Propagation in waveguides

Rectangular, cylindrical and coaxial waveguides

Alejandro García Lampérez

January 31, 2022

1 Problem formulation

Waveguide EM medium without sources with translation invariance in one direction.

Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(Faraday's law)}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{(Ampère's law + Maxwell term)}$$

$$\nabla \cdot \vec{D} = \rho \qquad \text{(Gauss's law)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(Gauss's law for magnetism)}$$

$\vec{E}$	electric field
$\vec{H}$	magnetizing field
$\vec{D}$	displacement field
$\vec{B}$	magnetic field

where $\vec{D} = \varepsilon\vec{E}$, $\vec{B} = \mu\vec{H}$.

Restricted version for waveguide analysis:

- Frequency domain: $\frac{\partial}{\partial t} \longrightarrow j\omega$
- Homogeneous medium: uniform μ, ε
- No sources: $\rho = 0$, $\vec{J} = 0$

$$\begin{aligned} \nabla \times \vec{E} &= -j\omega\mu\vec{H} \\ \nabla \times \vec{H} &= j\omega\varepsilon\vec{E} \\ \nabla \cdot \vec{E} &= 0 \\ \nabla \cdot \vec{H} &= 0 \end{aligned}$$

Apply curl to both terms of, e.g., first equation:

$$\nabla \times \nabla \times \vec{E} = \nabla(\cancelto{0}{\nabla \cdot \vec{E}}) - \nabla^2 \vec{E}$$
$$\nabla \times (-j\omega\mu\vec{H}) = -j\omega\mu(j\omega\varepsilon\vec{E}) = -\gamma_0^2 \vec{E}$$

where $\gamma_0^2 \triangleq -\omega^2\mu\varepsilon \;\Rightarrow\; \gamma_0 = \pm j\omega\sqrt{\mu\varepsilon}$ (free space propagation constant). Both terms form the Helmholtz equation in $\vec{E}$. Repeat with the second Maxwell equation for the Helmholtz equation in $\vec{H}$.

Helmholtz equations

$$\nabla^2 \vec{E} - \gamma_0^2 \vec{E} = 0$$
$$\nabla^2 \vec{H} - \gamma_0^2 \vec{H} = 0$$

Translation invariance

Separation of longitudinal and transverse components

$$\vec{E} = \vec{E}_t + \vec{E}_z = \vec{E}_t + \hat{z}E_z$$
$$\vec{H} = \vec{H}_t + \vec{H}_z = \vec{H}_t + \hat{z}H_z$$
$$\nabla = \nabla_t + \nabla_z = \nabla_t + \hat{z}\frac{\partial}{\partial z}$$
$$\Delta = \nabla^2 = (\nabla_t + \nabla_z)(\nabla_t + \nabla_z) = \nabla_t^2 + \frac{\partial^2}{\partial z^2}$$

Cartesian coordinates:

$$\vec{E}_t = \hat{x}E_x + \hat{y}E_y$$
$$\vec{H}_t = \hat{x}H_x + \hat{y}H_y$$
$$\nabla_t = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$$
$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Separation of variables (e.g., for E_z component and cartesian coordinates)

$$E_z(x, y, z) = F(x, y)Z(z) \;\equiv\; E_z = F\,Z$$

where:

- $F(x, y)$: transversal form function.
- $Z(z)$: longitudinal amplitude function.

Apply Laplacian:

$$\nabla^2 E_z = \left(\nabla_t^2 F\right) Z + F\frac{\partial^2 Z}{\partial z^2}$$

Helmholtz equation:

$$\frac{\nabla^2 E_z}{E_z} - \gamma_0^2 = 0 \quad \Rightarrow \quad \underbrace{\frac{\nabla_t^2 F}{F}}_{\text{Trans.}} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\text{Long.}} - \gamma_0^2 = 0$$

that is, the three terms are constant:

$$\frac{\nabla_t^2 F}{F} \triangleq \gamma_c^2 \quad \Rightarrow \quad \nabla_t^2 F - \gamma_c^2 F = 0 \qquad \text{Transversal Helmholtz equation}$$
$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} \triangleq \gamma^2 \quad \Rightarrow \quad \frac{d^2 Z}{dz^2} - \gamma^2 Z = 0 \qquad \text{Longitudinal Helmholtz equation}$$

But the problems are not independent, they are related through their constants:

$$\gamma_c^2 + \gamma^2 = \gamma_0^2$$

Propagation

General solution of the longitudinal problem (one dimensional ordinary differential equation):

$$Z(z) = Ae^{-\gamma z} + Be^{+\gamma z}$$

The general solution is a superposition of a forward (or *incident*) and a backward (or *reflected*) travelling waves. Therefore, the fields behave as waves. E.g., the longitudinal components are, if only the forward wave is considered:

$$\vec{E}_z = \hat{z}\, F_E(x,y)e^{-\gamma z}$$
$$\vec{H}_z = \hat{z}\, F_H(x,y)e^{-\gamma z}$$

Where F_E and F_H are the corresponding form functions. The variable γ, complex in general, is the **propagation constant**.

An additional note: since the longitudinal variation of all the fields and its components takes the form $E(x,y,z) = F(x,y)e^{-\gamma z}$, the corresponding derivative is always

$$\frac{\partial E(x,y,z)}{\partial z} = F(x,y)\frac{\partial e^{-\gamma z}}{\partial z} = -\gamma\, F(x,y)e^{-\gamma z} = -\gamma\, E(x,y,z)$$

Solution method

1. Solve the transversal problem $\nabla_t^2 F(x,y) - \gamma_c^2 F(x,y) = 0 \rightarrow F(x,y)$ and γ_c
2. Compute the propagation constant,

$$\gamma^2 = \gamma_0^2 - \gamma_c^2 = -\omega^2\mu\varepsilon - \gamma_c^2$$

3. Write the field component.

$$\vec{E}_z = \hat{z}\, F(x,y)e^{-\gamma z}$$

2 Types of modes

2.1 TEM modes

Definition $E_z = 0$, $H_z = 0$.

The fields are totally transversal, that is, $\vec{E} = \vec{E}_t$, $\vec{H} = \vec{H}_t$. The first Maxwell equation ($\nabla \times \vec{E} = -j\omega\mu\vec{H}$) is therefore:

$$\nabla \times \vec{E}_t = \left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \vec{E}_t = \underbrace{\nabla_t \times \vec{E}_t}_{\text{Long.}} + \underbrace{\hat{z} \times \frac{\partial \vec{E}_t}{\partial z}}_{\text{Trans.}} = \underbrace{-j\omega\mu\vec{H}_t}_{\text{Trans.}}$$

Longitudinal problem

$$\nabla_t \times \vec{E}_t(x,y,z) = 0$$

Using *separation of variables*, and considering only the forward wave,

$$\vec{E}_t(x,y,z) = \vec{F}(x,y)\, e^{-\gamma z}$$

and since the exponential is constant with respect to the transversal directions,

$$\nabla_t \times \vec{E}_t(x,y,z) = \big(\nabla_t \times \vec{F}(x,y)\big)\, e^{-\gamma z}$$

The resulting problem is a purely two-dimensional one,

$$\nabla_t \times \vec{F}(x,y) = 0$$

Any irrotational vector field can be represented as the gradient of a scalar field. In this case, the scalar field $\phi(x,y)$ is the electric potential.

