Lemma 1: Let $f(x)$ be a log-concave, symmetric, and twice differentiable pdf. Denote by $F(x)$ the corresponding CDF. Then for any $x_1 \geq \ldots \geq x_n \in \mathbb{R}$, we have

$$\frac{\left(\sum_{k=1}^{n}(-1)^{k+1}f(x_k)\right)^2}{\left(\sum_{k=1}^{n}(-1)^{k+1}F(x_k)\right)\left(1-\sum_{k=1}^{n}(-1)^{k+1}F(x_k)\right)} \leq \frac{2}{\pi}. \qquad (1)$$

Proof of Lem. 1: We use induction on $n \in \mathbb{N}$. For the base case $n=1$ we have

$$\eta(x) \triangleq \frac{f^2(x)}{F(x)(1-F(x))}. \qquad (2)$$

Since $\eta(x)$ is symmetric about the origin, it is enough to show that $\eta'(x) \leq$ for all $x>0$, and hence $x=0$ is a global maximum of $\eta(x)$. Since $F(x)$ is log-concave and twice differentiable, we may write $F(x)=e^{h(x)}$ with $h(x)$ concave and twice differentiable. Moreover, note that $h'(x)\geq 0$ because $e^{h(x)}$ is non-decreasing, $h''(x)\leq 0$ because $h(x)$ is concave, and $e^{h(-x)} = 1-e^{h(x)}$ because $f(x)$ is symmetric around 0. Moreover, since $f(x)$ is unimodal, for $x>0$ we have

$$0 \geq f'(x) = e^{h(x)}\left(h''(x)+(h'(x))^2\right),$$

and hence

$$0 \geq h''(x)+(h'(x))^2, \quad x\geq 0.$$

Similarly,

$$0 \leq h''(-x)+(h'(-x))^2, \quad x\geq 0.$$

Using these facts, we conclude that

$$\eta(x) = \frac{h'(x)h'(-x)e^{h(x)+h(-x)}}{e^{h(x)+h(-x)}} = h'(x)h'(-x),$$

thus

$$\eta'(x) = h''(x)h'(-x) - h''(-x)h'(x).$$

For $x>0$ we obtain

$$\begin{aligned}\eta'(x) &\leq -\left(h'(x)\right)^2 h'(-x) + \left(h'(-x)\right)^2 h'(x)\\ &= h'(x)h'(-x)\left(h'(-x)-h'(x)\right) \leq 0.\end{aligned}$$

Assume now that (1) holds for all integers up to some $n=N-1$ and consider the case $n=N$. The maximal value of (1) is attained for the same $(x_1,\ldots,x_N)\in\mathbb{R}^N$ that attains the maximal value of

$$\begin{aligned}&g(x_1,\ldots,x_N) \triangleq 2\log\left(\sum_{k=1}^{N}(-1)^{k+1}f(x_k)\right) -\\ &\log\left(\sum_{k=1}^{N}(-1)^{k+1}F(x_k)\right) - \log\left(1-\sum_{k=1}^{N}(-1)^{k+1}F(x_k)\right)\\ &= 2\log\delta_N - \log\Delta_N - \log(1-\Delta_N),\end{aligned}$$

where we denoted $\delta_N \triangleq \sum_{k=1}^{N}(-1)^{k+1}f(x_k)$ and $\Delta_N = \sum_{k=1}^{N}(-1)^{k+1}F(x_k)$. The derivative of $g(x_1,\ldots,x_N)$ with respect to x_k is given by

$$\frac{\partial g}{\partial x_k} = \frac{2(-1)^{k+1}f'(x_k)}{\delta_N} - \frac{(-1)^{k+1}f(x_k)}{\Delta_N} + \frac{(-1)^{k+1}\phi(x_k)}{1-\Delta_N}.$$

We conclude that the gradient of g vanishes only if

$$\frac{f'(x_k)}{f(x_k)} = g'(x_k) = \frac{\delta_N}{2}\left(\frac{1}{\Delta_N}-\frac{1}{1-\Delta_N}\right), \quad k=1,\ldots,N.$$

Due to our assumption that $g''(x)<0$, the function $g'(x)$ is strictly decreasing. As a result, $g'(x)$ is one to one and the condition above implies $x_1=\ldots=x_N$. If N is odd then for $x_1=\ldots=x_N$ we have that the LHS of (1) equals

$$\frac{f(x_1)^2}{F(x_1)(1-F(x_1))},$$

which was shown to be smaller than $4f^2(0)$. If N is even, then for any constant c the limit of the LHS of (1) as $(x_1,\ldots,x_N)\to(c,\ldots,c)$ exists and equals zero. Therefore, the maximum of the LHS of (1) is not attained at the line $x_1=\ldots=x_N$). We now consider the possibility that the LHS of (1) is maximized at the borders, as one or more of the coordinates of $(x_1,\ldots,x_N)$ approaches plus or minus infinity. For simplicity we only consider the cases where x_N goes to minus infinity or x_1 goes to plus infinity (the general case where the first m coordinates goes to infinity or the last m to minus infinity is obtained using similar arguments). Assume first $x_N\to-\infty$. Then the LHS of (1) equals

