Randomness Efficient Steganography

Abstract. Steganographic protocols enable one to embed covert messages into inconspicuous data over a public communication channel in such a way that no one, aside from the sender and the intended receiver, can even detect the presence of the secret message. In this paper, we provide a new provably-secure, private-key steganographic encryption protocol secure in the framework of Hopper et al [2]. We first present a "one-time stegosystem" that allows two parties to transmit messages of fixed length (depending on the length of the shared key) with information-theoretic security guarantees. Employing a pseudorandom generator (PRG) permits secure transmission of longer messages in the same way that such a generator allows the use of one-time pad encryption for long messages in a symmetric encryption framework. The advantage of our construction, compared to all previous work is randomness efficiency: in the information theoretic setting our protocol embeds a message of length n bits using a shared secret key of length (1+o(1))nbits while achieving security $2^{-n/\operatorname{polylog} n}$; simply put this gives a rate of key over message that is 1 as $n \to \infty$ (the previous best result [5] achieved a constant rate > 1 regardless of the security offered). In this sense, our protocol is the first truly randomness efficient steganographic system and breaks through a natural barrier imposed by bounded-round rejecting sampling. Furthermore, in our protocol, we can permit a portion of the shared secret key to be public while retaining precisely nprivate key bits. In this setting, by separating the public and the private randomness of the shared key, we achieve security of 2^{-n} . Our result comes as an effect of a novel application of randomness extractors to stegosystem design.

1 Introduction

The steganographic communication problem can be described using Simmons' [15] formulation of the problem: Alice and Bob are prisoners who wish to communicate securely in the presence of an adversary, called the "Warden." The warden monitors the communication channel to detect whether they exchange "conspicuous" messages. In particular, Alice and Bob are permitted to exchange messages that adhere to certain channel distributions that represent "inconspicuous" communication, but may not detectably stray from this distribution. By controlling the messages transmitted over such a channel, however, Alice and Bob may in fact exchange messages that cannot be detected by the Warden. There have been two approaches in formalizing this problem, one based on information theory [1, 17, 6] and one based on complexity theory [2, 5]. The latter approach is more concrete and has the potential of allowing more efficient constructions.

Most steganographic constructions supported by provable security guarantees are instantiations of the following basic procedure (often referred to as "rejection-sampling"). The problem specifies a family of message distributions (the "channel distributions") that provide a number of possible options for a so-called "covertext" to be transmitted. Additionally, the sender and the receiver possess some sort of private information (typically a keyed hash function, MAC, or other similar function) that maps channel messages to a single bit. In order to send a message bit m, the sender draws a covertext from the channel distribution, applies the function to the covertext and checks whether it happens to produce the "stegotext" m she originally wished to transmit. If this is the case, the covertext is transmitted. In case of failure, this procedure is repeated.

The complexity-theoretic approach to steganography considers the following experiment for the warden-adversary: The adversary selects a message to be embedded and receives either covertexts that embed the message or covertexts simply drawn from the channel distribution (without any embedding). The adversary is then asked to distinguish between the two cases. Clearly, if the probability of success is very close to 1/2 it is natural to claim that the stegosystem provides security against such (eavesdropping) adversarial activity. Formulation of stronger attacks (such as active attacks) is also possible.

Given the above framework, Kiayias et al. [4] (a full version appears in [5]) define a notion of one-time stegosystem: this is a steganographic protocol that is meant to be used for a single message transmission and is proven secure in an information-theoretic sense, provided that the key shared between the sender and the receiver is of sufficient length. This system is a natural analogue of a one-time pad for steganography. They then point out that this can be used to induce a system for longer messages using standard techniques. We shall adopt this same perspective, focusing on achieving optimal usage of randomness.

In this paper, we present a steganography protocol that embeds a message of length n using a shared secret key of length (1 + o(1))n bits while achieving security $2^{-n/\operatorname{polylog} n}$. In this sense, our protocol is **truly randomness efficient**: the rate of key over message approaches 1 for large values of n. In the previous best known protocol [4], the length of the shared secret key is at least (2+o(1))n bits, regardless of the security achieved. The key length requirement of (2+o(1))n bits was dictated by the fact that they perform "single" rejection sampling, in which case some of the randomness used to interrogate the channel during rejection sampling is discarded; as a result they must use a 2n-wise independent family of functions (where n is the length of the message).

Our improvement involves a number of technical elements: we introduce the use of randomness extractors in this context and perform a variant of rejection sampling which is more efficient in its use of the shared secret key. However, this randomness-efficient notion of rejection requires that we control significant new dependencies in the resulting distribution of covertexts. Thus, while the relative improvement in the number of random bits used by our protocol is not particularly impressive (we save a factor 1/2 over previous results), the constructions seems interesting because (i.) it achieves asymptotically optimal

usage of randomness and (ii.) develops a novel steganographic protocol. We remark, finally, that we can permit a portion of the shared secret key to be public while retaining precisely n private key bits. In this setting, by separating the public and the private randomness of the shared key, we can achieve security of 2^{-n} . We adopt the model of channel abstraction first defined by von Ahn [16] (and also used in [5]).

At the heart of our result is the pairing of the rejection sampling process with a randomness extractor. Extractors have been used widely in cryptographic applications and to the best of our knowledge, this is the first time extractors have been employed in the design of steganographic protocols. Given our one-time stegosystem, it is fairly straightforward now to construct provably secure steganographic encryption for longer messages by using a pseudorandom generator (PRG) to stretch a random seed that is shared by the sender and the receiver to sufficient length as shown in [5]. The resulting stegosystem is provably secure in the complexity theoretic model.

2 Preliminaries

We use the notation $x \leftarrow X$ to denote sampling an element x from a distribution X and the notation $x \in_R S$ to denote sampling an element x uniformly at random from a set S. For a function f and a distribution X on its domain, f(X) denotes the distribution that results from sampling x from X and applying f to x. The uniform distribution on $\{0,1\}^d$ is denoted by U_d . We use the notation |s| to stand for the number of symbols in a string s. For a probability distribution s0 with support s1, the notation s2 denoted by s3 denoted by s4. The cardinality of a set s5 is denoted by s5. The concatenation of string s5 and string s5 is denoted by s6. The logarithm base 2.

Pointwise ϵ -biased functions

Definition 1 ([16]). Let P be a distribution with a finite support X. A function $f: X \to Y$ is said to be pointwise ϵ -biased with respect to P if $\forall y \in Y$ $|\operatorname{Pr}_{x \leftarrow P}[f(x) = y] - 1/|Y| | < \epsilon$.

In this paper, we refer to such functions as ϵ -biased and drop the "pointwise" qualification for simplicity.

Min-entropy A distribution X is said to have min-entropy of t bits if the probability it assigns to each element in its range is bounded above by 2^{-t} . A distribution with min-entropy at least t is called a t-source.

Definition 2. The min-entropy of a random variable X, taking values in a set V, is the quantity $H_{\infty}(X) \triangleq \min_{v \in V} (-\log \Pr[X = v])$.

Statistical Distance We use statistical distance to measure the distance between two random variables. Shoup [14] presents a detailed discussion on statistical distance and its properties.

Definition 3. Let X and Y be random variables which both take values in a finite set S with probability distributions P_X and P_Y . The statistical distance between X and Y is defined as $\Delta[X,Y] \triangleq (1/2) \sum_{s \in S} |P_X(s) - P_Y(s)|$. We say that X and Y are ϵ -close if $\Delta[X,Y] \leq \epsilon$.

We will use the following properties of statistical distance which follow directly from the definition.

Fact 1 Let X, Y and Z be random variables taking values in a finite set S. We have (i.) $0 \le \Delta[X,Y] \le 1$ and (ii.) the triangle inequality: $\Delta[X,Z] \le \Delta[X,Y] + \Delta[Y,Z]$.