$$\vec{F}(x,y) = -\nabla_t\, \phi(x,y)$$

and since $\nabla \cdot \vec{E}_t = \nabla_t \cdot \vec{E}_t = 0$, and therefore $\nabla_t \cdot \vec{F} = 0$, the potential can be computed by solving the equation

$$\nabla_t^2\, \phi(x,y) = 0.$$

Propagation constant

Laplacian of the electric field:

$$\nabla^2 \vec{E}_t = \nabla_t^2 \vec{E}_t + \frac{\partial^2 \vec{E}_t}{\partial z^2} = \nabla_t^2 \vec{E}_t + \gamma^2 \vec{E}_t$$

since

$$\frac{\partial^2 \vec{E}_t(x,y,z)}{\partial z^2} = \vec{F}(x,y)\frac{\partial^2 e^{-\gamma z}}{\partial z^2} = \gamma^2 \vec{F}(x,y) e^{-\gamma z} = \gamma^2 \vec{E}_t(x,y,z)$$

Helmholtz equation

$$\nabla^2 \vec{E}_t - \gamma_0^2 \vec{E}_t = 0 \quad \Leftrightarrow \quad \nabla_t^2 \vec{E}_t - \big(\gamma_0^2 - \gamma^2\big)\vec{E}_t = 0$$

and representing the field as the gradient of the potential, $\vec{E}_t = -\nabla_t \phi\, e^{-\gamma z}$,

$$\nabla_t\Big(\cancelto{0}{\nabla_t^2 \phi} - \big(\gamma_0^2 - \gamma^2\big)\phi\Big) = 0 \quad \Rightarrow \quad \big(\gamma_0^2 - \gamma^2\big)\nabla_t \phi = 0$$

but since $-\nabla_t \phi = \vec{F} \neq 0$ (otherwise $\vec{E}_t = 0$ and there is no solution at all),

$$\gamma^2 = \gamma_0^2 = -\omega^2 \mu\varepsilon \quad \Rightarrow \quad \gamma = \gamma_0 = j\omega\sqrt{\mu\varepsilon}$$

For a TEM mode the propagation constant is just the free-space one, depending only on the material, not the geometry. It is purely imaginary, and proportional to the frequency. The electric field is therefore the following one (again, only forward wave)

$$\vec{E}_t(x,y,z) = -\nabla_t\, \phi(x,y)\, e^{-\gamma_0 z}$$

Transversal problem

$$\hat{z} \times \frac{\partial \vec{E}_t}{\partial z} = -j\omega\mu \vec{H}_t$$

but again, due to the variation of $\vec{E}_t$ with z $\left(\vec{E}_t(x,y,z) = \vec{F}(x,y)e^{-\gamma z}\right)$,

$$\frac{\partial \vec{E}_t(x,y,z)}{\partial z} = \vec{F}(x,y)\frac{\partial e^{-\gamma z}}{\partial z} = -\gamma\,\vec{F}(x,y)e^{-\gamma z} = -\gamma\,\vec{E}_t(x,y,z)$$

And the problem is simplified, where $\gamma = \gamma_0$,

$$-\gamma_0\,\hat{z} \times \vec{E}_t = -j\omega\mu \vec{H}_t$$

that provides the relationship between electric and magnetic fields,

$$\vec{H}_t = \frac{\gamma_0}{j\omega\mu}\hat{z} \times \vec{E}_t = \frac{\hat{z} \times \vec{E}_t}{Z_{\mathrm{TEM}}}$$

where Z_{TEM} is the **modal impedance**,

$$Z_{\mathrm{TEM}} \triangleq \frac{\left\|\hat{z} \times \vec{E}_t\right\|}{\left\|\vec{H}_t\right\|} = \frac{j\omega\mu}{\gamma_0} = \sqrt{\frac{\mu}{\varepsilon}} = \eta$$

The modal impedance of a TEM mode is just the intrinsic impedance of the medium. This is another similarity with free-space plane waves.

Summary: solution of a TEM mode

1. Propagation constant and modal impedance: $\gamma = \gamma_0$, $Z_{\mathrm{TEM}} = \eta$ (free-space with actual dielectric).
2. Solve the two-dimensional problem $\nabla_t^2 \phi(x,y) = 0$, using the boundary conditions (geometry).
3. Electric field: $\vec{E}_t(x,y,z) = \vec{F}(x,y)\,e^{-\gamma_0 z} = -\nabla_t\,\phi(x,y)\,e^{-\gamma_0 z}$.
4. Magnetic field: $\vec{H}_t(x,y,z) = \frac{\hat{z}\times\vec{E}_t(x,y,z)}{Z_{\mathrm{TEM}}} = \frac{\hat{z}\times\vec{F}_t(x,y)}{Z_{\mathrm{TEM}}}\,e^{-\gamma_0 z}$.

2.2 TM modes

Definition $H_z = 0$.

The magnetic field is transversal, $\vec{H} = \vec{H}_t$, but the electric field has a longitudinal component, $\vec{E} = \vec{E}_t + \hat{z}E_z$. To solve this component, $E_z(x,y,z)$, both the propagation constant γ and the transversal form function $F(x,y)$ must be solved,

$$E_z(x,y,z) = F(x,y)\,e^{-\gamma z}$$

The key is to solve the transversal Helmholtz equation for the form function,

$$\nabla_t^2 F(x,y) - \gamma_c^2 F(x,y) = 0$$

with the suitable boundary conditions (geometry). Both $F(x,y)$ and γ_c must be solved. Additionally, two partial results are required to fully reconstruct $\vec{E}$ and $\vec{H}$ from E_z.

Helmholtz equation of the transversal component

Transversal Helmholtz equation

$$\nabla_t^2 \vec{E} - \gamma_c^2 \vec{E} = 0$$

Expanded version

$$\nabla_t^2 \left(\vec{E}_t + \hat{z}E_z\right) - \gamma_c^2 \left(\vec{E}_t + \hat{z}E_z\right) = \nabla_t^2 \vec{E}_t - \gamma_c^2 \vec{E}_t + \hat{z}\cancelto{0}{\left(\nabla_t^2 F - \gamma_c^2 F\right)} e^{-\gamma z} = 0$$

Therefore, the Helmholtz equation is also verified for the transversal component

$$\nabla_t^2 \vec{E}_t - \gamma_c^2 \vec{E}_t = 0$$

Divergence of the electric field

$$\nabla \cdot \vec{E} = 0$$

Expanded version

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\vec{E}_t + \hat{z}E_z\right) = \nabla_t \cdot \vec{E}_t + \cancelto{0}{\hat{z} \cdot \frac{\partial \vec{E}_t}{\partial z}} + \cancelto{0}{\nabla_t \cdot \hat{z}E_z} + \hat{z} \cdot \hat{z}\frac{\partial E_z}{\partial z} = 0$$

since $\frac{\partial \vec{E}_t}{\partial z} = -\gamma \vec{E}_t$ is transversal, and $\nabla_t E_z = \nabla_t F e^{-\gamma z}$ is also transversal. As a result,

$$\nabla_t \cdot \vec{E}_t = -\frac{\partial E_z}{\partial z} = \gamma E_z = \gamma F e^{-\gamma z}$$

But the required relationship is the opposite: $\vec{E}_t$ in terms of E_z.

