Lemma 1: Let $f(x)$ be a log-concave, symmetric, and twice differentiable pdf. Denote by $F(x)$ the corresponding CDF. Then for any $x_1 \geq \ldots \geq x_n \in \mathbb{R}$, we have

$$\frac{\left(\sum_{k=1}^{n}(-1)^{k+1}f(x_k)\right)^2}{\left(\sum_{k=1}^{n}(-1)^{k+1}F(x_k)\right)\left(1-\sum_{k=1}^{n}(-1)^{k+1}F(x_k)\right)} \leq \frac{2}{\pi}. \quad (1)$$

Proof of Lem. 1: We use induction on $n \in \mathbb{N}$. For the base case $n=1$ we have

$$\eta(x) \triangleq \frac{f^2(x)}{F(x)\left(1-F(x)\right)}. \quad (2)$$

Since $f(x)$ is log-concave, positive, and twice differentiable, we may write $f(x) = e^{g(x)}$ with $g(x)$ concave and twice differentiable. Moreover, since symmetry of $f(x)$ implies symmetry of $g(x)$, and, since the latter is concave, it attains a single maximum at the symmetry point $x=0$. Therefore, $x=0$ is the only point for which $g'(x)=0$. We now show that $x=0$ is the point where $\eta(x)$ attains its maximum value. Since $\eta(x)$ is positive and goes to zero as $x \to \pm\infty$, the maximum of $\eta(x)$ cannot be attained as $x \to \pm\infty$. We therefore consider the derivative of $\eta(x)$, given by

$$\eta'(x) = 2\eta(x)e^{g(x)}\left(\frac{F(x)-1/2}{F(x)(1-F(x))}+g'(x)\right).$$

Since $f(x)$ is symmetric and integrates to 1, we have that $F(0)=0.5$ and hence $\eta'(0)=0$. It is left to verify that $\eta'(x) \neq 0$ for $x \neq 0$.

Taking the logarithm of (2) leads to

$$\begin{aligned} &2\log f(x) - \log F(x) - \log F(-x) \\ &= 2\log h'(x) + h(x) - h(-x). \end{aligned}$$

Note that since $F(x)$ is non-decreasing, so is $h(x)$. In addition, $h'(x)$ is non-increasing since $h(x)$, the maximal value of (2) is obtained as HERE!! finish proof by showing that (2) is smaller than $4f(0)$.

Assume now that (1) holds for all integers up to some $n = N-1$ and consider the case $n=N$. The maximal value of (1) is attained for the same $(x_1,\ldots,x_N) \in \mathbb{R}^N$ that attains the maximal value of

$$\begin{aligned} &g(x_1,\ldots,x_N) \triangleq 2\log\left(\sum_{k=1}^{N}(-1)^{k+1}f(x_k)\right) - \\ &\log\left(\sum_{k=1}^{N}(-1)^{k+1}F(x_k)\right) - \log\left(1-\sum_{k=1}^{N}(-1)^{k+1}F(x_k)\right) \\ &= 2\log\delta_N - \log\Delta_N - \log\left(1-\Delta_N\right), \end{aligned}$$

where we denoted $\delta_N \triangleq \sum_{k=1}^{N}(-1)^{k+1}f(x_k)$ and $\Delta_N = \sum_{k=1}^{N}(-1)^{k+1}F(x_k)$. The derivative of $g(x_1,\ldots,x_N)$ with respect to x_k is given by

$$\frac{\partial g}{\partial x_k} = \frac{2(-1)^{k+1}f'(x_k)}{\delta_N} - \frac{(-1)^{k+1}f(x_k)}{\Delta_N} + \frac{(-1)^{k+1}\phi(x_k)}{1-\Delta_N}.$$

We conclude that the gradient of g vanishes only if

$$\frac{f'(x_k)}{f(x_k)} = g'(x_k) = \frac{\delta_N}{2}\left(\frac{1}{\Delta_N}-\frac{1}{1-\Delta_N}\right), \quad k=1,\ldots,N.$$

Due to our assumption that $g''(x)<0$, the function $g'(x)$ is strictly decreasing. As a result, $g'(x)$ is one to one and the condition above implies $x_1 = \ldots = x_N$. If N is odd then for $x_1 = \ldots = x_N$ we have that the LHS of (1) equals

$$\frac{f(x_1)^2}{F(x_1)(1-F(x_1))},$$

which was shown to be smaller than $4f^2(0)$. If N is even, then for any constant c the limit of the LHS of (1) as $(x_1,\ldots,x_N) \to (c,\ldots,c)$ exists and equals zero. Therefore, the maximum of the LHS of (1) is not attained at the line $x_1 = \ldots = x_N$). We now consider the possibility that the LHS of (1) is maximized at the borders, as one or more of the coordinates of $(x_1,\ldots,x_N)$ approaches plus or minus infinity. For simplicity we only consider the cases where x_N goes to minus infinity or x_1 goes to plus infinity (the general case where the first m coordinates goes to infinity or the last m to minus infinity is obtained using similar arguments). Assume first $x_N \to -\infty$. Then the LHS of (1) equals