$$\frac{\left(\sum_{k=1}^{N-1}(-1)^{k+1}f(x_k)\right)^2}{\left(\sum_{k=1}^{N-1}(-1)^{k+1}F(x_k)\right)\left(1-\sum_{k=1}^{N-1}(-1)^{k+1}F(x_k)\right)},$$

which is smaller than $2/\pi$ by the induction hypothesis. Assume now that $x_1\to\infty$. Then the LHS of (1) equals

$$\begin{aligned}&\frac{\left(\sum_{k=2}^{N}(-1)^{k+1}f(x_k)\right)^2}{\left(1+\sum_{k=2}^{N}(-1)^{k+1}F(x_k)\right)\left(1-1-\sum_{k=2}^{N}(-1)^{k+1}F(x_k)\right)}\\ &= \frac{\left(-\sum_{m=1}^{N}(-1)^{m+1}f(x'_m)\right)^2}{\left(1-\sum_{m=1}^{N-1}(-1)^{m+1}F(x'_m)\right)\left(\sum_{m=1}^{N-1}(-1)^{m+1}F(x'_m)\right)},\end{aligned}$$

where $x'_m = x_{m+1}$. The last expression is also smaller than $4f(0)$ by the induction hypothesis. This proves Lem. 1.

Proof of Lem. **??***:* The Fisher information of M with respect to θ is given by

$$\begin{aligned}I_\theta &= \mathbb{E}\left[\left(\frac{d}{d\theta}\log P(M|\theta)\right)^2|\theta\right]\\ &= \frac{\left(\frac{d}{d\theta}P(M=1|\theta)\right)^2}{P(M=1|\theta)} + \frac{\left(\frac{d}{d\theta}P(M=-1|\theta)\right)^2}{P(M=-1|\theta)}\\ &= \frac{\left(\frac{d}{d\theta}\int_A\phi\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{P(M=1|\theta)} + \frac{\left(\frac{d}{d\theta}\int_A\phi\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{P(M=-1|\theta)}\\ &\overset{(a)}{=} \frac{\left(-\int_A\phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{\sigma^2 P(M=1|\theta)} + \frac{\left(-\int_A\phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{\sigma^2 P(M=-1|\theta)}\\ &= \frac{\left(\int_A\phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{\sigma^2 P(M=1|\theta)\left(1-P(M=1|\theta)\right)},\\ &= \frac{\left(\int_A\phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)\left(\int_A\phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)}{\sigma^2\left(\int_A\phi\left(\frac{x-\theta}{\sigma}\right)dx\right)\left(1-\int_A\phi\left(\frac{x-\theta}{\sigma}\right)dx\right)},\end{aligned} \qquad (3)$$

where differentiation under the integral sign in (a) is possible since $\phi(x)$ is differentiable with absolutely integrable derivative $\phi'(x) = -x\phi(x)$. Regularity of the Lebesgue measure implies that for any $\varepsilon > 0$, there exists a finite number k of disjoint open intervals $I_1,\dots I_k$ such that

$$\int_{A\setminus\cup_{j=1}^{k} I_j} dx < \varepsilon\sigma^2,$$

which implies that for any $\varepsilon' > 0$, the set A in (3) can be replaced by a finite union of disjoint intervals without increasing I_θ by more than ε'. It is therefore enough to proceed in the proof assuming that A is of the form

$$A = \cup_{j=1}^{k}(a_j, b_j),$$

with $\infty \le a_1 \le \dots a_k$, $b_1 \le b_k \le \infty$ and $a_j \le b_j$ for $j = 1,\dots,k$. Under this assumption we have

$$\begin{aligned}\mathbb{P}(M_n = 1|\theta) &= \sum_{j=1}^{k} \mathbb{P}(X_n \in (a_j, b_j)) \\ &= \sum_{j=1}^{k}\left(\Phi\left(\frac{b_j-\theta}{\sigma}\right) - \Phi\left(\frac{a_j-\theta}{\sigma}\right)\right),\end{aligned}$$

so (3) can be rewritten as

$$\begin{aligned}&= \frac{\left(\sum_{j=1}^{k}\phi\left(\frac{a_j-\theta}{\sigma}\right) - \phi\left(\frac{b_j-\theta}{\sigma}\right)\right)^2}{\sigma^2\left(\sum_{j=1}^{k}\Phi\left(\frac{b_j-\theta}{\sigma}\right) - \Phi\left(\frac{a_j-\theta}{\sigma}\right)\right)} \\ &\times \frac{1}{1-\left(\sum_{j=1}^{k}\Phi\left(\frac{b_j-\theta}{\sigma}\right) - \Phi\left(\frac{a_j-\theta}{\sigma}\right)\right)} \qquad (4)\end{aligned}$$

It follows from Lem. 1 that for any $\theta \in \mathbb{R}$ and any choice of the intervals endpoints, (4) is smaller than $2/(\sigma^2\pi)$. Therefore, the proof of Lem. **??** is now completed.

REFERENCES