Electric field transversal component in terms of the longitudinal one

First Maxwell equation:

$$\nabla \times \vec{E} = -j\omega\mu \vec{H}_t$$

or, expanded in transverse and longitudinal components,

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \left(\vec{E}_t + \hat{z}E_z\right) = \underbrace{\nabla_t \times \vec{E}_t}_{\text{Long.}} + \underbrace{\hat{z} \times \frac{\partial \vec{E}_t}{\partial z} + \nabla_t \times \hat{z}E_z}_{\text{Trans.}} + \cancelto{0}{\hat{z} \times \hat{z}\frac{\partial E_z}{\partial z}} = \underbrace{-j\omega\mu \vec{H}_t}_{\text{Trans.}}.$$

Therefore, $\nabla_t \times \vec{E}_t = 0$. Computing the two-dimensional curl of this expression results in

$$\nabla_t \times \nabla_t \times \vec{E}_t = \nabla_t(\nabla_t \cdot \vec{E}_t) - \nabla_t^2 \vec{E}_t = 0$$

Substituting from the previous results, $\nabla_t^2 \vec{E}_t = \gamma_c^2 \vec{E}_t$ and $\nabla_t \cdot \vec{E}_t = \gamma E_z$,

$$\vec{E}_t = \frac{\gamma}{\gamma_c^2} \nabla_t E_z$$

Magnetic field from the electric one: modal impedance

Second Maxwell equation:

$$\nabla \times \vec{H}_t = j\omega\varepsilon\vec{E}$$

or, expanded in transverse and longitudinal components,

$$\left(\nabla_t + \hat{z}\frac{\partial}{\partial z}\right) \times \vec{H}_t = \underbrace{\nabla_t \times \vec{H}_t}_{\text{Long.}} + \underbrace{\hat{z} \times \frac{\partial \vec{H}_t}{\partial z}}_{\text{Trans.}} = \underbrace{j\omega\varepsilon\vec{E}_t}_{\text{Trans.}} + \underbrace{j\omega\varepsilon E_z\,\hat{z}}_{\text{Long.}}$$

In particular, the equation of the longitudinal components is

$$\nabla_t \times \vec{H}_t = j\omega\varepsilon E_z\,\hat{z}$$

and computing its two-dimensional curl,

$$\nabla_t \times \nabla_t \times \vec{H}_t = \nabla_t(\nabla_t \cdot \vec{H}_t) - \nabla_t^2\vec{H}_t = j\omega\varepsilon\nabla_t \times \hat{z}E_z$$

The substitutions are analogous to the ones for the electric field, $\nabla_t^2\vec{H}_t = \gamma_c^2\vec{H}_t$ and $\nabla_t \cdot \vec{H}_t = 0$. Therefore, the magnetic field can be computed from the longitudinal electric field,

$$\vec{H}_t = -\frac{j\omega\varepsilon}{\gamma_c^2}\nabla_t \times \hat{z}E_z = -\frac{j\omega\varepsilon}{\gamma_c^2}\nabla_t E_z \times \hat{z}$$

Alternatively, the transversal components of the Maxwell equation provide the relationship between $\vec{E}_t$ and $\vec{H}_t$ (remember, again, that fields vary as $e^{-\gamma z}$)

$$\hat{z} \times \frac{\partial \vec{H}_t}{\partial z} = -\gamma\big(\hat{z} \times \vec{H}_t\big) = j\omega\varepsilon\vec{E}_t$$

or, solving for $\vec{H}_t$,

$$\vec{H}_t = \frac{j\omega\varepsilon}{\gamma}\hat{z} \times \vec{E}_t = \frac{\hat{z} \times \vec{E}_t}{Z_{\mathrm{TM}}}$$

where the modal impedance is

$$Z_{\mathrm{TM}} \triangleq \frac{\big\|\hat{z} \times \vec{E}_t\big\|}{\big\|\vec{H}_t\big\|} = \frac{\gamma}{j\omega\varepsilon}$$

that depends on both the material properties and the waveguide geometry. Notice that $\hat{z}$, $\vec{E}_t$ and $\vec{H}_t$ form a positively oriented trihedron, related to the fact that the propagation is towards $+z$.

Summary: solution of a TM mode

1. Solve the two-dimensional Helmholtz equation $\nabla_t^2 F - \gamma_c^2 F = 0$, using the boundary conditions (geometry). The results are the transversal form function $F(x, y)$ and the transversal constant γ_c.
2. Compute the propagation constant $\gamma^2 = -\gamma_c^2 + \gamma_0^2$ and the modal impedance $Z_{\mathrm{TM}} = \frac{\gamma}{j\omega\varepsilon}$.
3. Longitudinal component of the electric field: $E_z(x, y, z) = F(x, y)\,e^{-\gamma z}$.
4. Transversal component of the electric field: $\vec{E}_t = \frac{\gamma}{\gamma_c^2}\nabla_t E_z$.
5. Magnetic field (purely transversal): $\vec{H}_t = \frac{\hat{z}\times\vec{E}_t}{Z_{\mathrm{TM}}}$.

2.3 TE modes

Definition $E_z = 0$.

The electric field is transversal, $\vec{E} = \vec{E}_t$, but the magnetic field has a longitudinal component, $\vec{H} = \vec{H}_t + \hat{z}H_z$. To solve this component $H_z(x, y, z)$ both the propagation constant γ and the transversal form function $F(x, y)$ must be solved,

$$H_z(x, y, z) = F(x, y)\, e^{-\gamma z}$$

From here the transversal component of the magnetic field and the electric field are computed,

$$\vec{H}_t = \frac{\gamma}{\gamma_c^2} \nabla_t H_z$$

$$\vec{E}_t = \frac{j\omega\mu}{\gamma_c^2} \nabla_t \times \hat{z} H_z$$

that are related through the modal impedance,

$$Z_{\mathrm{TE}} \triangleq \frac{\left\| \hat{z} \times \vec{E}_t \right\|}{\left\| \vec{H}_t \right\|} = \frac{j\omega\mu}{\gamma}$$

Summary: solution of a TE mode

1. Solve the two-dimensional Helmholtz equation $\nabla_t^2 F - \gamma_c^2 F = 0$, using the boundary conditions (geometry). The results are the transversal form function $F(x, y)$ and the transversal constant γ_c.
2. Compute the propagation constant $\gamma^2 = -\gamma_c^2 + \gamma_0^2$ and the modal impedance $Z_{\mathrm{TE}} = \frac{j\omega\mu}{\gamma}$.
3. Longitudinal component of the magnetic field: $H_z(x, y, z) = F(x, y)\, e^{-\gamma z}$.
4. Transversal component of the magnetic field: $\vec{H}_t = \frac{\gamma}{\gamma_c^2} \nabla_t H_z$.
5. Electric field (purely transversal): $\vec{E}_t = Z_{\mathrm{TE}} \big(\vec{H}_t \times \hat{z} \big)$.

$$\vec{H}_t = \frac{\hat{z} \times \vec{E}_t}{Z_{\mathrm{TE}}} \quad \Rightarrow \quad Z_{\mathrm{TE}} \big(\vec{H}_t \times \hat{z} \big) = \big(\hat{z} \times \vec{E}_t \big) \times \hat{z} = (\hat{z} \cdot \hat{z}) \vec{E}_t - \cancel{(\vec{E}_t \cdot \hat{z})} \hat{z} = \vec{E}_t$$

3 Wave definitions

The general solution is a superposition of (infinite) TEM, TM and TE modes. Lateral boundary conditions of the transversal problem (Helmholtz equation) define:

- Which modes appear (possible solutions).
- For each mode, field distribution in a cross section ($F(x, y)$).
- For each mode, the propagation characteristics in terms of the frequency (γ).