$$\frac{\left(\sum_{k=1}^{N-1}(-1)^{k+1}f(x_k)\right)^2}{\left(\sum_{k=1}^{N-1}(-1)^{k+1}F(x_k)\right)\left(1-\sum_{k=1}^{N-1}(-1)^{k+1}F(x_k)\right)},$$

which is smaller than $2/\pi$ by the induction hypothesis. Assume now that $x_1 \to \infty$. Then the LHS of (1) equals

$$\begin{aligned} &\frac{\left(\sum_{k=2}^{N}(-1)^{k+1}f(x_k)\right)^2}{\left(1+\sum_{k=2}^{N}(-1)^{k+1}F(x_k)\right)\left(1-1-\sum_{k=2}^{N}(-1)^{k+1}F(x_k)\right)} \\ &= \frac{\left(-\sum_{m=1}^{N}(-1)^{m+1}f(x'_m)\right)^2}{\left(1-\sum_{m=1}^{N-1}(-1)^{m+1}F(x'_m)\right)\left(\sum_{m=1}^{N-1}(-1)^{m+1}F(x'_m)\right)}, \end{aligned}$$

where $x'_m = x_{m+1}$. The last expression is also smaller than $4f(0)$ by the induction hypothesis. This proves Lem. 1.

Proof of Lem. ??: The Fisher information of M with respect to θ is given by

$$\begin{aligned} I_\theta &= \mathbb{E}\left[\left(\frac{d}{d\theta}\log P(M|\theta)\right)^2|\theta\right] \\ &= \frac{\left(\frac{d}{d\theta}P(M=1|\theta)\right)^2}{P(M=1|\theta)} + \frac{\left(\frac{d}{d\theta}P(M=-1|\theta)\right)^2}{P(M=-1|\theta)} \\ &= \frac{\left(\frac{d}{d\theta}\int_A \phi\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{P(M=1|\theta)} + \frac{\left(\frac{d}{d\theta}\int_A \phi\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{P(M=-1|\theta)} \\ &\overset{(a)}{=} \frac{\left(-\int_A \phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{\sigma^2 P(M=1|\theta)} + \frac{\left(-\int_A \phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{\sigma^2 P(M=-1|\theta)} \\ &= \frac{\left(\int_A \phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)^2}{\sigma^2 P(M=1|\theta)\left(1-P(M=1|\theta)\right)}, \\ &= \frac{\left(\int_A \phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)\left(\int_A \phi'\left(\frac{x-\theta}{\sigma}\right)dx\right)}{\sigma^2\left(\int_A \phi\left(\frac{x-\theta}{\sigma}\right)dx\right)\left(1-\int_A \phi\left(\frac{x-\theta}{\sigma}\right)dx\right)}, \end{aligned} \quad (3)$$

where differentiation under the integral sign in (a) is possible since $\phi(x)$ is differentiable with absolutely integrable derivative $\phi'(x) = -x\phi(x)$. Regularity of the Lebesgue measure implies that for any $\varepsilon > 0$, there exists a finite number k of disjoint open intervals $I_1, \dots I_k$ such that

$$\int_{A \setminus \cup_{j=1}^k I_j} dx < \varepsilon \sigma^2,$$

which implies that for any $\varepsilon' > 0$, the set A in (3) can be replaced by a finite union of disjoint intervals without increasing I_θ by more than ε'. It is therefore enough to proceed in the proof assuming that A is of the form

$$A = \cup_{j=1}^k (a_j, b_j),$$

with $\infty \leq a_1 \leq \dots a_k$, $b_1 \leq b_k \leq \infty$ and $a_j \leq b_j$ for $j = 1, \dots, k$. Under this assumption we have

$$\begin{aligned}\mathbb{P}(M_n = 1|\theta) &= \sum_{j=1}^k \mathbb{P}(X_n \in (a_j, b_j)) \\ &= \sum_{j=1}^k \left(\Phi\left(\frac{b_j - \theta}{\sigma}\right) - \Phi\left(\frac{a_j - \theta}{\sigma}\right) \right),\end{aligned}$$

so (3) can be rewritten as

$$\begin{aligned}&= \frac{\left(\sum_{j=1}^k \phi\left(\frac{a_j-\theta}{\sigma}\right) - \phi\left(\frac{b_j-\theta}{\sigma}\right)\right)^2}{\sigma^2\left(\sum_{j=1}^k \Phi\left(\frac{b_j-\theta}{\sigma}\right) - \Phi\left(\frac{a_j-\theta}{\sigma}\right)\right)} \\ &\times \frac{1}{1-\left(\sum_{j=1}^k \Phi\left(\frac{b_j-\theta}{\sigma}\right) - \Phi\left(\frac{a_j-\theta}{\sigma}\right)\right)} \qquad (4)\end{aligned}$$

It follows from Lem. 1 that for any $\theta \in \mathbb{R}$ and any choice of the intervals endpoints, (4) is smaller than $2/(\sigma^2\pi)$. Therefore, the proof of Lem. **??** is now completed.

REFERENCES