathbf{D}(\mathbf{M}^*)^T\mathbf{U}_p^T\mathbf{\Lambda}^{-1}\mathbf{F}^T$, where $\mathbf{E} \in \mathbb{R}^{m\times m}$ is an arbitrary unitary matrix and $\mathbf{D} \in \mathbb{R}^{m\times m}$ is an arbitrary diagonal positive-definite matrix, we have $\mathbf{T}^* = \operatorname{argmax}_{\mathbf{T}\in\mathcal{S}_{\mathbf{T}}} J(\mathbf{T})$.

Proof: See Appendix III. ∎

Proposition 4.2 allows us to deduce the existence of $\mathbf{T}_{opt}$ with the sufficiency of the existence of $\mathbf{M}^*$. Then, in order to find an optimal linear transformation, which is the main goal of this section, we first need to show the existence of $\mathbf{M}^*$, and then construct $\mathbf{T}_{opt}$ using $\mathbf{M}^*$ that is the solution for the reduced problem (4.5).

Proposition 4.3: A set of solutions for (4.5) is given by

$$\mathcal{M} = \left\{\mathbf{M}\in\mathcal{S}_{\mathbf{M}} \;\middle|\; \mathbf{M}=\mathbf{Q}^T\left[\begin{array}{c}\mathbf{\Gamma}_m\\\mathbf{0}_{(n-m)\times m}\end{array}\right]\right\},$$

where $\mathbf{\Gamma}_m \in \mathbb{R}^{m\times m}$ is a unitary matrix, $\mathbf{Q} \in \{0,1\}^{n\times n}$ denotes a permutation matrix s.t. the eigenvalues of $\mathbf{Q}\hat{\mathbf{\Lambda}}_p\mathbf{Q}^T$ are in non-decreasing order. Moreover,

$$\max_{\mathbf{M}\in\mathcal{S}_{\mathbf{M}}} \prod_{i=1}^m \left[1+\frac{1}{\lambda_i\left(\mathbf{M}^T\hat{\mathbf{\Lambda}}_p\mathbf{M}\right)}\right] = \prod_{i\in\mathcal{I}} \left[1+\frac{1}{\lambda_i\left(\hat{\mathbf{\Lambda}}_p\right)}\right], \quad (4.6)$$

where $\mathcal{I} \subseteq \{1,2,\ldots,n\}$ denotes the cardinality-m index set corresponding to the m-smallest eigenvalues of $\hat{\mathbf{\Lambda}}_p$.

Proof: See Appendix IV. ∎

Theorem 4.1: A set of optimal linear transforms, in the sense of expected Chernoff bound on the probability of error P_e, for communication with partial information in the Gaussian setup is given by

$$\mathcal{T} = \left\{\mathbf{T}\in\mathcal{S}_{\mathbf{T}} \mid \mathbf{T}=\mathbf{E}\mathbf{D}\mathbf{M}^T\mathbf{U}_p^T\mathbf{\Lambda}^{-1}\mathbf{F}^T\right\} \quad (4.7)$$

where $\mathbf{E} \in \mathbb{R}^{m\times m}$ is unitary, $\mathbf{D} \in \mathbb{R}^{m\times m}$ is diagonal, $\mathbf{M} \in \mathcal{M}$, $\mathcal{S}_{\mathbf{T}} = \{\mathbf{T}\in\mathbb{R}^{m\times n} \mid r(\mathbf{T})=m\}$, $\mathcal{M}$ is given by Proposition 4.3 and $\mathbf{F}$, $\mathbf{\Lambda}$ and $\mathbf{U}_p$ denote matrix of eigenvectors and diagonal matrix of eigenvalues of $\mathbf{\Sigma}_x$ and the matrix of eigenvectors of $\mathbf{P} = \mathbf{\Lambda}^{-1}\mathbf{F}^T\left(\mathbf{\Sigma}_x+\mathbf{\Sigma}_e\right)\mathbf{F}\mathbf{\Lambda}^{-1}$, respectively.

Proof: By Proposition 4.2 we know that $\mathcal{T} \neq \emptyset$. We also know for a given $\mathbf{M}^*$, i.e. $\mathbf{M}$ satisfying (4.5), $\mathbf{T} = \mathbf{E}\mathbf{D}\mathbf{M}^{*T}\mathbf{U}_p^T\mathbf{\Lambda}^{-1}\mathbf{F}^T$ satisfies (4.4), i.e., $\mathbf{T} = \mathbf{T}_{opt}$ (cf. Appendix III). Moreover, a set of $\mathbf{M}$ satisfying (4.5), namely $\mathcal{M}$, is given by Proposition 4.3. This clearly implies that $\mathcal{T}$, induced by $\mathcal{M}$, is a set of optimal linear transforms, in the sense of expected Chernoff bound on the probability of error P_e. ∎