Fact 2 ([14]) If S and T are finite sets, X and Y are random variables taking values in the set S and $f: S \to T$ is a function, then $\Delta[f(X), f(Y)] \leq \Delta[X, Y]$.

Lemma 1. Consider two random variables (X,Y) and (X',Y'), both taking values in $X \times Y$. For a particular value $x \in X$ in the support of X, we let Y_x denote the random variable Y conditioned on the event X = x and define Y'_x likewise. Then $\Delta[(X,Y),(X',Y')] \leq \Delta[X,X'] + \mathbb{E}_X[\Delta[Y_X,Y_X']]$.

We include the proof in Appendix C for completeness.

2.1 Extractors

Extractors are deterministic functions that operate on arbitrary distributions with sufficient randomness and output "almost" uniformly distributed, independent random bits. Extractors require an additional input: a short seed of truly random bits as a catalyst to "extract" randomness from such distributions. Thus the input to an extractor contains two independent sources of randomness: the actual distribution (the source) and the seed. Extractors were first defined by Nisan and Zuckerman [9].

Definition 4. A (t, ϵ) -extractor is a function $\operatorname{Ext}: \{0, 1\}^{\nu} \times \{0, 1\}^{d} \to \{0, 1\}^{\mu}$ such that for every distribution X on $\{0, 1\}^{\nu}$ with $H_{\infty}(X) \geq t$, the distribution $\operatorname{Ext}(X, U_d)$ is ϵ -close to the uniform distribution on $\{0, 1\}^{\mu}$.

For our application, we require a stronger property from the extractor. We need the output of the extractor to remain essentially uniform even given the knowledge of the seed used. In other words, we require the extractor to extract randomness only from the source and not from the seed. A way of enforcing this condition is to demand that when the seed is concatenated to the output, the resulting distribution is still ϵ -close to uniform. Such an extractor is called a strong extractor to distinguish from the weaker notion of extractors defined above. The extractors defined above guarantee to extract randomness from t-sources on an average seed while strong extractors guarantee to extract randomness for most seeds. In this paper, we use the term extractor to refer to a strong extractor.

Definition 5. A (t, ϵ) -strong extractor is a function $\operatorname{Ext}: \{0, 1\}^{\nu} \times \{0, 1\}^{d} \to \{0, 1\}^{\mu}$ such that for every distribution X on $\{0, 1\}^{\nu}$ with $H_{\infty}(X) \geq t$, the distribution $S \circ \operatorname{Ext}(X, S)$ is ϵ -close to the uniform distribution on $\{0, 1\}^{\mu+d}$ where S is distributed according to U_d .

We refer to ν as the length of the source, t as the min-entropy threshold, ϵ as the error of the extractor, the ratio t/ν as the entropy rate of the source X and to the ratio μ/t as the fraction of randomness extracted by the extractor. The entropy loss of the extractor is defined as $t+d-\mu$. The two inputs of the extractor have a total min-entropy of at least t+d and the entropy loss measures how much of this randomness was "lost" in the extraction process. Radhakrishnan and Ta-shma [10] showed that no non-trivial (t,ϵ) -extractor can extract all the randomness present in its inputs and must suffer an entropy loss of $2\log(1/\epsilon) + O(1)$. For our application, we need efficient, explicit strong extractor constructions as defined below.

Definition 6 ([13]). For functions $t(\nu)$, $\epsilon(\nu)$, $d(\nu)$, $\mu(\nu)$ a family $\operatorname{Ext} = \{\operatorname{Ext}_{\nu}\}$ of functions $\operatorname{Ext}_{\nu} : \{0,1\}^{\nu} \times \{0,1\}^{d(\nu)} \to \{0,1\}^{\mu(\nu)}$ is an explicit (t,ϵ) -strong extractor if $\operatorname{Ext}(x,y)$ can be computed in polynomial time in its input length and for every ν , Ext_{ν} is a $(t(\nu), \epsilon(\nu))$ -extractor.

An important property of strong extractors which makes it attractive for our application is that for any t-source, a $(1 - \epsilon)$ fraction of the seeds extract randomness from that source.

Remark ([12]). Let Ext: $\{0,1\}^{\nu} \times \{0,1\}^{d} \to \{0,1\}^{\mu}$ be a (t,ϵ) -strong extractor. From the definition of a strong extractor, we know that $\mathbb{E}_{s}\left[\Delta\left[\operatorname{Ext}(X,s),U_{\mu}\right]\right] \leq \epsilon$ where $s \in_{R} \{0,1\}^{d}$. By applying Markov's inequality, we can see that $\Pr_{s}[\Delta\left[\operatorname{Ext}(X,s),U_{\mu}\right] \geq \epsilon \cdot r] \leq 1/r$.

See the survey articles by Shaltiel [13], Nisan [7], and Nisan and Ta-Shma [8] for more details on extractors and their properties. In this paper, we use the explicit strong extractor construction by Raz, Reingold and Vadhan [11] which works on sources of any min-entropy. It extracts all the min-entropy using $O(\log^3 \nu)$ additional random seed bits while achieving an optimal entropy loss (up to an additive constant) of $\chi = 2\log(1/\epsilon) + O(1)$ bits.

Theorem 1 (RRV Extractor [11]). For every ν , $t \in \mathbb{N}$, and $\epsilon > 0$ such that $t \leq \nu$, there are explicit (t, ϵ) -strong extractors $\operatorname{Ext} : \{0, 1\}^{\nu} \times \{0, 1\}^{d} \to \{0, 1\}^{t-\chi}$ with entropy loss $\chi = 2\log(1/\epsilon) + O(1)$ bits and requiring seeds of length

$$d = O(\log^2 \nu \cdot \log(1/\epsilon) \cdot \log t)$$
 bits.

2.2 The channel model

The security of a steganography protocol is measured by the adversary's ability to distinguish between "normal" and "covert" message distributions over a communication channel. To characterize normal communication we define and

formalize the *communication channel* following standard terminology used in the literature [2, 1, 16, 5, 3]. We let Σ denote the symbols of an alphabet and treat the *channel* as a family of distributions $\mathfrak{C} = \{C_h\}_{h \in \Sigma^*}$; each C_h is supported on Σ . These channel distributions model a history-dependent notion of channel data.

We adopt the model of channel abstraction first defined by von Ahn and Hopper [16]. Here, Alice is provided with a means for sampling "deep into the channel." In particular, Alice and, consequently, the steganographic encoding protocol, has access to a channel oracle that can sample from the channel for any history. Formally, during the embedding process, Alice may sample from $C_{h_1 \circ ... \circ h_{\ell}}$ for any history she wishes (though Alice is constrained to be efficient and so can make no more than polynomially many queries of polynomial length). This model allows Alice to transform a channel C with min-entropy δ into a channel C^{π} with min-entropy $\pi\delta$. Specifically, the channel C^{π} is defined over the alphabet Σ^{π} , whose elements we write as vectors $\mathbf{h} = (h_1, \dots, h_{\pi})$. The distribution $C^{\pi}_{\mathbf{h}^{1},\dots,\mathbf{h}^{v}}$ is determined by the channel C with history $h_{1}^{1}\circ\cdots\circ$ $h_{\pi}^{1} \circ h_{1}^{2} \circ \cdots \circ h_{\pi}^{v}$. This definition captures the adaptive nature of the channel by taking into account the dependence between symbols as is typical in real world communications. We assume that the channel satisfies a min-entropy constraint for all histories. We say that a channel has min-entropy δ if $\forall h \in \Sigma^*, H_{\infty}(C_h) \geq$ δ . Observe that this implies that $H_{\infty}(C_h^{\pi}) \geq \delta \pi$ due to the additive nature of marginal min-entropy.

2.3 One-time stegosystem

Here, we give the definition of a *one-time stegosystem*, a steganographic system that enables the one-time steganographic transmission of a message provided that the two parties share a suitable key. We adopt the definitions used by Kiayias et al. [5].