Propagation constant

- TEM modes: $\gamma = \gamma_0 = j\omega\sqrt{\mu\varepsilon}$
  - Purely imaginary.
  - Linear with frequency.
  - Only depends on the properties of the medium.
- TM, TE modes: $\gamma = \sqrt{\gamma_0^2 - \gamma_c^2} = \sqrt{-\omega^2\mu\varepsilon - \gamma_c^2}$
  - Complex (it will be shown that sometimes it is real or imaginary).
  - Nonlinear with frequency.
  - Depends on both the properties of the medium and the geometry.

$$\gamma(\omega) = \alpha(\omega) + j\beta(\omega) \qquad \longrightarrow \qquad e^{-\gamma z} = e^{-\alpha z}e^{-j\beta z}$$

- **Attenuation constant**: α (Np/m)
- **Phase constant**: β (rad/m)
- Separation of two consecutive points with same phase, or **wavelength**,

$$\lambda = \frac{2\pi}{\beta}$$

- Velocity of propagation of constant phase points, or **phase velocity**,

$$v_p = \frac{\omega}{\beta}$$

- Velocity of propagation of the envelope of a narrow band signal, or **group velocity**,

$$v_g = \frac{\partial\omega}{\partial\beta}$$

Two cases:

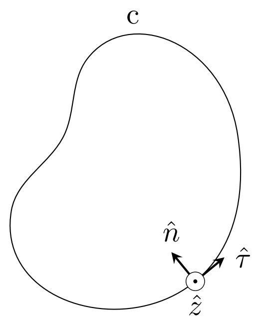
1. The mode does not propagate: $\alpha > \beta$ (the extreme case is $\beta = 0$, and the mode just attenuates exponentially with distance).
2. The mode propagates: $\alpha < \beta$ (the extreme case is $\alpha = 0$, with the mode propagating indefinitely with no attenuation at all).

4 Lateral boundary conditions

For now, only perfect conductor boundary conditions:

$$\hat{n}\times\vec{E}\,\Big|_c = 0 = \hat{n}\times\left(\vec{E}_t+\hat{z}E_z\right)\Big|_c$$

$$\begin{cases}\hat{n}\times\vec{E}_t \parallel \hat{z}\\ \hat{n}\times\hat{z}E_z=\hat{\tau}E_z\perp\hat{z}\end{cases} \quad\Longrightarrow\quad \begin{cases}\hat{n}\times\vec{E}_t\,\Big|_c=0\\ E_z\,\big|_c=0\end{cases}$$



4.1 TEM modes

Always, $E_z=0$. Additionally,

$$\hat{n}\times\vec{E}_t\,\Big|_c = -\hat{n}\times\nabla_t\phi\,\big|_c = 0.$$

But

$$\nabla_t\phi = \hat{n}\frac{\partial\phi}{\partial n}+\hat{\tau}\frac{\partial\phi}{\partial\tau} \quad\Rightarrow\quad \hat{n}\times\nabla_t\phi = \cancel{\hat{n}\times\hat{n}}\frac{\partial\phi}{\partial n}+\hat{n}\times\hat{\tau}\frac{\partial\phi}{\partial\tau} = -\hat{z}\frac{\partial\phi}{\partial\tau}$$

therefore, the condition is

$$\frac{\partial\phi}{\partial\tau}\bigg|_c = 0 \quad\Leftrightarrow\quad \boxed{\phi\,\big|_c = \text{constant}}$$

But if $\phi|_c$ is constant for all the conductors, the solution of $\nabla_t^2\phi=0$ is just $\phi=\phi|_c$ and the field is zero. TEM modes require a difference of potential, and therefore two or more conductors. The potential is constant on each conductor (and the boundary condition is fulfilled), $\phi_1=\phi|_{c1}$, $\phi_2=\phi|_{c2}$, but different between conductors, $\phi_1\neq\phi_2$, in order to have a non-zero TEM mode solution. That is, the difference of potential $\phi_1-\phi_2\neq 0$ is what defines the existence of the TEM mode.

In fact, each couple of conductors produce a TEM mode, and the total number of supported TEM modes is $n-1$, if n is the number of non-connected boundary parts.

> Consider a waveguide with three disjoint conductors ($n=3$), A, B and C. Why are not considered three TEM modes, with differences of potential $\phi_A-\phi_B$, $\phi_B-\phi_C$, and $\phi_C-\phi_A$, for example?

Finally, it should be noted that the difference of potential, or voltage, and the current associated to a TEM mode can be defined as line integrals of $\vec{E}$ and $\vec{H}$, respectively, along paths contained in the waveguide cross-section:

- Voltage (path that connects both conductors):

$$V_{12}=\phi_1-\phi_2=\int_{c1}^{c2}\vec{E}_t\cdot d\vec{l}$$

- Current (closed path splitting the cross-section in two, each one with one conductor):

$$I_1=-I_2=\oint_c\vec{H}_t\cdot d\vec{l}$$

Both integrals are independent of the particular integration path.

4.2 TE modes

Always, $E_z = 0$. The condition is just $\vec{n} \times \vec{E}_t \big|_c = 0$. But $\vec{E}_t = Z_{\mathrm{TE}}\, \vec{H}_t \times \hat{z}$, and it is equivalent to

$$\hat{n} \times \left(\vec{H}_t \times \hat{z}\right)\Big|_c = 0 = \vec{H}_t \cancelto{0}{(\hat{n} \cdot \hat{z})} - \hat{z}(\hat{n} \cdot \vec{H}_t)\Big|_c$$

But then

$$\hat{n} \cdot \vec{H}_t = \hat{n} \cdot \frac{\gamma}{\gamma_c^2} \nabla_t H_z = \hat{n} \cdot \frac{\gamma}{\gamma_c^2} \left(\frac{\partial H_z}{\partial n} \hat{n} + \frac{\partial H_z}{\partial \tau} \hat{\tau} \right) = \frac{\gamma}{\gamma_c^2} \frac{\partial H_z}{\partial n} = \frac{\gamma}{\gamma_c^2} \frac{\partial F}{\partial n} e^{-\gamma z}.$$

And the condition becomes

$$\boxed{\left.\frac{\partial F}{\partial n}\right|_c = 0}$$

4.3 TM modes

Imposing $E_z\big|_c = 0$ has two consequences:

1. First,

   $$\vec{E}_t = \frac{\gamma}{\gamma_c^2} \nabla E_z = \frac{\gamma}{\gamma_c^2} \left(\hat{n} \frac{\partial E_z}{\partial n} + \hat{\tau} \frac{\partial E_z}{\partial \tau} \right)$$

   but, on the conductor, $E_z\big|_c$ is constant (it is zero!) and therefore $\left.\frac{\partial E_z}{\partial \tau}\right|_c = 0$. As a result, $\vec{E}_t\big|_c \parallel \hat{n}$, and the condition $\hat{n} \times \vec{E}_t \big|_c = 0$ is automatically met.