V. NUMERICAL RESULTS

Optimality of $\mathbf{T}^*$: Theorem 4.1 gives a set of optimal linear transforms, however does not address the "denseness" of $\mathcal{T}$ in $\mathcal{S}_\mathbf{T}$: "is it easy to find an optimal transform in $\mathcal{S}_\mathbf{T}$ randomly, and how much is the performance of transforms in $\mathcal{S}_\mathbf{T} \backslash \mathcal{T}$ separated from that of optimal transforms?". The computational provided in Fig. 2 provide an experimental basis. In Fig. 2, the simulations are performed with Σ_x and Σ_e having uniformly distributed eigenvalues, and the result is given using the reciprocal of the Chernoff bound on P_e to improve visibility. The first observation is that it is not "easy" to guess an element of $\mathcal{T}$ randomly (we actually simulated over much larger number of trials, however give here the result for a set of 1000 trials for illustrative purposes). This is clear by observing that none of the transforms chosen randomly from $\mathcal{S}_\mathbf{T}$ achieves the optimal value calculated from (4.6) in Proposition 4.3, except $\mathbf{T}_{opt}$ constructed by (4.7) and indicated as the transform in the middle of set of transforms, i.e. $\mathbf{T}_{500}$. Also, the minimum value of the bound on P_e achieved by arbitrary choices is not even close to that achieved by $\mathbf{T}_{opt}$, it is around 4 times larger than the minimum bound on P_e. Thus, we experimentally conjecture that $\mathcal{T}$ is not "dense" in $\mathcal{S}_\mathbf{T}$.
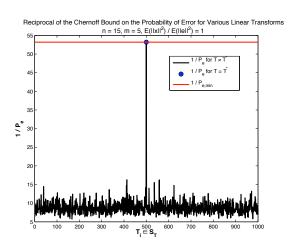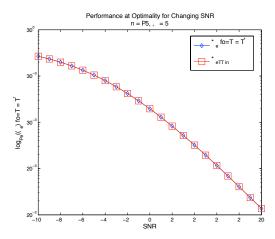

Fig. 3. Performance of $\mathbf{T}_{opt}$ vs. SNR (dB), P_e indicates Chernoff bound on expected probability of error here

sense of expected Chernoff bound on P_e. Results are shown in Fig. 4. As expected, the capability of the detector improves as the amount of partial information increases. Also, as m tends to n, the performance at optimality converges to that for $m = n$, which is the Gaussian bound (the case when $\mathbf{T}$ is invertible).
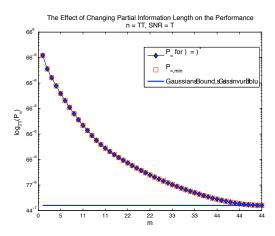

Fig. 2. Performance of $\mathbf{T}_{opt}$ compared to arbitrary $\mathbf{T} \in \mathcal{S}_\mathbf{T}$

Fig. 4. Performance of $\mathbf{T}_{opt}$ vs. m (length of partial information)

P_e *vs.* $E\left(\|\mathbf{x}\|^2\right)/E\left(\|\mathbf{e}\|^2\right)$: In this part we observe the effect of $SNR = \mathrm{E}\left(\|\mathbf{x}\|^2\right)/\mathrm{E}\left(\|\mathbf{e}\|^2\right)$ on the optimality of $\mathbf{T}_{opt}$. Fig. 3 is given to discuss this effect. Similar to the setup of top-left panel, the simulations are performed with Σ_x and Σ_e having uniformly distributed eigenvalues. As expected, the performance at optimality improves with increasing SNR since it gets easier to differentiate $\mathbf{z}_0$ from $\mathbf{z}_1$ in that case.

P_e *vs.* m: In this case, we study the effects of the amount of partial information shared by the detector side on the bound on the expected performance of the detector. This case is studied for Σ_x and Σ_e having uniformly distributed eigenvalues and $SNR = 1$. For $n = 50$, we construct $\mathbf{T}_{opt}$ for particular values of m and evaluate its performance in the

P_e *vs.* n: In this part we study the effect of changes in signal length on the performance of $\mathbf{T}_{opt}$. The simulation results, for various Σ_x and Σ_e all having uniformly distributed eigenvalues, are shown in Fig. 5. At first glance, the results might seem counter-intuitive. The crucial point is that since m (the dimension of the partial information space) is constant, as n increases we get more degrees of freedom to construct $\mathbf{T}_{opt}$ (i.e. the number of eigenvalues of $\mathbf{P}$ increases and so does (4.6), improving the detector performance).