Definition 7. A one-time stegosystem consists of three probabilistic polynomial time algorithms S = (SK, SE, SD), where:

- SK is the key generation algorithm; we write $SK(1^k) = \kappa$. It produces a key κ of length k.
- SE is the embedding procedure and has access to the channel; SE $(\kappa, m; 0) = s \in \Sigma^*$. The embedding procedure takes into account the history h of communication that has taken place between Alice and Bob thus far and begins its operation corresponding to this history. It takes as input the key κ of length k, a message m of length n = n(k) and accesses the channel through an (probabilistic) oracle 0. The oracle 0 accepts as input any polynomial length history $h' \in \Sigma^*$ and allows SE to draw independent samples repeatedly from $C_{h\circ h'}$. The output is the stegotext $s \in \Sigma^*$. Observe that in a one-time stegosystem, once a security parameter k is chosen, the length of the message n is a fixed function of k. In our model of channel abstraction, SE can access the channel for any history.

- SD is the extraction procedure; $SD(\kappa, c) = m$. It takes as input the key κ of length k, and some $c \in \Sigma^*$. The output is a message m.

We next define a notion of correctness for a one-time stegosystem.

Definition 8 (Correctness). A one-time stegosystem (SK, SE, SD) is said to be (ϵ, δ) -correct provided that for all channels \mathfrak{C} of min-entropy δ , it holds that $\forall h \in \Sigma^*$

$$\forall m \in \{0,1\}^{n(k)} \ \Pr[SD(\kappa, SE(\kappa, m; \mathfrak{O})) \neq m \mid \kappa \leftarrow SK(1^k)] \leq \epsilon \ .$$

In general, we treat both $\epsilon = \epsilon(k)$ and $\delta = \delta(k)$ as functions of k, the security parameter and the oracle O as a function of the history h.

One-time stegosystem security is based on the indistinguishability between a transmission that contains a steganographically embedded message and a transmission that contains no embedded messages. The adversarial game discussed next is meant to model the behavior of a warden in the Simmons' formulation of the problem discussed earlier.

An adversary \mathcal{A} against a one-time stegosystem S = (SK, SE, SD) is a pair of algorithms $\mathcal{A} = (SA_1, SA_2)$, that plays the following game, denoted $G^{\mathcal{A}}(1^k)$:

- 1. A key κ is generated by $SK(1^k)$.
- 2. Algorithm SA_1 receives as input the security parameter k and outputs a triple $(m^*, \text{aux}, h_c) \in M_n \times \{0, 1\}^* \times \Sigma^*$, where aux is some auxiliary information that will be passed to SA_2 . SA_1 is provided access to the channel via the oracle \mathcal{O} , which takes the history h as input. The channel oracle \mathcal{O} , on input h, returns to SA_1 an element c selected according to C_h . This way, the adversary can learn about the channel distribution for any history.
- 3. A bit b is chosen uniformly at random.
 - If b = 0, let $c^* \leftarrow SE(\kappa, m^*; 0)$, so c^* is a stegotext.
 - If b = 1, let $c^* = c_1 \circ \cdots \circ c_{\lambda}$ where $\lambda = |SE(\kappa, m^*; 0)|$ and $c_i \stackrel{r}{\leftarrow} C_{h \circ c_1 \circ \cdots \circ c_{i-1}}$. In this case, $c^* \leftarrow C_h^{\lambda}$.
- 4. The input for SA_2 is 1^k , h_c , c^* and aux. SA_2 outputs a bit b'. If b' = b then we say that (SA_1, SA_2) succeeded and write $G^{\mathcal{A}}(1^k) =$ success.

The advantage of the adversary \mathcal{A} over a stegosystem S is defined as: $\mathbf{Adv}_{S}^{\mathcal{A}}(k) = |\Pr[G^{\mathcal{A}}(1^{k}) = \text{success}] - 1/2|.$

The probability includes the coin tosses of A and SE. The (information-theoretic) insecurity of the stegosystem is defined as

$$\mathbf{InSec}_{S}(k) = \max_{\mathcal{A}} \{ \mathbf{Adv}_{S}^{\mathcal{A}}(k) \},$$

this maximum taken over all (time unbounded) adversaries A.

Definition 9 (Security). We say that a stegosystem is (ϵ, δ) -secure if for all channels with min-entropy δ we have $\mathbf{InSec}_S(k) \leq \epsilon$.

Overhead. The overhead of a one-time stegosystem is judged by the relation of the key length k and message length n. We adopt the ratio k/n as the measure of overhead as first defined by Kiayias et al. [5].

2.4 Rejection Sampling

As noted before, a common method used in steganography employing a channel distribution is that of rejection sampling (cf. [1, 2, 5]). We use a variant of rejection sampling to transmit bit vectors as opposed to a single bit. To transmit bit vectors, we amplify the entropy of the channel as discussed before and apply ρ -rejection sampling described below. More precisely, we transform a channel C with min-entropy δ into a channel C^{π} with min-entropy $\pi\delta$, defined over the alphabet Σ^{π} . We now perform ρ -rejection sampling over C^{π} as described: Assuming that one wishes to transmit a bit vector $\mathbf{m} \in \{0,1\}^{\eta}$ and employs a random function $f: \Sigma^{\pi} \to \{0,1\}^{\eta}$, one performs the following "rejection sampling" process:

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\begin{aligned} \operatorname{REJ}_h^f(\boldsymbol{m}, \rho) \\ \text{let } j &= 0 \\ & \text{repeat:} \\ & \text{sample } \boldsymbol{c} \leftarrow C_h^\pi \text{ , increment } j \\ & \text{until } f(\boldsymbol{c}) = \boldsymbol{m} \text{ or } (j > \rho) \\ \text{output: } \boldsymbol{c} \end{aligned}
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For a given history h, the procedure $\operatorname{ReJ}_h^f(\boldsymbol{m},\rho)$ draws independent samples from the channel distribution C_h^π in rounds until $f(\boldsymbol{c}) = \boldsymbol{m}$ or $j > \rho$. As there are at most a total of $\rho+1$ rounds, if none of the first ρ samples drawn map to the target bit vector, the sample drawn at round $\rho+1$ is returned by the procedure. Here, as defined before, Σ^π denotes the output alphabet of the channel, h denotes the history of the channel at the start of the process, and C_h^π denotes the marginal distribution on sequences of π symbols given by the channel after history h. The receiver (also privy to the function f) applies the function to the received message $\boldsymbol{c} \in \Sigma^\pi$ and recovers \boldsymbol{m} with a certain probability of success. Note that the above process performs $\rho+1$ draws from the channel with the same history. These draws are assumed to be independent. One basic property of rejection sampling that we use is:

Lemma 2 ([16]). If the function f is ϵ -biased on C_h^{π} for history h, then for any ρ and uniformly random $\mathbf{m} \in_R \{0,1\}^{\eta}$:

$$\Delta\left[\operatorname{ReJ}_h^f(\boldsymbol{m},\rho),C_h^{\pi}\right] \leq \epsilon.$$