2. Since E_z is proportional to F, the remaining condition, that must be imposed, is

   $$\boxed{F\big|_c = 0}$$

5 Cutoff frequency

For the following analysis, everything is two-dimensional, transversal. Define the vector field $\vec{A}(x, y)$ from the scalar field $F(x, y)$ (the transversal form function of E_z or H_z):

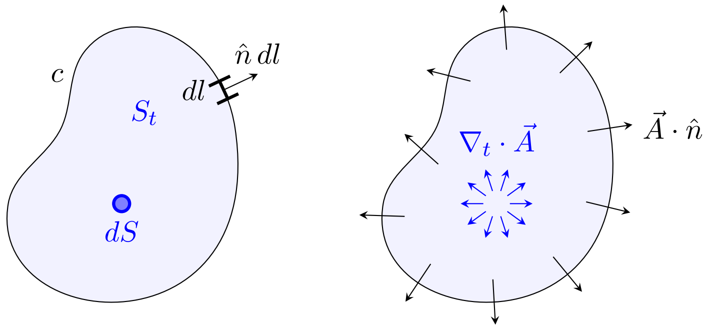
$$\vec{A} \triangleq F^* \, \nabla_t F$$

Divergence:

$$\nabla_t \cdot \vec{A} = \nabla_t F^* \cdot \nabla_t F + F^* \nabla_t^2 F$$

Green's theorem (bidimensional version of the divergence, or Gauss's, theorem):

$$\int_{S_t} \left(\nabla_t \cdot \vec{A}\right) dS = \oint_c \left(\vec{A} \cdot \hat{n}\right) dl$$



I.e. the integral of the divergence in an enclosed area (the field that is generated in this area) is equal to the flux of field across its boundary (the field that escapes the area).

Applying the theorem to the vector field $\vec{A}$ defined above, being S_t the waveguide cross-section and c its perfect conductor boundary:

$$\int_{S_t} \left(\nabla_t F^* \cdot \nabla_t F + F^* \nabla_t^2 F\right) dS = \oint_c \left(F^*\, \nabla_t F \cdot \hat{n}\right) dl = \oint_c F^* \frac{\partial F}{\partial n}\, dl = 0$$

since, in the last term, $F = 0$ or $\frac{\partial F}{\partial n} = 0$. And since $\nabla_t^2 F = \gamma_c^2 F$, the previous equation is equivalent to

$$\int_{S_t} \left(\nabla_t F^* \cdot \nabla_t F\right) dS = -\gamma_c^2 \int_{S_t} \left(F^*\, F\right) dS$$

or, equivalently

$$\int_{S_t} \left|\nabla_t F\right|^2 \, dS = -\gamma_c^2 \int_{S_t} \left|F\right|^2 \, dS$$

Therefore γ_c^2 is real and $-\gamma_c^2 \geq 0$. As a consequence, the propagation constant squared γ^2 is real,

$$\gamma^2 = -\gamma_c^2 + \gamma_0^2 = -\gamma_c^2 - \omega^2 \mu\varepsilon$$

Since $-\gamma_c^2$ does not depend with the frequency (it results from the solution of the transversal Helmholtz equation, that is purely geometrical), γ^2 is positive for low frequencies, and negative for high enough frequencies. The transition between both situations corresponds to the **cutoff frequency**, with no propagation at all,

$$\gamma = 0 \quad \Leftrightarrow \quad \omega_c^2 \mu\varepsilon = -\gamma_c^2$$

that is,

$$\omega_c = \frac{\sqrt{-\gamma_c^2}}{\sqrt{\mu\varepsilon}}, \qquad\qquad f_c = \frac{\sqrt{-\gamma_c^2}}{2\pi\sqrt{\mu\varepsilon}} = \frac{c}{2\pi}\sqrt{-\gamma_c^2}$$

Notice that $1/\sqrt{\mu\varepsilon} = c$ is the speed of light in the medium. The cutoff frequency is determined just by $-\gamma_c^2$ (and the material properties). Notice that TEM modes are the special case with $\gamma_c = 0$.

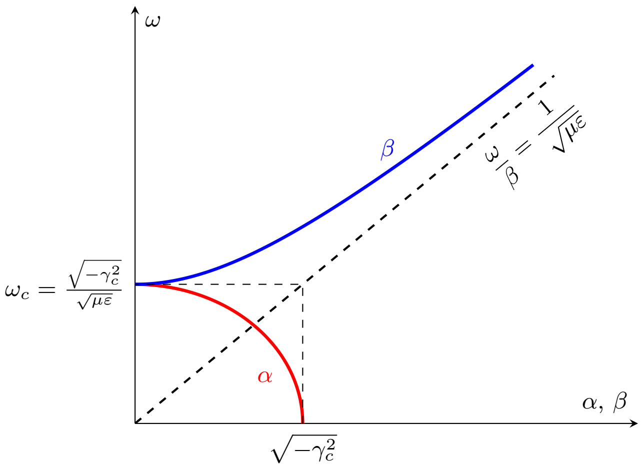
Change of behaviour of the propagation with the cutoff frequency:

- $\omega > \omega_c \;\Rightarrow\; \gamma^2 < 0 \;\Rightarrow\; \gamma = j\beta$ propagating mode
- $\omega < \omega_c \;\Rightarrow\; \gamma^2 > 0 \;\Rightarrow\; \gamma = \alpha$ cutoff mode

Notice that $\omega^2\mu\varepsilon + \gamma^2 = -\gamma_c^2$ (where $-\gamma_c^2 > 0$) has simple geometrical interpretations for both cases $\gamma = \alpha$ and $\gamma = j\beta$, if we consider (α, ω) or (β, ω) as coordinate pairs on a plane.

- $\omega^2\mu\varepsilon + \alpha^2 = -\gamma_c^2$: equation of an ellipse with semi-axes $\sqrt{-\gamma_c^2}$ and $\sqrt{-\gamma_c^2/\mu\varepsilon} = \omega_c$.
- $\omega^2\mu\varepsilon - \beta^2 = -\gamma_c^2$: equation of a hyperbola with semi-axes $\sqrt{-\gamma_c^2}$ and ω_c. The asymptote of this hyperbola is $\beta = \omega\sqrt{\mu\varepsilon}$.

The graphical representation of these curves is called **dispersion diagram**.



Notice several interesting points about the dispersion diagram:

- Both phase velocity $v_p = \frac{\omega}{\beta}$ and group velocity $v_g = \frac{\partial\omega}{\partial\beta}$ can be computed graphically from the dispersion diagram, as the ratio of coordinates ω/β and the slope of the $\omega(\beta)$ curve, respectively.
- Asymptotically, for high frequencies, the TE or TM modes behave as TEM modes (the phase constant tends to the dashed line). Notice that the phase velocity converges to the speed of light in the medium, $c = 1/\sqrt{\mu\varepsilon}$.
- The cutoff frequency and $\sqrt{-\gamma_c^2}$ can be directly read from the diagram. Their quotient is the speed of light in the medium, $1/\sqrt{\mu\varepsilon}$.
- In fact, the dispersion diagram is fully determined just by $\sqrt{-\gamma_c^2}$ (geometry and mode) and $1/\sqrt{\mu\varepsilon}$ (material).