VI. CONCLUSIONS

We introduce the concept of communication with partial information. The main idea is that the codebooks used by the transmitter and the receiver are different. This concept is
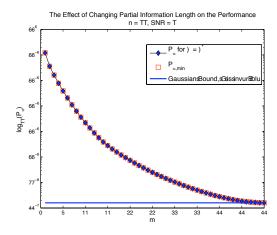
Fig. 5. Performance of $\mathbf{T}_{opt}$ vs. n (signal length)

different from that of communication with side information, where the utilized codebooks are the same but there is extra information available to one of the communicating parties.

Within the context of communication with partial information, we particularly concentrate on a binary detection theoretic scenario. The transmitter sends one of the two codewords (which are independent realizations of a colored multivariate Gaussian distribution) to the additive colored Gaussian noise channel. The receiver acts as a detector, using *dimensionality reduced versions* of the encoder codewords, where the dimensionality reduction is achieved via a linear transform. We first find the optimal (in the sense of probability of error) detection rule. Then we derive the optimal class of linear transforms in the sense of the expected value of the Chernoff bound on the conditional probability of error of the detector.

Although the focus here is on binary detection, we believe that the proposed "communication with partial information" covers several setups of interest, especially the cases where there is an inherent asymmetry between the transmitter and the receiver due to the unbalanced limitations on the physical resources, such as memory and computational power. In our future research, we plan to explore various communication theoretic setups where asymmetry is a crucial feature.

APPENDIX I
PROOF OF THEOREM 3.1

Throughout the proof, we use the definitions of $\mathbf{y}_i \triangleq \mathbf{x}_i + \mathbf{e}$ for $i \in \{0,1\}$. Accordingly, we use $\mu_{y_i|z_i} = \mathrm{E}\,(y_i|z_i)$ and $\mathbf{\Sigma}_{y_i|z_i} = \mathrm{Cov}\,(y_i|z_i)$. We start with the following lemma.

Lemma 1.1: For $i \in \{0,1\}$, conditioned on $\mathbf{z}_i$, $\mathbf{y}_i$ is a normal random vector. Furthermore

$$\mathbf{\Sigma}_{y_i|z_i} = \mathbf{\Sigma}_x + \mathbf{\Sigma}_e - \mathbf{\Sigma}_x \mathbf{T}^T (\mathbf{T}\mathbf{\Sigma}_x \mathbf{T}^T)^{-1}\mathbf{T}\mathbf{\Sigma}_x, \quad \text{(I.1)}$$

is independent of i and positive definite.

Proof: The crucial point is to show that, for $i \in \{0,1\}$, $\mathbf{y}_i$ and $\mathbf{z}_i$ are jointly normal with a positive definite covariance matrix. First, consider $\begin{bmatrix} \mathbf{x}_i \\ \mathbf{e} \end{bmatrix} \in \mathbb{R}^{2n}$. Since $\mathbf{x}_i$ and $\mathbf{e}$ are both normal and are independent, they are also

jointly normal with zero mean and the covariance matrix of $\mathbf{H} \triangleq \begin{bmatrix} \mathbf{\Sigma}_x & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_e \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$. Note that, $\mathbf{H}$ is clearly positive definite, since for any $\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix} \in \mathbb{R}^{2n}$ where $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$, $\mathbf{v}^T\mathbf{H}\mathbf{v} = \mathbf{v}_1^T\mathbf{\Sigma}_x\mathbf{v}_1 + \mathbf{v}_2^T\mathbf{\Sigma}_e\mathbf{v}_2 \geq 0$ by the positive definiteness of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_e$ (that we assumed). By the same token, $[\mathbf{v}_1^T\mathbf{\Sigma}_x\mathbf{v}_1 + \mathbf{v}_2^T\mathbf{\Sigma}_e\mathbf{v}_2 = 0] \iff [\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}] \iff [\mathbf{v} = \mathbf{0}]$, yielding the positive definiteness of $\mathbf{H}$.

Now, consider the linear transformation from the normal random vector $\begin{bmatrix} \mathbf{x}_i \\ \mathbf{e} \end{bmatrix} \in \mathbb{R}^{2n}$ to the vector $\begin{bmatrix} \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix} \in \mathbb{R}^{n+m}$ represented by $\mathbf{F} = \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \\ \mathbf{T} & \mathbf{0}_{m \times n} \end{bmatrix} \in \mathbb{R}^{(n+m) \times 2n}$, where $\mathbf{0}_{m \times n}$ denotes the $m \times n$ zero matrix. This linear transform establishes the normality of $\begin{bmatrix} \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix} \in \mathbb{R}^{n+m}$ (by the properties of jointly normal random vectors) with zero mean and the covariance matrix of $\mathbf{FHF}^T = \begin{bmatrix} \mathbf{\Sigma}_x + \mathbf{\Sigma}_e & \mathbf{\Sigma}_x T^T \\ \mathbf{T}\mathbf{\Sigma}_x & \mathbf{T}\mathbf{\Sigma}_x\mathbf{T}^T \end{bmatrix}$. To deduce the positive definiteness of this covariance matrix, i.e., $\mathbf{FHF}^T$, it is sufficient to show that $\mathbf{F}$ is full rank. This stems from the fact that if $\mathbf{F}$ is full rank (i.e., if $r\,(\mathbf{F}) = m+n$ since $m < n$), for any nonzero vector $\mathbf{s} \in \mathbb{R}^{m+n}$ we have $\mathbf{F}^T\mathbf{s} = \mathbf{w} \neq \mathbf{0} \in \mathbb{R}^{2n}$ since $\mathbf{F}^T$ has a trivial *null-space*, so we end-up with $\mathbf{s}^T\mathbf{FHF}^T\mathbf{s} = \mathbf{w}^T\mathbf{Hw} > 0$ by the positive definiteness of $\mathbf{H}$.