Proof. Let us denote the samples drawn by the procedure $\text{ReJ}_h^f(\boldsymbol{m}, \rho)$ as $\boldsymbol{c_i}, i = 1, \cdots, \rho + 1$. Suppose the target bit vector \boldsymbol{m} was chosen with the probability $P_f^{(\boldsymbol{m})} \triangleq \Pr[f(C_h^{\pi}) = \boldsymbol{m}]$, i.e, $\boldsymbol{m} \leftarrow P_f^{(\boldsymbol{m})}$, we first show that the output from $\text{ReJ}_h^f(\boldsymbol{m}, \rho)$ is distributed identically to C_h^{π} . For simplicity of notation, let us

define $p_m \triangleq \Pr_{P_f^{(m)}}[m]$. Let p_c denote the probability of drawing c from the channel distribution C_h^{π} , i.e., $p_c \triangleq \Pr_{C_h^{\pi}}[c]$. For $c \in C_h^{\pi}$, the probability of observing c under the $\operatorname{ReJ}_h^f(m, \rho)$ procedure is then given by

$$\Pr[\operatorname{ReJ}_{h}^{f}(\boldsymbol{m}, \rho) = \boldsymbol{c}] \\
= \Pr_{\boldsymbol{c}_{1} \leftarrow C_{h}^{\pi}} [\boldsymbol{c}_{1} = \boldsymbol{c}] \cdot \Pr[f(\boldsymbol{c}_{1}) = \boldsymbol{m}] + \Pr_{\boldsymbol{c}_{2} \leftarrow C_{h}^{\pi}} [\boldsymbol{c}_{2} = \boldsymbol{c}] \cdot \Pr[f(\boldsymbol{c}_{2}) = \boldsymbol{m}] \cdot \Pr[f(\boldsymbol{c}_{1}) \neq \boldsymbol{m}] \\
+ \Pr_{\boldsymbol{c}_{3} \leftarrow C_{h}^{t}} [\boldsymbol{c}_{3} = \boldsymbol{c}] \cdot \Pr[f(\boldsymbol{c}_{3}) = \boldsymbol{m}] \cdot \Pr[f(\boldsymbol{c}_{1}) \neq \boldsymbol{m} \land f(\boldsymbol{c}_{2}) \neq \boldsymbol{m}] + \cdots \\
= p_{c}p_{m} + p_{c}p_{m} (1 - p_{m}) + \cdots + p_{c}p_{m} (1 - p_{m})^{\rho-1} + p_{c} (1 - p_{m})^{\rho} \\
= p_{c}p_{m} \left(\frac{1 - (1 - p_{m})^{\rho}}{p_{m}}\right) + p_{c} (1 - p_{m})^{\rho} = p_{c}.$$

From the above discussion, we can see that when the target bit vector \boldsymbol{m} was chosen from the distribution $P_f^{(\boldsymbol{m})}$, the output from $\operatorname{ReJ}_h^f(\boldsymbol{m},\rho)$ is distributed identically to C_h^{π} . Since f is ϵ -biased, $\Delta \left[U_{\eta}, P_f^{(\boldsymbol{m})} \right] \leq \epsilon$. Hence,

$$\Delta\left[\operatorname{ReJ}_h^f(\boldsymbol{m}\leftarrow U_{\eta},\rho),\operatorname{ReJ}_h^f(\boldsymbol{m}\leftarrow P_f^{(\boldsymbol{m})},\rho)\right] \leq \epsilon$$

by Fact 2 which gives us the statement of the lemma.

3 The construction

In this section, we outline our construction of a one-time stegosystem as an interaction between Alice (the sender) and Bob (the receiver). Alice and Bob wish to communicate over a channel C_h^{π} with history h. We also assume that the support of \mathcal{C}_h is $\{0,1\}^b$, i.e, $|\Sigma|=2^b$.

3.1 A one-time stegosystem

Let $m \in \{0,1\}^n$ be the message to be embedded. Our stegosystem uses the RRV strong-extractor construction as described in Theorem 1 which extracts randomness from the distribution C_h^{π} supported on $\{0,1\}^{\pi \cdot b}$ by rejection sampling as described in Section 2.4. Specifically, we will use the extractor with the seed s as the function f in the rejection sampling procedure. Alice and Bob agree on the following:

Extractor Construction. Alice and Bob agree to use the explicit RRV strong-extractor construction as described in Theorem 1. They use a seed $s \in_R \{0,1\}^d$ for the extractor. The length of the seed d will be determined later as a function of δ , n, b and security ϵ . The notation E_s stands for the extractor used with the seed s i.e., $E(\cdot,s)$. Here, we treat the seed s as private and in Section 3.4 we show that the seed s may be public and discuss the implications of this choice.

One-Time Pad. Alice and Bob also use a shared one-time pad secret key $\kappa^{\text{otp}} \in_R \{0,1\}^n$ effectively transmitting $\mathbf{m}' = \kappa^{\text{otp}} \oplus \mathbf{m}$.

Shared Secret Key. The secret key that they now share is $\kappa = (\kappa^{\text{otp}}, s)$ of length k = n + d.

Key generation consists of generating the one-time pad secret key $\kappa^{\text{otp}} \in_R \{0,1\}^n$ and the random seed s of length d to be used with the extractor. The encoding procedure accepts an input message m of length n bits and outputs a stegotext of length λ . We will analyze the stegosystem below in terms of the parameters π , d, λ , ρ and some constant c > 1 relegating discussion of how these parameters determine the overall efficiency of the system to Section 3.4.

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PROCEDURE SE:
                                                                                             PROCEDURE SD:
Input: Key \kappa = (\kappa^{	exttt{otp}}, s); \ oldsymbol{m} \in \{0,1\}^n,
                                                                                             Input: Key \kappa = (\kappa^{\text{otp}}, s)
                    history h \in \Sigma
                                                                                                                 stegotext cstego
let oldsymbol{m}' = \kappa^{	exttt{otp}} \oplus oldsymbol{m}
parse m' as m' = m'_1 m'_2 \dots m'_{\lceil n/c \log n 
ceil}
                                                                                           parse: \mathsf{c}_{\mathsf{stego}} = c_1 c_2 \dots c_{\lceil n/c \log n \rceil} for i = 1 to \lceil n/c \log n \rceil do \{
for i=1 to \lceil n/c\log n \rceil {
             oldsymbol{c_i} \leftarrow \mathrm{Rej}_h^{E_s}(oldsymbol{m_i'}, 
ho)
                                                                                                            set m_i{}' = E_s(c_i)
               set h \leftarrow h \circ c_i
                                                                                            set m' = m_1' m_2' \dots m_{\lceil n/c \log n 
ceil}'
                                                                                            Output: oldsymbol{m}' \oplus \kappa^{	exttt{otp}}
\lceil \texttt{Output: } \mathsf{c}_{\mathsf{stego}} = c_1 c_2 \dots c_{\lceil n/c \log n \rceil} \in \mathsf{\Sigma}^{\lambda}
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Fig. 1. Encryption and Decryption algorithms for the one-time stegosystem of 3.1.

Alice and Bob communicate using the algorithm SE for steganographic embedding and SD for decoding as described in Figure 1. In SE, after applying the one-time pad to randomize her message m, Alice obtains $m' = \kappa^{\text{otp}} \oplus m$. She then parses m' into $\lceil n/c \log n \rceil$ blocks, each block of length $c \log n$ for some constant c > 1, i.e., $m' = m_1' m_2' \dots m'_{\lceil n/c \log n \rceil}$. She then applies the procedure $\operatorname{ReJ}_h^{E_s}(\boldsymbol{m}_i', \rho)$ to obtain an element $\boldsymbol{c_i} \in \Sigma^{\pi}$ for each block $\boldsymbol{m_i}', i =$ $1, \dots, \lceil n/c \log n \rceil$ of the randomized message. Here, the history h represents the current history at the time of the rejection sampling procedure which is updated after the completion of the procedure. Recall that the notation E_s stands for the extractor used with the seed s i.e., $E(\cdot, s)$. The resulting stegotext, denoted by c_{stego} that is transmitted to Bob is $c_{\text{stego}} = c_1 c_2 \dots c_{\lceil n/c \log n \rceil}$. In SD, the received stegotext is first parsed into $\lceil n/c \log n \rceil$ blocks as shown and then evaluated using the extractor with seed s for each block; this results in a message block. After performing this for each received block, a bit string of length n is obtained, which is subjected to the one-time pad decoding to obtain the original message. The detailed security and correctness analysis follow in the next two sections.

