Mean Estimation from One-bit Measurements

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Motivation

Point estimation under communication constraints:

$$p_{\theta}(x) \to X_1, \ldots, X_n \longrightarrow \boxed{\begin{array}{l}\text{limited bits}\\ \text{per sample}\end{array}} \longrightarrow \widehat{\theta}$$

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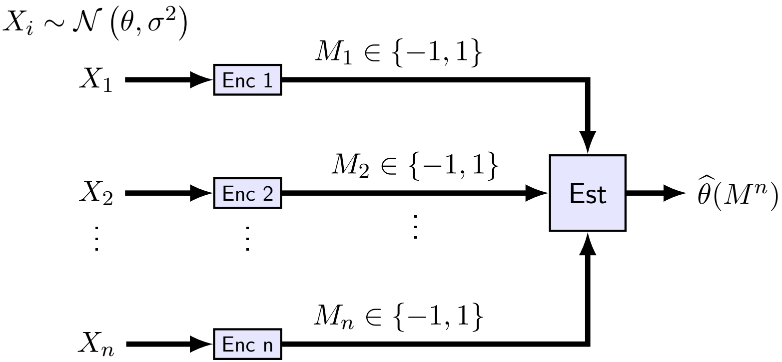
Relevant scenarios:

- big data
- low-power sensors
- distributed computing / optimization

This talk:

Estimating the mean θ of a normal distribution $\mathcal{N}(\theta, \sigma^2)$ from one-bit per sample (σ is known)

Three Encoding Scenarios



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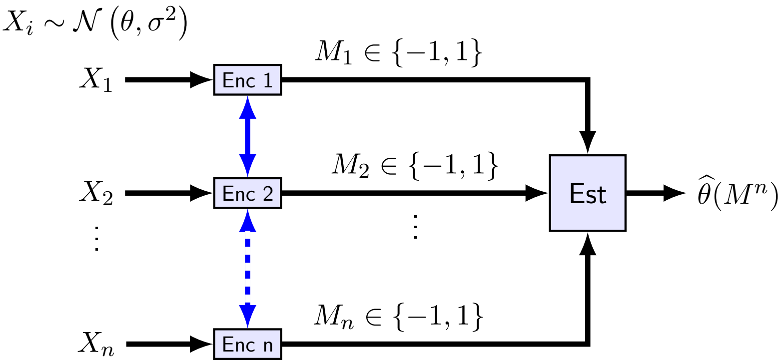

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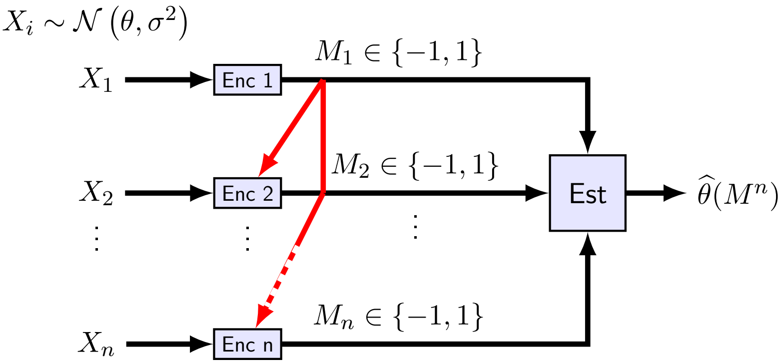

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- Distributed hypothesis testing under quantization [Tsitsiklis '88] (distributed)
- Remote multiterminal source coding (CEO) [Berger, Zhang, Wiswanathan '96], [Oohama '97] (distributed)

Consistency

Q: in what setting consistent estimation is possible?

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A: all !

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A: all ! **Proof:**

$$M_i = \mathbf{1}(X_i > 0), \quad i = 1, \ldots, n$$

(as in the distributed setting)

$$\frac{1}{n}\sum_{i=1}^{n} M_i \to \mathbb{P}\left(X > 0\right) = \Phi\left(\theta/\sigma\right)$$

Efficiency

Definition: *asymptotic relative efficiency (ARE)* of an estimator:

$$\mathrm{ARE}(\widehat{\theta}) \triangleq \lim_{n\to\infty} \mathbb{E}\left[\left(\widehat{\theta} - \theta\right)^2\right] / \left(\sigma^2/n\right)$$

(relative to minimax risk without bit constraint)

ARE under Centralized Encoding

Proposition

If the parameter space Θ *is bounded, then the ARE under centralized encoding is* 1