To establish the full-rank property of $\mathbf{F}$ (equivalent to having "$\mathbf{F}^T$ has a trivial null-space"), consider $\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} \in \mathbb{R}^{m+n}$ where $\mathbf{a}_1 \in \mathbb{R}^n$ and $\mathbf{a}_2 \in \mathbb{R}^m$. In this case, $\mathbf{F}^T\mathbf{a} = \begin{bmatrix} \mathbf{a}_1 + \mathbf{T}^T\mathbf{a}_2 \\ \mathbf{a}_1 \end{bmatrix}$. Suppose there exists some $\mathbf{a} \neq \mathbf{0}$ such that $\mathbf{F}^T\mathbf{a} = \mathbf{0}$. This implies, $\mathbf{a}_1 = \mathbf{0}$ and $\mathbf{T}^T\mathbf{a}_2 = \mathbf{0}$. However, since $r(\mathbf{T}) = m$, $[\mathbf{T}^T\mathbf{a}_2 = \mathbf{0}] \iff [\mathbf{a}_2 = \mathbf{0}]$. Therefore, $[\mathbf{F}^T\mathbf{a} = \mathbf{0}] \iff [\mathbf{a} = \mathbf{0}]$ and hence contradiction. Thus, $\mathbf{F}$ is necessarily full-rank implying positive-definiteness of the covariance matrix of $\begin{bmatrix} \mathbf{y}_i \\ \mathbf{z}_i \end{bmatrix}$, i.e., $\mathbf{FHF}^T$.

Finally, the normality of $[\mathbf{y}_i \mid \mathbf{z}_i]$ follows from the properties of normal distributed random variables. The positive definiteness of the corresponding covariance matrix $\mathbf{\Sigma}_{y_i|z_i} = \mathbf{\Sigma}_x + \mathbf{\Sigma}_e - \mathbf{\Sigma}_x\mathbf{T}^T(\mathbf{T}\mathbf{\Sigma}_x\mathbf{T}^T)^{-1}\mathbf{T}\mathbf{\Sigma}_x$ follows from the fact that it is the inverse of a principal submatrix of the inverse of $\mathbf{FHF}^T$, which is positive definite (see (7.1.2) and (7.7.5) in [11]). Also, $\mathbf{\Sigma}_{y_i|z_i}$ is clearly independent of $i \in \{0,1\}$. ∎

Per Lemma 1.1, since $\mathbf{\Sigma}_{y|z} = \mathbf{\Sigma}_{y_i|z_i}$ is positive definite, it is invertible and it has an invertible square root.

Remark 1.1: From properties of normal random vectors, we have

$$\mu_{y_i|z_i} = \mathrm{E}\,(\mathbf{y}_i \mid \mathbf{z}_i) = \mathbf{\Sigma}_x\mathbf{T}^T(\mathbf{T}\mathbf{\Sigma}_x\mathbf{T}^T)^{-1}\mathbf{z}_i. \quad \text{(I.2)}$$

Now let $\beta \triangleq [(2\pi)^{n/2}\det(\mathbf{\Sigma}_{y|z})^{1/2}]^{-1}$, $\theta \triangleq (\mu_{y_0|z_0} - \mu_{y_1|z_1})^T\mathbf{\Sigma}_{y|z}^{-1}\mathbf{y}$, and $\alpha_i \triangleq (\mathbf{y} - \mu_{y_i|z_i})^T\mathbf{\Sigma}_{y|z}^{-1}(\mathbf{y} - \mu_{y_i|z_i})$,