3.2 Security

In this section, we argue about the security of our one-time stegosystem. Specifically, we establish an upper bound on the statistical distance between the "normal" and "covert" message distributions over the communication channel. First, by Lemma 2, observe that if the function f is ϵ -biased on C_h^{π} for history h, then for any ρ , $\mathbf{m}' \in_R \{0,1\}^{\eta}$: $\Delta[\operatorname{REJ}_h^f(\mathbf{m}',\rho), C_h^{\pi}] \leq \epsilon$. Now, consider the strong extractor $\operatorname{Ext}: \{0,1\}^{\nu} \times \{0,1\}^{d} \to \{0,1\}^{\mu}$ used in the rejection sampling procedure. Denote the error of extractor by ϵ_{ext} . Recall from the remark in Section 2.1 that, for a uniformly chosen seed $s \in_R \{0,1\}^d$, $\operatorname{Pr}_s[\Delta[\operatorname{Ext}(X,s),U_{\mu}] \geq \sqrt{\epsilon_{ext}}] \leq \sqrt{\epsilon_{ext}}$. From this we can see that Ext fails to be a $\sqrt{\epsilon_{ext}}$ -biased function with probability no more than $\sqrt{\epsilon_{ext}}$ in the choice of the seed s. Thus, for a random m' and s,

$$\Delta[\operatorname{ReJ}_h^{E_s}(\boldsymbol{m}', \rho), C_h^{\pi}] \leq 1 \cdot \sqrt{\epsilon_{ext}} + \sqrt{\epsilon_{ext}} \cdot 1 \leq 2\sqrt{\epsilon_{ext}}$$

We obtain the above inequality by upper bounding the probability of the extractor being a $\sqrt{\epsilon_{ext}}$ -biased function by 1 and observing that the statistical distance is also upper bounded by 1 by Fact 1. Suppose that in our stegosystem construction, we had used an independent and uniformly chosen seed $s_i \in_R \{0,1\}^d$ for each message block $i=1,2,\cdots,\lceil n/c\log n\rceil$, the statistical distance between C_h^{λ} and the output of the procedure SE would then be

$$\Delta \left[SE(\kappa, \boldsymbol{m}; \mathfrak{O}), C_h^{\lambda} \right] \le 2\sqrt{\epsilon_{ext}} \left\lceil n/c \log n \right\rceil.$$

However, employing an independent and uniformly chosen seed for each message block would require too much randomness. In our scheme, we employ a single seed s over all the message blocks and so we need to manage the dependencies between the output covertexts; this is the major technical issue in the proof, which is relegated to Appendix A for lack of space. In particular, for any message $\mathbf{m} \in \{0,1\}^n$, we present an upper bound on $\Delta \left[SE(\kappa, \mathbf{m}; \mathcal{O}), C_h^{\lambda}\right]$ when using a single seed $s \in_R \{0,1\}^d$ over all the message blocks. We record the theorem below; the proof appears in Appendix A.

Theorem 2. For any $\epsilon, \delta > 0$, message $\mathbf{m} \in \{0,1\}^n$ consider the stegosystem (SK, SE, SD) of Section 3.1 under the parameter constraint $\epsilon_{ext} \leq \left(\frac{\epsilon}{3\ell}\right)^3$. Then it holds that the stegosystem is (ϵ, δ) -secure where ϵ_{ext} is the extractor error and $\ell = \lceil n/c \log n \rceil$ for some constant c > 1.

3.3 Correctness

In this section we obtain an upper bound on the soundness of our stegosystem. We focus on the mapping between $\{0,1\}^n$ and Σ^λ determined by the SE procedure of the one-time stegosystem. We would like to bound the probability of the stego decoding procedure's inability to faithfully recover the encoded message.

Theorem 3. For any $\epsilon, \delta > 0$, message $\mathbf{m} \in \{0,1\}^n$ consider the stegosystem (SK, SE, SD) of Section 3.1 under the parameter constraints $\epsilon_{ext} \leq \left(\frac{\epsilon}{6\ell^2}\right)^3$ and

 $\rho \geq 2n^c \log(3\ell\epsilon^{-1})$ for some constant c > 1. Then it holds that the stegosystem is (ϵ, δ) -correct where ϵ_{ext} is the extractor error and $\ell = \lceil n/c \log n \rceil$ for some constant c > 1.

Proof. Recall that the first step of the procedure SE is to randomize the message m to get $m' = m \oplus \kappa^{\text{otp}}$. SE then proceeds to parse m' into blocks: $m' = m'_1 m'_2 \dots m'_\ell$, $\ell = \lceil n/c \log n \rceil$. Let F be the event that SD is unable to correctly decode the message encoded by SE. We seek to upper bound the probability of F. We proceed to first estimate the probability of failure for one message block m_i . Let us denote this event by F'.

Recall that we pick a seed $s \in_R \{0,1\}^d$ for the extractor we use in our construction and let ϵ_{ext} denote the error of the extractor. We say that a seed s is good if $\forall \tau$, $\mu(G_s^\tau) \geq 1 - \sqrt[3]{\epsilon_{ext}}$, $\tau = 1, 2, \cdots, \ell$. We show in Appendix A that the probability of seed s to be good is given by $\Pr_s\left[\forall \tau \mid \mu(G_s^\tau) \geq 1 - \sqrt[3]{\epsilon_{ext}}\right] \geq 1 - \ell\sqrt[3]{\epsilon_{ext}}$. (This follows from straightforward applications of Markov's inequality.) Thus that the probability that the seed s is not good is no more than $\ell\sqrt[3]{\epsilon_{ext}}$. By the union bound this yields

$$\Pr[F] = \ell \cdot (\Pr[F' \mid s \text{ good}] \cdot \Pr[s \text{ good}] + \Pr[F' \mid s \text{ not good}] \cdot \Pr[s \text{ not good}])$$

$$\leq \ell \cdot (\Pr[F' \mid a \text{ good}] \cdot 1 + 1 \cdot (\ell \sqrt[3]{\epsilon_{ext}})).$$

We proceed to bound $\Pr[F' \mid s \text{ is good}]$. We know that when the seed s is good for no more than $\sqrt[3]{\epsilon_{ext}}$ fraction of distributions in every level $\tau = 1, 2, \dots, \ell$, the extractor coupled with the seed s is not a $\sqrt[3]{\epsilon_{ext}}$ -biased function with probability no more than $\sqrt[3]{\epsilon_{ext}^2}$. So, we get

$$\Pr[F' \mid s \text{ good}] \le 1 \cdot \left(1 - \left(\frac{1}{2^{|\boldsymbol{m}_i|}} - \sqrt[3]{\epsilon_{ext}}\right)\right)^{\rho} + \sqrt[3]{\epsilon_{ext}} \cdot 1 + \sqrt[3]{\epsilon_{ext}^2} \cdot 1$$

where ρ is the bound on the number of iterations performed by the rejection sampling procedure. Setting $\epsilon_{ext} \leq 1/(8 \cdot 2^{3|m_i|}) = 1/(8 \cdot n^{3c})$ and

$$\rho = 2 \cdot 2^{|\boldsymbol{m_i}|} \cdot \log(3\ell\epsilon^{-1}) = 2n^c \log(3\ell\epsilon^{-1})$$

(since in our construction $|m_i| = c \log n$, and as ρ is exponential in the block length, we choose the message block length to be $c \log n$), we have

$$\Pr[F' \mid s \text{ good}] \le \frac{\epsilon}{3\ell} + 2\sqrt[3]{\epsilon_{ext}}.$$

From the statement of the theorem we have that $\epsilon_{ext} \leq \left(\frac{\epsilon}{6\ell^2}\right)^3$ and hence

$$\Pr[F] \leq \ell \cdot (\Pr[F' \mid s \text{ good}] \cdot 1 + 1 \cdot (\ell \sqrt[3]{\epsilon_{ext}})) \leq \epsilon$$

and the statement of the theorem follows.