> Sketch and compare the dispersion diagram of two modes of the same waveguide with different cutoff frequencies. Also, compare the dispersion diagram of the fundamental modes of two waveguides with identical geometries, but filled with different dielectrics, with ε_{r1} and ε_{r2}.

Finally, everything can be expressed in terms of the cutoff frequency, starting from the propagation constant:

$$\gamma^2 = -\omega^2\mu\varepsilon - \gamma_c^2 = -\omega^2\mu\varepsilon\left(1 - \frac{\omega_c^2}{\omega^2}\right) = \gamma_0^2\left(1 - \frac{\omega_c^2}{\omega^2}\right) \quad \Leftrightarrow \quad \gamma = \pm\gamma_0\sqrt{1 - \frac{f_c^2}{f^2}}$$

Mode impedances:

$$Z_{\mathrm{TE}} = \frac{j\omega\mu}{\gamma} = \frac{\gamma_0}{\gamma}\eta = \frac{\eta}{\sqrt{1 - f_c^2/f^2}}$$

$$Z_{\mathrm{TM}} = \frac{\gamma}{j\omega\varepsilon} = \frac{\gamma}{\gamma_0}\eta = \eta\sqrt{1 - f_c^2/f^2}$$

TEM modes

$$\gamma = j\beta = j2\pi f\sqrt{\mu\varepsilon}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{c}{f}$$

$$v_p = \frac{\omega}{\beta} = c$$

$$v_g = \frac{\partial\omega}{\partial\beta} = c$$

TE, TM modes

$$\gamma = j\beta = j2\pi\sqrt{f^2 - f_c^2}\sqrt{\mu\varepsilon} = \gamma_0\sqrt{1 - f_c^2/f^2}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{c}{\sqrt{f^2 - f_c^2}} = \frac{\lambda_0}{\sqrt{1 - f_c^2/f^2}}$$

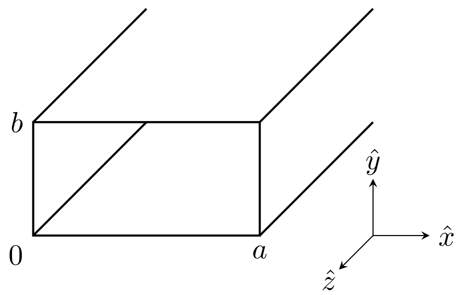
$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - f_c^2/f^2}}$$

$$v_g = \frac{\partial\omega}{\partial\beta} = c\sqrt{1 - f_c^2/f^2}$$

Notice that $v_p\, v_g = c = \frac{1}{\sqrt{\mu\varepsilon}}$.

6 Rectangular waveguide

Geometry and coordinate system (Cartesian):



Invariance in x and y dimensions (Cartesian coordinates): separation of variables.

$$F(x, y) = X(x)\, Y(y)$$

Helmholtz equation $\left(\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$$\nabla_t^2 F - \gamma_c^2 F = 0 \quad \Leftrightarrow \quad Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} - \gamma_c^2 XY = 0$$

Dividing every term by XY:

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{x \text{ only}} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{y \text{ only}} - \underbrace{\gamma_c^2}_{\text{const.}} = 0$$

showing that each term is in fact constant,

$$-k_x^2 - k_y^2 - \gamma_c^2 = 0$$

As a result, the original equation can be separated into two harmonic equations of just one variable,

$$\frac{d^2X(x)}{dx^2} + k_x^2\,X(x) = 0 \qquad\qquad \frac{d^2Y(y)}{dy^2} + k_y^2\,Y(y) = 0$$

related by $-\gamma_c^2 = k_x^2 + k_y^2$. The general solutions of these harmonic equations are:

$$\begin{aligned} X(x) &= A\sin(k_x x) + B\cos(k_x x) \\ Y(y) &= C\sin(k_y y) + D\cos(k_y y) \end{aligned}$$

6.1 Perfect conductor boundary conditions

TE mode

$$\left.\frac{\partial F}{\partial n}\right|_c = 0 \quad \Rightarrow \quad \begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{dX}{dx}Y = 0 & \text{in } x = 0 \text{ and } x = a \\ \dfrac{\partial F}{\partial y} = X\dfrac{dY}{dy} = 0 & \text{in } y = 0 \text{ and } y = b \end{cases}$$

and therefore

$$\begin{aligned} X(x) &= B\cos(k_x x) \quad \text{with } k_x = \frac{m\pi}{a} \\ Y(y) &= D\cos(k_y y) \quad \text{with } k_y = \frac{n\pi}{b} \end{aligned}$$

where B and D are unknown amplitudes. Putting everything together,

$$F(x,y) = P\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad\qquad -\gamma_{c,mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

TM mode

$$F|_c = XY|_c = 0 \quad \Rightarrow \quad \begin{cases} X = 0 & \text{in } x = 0 \text{ and } x = a \\ Y = 0 & \text{in } y = 0 \text{ and } y = b \end{cases}$$

and therefore

$$\begin{aligned} X(x) &= A\sin(k_x x) \quad \text{with } k_x = \frac{m\pi}{a} \\ Y(y) &= C\sin(k_y y) \quad \text{with } k_y = \frac{n\pi}{b} \end{aligned}$$

where A and C are unknown amplitudes. Putting everything together,

$$F(x,y) = P\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \qquad\qquad -\gamma_{c,mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Summary

- TE_{mn} mode:

$$H_{z,mn} = P\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{-\gamma_{mn}z} \qquad \text{with } m > 0 \textbf{ or } n > 0.$$

- TM_{mn} mode:

$$E_{z,mn} = P \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) e^{-\gamma_{mn} z} \qquad \text{with } m > 0 \textbf{ and } n > 0.$$

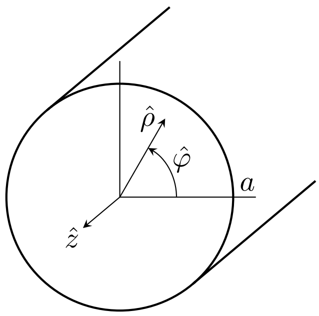
$$\gamma_{mn} = \sqrt{\gamma_0^2 - \gamma_{c,mn}^2} = \sqrt{-\omega^2 \mu\varepsilon + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$f_{c,mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} = \frac{c}{2\pi} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

The fundamental mode of a rectangular waveguide as defined is TE_{10}. Compute all the information about this mode, including the cutoff frequency and the different field components.