$\kappa_i \triangleq \mu_{y_i|z_i}^T \Sigma_{y|z}^{-1} \mu_{y_i|z_i}$ for $i = 0, 1$; where $\Sigma_{y|z}$ and $\mu_{y_i|z_i}$ are given in (I.1) and (I.2), respectively. Then, using Lemma 1.1 and Remark 1.1, given $\mathbf{y}$ is observed we have

$$p\left(\mathbf{y}_i | \mathbf{z}_i\right)\Big|_{y_i = y} = \beta \exp\left[-\frac{\alpha_i}{2}\right]. \tag{I.3}$$

Then, using the above distribution of $[\mathbf{y}_i \mid \mathbf{z}_i]$ the maximum likelihood detection rule (3.2) can be written as

$$\beta \exp\left[-\frac{\alpha_0}{2}\right] \overset{H_0}{\underset{H_1}{\gtrless}} \beta \exp\left[-\frac{\alpha_1}{2}\right]$$

which is equivalent to (3.3) since $\det\left(\Sigma_{y|z}\right) \neq 0$ and $\exp(.)$ is a strictly increasing function in its argument. Moreover,

$$\begin{aligned}
P_{e|H_0} &= \Pr\left[\alpha_0 > \alpha_1 \mid \mathbf{y} \sim \mathcal{N}\left(\mu_{y_0|z_0}, \Sigma_{y|z}\right)\right] \\
&= \Pr\left[\theta < \frac{\kappa_0 - \kappa_1}{2} \mid \mathbf{y} \sim \mathcal{N}(\mu_{y_0|z_0}, \Sigma_{y|z})\right],
\end{aligned}$$

where $P_{e|H_0}$ denotes the probability of error conditioned on H_0. Here, conditioned on H_0, the random variable θ is normal since $\left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right)^T \Sigma_{y|z}^{-1}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$ and $\mathbf{y}|H_0$ is normal. Conditioned on H_0, the mean and variance of θ are given by

$$\begin{aligned}
\mu_{\theta|H_0} &= \left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right)^T \Sigma_{y|z}^{-1} \mu_{y_0|z_0}, \\
\sigma_{\theta|H_0}^2 &= \left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right)^T \Sigma_{y|z}^{-1} \left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right).
\end{aligned}$$

Then, $\Pr\left[\text{error} \mid H_0\right]$ is given in terms of the standard Q-function. As a result, after some algebraic manipulations we get

$$P_{e|H_0} = Q\left(\frac{\|\Sigma_{y|z}^{-1/2}\left(\mu_{y_0|z_0} - \mu_{y_1|z_1}\right)\|}{2}\right) = Q\left(\frac{\sigma_{\theta|H_0}}{2}\right).$$

Furthermore, from symmetry, we have $P_{e|\mathbf{z}_0,\mathbf{z}_1} = P_{e|H_0}$. $\square$

APPENDIX II
PROOF OF PROPOSITION 4.1

First, we recall the standard Chernoff bound on $Q(\cdot)$ function: $Q(x) \leq \frac{1}{2}\exp\left(-\frac{x^2}{2}\right)$ for $x \geq 0$ [12]. Then, (4.2) is obvious via using it in (3.4). Next, we have

$$\begin{aligned}
P_e &\leq \mathrm{E}_{\{\mathbf{z}_0,\mathbf{z}_1\}}\left[\frac{1}{2}\exp\left(-\frac{\sigma_{\theta|H_0}^2}{8}\right)\right], \tag{II.4} \\
&= \mathrm{E}_{\{\mathbf{z}_0,\mathbf{z}_1\}}\left[\frac{1}{2}\exp\left(-\frac{[\mathbf{A}\gamma]^T \mathbf{B}^{-1}[\mathbf{A}\gamma]}{8}\right)\right], \tag{II.5} \\
&= \mathrm{E}_\gamma\left[\frac{1}{2}\exp\left(-\frac{[\mathbf{A}\gamma]^T \mathbf{B}^{-1}[\mathbf{A}\gamma]}{8}\right)\right], \tag{II.6} \\
&= \int_{\mathbb{R}^m} \frac{1/2}{(2\pi)^{\frac{m}{2}} \det(2\Sigma_z)^{\frac{1}{2}}} \\
&\quad \exp\left(-\frac{1}{2}\gamma^T\left[(2\Sigma_z)^{-1} + \frac{\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}}{4}\right]\gamma\right) d\gamma, \tag{II.7}
\end{aligned}$$

where (II.4) follows from using (4.2) in (4.1), (II.5) follows from using the definitions of $\Sigma_{y|z}$, $\mu_{y_0|z_0}$, $\mu_{y_0|z_0}$

(cf. Theorem 3.1) and defining $\mathbf{A} \triangleq \Sigma_x \mathbf{T}^T(\mathbf{T}\Sigma_x\mathbf{T}^T)^{-1}$, $\mathbf{B} \triangleq \Sigma_{y|z}$, $\gamma \triangleq \mathbf{z}_0 - \mathbf{z}_1$, (II.6) follows since the only source of randomness is due to γ per our reparametrization, (II.7) follows since $\gamma \sim \mathcal{N}(\mathbf{0}, 2\Sigma_z)$ where $\Sigma_z = \mathrm{Cov}(\mathbf{z}_0) = \mathrm{Cov}(\mathbf{z}_1) = \mathbf{T}\Sigma_x\mathbf{T}^T$.