We record the security and correctness theorem below.

Theorem 4. For any $\epsilon_{ext} \leq 1/8n^{3c}$, $\delta > 0$, message $m \in \{0,1\}^n$, and $\rho \geq 2n^c \log(\epsilon_{ext}^{-1/3})$, the stegosystem (SK, SE, SD) of Section 3.1 is (ϵ_{cor}, δ) -correct and (ϵ_{sec}, δ) -secure, where $\epsilon_{cor} \leq 4\ell^2 \sqrt[3]{\epsilon_{ext}}$ and $\epsilon_{sec} \leq 3\ell \sqrt[3]{\epsilon_{ext}}$. Here, ϵ_{ext} is the extractor error and $\ell = \lceil n/c \log n \rceil$ for some constant c > 1.

3.4 Putting it all together

The objective of this section is to integrate the results of the previous sections of the paper. We first show that our steganography protocol embeds a message of length n bits using a shared secret key of length (1+o(1))n bits while achieving security $2^{-n/\operatorname{polylog} n}$. In this sense, our protocol is randomness efficient in the shared key. We next show that by permitting a portion of the shared secret key to be public while retaining n private key bits, we can achieve security of 2^{-n} . Let us first start our discussion by considering the parameters of the extractor construction we employ in our protocol.

Extractor Parameters Recall that π is the parameter that dictates how many copies of the channel Alice decides to use in order to transform the channel C with min-entropy δ into a channel C^{π} with min-entropy $\pi\delta$. If we let $\pi = \delta^{-1} \cdot (c \log n + 2 \log (1/\epsilon_{ext}) + O(1))$ for some constant c > 1, the channel distribution C_h^{π} supported on $\{0,1\}^{\delta^{-1} \cdot (c \log n + 2 \log (1/\epsilon_{ext}) + O(1)) \cdot b}$ has a minentropy of at least $t = c \log n + 2 \log (1/\epsilon_{ext}) + O(1)$. To put this all together, the RRV strong-extractor is a function $Ext : \{0,1\}^{\nu} \times \{0,1\}^{d} \to \{0,1\}^{t-\Delta}$ where

$$\nu = \delta^{-1} \cdot (c \log n + 2 \log (1/\epsilon_{ext}) + O(1)) \cdot b$$

$$d = O\left(\log^2 \left(\delta^{-1} \cdot (c \log n + 2 \log (1/\epsilon_{ext}) + O(1)) \cdot b\right) \cdot \log (1/\epsilon_{ext}) \cdot \log t\right)$$

$$t = c \log n + 2 \log (1/\epsilon_{ext}) + O(1)$$

$$\Delta = 2 \log (1/\epsilon_{ext}) + O(1) \text{ and }$$

$$t - \Delta = c \log n$$

We can immediately see from the preceding discussion that our stegotext is of length

$$\frac{n}{c\log n} \cdot \delta^{-1} \cdot \left(c\log n + 2\log\left(1/\epsilon_{ext}\right) + O\left(1\right)\right) \cdot b = \frac{n}{\delta} \left(1 + \frac{2\log\left(1/\epsilon_{ext}\right)}{c\log n} + o\left(1\right)\right) \cdot b$$

bits to embed n bits of message.

Randomness Efficiency Recall that the shared secret key between Alice and Bob is comprised of the one-time pad $\kappa^{\text{otp}} \in_R \{0,1\}^n$ of length n and the extractor seed $s \in_R \{0,1\}^d$ of length d bits, i.e., $\kappa = (\kappa^{\text{otp}}, s)$. Also, the length of the seed from the above discussion is given by

$$d = O\left(\log^2\left(\delta^{-1} \cdot \left(c\log n + 2\log\left(1/\epsilon_{ext}\right) + O\left(1\right)\right) \cdot b\right) \cdot \log\left(1/\epsilon_{ext}\right) \cdot \log t\right) \,.$$

Notice the relationship between the error of the extractor ϵ_{ext} and the desired security from our stegosystem ϵ is given by $\epsilon_{ext} \leq \left(\frac{\epsilon}{3\ell}\right)^3$ from Theorem 2. When we let $\epsilon = 2^{-n/\log^{O(1)} n}$, we can see that the length of the seed d = o(n). Thus we can embed a message of length n bits using a shared secret key of length (1 + o(1))n bits while achieving security $2^{-n/\log^{O(1)} n}$. Suppose, we were to let

the extractor seed of length d be public, observe now that we can attain $\epsilon = 2^{-n}$ security in the length of the shared private key of length n. The seed length can now be given by $d = O(n \log n \log^2(\delta^{-1}bn))$. For small ϵ , the relationship between the seed length d and security ϵ can be given by $d = O\left(\log^3\left(\log\left(\epsilon^{-3}\right)\right)\log\left(\epsilon^{-3}\right)\right)$. We would like to note that our protocol offers a non-trivial improvement over the protocol offered by Kiayias et al. [5] as in their protocol, they need O(n) secret bits regardless of the security achieved.

Also, when we elect to make use of the public randomness for the d bits for the extractor seed, we obtain constant overhead as well. In particular, the length of the shared secret key is equal to the length of the message, n bits while attaining 2^{-n} security.

In this context of making the seed of the extractor public, we would like to explain our model and clarify the implications of making the seed public. In our model for steganography, we assume that the communication channel is not adversarially controlled. In particular, the adversary is not allowed to reconfigure the channel distributions once the seed has been made public. In this sense, the channel is chosen and fixed first, then a seed s is chosen uniformly at random and made public. In other words, we require that the randomness in the seed s is independent of the channel. Indeed, in a stronger model where the adversary does have the ability to readapt the channel distributions, we would need to keep the seed private. From our above discussion, we can see that our stegosytem of Section 3.1 is still (ϵ, δ) -correct and (ϵ, δ) -secure when the seed s is public.

Theorem 5. For any $\epsilon, \delta > 0$, message $m \in \{0,1\}^n$ consider the stegosystem (SK, SE, SD) of Section 3.1 under the parameter constraints $\epsilon_{ext} \leq \left(\frac{\epsilon}{6\ell^2}\right)^3$ and $\rho \geq 2n^c \log(3\ell\epsilon^{-1})$ for some constant c > 1. Then for every channel, if the key $\kappa^{otp} \in_R \{0,1\}^n$ is private and the seed $s \in_R \{0,1\}^n$ is public, then it holds that the stegosystem is (ϵ,δ) -correct and (ϵ,δ) -secure. Here, ϵ_{ext} is the extractor error and $\ell = \lceil n/c \log n \rceil$ for some constant c > 1. The stegosystem exhibits O(1) overhead, the length of the shared private key is equal to the length of the message.

4 A provably secure stegosystem for longer messages

In Appendix B we show how to apply the "one-time" stegosystem of Section 3.1 together with a pseudorandom generator so that longer messages can be transmitted as shown by Kiayias et al. [5].

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A Security Proof

In this section, we provide the proof for Theorem 2 from Section 3.2.

Theorem 6. For any $\epsilon, \delta > 0$, message $m \in \{0,1\}^n$ consider the stegosystem (SK, SE, SD) of Section 3.1 under the parameter constraint $\epsilon_{ext} \leq \left(\frac{\epsilon}{3\ell}\right)^3$. Then it holds that the stegosystem is (ϵ, δ) -secure where ϵ_{ext} is the extractor error and $\ell = \lceil n/c \log n \rceil$ for some constant c > 1.

Proof. We start the encoding procedure SE with history h which embeds message blocks into the channel using rejection sampling. We want to show that the statistical distance between the output of SE and C_h^{λ} is given by

$$\Delta\left[SE(\kappa, \boldsymbol{m}; 0), C_h^{\lambda}\right] \leq \epsilon$$

where λ is the length of the output by procedure SE.