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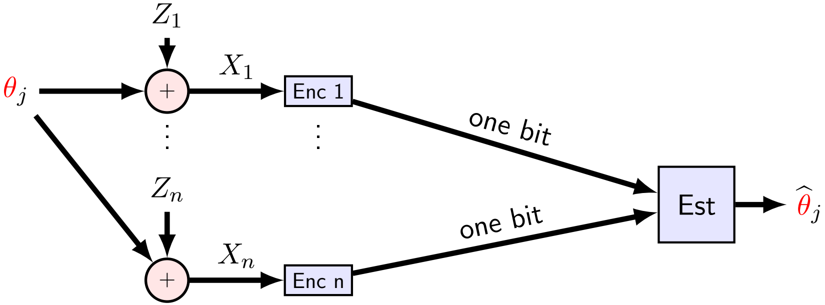
Note: mean of $\bar{\theta}$ is unknown – hard to derive a globally optimal strategy

Relation to CEO



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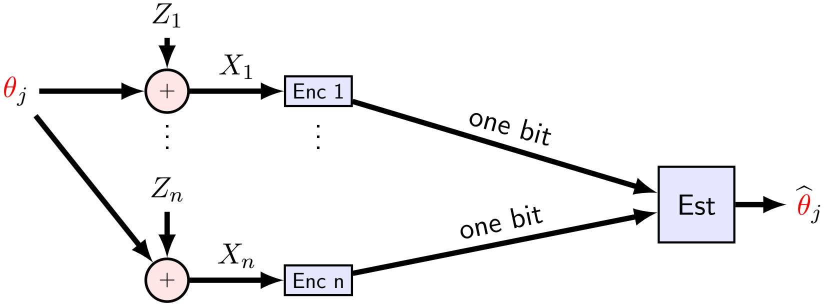

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- $Z_1, \dots, Z_n \overset{\text{i.i.d}}{\sim} \mathcal{N}(0, \sigma^2)$

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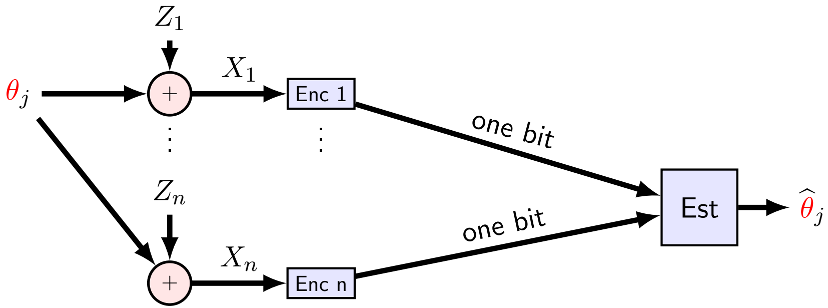

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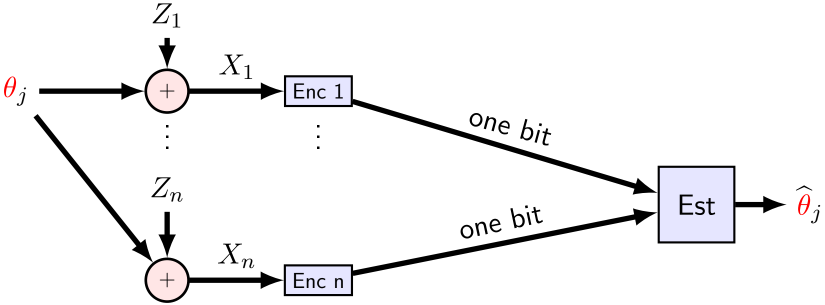

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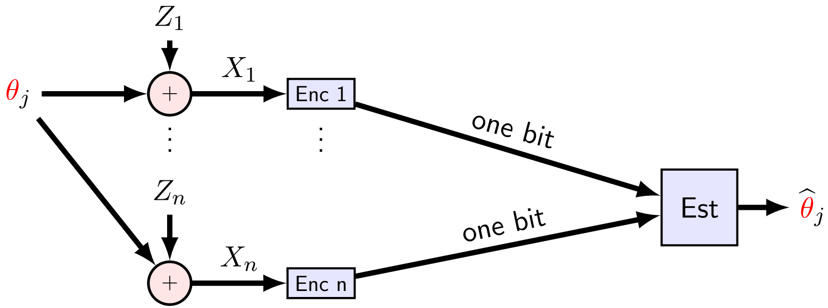

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- Block encode $X_{j,1}, \ldots, X_{j,n}$ using k bits, $j = 1, \ldots, k$
- $D_{CEO} = \inf_{k,\widehat{\theta}} \frac{1}{k} \sum_{j=1}^{k} \mathbb{E}\left(\theta_j - \widehat{\theta}_j\right)^2$