Next, we proceed by showing the positive definiteness of the matrix $\left[(2\Sigma_z)^{-1} + \frac{\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}}{4}\right]^{-1}$, which would ensure that it is a valid covariance matrix. First, by assumption, Σ_x is positive definite and $\mathbf{T}$ is full-rank. Hence, using similar steps to the ones that are used in the proof of positive definiteness of $\mathbf{FHF}^T$ within the proof of Lemma 1.1, we conclude that $\Sigma_z = \mathbf{T}\Sigma_x\mathbf{T}^T$ is positive definite. Furthermore, $[\Sigma_z = \mathbf{T}\Sigma_x\mathbf{T}^T > 0] \iff [2\Sigma_z > 0] \iff \left[(2\Sigma_z)^{-1} > 0\right]$. Next, note that $\mathbf{A}$ is full-rank using straightforward linear algebra. Using this result and the positive definiteness of $\mathbf{B}$, and applying similar arguments to those above, we conclude that $\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}$ is positive definite as well. Thus, $\left[(2\Sigma_z)^{-1} + \frac{\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}}{4}\right]^{-1}$ is positive definite since it is the inverse of the sum of two positive definite matrices, which is itself positive definite. As a result, the quantity $\left[(2\Sigma_z)^{-1} + \frac{\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}}{4}\right]^{-1}$ is a valid covariance matrix and the integral (II.7) converges, yielding

$$P_e \leq \frac{1}{2}\left\{\det\left(\mathbf{I}_m + \frac{\Sigma_z\mathbf{A}^T\mathbf{B}^{-1}\mathbf{A}}{2}\right)\right\}^{-\frac{1}{2}},$$

by properties of determinants; hence the proof. $\square$

APPENDIX III
PROOF OF PROPOSITION 4.2

Our first goal is to show that $\mathbf{T}^* \in \mathcal{S}_\mathbf{T}$. First, note that, $\mathbf{T}^*$ is a $m \times n$ matrix by construction. Next, observe that, by definition $\mathbf{ED}$ is a $m \times m$, non-singular matrix and $\mathbf{U}_p^T \mathbf{\Lambda}^{-1}\mathbf{F}^T$ is a $n \times n$, non-singular matrix. Furthermore, $\mathbf{M}^*$ is of size $n \times m$ and $r(\mathbf{M}^*) = m$, i.e., it is full-rank by definition. Hence, $\mathbf{T}^*$ is also of rank-m, implying that $\mathbf{T}^* \in \mathcal{S}_\mathbf{T}$. Next, using $\Sigma_x = \mathbf{F}\mathbf{\Lambda}^2\mathbf{F}^T$ and the definition of $\mathbf{T}^*$, after some algebraic manipulations we get

$$\begin{aligned}
\mathbf{F}\mathbf{\Lambda}\mathbf{U}_p\mathbf{M}^* &= \Sigma_x(\mathbf{T}^*)^T \mathbf{ED}^{-1}, \tag{III.8} \\
\mathbf{ED}^{-2}\mathbf{E}^T &= \left(\mathbf{T}^*\Sigma_x(\mathbf{T}^*)^T\right)^{-1}. \tag{III.9}
\end{aligned}$$

Lemma 3.1: For any $\mathbf{M} \in \mathcal{S}_\mathbf{M}$, letting $\mathbf{T} = \mathbf{EDM}^T\mathbf{U}_p^T\mathbf{\Lambda}^{-1}\mathbf{F}^T$, where $\mathbf{E} \in \mathbb{R}^{m \times m}$ is an arbitrary unitary matrix and $\mathbf{D} \in \mathbb{R}^{m \times m}$ is an arbitrary diagonal positive-definite matrix, we have $G(\mathbf{M}) = J(\mathbf{T})$.