First, we define some notation to capture the operation of the procedure SE. Let C_1 denote the distribution at depth 1 that results by sampling $c_1 \leftarrow C_h^{\pi}$; C_2 denotes the distribution at depth 2 that results by sampling $c_1 \leftarrow C_h^{\pi}$ and $c_2 \leftarrow C_{h \circ c_1}^{\pi}$. We likewise define C_{τ} for $\tau \leq \ell$. We define the random variables R_1, \dots, R_{τ} obtained by rejection sampling in the same fashion. To be precise, for a message $m' = \kappa^{\text{otp}} \oplus m = m_1' \circ m_2' \circ \dots \circ m_{\ell}'$ and $|m_{\tau}'| = c \log n$ we define

$$C_1 \triangleq C_h^{\pi}, \quad C_{\tau} \triangleq C_{h \circ C_1 \circ \cdots \circ C_{\tau-1}}^{\pi},$$

for $\tau \in \{2, \ldots, \ell\}$. Likewise, we define the random variables R_{τ} :

$$R_1 \triangleq \operatorname{REJ}_h^{E_s(\cdot)}(\boldsymbol{m_1}', \rho) , \quad R_{\tau} \triangleq \operatorname{REJ}_{h \circ R_1 \circ \cdots \circ R_{\tau-1}}^{E_s(\cdot)}(\boldsymbol{m_{\tau}}', \rho) .$$

Finally, in anticipation of the proof below, we define a "hybrid" random variable

$$H_{\tau} = \operatorname{Rej}_{h \circ C_1 \circ \cdots \circ C_{\tau-1}}^{E_s(\cdot)} (\boldsymbol{m_{\tau}}', \rho)$$

which corresponds to the distribution obtained by selecting $C_1, \ldots, C_{\tau-1}$ from the natural channel distribution, and then selecting the τ th channel element via rejection sampling.

Now, let us analyze the implications of picking a uniformly random seed $s \in_R \{0,1\}^d$ for the extractor as we do in our construction. Recall that ϵ_{ext} denotes the error of the extractor. First, we show that for each depth τ , the probability mass of distributions for which the extractor coupled with the seed s yields a $\sqrt[3]{\epsilon_{ext}}$ -biased function is large.

We say that a channel distribution C is $(s, \sqrt[3]{\epsilon_{ext}})$ -good if E_s is $\sqrt[3]{\epsilon_{ext}}$ -biased on C. Otherwise we say that the distribution C is $(s, \sqrt[3]{\epsilon_{ext}})$ -bad. With this definition in place, recall that a strong extractor has the property that for any distribution C on the right domain with sufficient min-entropy,

$$\Pr_{s}[C \text{ is } (s, \sqrt[3]{\epsilon_{ext}}) \text{ -bad}] \le \epsilon_{ext}^{2/3}.$$
 (1)

Define now the following sets for $\tau \in \{0, \dots, \ell - 1\}$:

$$G_s^{\tau} = \{ (c_1, c_2, \cdots, c_{\tau}) \mid C_{h \circ c_1 \circ c_2 \circ \cdots \circ c_{\tau}}^{\pi} \text{ is } (s, \sqrt[3]{\epsilon_{ext}}) \text{-good} \}$$

and

$$B_s^{\tau} = \left\{ (c_1, c_2, \cdots, c_{\tau}) \mid C_{h \circ c_1 \circ c_2 \circ \cdots \circ c_{\tau}}^{\pi} \text{ is } (s, \sqrt[3]{\epsilon_{ext}}) \text{-bad} \right\},$$

where $|c_i| = \pi$. The two sets G_s^{τ} and B_s^{τ} denote the collection of $\left(s, \sqrt[3]{\epsilon_{ext}}\right)$ -good and $\left(s, \sqrt[3]{\epsilon_{ext}}\right)$ -bad distributions at depth τ , respectively. Let $\mu\left(B_s^{\tau}\right)$ denote $\Pr\left[C_h^{\tau\pi} \in B_s^{\tau}\right]$, the total probability mass of the set B_s^{τ} . Define $\mu\left(G_s^{\tau}\right)$ similarly. Observe that in light of Equation 1 above, the expected mass of B_s^{τ} over the choice of a uniform seed s is $\mathbb{E}_s\left[\mu\left(B_s^{\tau}\right)\right] \leq \epsilon_{ext}^{2/3}$. By Markov's inequality $\Pr_s\left[\mu\left(B_s^{\tau}\right) \geq \sqrt[3]{\epsilon_{ext}}\right] \leq \sqrt[3]{\epsilon_{ext}}$ and, then, by the union bound we conclude

$$\Pr_{s} \left[\exists \tau < \ell \mid \mu \left(B_{s}^{\tau} \right) \ge \sqrt[3]{\epsilon_{ext}} \right] \le \ell \sqrt[3]{\epsilon_{ext}} \ .$$

where $\ell = \lceil n/c \log n \rceil$, the number of message blocks. We say that a seed s is good if $\forall \tau \in \{1, 2, \dots, \ell\}, \ \mu(G_s^{\tau}) \geq 1 - \sqrt[3]{\epsilon_{ext}}$. To summarize the discussion above, for randomly chosen s,

$$\Pr_{s}[s \text{ is good}] \geq 1 - \ell \sqrt[3]{\epsilon_{ext}}$$
.

Now, fix a good seed s. We will now prove that for a good seed s,

$$\Delta\left[\left(C_{1}, C_{2}, \cdots, C_{\ell}\right), \left(R_{1}, R_{2}, \cdots, R_{\ell}\right)\right] \leq \ell \cdot \left(3\sqrt[3]{\epsilon_{ext}}\right) . \tag{2}$$

We prove this by induction on τ , the number of message blocks. When $\tau = 1$,

$$\Delta\left[C_1, R_1\right] \le 2\sqrt{\epsilon_{ext}} \le 2\sqrt[3]{\epsilon_{ext}}$$
,

as desired. In general, assuming

$$\Delta\left[\left(C_{1}, C_{2}, \cdots, C_{\tau}\right), \left(R_{1}, R_{2}, \cdots, R_{\tau}\right)\right] \leq \tau \cdot \left(2\sqrt[3]{\epsilon_{ext}}\right)$$

for a particular value τ , we wish to establish the inequality for $\tau+1$. Observe that

$$\Delta[(C_{1}, C_{2}, \cdots, C_{\tau+1}), (R_{1}, R_{2}, \cdots, R_{\tau+1})] \\
\leq \Delta[(C_{1}, C_{2}, \cdots, C_{\tau}), (R_{1}, R_{2}, \cdots, R_{\tau})] + \underset{C_{1}, \dots, C_{\tau}}{\mathbb{E}} \left[\Delta[C_{\tau+1}, H_{\tau+1}]\right] \text{(Lemma 1)} \\
\leq \tau \cdot (2\sqrt[3]{\epsilon_{ext}}) + \underset{C_{1}, \dots, C_{\tau}}{\mathbb{E}} \left[\Delta[C_{\tau+1}, H_{\tau+1}]\right] \text{ (by induction.)}$$