7 Circular waveguide

Geometry and coordinate system (cylindrical):



Invariance in ρ and φ dimensions (cylindrical coordinates): separation of variables.

$$F(\rho, \varphi) = R(\rho)P(\varphi)$$

Helmholtz equation $\left(\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)$

$$\nabla_t^2 F - \gamma_c^2 F = 0 \quad \Leftrightarrow \quad \frac{P}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{R}{\rho^2}\frac{\partial^2 P}{\partial\varphi^2} - \gamma_c^2 RP = 0$$

but

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) = \rho\frac{\partial^2 R}{\partial\rho^2} + \frac{\partial R}{\partial\rho}$$

Then, substituting and multiplying all the terms by ρ^2/RP,

$$\underbrace{\frac{\rho^2}{R}\frac{\partial^2 R}{\partial\rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial\rho} - \gamma_c^2\rho^2}_{\rho \text{ only}} = \underbrace{-\frac{1}{P}\frac{\partial^2 P}{\partial\varphi^2}}_{\varphi \text{ only}} \quad \Rightarrow \quad \begin{cases} -\dfrac{1}{P}\dfrac{d^2 P}{d\varphi^2} = k_\varphi^2 \\ \dfrac{\rho^2}{R}\dfrac{\partial^2 R}{\partial\rho^2} + \dfrac{\rho}{R}\dfrac{\partial R}{\partial\rho} - \gamma_c^2\rho^2 = k_\varphi^2 \end{cases}$$

General solution

$$P(\varphi) = A\sin(k_\varphi \varphi) + B\cos(k_\varphi \varphi)$$
$$R(\rho) = C\,J_{k_\varphi}(k_c\rho) + \cancel{D\,Y_{k_\varphi}(k_c\rho)} \quad \text{with } k_c = \sqrt{-\gamma_c^2}$$

where J_{k_φ} and Y_{k_φ} are the Bessel functions of the first and second kinds, with order k_φ. Since $Y_{k_\varphi}(0) \to \infty$, the last term has no physical meaning and $D = 0$. Additionally, $P(\varphi)$ must be periodical, $P(\varphi) = P(\varphi + 2\pi)$, so that $k_\varphi = n$.

Therefore,

$$F(\rho, \varphi) = \big(A\sin(n\varphi) + B\cos(n\varphi)\big)\,J_n(k_c\rho)$$

7.1 Perfect conductor boundary conditions

TE mode

$$\left.\frac{\partial F}{\partial n}\right|_c = \left.\frac{\partial F}{\partial \rho}\right|_{\rho=a} = 0 \quad \Leftrightarrow \quad J_n'(k_c a) = 0$$

- p'_{nm}: m-th zero of the derivative of the Bessel function of the first kind of order n

$$k_{c,nm} = \frac{p'_{nm}}{a} \quad \Leftrightarrow \quad -\gamma_{c,nm}^2 = \left(\frac{p'_{nm}}{a}\right)^2$$

$$\gamma_{nm} = \sqrt{-\omega^2\mu\varepsilon + \left(\frac{p'_{nm}}{a}\right)^2}$$

$$f_{c,nm} = \frac{p'_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

n	p'_{n1}	p'_{n2}	p'_{n3}
0	3.832	7.016	10.174
1	1.841	5.331	8.536
2	3.054	6.706	9.970

TM mode

$$\left.F\right|_c = \left.F\right|_{\rho=a} = 0 \quad \Leftrightarrow \quad J_n(k_c a) = 0$$

- p_{nm}: m-th zero of the Bessel function of the first kind of order n

$$k_{c,nm} = \frac{p_{nm}}{a} \quad \Leftrightarrow \quad -\gamma_{c,nm}^2 = \left(\frac{p_{nm}}{a}\right)^2$$

$$\gamma_{nm} = \sqrt{-\omega^2\mu\varepsilon + \left(\frac{p_{nm}}{a}\right)^2}$$

$$f_{c,nm} = \frac{p_{nm}}{2\pi a\sqrt{\mu\varepsilon}}$$

n	p_{n1}	p_{n2}	p_{n3}
0	2.405	5.520	8.654
1	3.832	7.016	10.174
2	5.135	8.417	11.620

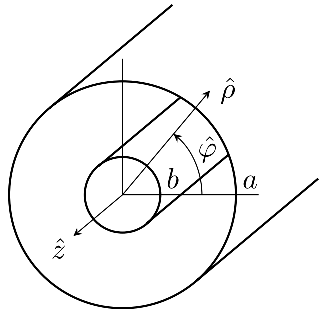
Notice that there are two indexes (as in any waveguide, since the cross section is two-dimensional) but their meaning is not the same as the rectangular waveguide ones.

- n: number of field maxima along the angular direction φ (that is, rotating around the cylinder axis). Also, order of the Bessel function. It can be zero, so both TE_{0m} and TM_{0m} are valid modes.
- m: number of field maxima along the radial direction ρ (that is, from the axis to the cylinder surface), or, equivalently, zero of $J_n(k_c a)$ or $J_n'(k_c a)$. Since the count starts at 1, it cannot be 0.

Compare the single mode band of a circular waveguide and a rectangular one (with $a = 2b$), if the fundamental modes of both waveguides have the same cutoff frequency.

8 Coaxial waveguide

Geometry and coordinate system (cylindrical):



Higher order modes (TE and TM) can be computed just as for cylindrical waveguides, with an additional boundary condition. Here, we are interested in the TEM mode that is supported by the two non-connected conductors. Since this is a TEM mode, $\gamma = \gamma_0 = j\omega\sqrt{\mu\varepsilon}$ and $Z_{\text{TEM}} = \sqrt{\mu/\varepsilon} = \eta$, as is already known. The TEM mode is associated to a difference of potential between the conductors. Thus, consider that the potentials of the external and internal conductors are 0 and V_0, respectively (that is, the external conductor is the reference, or ground) for $z = 0$.

> One difference between the cylindrical waveguide and the coaxial higher order modes is that in the latter problem the Bessel function of the second kind $Y_n(k_c\rho)$ is not canceled. Why?

The transversal problem to be solved for a TEM mode is the Laplace's equation of its scalar potential ϕ:

$$\nabla_t^2 \phi = 0$$

that, expressed in cylindrical coordinates, becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

This problem can be solved by applying the same separation of variables already used for cylindrical waveguides, $\phi(\rho, \varphi) = R(\rho)P(\varphi)$, with identical general solution,

$$\begin{aligned} P(\varphi) &= A \sin(n\varphi) + B \cos(n\varphi) \\ R(\rho) &= C\, J_n(k_c\rho) + D\, Y_n(k_c\rho) \end{aligned}$$

But now, the boundary conditions impose that $\phi(a, \varphi)$ and $\phi(b, \varphi)$ are constant, and therefore $P(\varphi)|_{\rho=a}$ and $P(\varphi)|_{\rho=b}$ are also constant. Consequently, $P(\varphi)$ is itself constant (i.e., $n = 0$). Without loss of generality we consider $P(\varphi) = 1$ (the constant B is absorbed by other terms). In other words, the potential has no angular variation, and $\phi(\rho, \varphi) = \phi(\rho) = R(\rho)$.

The Laplace's equation is now one-dimensional, and can be easily solved by integrating twice,

$$\frac{d}{d\rho}\left(\rho \frac{d\phi}{d\rho}\right) = 0 \;\Rightarrow\; \frac{d\phi}{d\rho} = \frac{c_1}{\rho} \;\Rightarrow\; \phi(\rho) = c_1 \ln \rho + c_2$$

where c_1 and c_2 are coefficients that must be solved by enforcing the boundary conditions,

$$\left.\begin{aligned} \phi(a) = c_1 \ln a + c_2 = 0 \\ \phi(b) = c_1 \ln b + c_2 = V_0 \end{aligned}\right\} \Rightarrow \quad c_1 = \frac{V_0}{\ln b/a}, \quad c_2 = -c_1 \ln a$$

The potential field is, therefore,

$$\phi(\rho) = V_0 \,\frac{\ln \rho/a}{\ln b/a}$$

Both the electric and the magnetic fields can be directly computed from the potential. Remember that they are fully transversal. They do not vary with φ, just like the potential.