Proof: We have

$$\begin{aligned}
G(\mathbf{M}) &= \left(\frac{1}{2}\right)^m \det\left[\mathbf{I}_m + \left(\mathbf{M}^T\hat{\mathbf{\Lambda}}_p\mathbf{M}\right)^{-1}\right], \tag{III.10} \\
&= \det\left[\mathbf{I}_m + \frac{1}{2}\mathbf{EDM}^T\mathbf{U}_p^T\mathbf{\Lambda}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{\Lambda}\mathbf{\Lambda}\mathbf{F}^T \right. \\
&\quad \left[\Sigma_x + \Sigma_e - \Sigma_x\mathbf{T}^T\mathbf{ED}^{-2}\mathbf{E}^T\mathbf{T}\Sigma_x\right]^{-1} \\
&\quad \left. \Sigma_x\mathbf{T}^T\mathbf{ED}^{-2}\mathbf{E}^T\right] \tag{III.11}
\end{aligned}$$

$$= J(\mathbf{T}) \qquad \text{(III.12)}$$

where (III.10) follows from the definition of determinant and properties of positive definite matrices; (III.11) follows from our auxiliary definitions, properties of the defined matrices and the matrix inversion lemma; (III.12) follows from the substitution of the auxiliary matrices in (III.11). ∎

Lemma 3.2: For any $\mathbf{T} \in \mathcal{S}_\mathbf{T}$, there exists $\mathbf{M} \in \mathcal{S}_\mathbf{M}$, such that $J(\mathbf{T}) = G(\mathbf{M})$.

Proof: For any $\mathbf{T} \in \mathcal{S}_\mathbf{T}$, let $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{D}}$ be given by the SVD of $\mathbf{T}\boldsymbol{\Sigma}_x\mathbf{T}^T$, i.e., $\mathbf{T}\boldsymbol{\Sigma}_x\mathbf{T}^T = \tilde{\mathbf{E}}\tilde{\mathbf{D}}^2\tilde{\mathbf{E}}^T$. Naturally, $\tilde{\mathbf{E}} \in \mathbb{R}^{m \times m}$ and $\tilde{\mathbf{D}} \in \mathbb{R}^{m \times m}$ are unitary and positive-definite diagonal, respectively. Then let $\mathbf{M} \triangleq \mathbf{U}_p^T\boldsymbol{\Lambda}\mathbf{F}^T\mathbf{T}^T\tilde{\mathbf{E}}\tilde{\mathbf{D}}^{-1}$. First, we show that $\mathbf{M} \in \mathcal{S}_\mathbf{M}$. Clearly, $\mathbf{M} \in \mathbb{R}^{n \times m}$. Here,

$$
\begin{aligned}
\mathbf{M}^T\mathbf{M} &= \mathbf{D}^{-1}\tilde{\mathbf{E}}^T\mathbf{T}\mathbf{F}\boldsymbol{\Lambda}\mathbf{U}_p\mathbf{U}_p^T\boldsymbol{\Lambda}\mathbf{F}^T\mathbf{T}^T\tilde{\mathbf{E}}\mathbf{D}^{-1} \\
&= \mathbf{D}^{-1}\tilde{\mathbf{E}}^T\mathbf{T}\mathbf{F}\boldsymbol{\Lambda}^2\mathbf{F}^T\mathbf{T}^T\tilde{\mathbf{E}}\mathbf{D}^{-1} \\
&= \mathbf{D}^{-1}\tilde{\mathbf{E}}^T\mathbf{T}\boldsymbol{\Sigma}_x\mathbf{T}^T\tilde{\mathbf{E}}\mathbf{D}^{-1}, \\
&= \mathbf{D}^{-1}\tilde{\mathbf{E}}^T\tilde{\mathbf{E}}\tilde{\mathbf{D}}^2\tilde{\mathbf{E}}^T\tilde{\mathbf{E}}\mathbf{D}^{-1} = \mathbf{I}_m,
\end{aligned}
$$

implying $\mathbf{M} \in \mathcal{S}_\mathbf{M}$. Now, note that we have $\mathbf{T} = \tilde{\mathbf{E}}\tilde{\mathbf{D}}\mathbf{M}^T\mathbf{U}_p^T\boldsymbol{\Lambda}^{-1}\mathbf{F}^T$ due to the way $\mathbf{M}$ was defined; this means $\mathbf{T}$ is of the functional form given in the statement of Prop. 4.2. Also, $\tilde{\mathbf{E}}$ is unitary and $\tilde{\mathbf{D}}$ is diagonal and positive definite. Therefore, we necessarily have $G(\mathbf{M}) = J(\mathbf{T})$ per Lemma 3.1. Hence the proof. ∎

Now, we go back to the proof of Prop. 4.2 and use proof by contradiction. Suppose, there exists some $\bar{\mathbf{T}} \in \mathcal{S}_\mathbf{T}$ such that $J(\bar{\mathbf{T}}) > J(\mathbf{T}^*)$. By Lemma 3.1, we necessarily have $J(\mathbf{T}^*) = G(\mathbf{M}^*)$. Furthermore, by Lemma 3.2, there exists $\bar{\mathbf{M}} \in \mathcal{S}_M$ such that $J(\bar{\mathbf{T}}) = G(\bar{\mathbf{M}})$. But this implies $G(\bar{\mathbf{M}}) = J(\bar{\mathbf{T}}) > J(\mathbf{T}^*) = G(\mathbf{M}^*)$ which contradicts with the way $\mathbf{M}^*$ was defined in the first place. Hence contradiction and proof. □

