We consider the distributed encoding setting in which each single-bit message is independent of the other messages.

Definitions

$$\eta(\theta) \triangleq \frac{f^2(\theta)}{F(\theta)\,(1-F(\theta))}, \tag{1}$$

where f and F are the PDF and CDF of a log-concave distribution, respectively.

Proposition 1: Let $f(x)$ be log-concave, symmetric and differntiable PDF. Then

(i) The function

$$\eta(x) = \frac{f^2(x)}{F(x)\,(1-F(x))}$$

is symmetric and strictly decreasing for any $x > 0$ in the support of $f(x)$. In particular, $\eta(x) \leq \eta(0) = 4f^2(0)$.

(ii) $\eta(x)$ goes to 0 as $|x|$ goes to infinity.

Local Asymptotic Normality of Messages

We first provides conditions under which the messages $M_1, M_2, \ldots$ constitute a local asymptotic normal family.

Theorem 2: Let

$$M_n = \begin{cases} 1 & X_n \in A_n, \\ -1 & X_n \notin A_n, \end{cases} \quad n \in \mathbb{N},$$

where $\{A_n\}_{n=1}^{\infty}$ is a sequence of Borel sets. Assume that as n goes to infinity, for any $\theta \in \Theta$,

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\left(\frac{d}{d\theta}\Pr(X_i \in A_i)\right)^2}{\Pr(X_i \in A_i)\Pr(X_i \notin A_i)} \tag{2}$$

converges to a positive number $K(\theta)$ denoted as the *precision parameter*. Then for any $\theta \in \Theta$ and $h \in \mathbb{R}$,

$$\log \frac{\mathbb{P}_{\theta + h/\sqrt{n}}(M_1, \ldots, M_n)}{\mathbb{P}_{\theta}(M_1, \ldots, M_n)} \xrightarrow{D} \mathcal{N}\left(-\frac{1}{2}h^2 K(\theta), h^2 K(\theta)\right).$$

Non-existence of Uniformly Optimal Strategy

We now show that no distributed estimation scheme can attain normalized risk of $1/\eta(0)$ for all $\theta \in \Theta$.

Theorem 3: Let $M_1, M_2, \ldots$ be a sequence satisfying the condition in Theorem 2. Let $\Xi \subset \Theta$ be the set of points for which there exists an estimator $\hat{\theta}_n = \hat{\theta}(M_1, \ldots, M_n)$ such that

$$\lim_{n\to\infty} \mathbb{E}\left[n\left(\theta - \hat{\theta}_n\right)^2\right] = 1/\eta(0).$$

Then Ξ has Lebesgue measure zero.

Proof: The proof requires the following lemmas:

Lemma 4: Let $\varepsilon > 0$ and $x_1, \ldots, x_m \in \mathbb{R}$. If $\{x_1, \ldots, x_m\} \cap (-\varepsilon, \varepsilon) = \emptyset$, then

$$\frac{\left(\sum_{i=1}^m (-1)^{i+1} f(x_i)\right)^2}{\left(\sum_{i=1}^m (-1)^{i+1} F(x_i)\right)\left(1 - \sum_{i=1}^m (-1)^{i+1} F(x_i)\right)} \leq \eta(\varepsilon).$$

Proof of Lemma 4: By induction on m.

Theorem 2 says that $M_1, M_2, \ldots$ defines a LAN family with precision parameter $K(\theta)$. The local asymptotic minimax property of estimators in LAN models implies that for any estimator $\hat{\theta}_n = \hat{\theta}_n(M_1, \ldots, M_n)$ and $\delta > 0$, there exists n large enough such that

$$\mathbb{E}\left[n(\theta - \hat{\theta}_n)^2\right] + \delta \geq \frac{1}{K(\theta)},$$

where $K(\theta)$ is the precision parameter (2). It follows that for all $\theta \in \Xi$ we have $K(\theta) = 1/\eta(0)$. It follows from Proposition 1 that if $\varepsilon > 0$, then $\eta(\varepsilon) < \eta(0)$. Since the only defining property of the sequence of messages $\{M_i\}_{i=1}^{\infty}$ is the precision parameter $K(\theta)$, we may assume without loss of generality that each A_i is a finite union of M_i intervals

$$A_i = \bigcup_{k=1}^{M_i} (b_{i,k}, a_{i,k}), \tag{3}$$

where $b_{i,k+1} > a_{i,k+1} > b_{i,k} > a_{i,k}$ for every i and k. Indeed, for any sequence of Borel measurable sets $\{A_i\}_{i=1}^{\infty}$ for which (2) converge to $K(\theta)$ for any $\theta \in \Theta$, has a sequence $\{A_i'\}_{i=1}^{\infty}$ converging to the same limit in which A_i' is of the form (3). With this assumption, (2) is given by

$$\frac{1}{n}\sum_{i=1}^{n} \frac{\left(\sum_{k=1}^{M_i} f(b_{i,k}) - f(a_{i,k})\right)^2}{\sum_{k=1}^{M_i}\left(F(b_{i,k}) - F(a_{i,k})\right)\left(1 - \sum_{k=1}^{M_i}\left(F(b_{i,k}) - F(a_{i,k})\right)\right)} \tag{4}$$

Next, Lemma 4 implies that for any θ such that $K(\theta) \to \eta(0)$, any interval of the form $(\theta - \varepsilon, \theta + \varepsilon)$ contains an infinite number of elements from $\{a_{i,k}, b_{i,k}, i \in \mathbb{N}, k = 1, \ldots, M_i\}$ (otherwise, there exists $\rho > 0$ such that

$$\{a_{i,k}, b_{i,k}, i \in \mathbb{N}, k = 1, \ldots, M_i\} \cap (-\rho + \theta, \theta + \rho) = \emptyset,$$

so that Lemma 4 implies $K(\theta) \leq \eta(\rho) < \eta(0)$). We conclude that each point in Ξ is a limit points for the set

$$\left\{b_{i,k}, a_{i,k}, k = 1, \ldots, M_i, i = 1, \ldots, \mathbb{N}\right\} = \bigcup_{i \in \mathbb{N}} \bigcup_{k=1}^{M_i} \{a_{i,k}, b_{i,k}\}. \tag{5}$$

However, (5) is a countable union of countable sets, and hence has no interior point (by Baire's Category Theorem). In particular, it Lebesgue measure is zero. $\square$