As for the expectation $\mathbb{E}_{C_1,\ldots,C_{\tau}}\left[\Delta\left[C_{\tau+1},H_{\tau+1}\right]\right]$, observe that

$$\begin{split} \mathbb{E}_{C_{1},...,C_{\tau}} \left[\Delta \left[C_{\tau+1}, H_{\tau+1} \right] \right] \\ &\leq \Pr[(C_{1},...,C_{\tau}) \in G_{s}^{\tau}] \cdot \mathbb{E} \left[\Delta \left[C_{\tau+1}, H_{\tau+1} \right] \mid (C_{1},...,C_{\tau}) \in G_{s}^{\tau} \right] \\ &+ \Pr[(C_{1},...,C_{\tau}) \in B_{s}^{\tau}] \cdot \mathbb{E} \left[\Delta \left[C_{\tau+1}, H_{\tau+1} \right] \mid (C_{1},...,C_{\tau}) \in G_{s}^{\tau} \right] \\ &\leq \mathbb{E} \left[\Delta \left[C_{\tau+1}, H_{\tau+1} \right] \mid (C_{1},...,C_{\tau}) \in G_{s}^{\tau} \right] + \Pr[(C_{1},...,C_{\tau}) \in B_{s}^{\tau}] \\ &< \sqrt[3]{\epsilon_{ext}} + \sqrt[3]{\epsilon_{ext}}, \end{split}$$

as s is good. We can conclude that for a good seed s,

$$\Delta[(C_1, C_2, \cdots, C_{\tau}), (R_1, R_2, \cdots, R_{\tau})] \leq \tau \cdot (2\sqrt[3]{\epsilon_{ext}}),$$

for any $\tau \leq \ell$. The total statistical distance is now given by

$$\begin{split} &\Delta\left[\left(C_{1},C_{2},\cdots,C_{\ell}\right),\left(R_{1},R_{2},\cdots,R_{\ell}\right)\right]\\ &=\Delta\left[\left(C_{1},C_{2},\cdots,C_{\ell}\right),\left(R_{1},R_{2},\cdots,R_{\ell}\right)\right]|_{s\ good}\cdot\Pr[s\ good]+\\ &\Delta\left[\left(C_{1},C_{2},\cdots,C_{\ell}\right),\left(R_{1},R_{2},\cdots,R_{\ell}\right)\right]|_{s\ not\ good}\cdot\Pr[s\ not\ good]\\ &\leq\ell\cdot\left(2\sqrt[3]{\epsilon_{ext}}\right)\cdot1+1\cdot\left(\ell\sqrt[3]{\epsilon_{ext}}\right)\leq3\ell\sqrt[3]{\epsilon_{ext}}\leq\epsilon\,. \end{split}$$

The last inequality is because of the fact that $\epsilon_{ext} \leq \left(\frac{\epsilon}{3\ell}\right)^3$. Thus,

$$\Delta\left[SE(\kappa, \boldsymbol{m}; \boldsymbol{0}), C_h^{\lambda}\right] \leq \epsilon$$

and the theorem follows by the definition of insecurity.

B A provably secure stegosystem for longer messages

In this section we show how to apply the "one-time" stegosystem of Section 3.1 together with a pseudorandom generator so that longer messages can be transmitted as shown by Kiayias et al. [5].

Definition 10. Let U_k denote the uniform distribution over $\{0,1\}^k$. A polynomial time deterministic algorithm G is a pseudorandom generator (PRG) if the following conditions are satisfied:

Variable output For all seeds $x \in \{0,1\}^*$ and $y \in \mathbb{N}$, $|G(x,1^y)| = y$. Pseudorandomness For every polynomial p the set of random variables $\{G(U_k, 1^{p(k)})\}_{k \in \mathbb{N}}$ is computationally indistinguishable from the uniform distribution $\{U_{p(k)}\}_{k \in \mathbb{N}}$.

For a PRG G and 0 < k < k', if A is some statistical test, we define the advantage of A over the PRG as follows:

$$\mathbf{Adv}_{G}^{A}(k,k') = \left| \Pr_{w \leftarrow G(U_{k},1^{k'})} [A(w) = 1] - \Pr_{w \leftarrow U_{k'}} [A(w) = 1] \right|.$$

The insecurity of the above PRG G against all statistical tests A computable by circuits of size $\leq P$ is then defined as

$$\mathbf{InSec}_G(k, k'; P) = \max_{A \in A_P} \{ \mathbf{Adv}_G^A(k, k') \}$$

where \mathcal{A}_P is the collection of statistical tests computable by circuits of size $\leq P$. It is convenient for our application that typical PRGs have a procedure G' such that if $z = G(x, 1^y)$, it holds that $G(x, 1^{y+y'}) = G'(x, z, 1^{y'})$ (i.e., if one maintains z, one can extract the y' bits that follow the first y bits without starting from the beginning).

Consider now the following stegosystem S' = (SK', SE', SD') that can be used for steganographic transmission of longer messages using the one-time stegosystem S = (SK, SE, SD) as defined in Section 3.1. S' can handle messages of length polynomial in the security parameter k and employs a PRG G. The two players Alice and Bob, share a key of length k denoted by x. The function SE' is given input x and the message $m \in \{0,1\}^{\nu}$ to be transmitted of length $\nu = p(k)$ for some fixed polynomial p. SE' in turn employs the PRG G to extract k' bits (it computes $\kappa = G(x, 1^{k'})$, $|\kappa| = k'$). The length k' is selected to match the number of key bits that are required to transmit the message m using the one-time stegosystem of Section 3.1. Once the key κ of length k' is produced by the PRG, the procedure SE' invokes the one-time stegosystem on input κ , m, h. The function SD' is defined in a straightforward way based on SD

The computational insecurity of the stegosystem S' is defined by adapting the definition of information theoretic stegosystem security from Section 2.3 for the computationally bounded adversary as follows:

$$\mathbf{InSec}_{S'}(k, k'; P) = \max_{\mathcal{A} \in \mathcal{A}_P} \{ \mathbf{Adv}_{S'}^{\mathcal{A}}(k, k') \},$$

this maximum taken over all adversaries \mathcal{A} , where SA_1 and SA_2 have circuit size $\leq P$ and the definition of advantage $\mathbf{Adv}_{S'}^{\mathcal{A}}(k,k')$ is obtained by suitably modifying the definition of $\mathbf{Adv}_{S}^{\mathcal{A}}(k)$ in Section 2.3. In particular, we define a new adversarial game $G^{\mathcal{A}}(1^k,1^{k'})$ which proceeds as the previous game $G^{\mathcal{A}}(1^k)$ in Section 2.3 except that in this new game $G^{\mathcal{A}}(1^k,1^{k'})$, algorithms SA_1 and SA_2 receive as input the security parameter k' and SE' invokes SE as $SE(\kappa,m^*;\mathcal{O})$ where $\kappa = G(x,1^{k'})$.

Theorem 7. The stegosystem S' = (SK', SE', SD') is provably secure in the model of [2] (steganographically secret against chosen hiddentext attacks); in particular employing a PRG G to transmit a message m we get $\mathbf{InSec}_{S'}(k, k'; P) \leq \mathbf{InSec}_{S'}(k, k'; P) + \mathbf{InSec}_{S'}(k')$ where $\mathbf{InSec}_{S'}(k')$ is the information theoretic insecurity defined in Section 2.3 and $|m| = \ell(k')$.

C Omitted proofs

Proof (Lemma 1). For $x \in X$ denote $\Pr[X = x]$ by P_x and $\Pr[Y_x = y]$ by $P_{y|x}$. Define P'_x and $P'_{y|x}$ similarly. Then we may compute

$$\Delta [(X,Y),(X',Y')] = \frac{1}{2} \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \left| P_x \cdot P_{y|x} - P'_x \cdot P'_{y|x} \right|$$

$$\leq \frac{1}{2} \sum_{x,y} \left| P_x \cdot P_{y|x} - P_x \cdot P'_{y|x} \right| + \frac{1}{2} \sum_{x,y} \left| P_x \cdot P'_{y|x} - P'_x \cdot P'_{y|x} \right|$$

$$= \frac{1}{2} \sum_{x,y} P_x \cdot \left| P_{y|x} - P'_{y|x} \right| + \frac{1}{2} \sum_{x,y} P'_{y|x} \cdot |P_x - P'_x|$$

$$= \mathbb{E} [\Delta [Y_X, Y'_X]] + \Delta [X, X'].$$