APPENDIX IV
PROOF OF PROPOSITION 4.3

First, observe that, $G(\mathbf{M})$ is a product of positive real numbers since $\lambda_i\left(\mathbf{M}^T\hat{\boldsymbol{\Lambda}}_p\mathbf{M}\right) > 0$ for all i by the positive definiteness of $\mathbf{M}^T\hat{\boldsymbol{\Lambda}}_p\mathbf{M}$ (because $\mathbf{M}$ is orthonormal, full-rank and $\hat{\boldsymbol{\Lambda}}_p$ is positive definite). So, in order to maximize $G(\mathbf{M})$, we follow the strategy of maximizing each positive factor $\left(1 + \frac{1}{\lambda_i(\mathbf{M}^T\hat{\boldsymbol{\Lambda}}_p\mathbf{M})}\right)$ for all i, which clearly is equivalent to minimizing $\lambda_i\left(\mathbf{M}^T\hat{\boldsymbol{\Lambda}}_p\mathbf{M}\right)$ for all i. Here, let $\mathbf{Q} \in \mathbb{R}^{m \times m}$ denote a permutation matrix such that $\hat{\boldsymbol{\Lambda}}_p = \mathbf{Q}^T\hat{\hat{\boldsymbol{\Lambda}}}_p\mathbf{Q}$, the matrix $\hat{\hat{\boldsymbol{\Lambda}}}_p$ is diagonal, and its eigenvalues (i.e., the diagonal entries) are in non-decreasing order [2]. Then, $G(\mathbf{M})$ can be rewritten as

$$G(\mathbf{M}) \triangleq 2^{-m} \prod_{i=1}^{m}\left[1 + \frac{1}{\lambda_i\left(\mathbf{M}^T\mathbf{Q}^T\hat{\hat{\boldsymbol{\Lambda}}}_p\mathbf{Q}\mathbf{M}\right)}\right]. \qquad \text{(IV.13)}$$

---

[2]See [11] for the existence of such a $\mathbf{Q}$. Note that, such a $\mathbf{Q}$ is unique iff the eigenvalues of $\hat{\boldsymbol{\Lambda}}_p$ are distinct.

Next, we recall the *Poincaré seperation theorem* (see [11], pp. 190–191) which is crucial in completing the proof.

Theorem 4.1: Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric, and let m be a given integer with $1 \le m \le n$, and $\mathbf{B}_m = \mathbf{U}^T\mathbf{A}\mathbf{U}$, where $\mathbf{U} \in \mathbb{R}^{n \times m}$ is orthonormal. If eigenvalues of $\mathbf{A}$ and $\mathbf{B}_m$ are arranged in non-decreasing order, we have

$$\lambda_i(\mathbf{A}) \le \lambda_i(\mathbf{B}_m) \le \lambda_{i+n-m}(\mathbf{A}) \quad i = 1, 2, ..., m \quad \text{(IV.14)}$$

Using (IV.14) in (IV.13), we get

$$G(\mathbf{M}) \le 2^{-m} \prod_{i=1}^{m}\left[1 + \frac{1}{\lambda_i\left(\hat{\hat{\boldsymbol{\Lambda}}}_p\right)}\right]. \qquad \text{(IV.15)}$$

Choosing $\mathbf{QM} = \begin{bmatrix}\mathbf{I}_m & \mathbf{0}_{m \times (n-m)}\end{bmatrix}^T$ clearly satisfies $\lambda_i\left(\mathbf{M}^T\mathbf{Q}^T\hat{\hat{\boldsymbol{\Lambda}}}_p\mathbf{QM}\right) = \lambda_i\left(\hat{\hat{\boldsymbol{\Lambda}}}_p\right)$ for $1 \le i \le m$, thereby achieving (IV.15) with equality. Furthermore, since eigenvalues are invariant under similarity transformations, for any unitary $\boldsymbol{\Gamma} \in \mathbb{R}^{m \times m}$ choosing $\mathbf{QM} = \begin{bmatrix}\boldsymbol{\Gamma}^T_{m \times m} & \mathbf{0}_{m \times (n-m)}\end{bmatrix}^T$ also satisfies (IV.15) with equality. Also, the resulting $\mathbf{M} = \mathbf{Q}^T \begin{bmatrix}\boldsymbol{\Gamma}^T_{m \times m} & \mathbf{0}_{m \times (n-m)}\end{bmatrix}^T$ clearly satisfies $\mathbf{M} \in \mathcal{S}_\mathbf{M}$. Hence, any such $\mathbf{M}$ is a solution to (4.5) where the maximum value is the RHS of (IV.15). □

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