Quadratic Gaussian CEO under optimal rate allocation [Chen, Zhang, Berger, Wicker '04] :

$$D_{CEO} \geq \frac{4}{3}\frac{\sigma^2}{n} + o(1/n)$$

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- In fact [K., Rini, Goldsmith '17]:

$$D_{CEO} \leq \frac{4}{3}\frac{\sigma^2}{n} + \frac{\sigma_0^2}{3n} + o(1/n)$$

(ARE of $4/3$ is achievable with coding over blocks)

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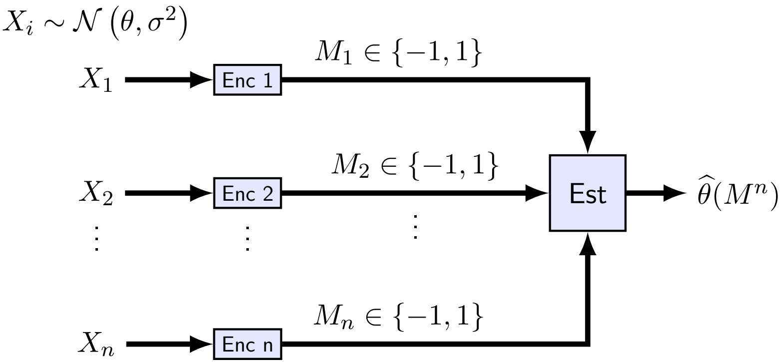

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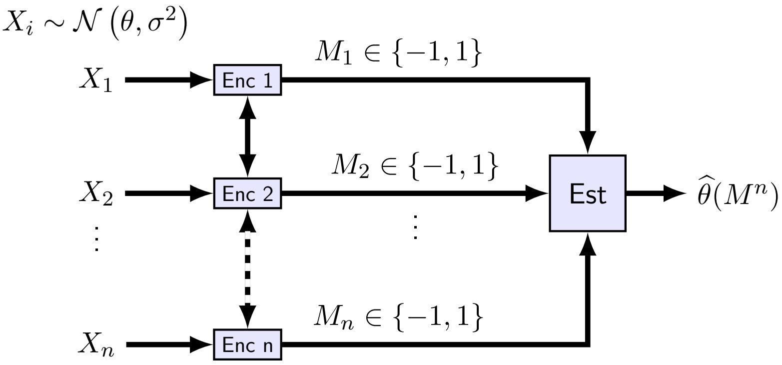

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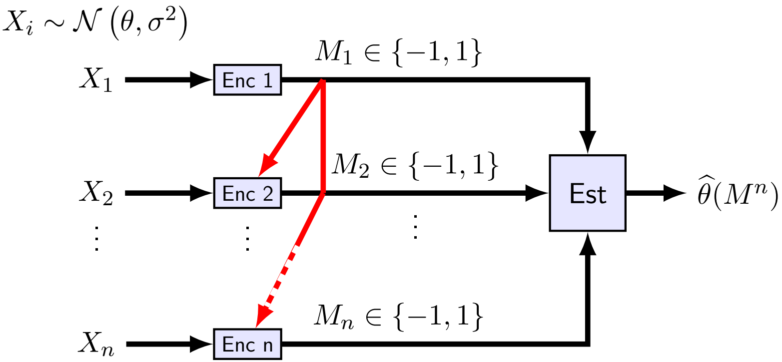

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Main Results (adaptive encoding)

Theorem (achievability)

There exists an estimator with ARE $\pi/2$

Theorem (converse)

No estimator have ARE lower than $\pi/2$

Achievability

existence of an estimator with ARE $= \pi/2$

(i) For $X \sim \mathcal{N}(\theta, \sigma^2)$, $\mathrm{med}(X) = \theta$

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(iii) Stochastic gradient descent on $\mathbb{E}\,|X - \theta|$:

$$\theta_n = \theta_{n-1} + \gamma_n \mathsf{sign}(X_n - \theta_n)$$

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From [Polyak & Juditsky '92] (under conditions on (γ_n)):

$$\sqrt{n}(\theta - \widehat{\theta}_n) \to \mathcal{N}\left(0, \sigma^2 \pi/2\right)$$

Converse

ARE $\geq \pi/2$

The van-Trees inequality (e.g. [Tsybakov '08]) implies

$$\mathbb{E}\left(\theta - \widehat{\theta}\right)^2 \geq \frac{1}{\mathbb{E}I_\theta(M^n) + I_\pi} \geq \frac{1}{\sum_{i=1}^n I_\theta(M_i|M^{i-1}) + I_\pi}$$

I_π is the location Fisher information with w.r.t. some prior $\pi(d\theta)$ on Θ

Lemma (K. & Duchi '17)

$$I_\theta(M_i|M^{i-1}) \leq \frac{2}{\pi\sigma^2}$$

Proof:
Stein identity implies that portion maximizing the information is a threshold: $M_i^{-1}(1) = (\theta, \infty)$
The rest follows by induction over number of portions

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Threshold Detection

We consider only messages of the form

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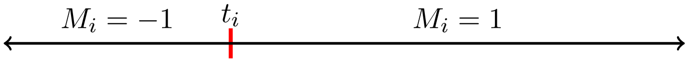

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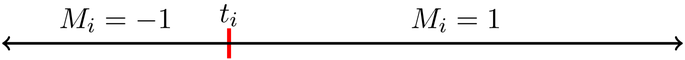

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Example: $t_1, \ldots, t_n$ drawn i.i.d. from a distribution λ on $\mathbb{R}$

Main Results (distributed encoding)

Theorem

(i) *The Maximum likelihood estimator $\widehat{\theta}_{ML}$ satisfies*

$$\sqrt{n}(\theta - \widehat{\theta}_{ML}) \to \mathcal{N}\left(0, \sigma^2 / K_\lambda(\theta)\right)$$

where:

$$K_\lambda(\theta) = \int_{\mathbb{R}} \eta\left(\frac{t-\theta}{\sigma}\right) \lambda(dt)$$

$$\eta(x) = \frac{\phi^2(x)}{\Phi(x)\Phi(-x)}$$

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(ii) *For any estimator* $\widehat{\theta}(M_1, \ldots, M_n)$*:*

$$\liminf_{c\to\infty}\liminf_{n\to\infty}\sup_{\tau\,:\,|\tau-\theta|\leq\frac{c}{\sqrt{n}}} n\mathbb{E}\left(\widehat{\theta}-\tau\right)^2 \geq \sigma^2/K_\lambda(\theta),$$

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  (equality iff $\lambda = \delta_\theta$)
- Minimax λ and ARE can be found using a convex program – depends on radius of Θ

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Open question

Is there a distributed encoding scheme with ARE that is both finite and independent of size of parameter space ?

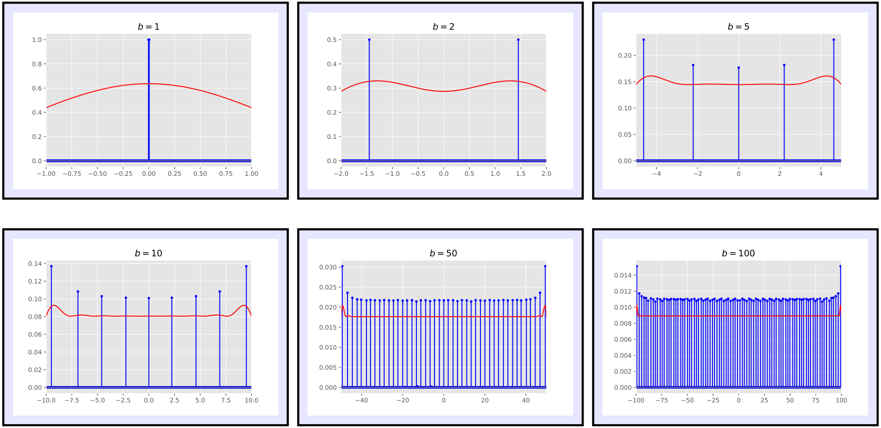
Minimax threshold density

Minimax λ for $\theta \in (-b\sigma, b\sigma)$:

$$\begin{aligned} \text{maximize} \quad & \inf_{\tau \in (-b,b)} \int \eta(t-\tau)\lambda(dt) \\ \text{subject to} \quad & \lambda(dt) \geq 0, \quad \int \lambda(dt) \leq 1. \end{aligned}$$

Minimax λ

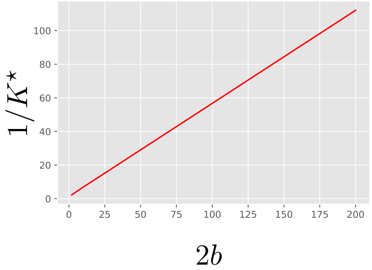
support of optimal threshold density $\lambda^\star$

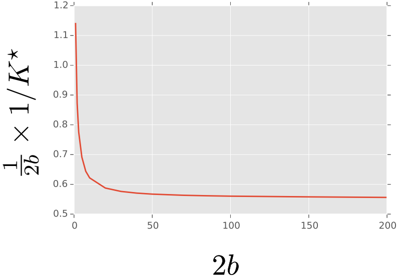


$$K^\star = \inf_\theta K^\star(\theta) = \inf_\theta \int \eta(t-\theta)\lambda^\star(dt)$$

Minimax λ

Minimax ARE vs size of parameter space





- ARE increases with size of parameter space