$$\vec{E}(\rho, z) = -\nabla_t \phi(\rho)\, e^{-\gamma_0 z} = -\hat{\rho}\,\frac{d\phi}{d\rho}\, e^{-\gamma_0 z} = -\hat{\rho}\,\frac{V_0}{\ln b/a}\frac{1}{\rho}\, e^{-\gamma_0 z}$$

$$\vec{H}(\rho, z) = \frac{\hat{z} \times \vec{E}(\rho, z)}{Z_{\mathrm{TEM}}} = -\hat{\varphi}\,\frac{V_0}{\eta \ln b/a}\frac{1}{\rho}\, e^{-\gamma_0 z}$$

It is interesting to compute the current associated to the TEM mode, as the line integral of $\vec{H}$ along any closed path c that fully separates both conductors. Since the result does not depend on the path, we choose a circular path of fixed ρ, with $b < \rho < a$, and $z = 0$.

$$I_0 = \oint_c \vec{H}(\rho, 0) \cdot \vec{dl} = \int_0^{2\pi} \left(-\hat{\varphi}\frac{V_0}{\eta \ln b/a}\frac{1}{\rho}\right) \cdot (-\hat{\varphi}\rho\, d\varphi) = \frac{2\pi V_0}{\eta \ln b/a}$$

Both the difference of potential (or *voltage*) and the current are computed at $z = 0$ but, of course, the result is also the same for other values of z, due to the problem invariance.

Why the minus sign in the definition of $\vec{dl}$?

The relationship between the voltage and the current is, by definition, the **characteristic impedance**:

$$Z_0 \triangleq \frac{V_0}{I_0} = \frac{\eta}{2\pi} \ln \frac{b}{a} = \eta\, K$$

Notice that $Z_0 = \eta\, K$ where K is a factor that depends on the geometry, but not the scale. Notice also that Z_0 is independent of the coordinate z where it is calculated, due to the invariance. These statements are general results for every TEM mode.

Verify that the voltage can be defined as an integral of $\vec{E}$ along a path between the conductors. Consider a path with fixed ϕ from $\rho = a$ to $\rho = b$ (and $z = 0$). In other words, prove that

$$\int_{\rho=a}^{\rho=b} \vec{E}(\rho, 0) \cdot \vec{dl} = V_0$$

Finally the variation of the fields as $e^{-\gamma_0 z}$ (taking into account only the incident wave) allows us to write,

$$\vec{E}(\rho, z) = \vec{E}(\rho, 0)\, e^{-\gamma_0 z} \qquad \vec{H}(\rho, z) = \vec{H}(\rho, 0)\, e^{-\gamma_0 z}$$

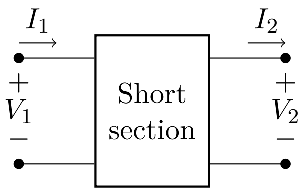
and since the voltage and current integrals are computed for constant z, it can be concluded that they also behave as waves,

$$V(z) = V_0\, e^{-\gamma_0 z} \qquad\qquad I(z) = I_0\, e^{-\gamma_0 z}$$

Again, this last result is valid for any TEM mode, not only coaxial waveguide ones.

8.1 Short section approximation

Consider a short section ($|\gamma_0 l| \ll 1$) of waveguide supporting the forward wave of a TEM mode, as a two-port network. The model is valid since voltages $V_1 = V(z)$, $V_2 = V(z+l)$ and currents $I_1 = I(z)$, $I_2 = I(z+l)$ are well defined.

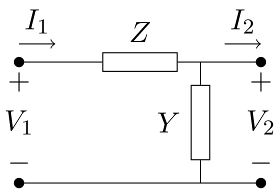


$$
\begin{aligned}
V_2 &= V_1 e^{-\gamma_0 l} = V_1 \left(1 - \gamma_0 l + \frac{(\gamma_0 l)^2}{2} - \cdots\right) \approx V_1\big(1 - \gamma_0 l\big) \\
I_1 &= I_2 e^{\gamma_0 l} = I_2 \left(1 + \gamma_0 l + \frac{(\gamma_0 l)^2}{2} + \cdots\right) \approx I_2\big(1 + \gamma_0 l\big)
\end{aligned}
$$

and by definition of the characteristic impedance, $V_1/I_1 = V_2/I_2 = Z_0$. Therefore,

$$
\begin{aligned}
V_1 - V_2 &= \gamma_0 l\, V_1 = Z_0 \gamma_0 l\, I_1 \\
I_1 - I_2 &= \gamma_0 l\, I_2 = \frac{\gamma_0 l}{Z_0} V_2
\end{aligned}
$$

But this is the behaviour of a simple series-parallel association of impedances,



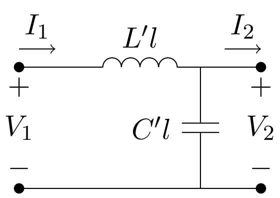
$$
\begin{aligned}
V_1 - V_2 &= Z I_1 \\
I_1 - I_2 &= Y V_2
\end{aligned}
\implies Z = Z_0 \gamma_0 l, \qquad Y = \frac{\gamma_0 l}{Z_0}
$$

Check that this model is also valid if the waveguide supports the backward wave of the TEM mode, if the currents I_1, I_2 are defined in the opposite direction. Since the network is linear, this makes the model valid for any arbitrary solution, that is superposition of both the forward and backward waves, as long as we take into account this definition of their currents with opposite directions.

But for a TEM mode,

$$
\begin{aligned}
Z_0 &= \eta K = K\sqrt{\frac{\mu}{\varepsilon}} \\
\gamma_0 &= j\omega\sqrt{\mu\varepsilon}
\end{aligned}
\implies
\begin{aligned}
Z &= j\omega\mu K l \\
Y &= j\omega\frac{\varepsilon}{K} l
\end{aligned}
$$

For a lossless waveguide (perfect conductor, real ε),



Primary line constants:

- $L' \equiv \mu K$: inductance per unit length (H/m)
- $C' \equiv \frac{\varepsilon}{K}$: capacitance per unit length (F/m)

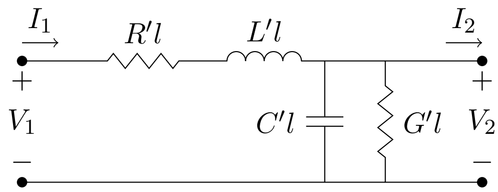
since $Z = j\omega L' l$ and $Y = j\omega C' l$ correspond to an inductor ($L'l$) and a capacitor ($C'l$), respectively.

Notice that Z_0, γ_0 can be extracted from the primary line constants

$$
Z_0 = \sqrt{\frac{L'}{C'}} \qquad \gamma_0 = j\omega\sqrt{L'C'}
$$

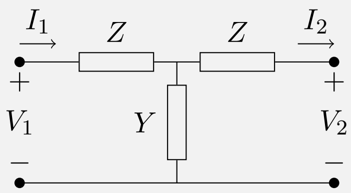
and are sometimes called *secondary line constants*. Since $c = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{L'C'}}$, the inductance p.u.l. can be extracted from the capacitance p.u.l. if the dielectric is known. Therefore, the TEM mode can be fully characterized by computing the capacitance at f_0 (quasi-static approximation).

For lossy waveguides, $\varepsilon = \varepsilon' - j\varepsilon''$, the conductors are resistive, and two additional primary line constants are necessary:



Primary line constants:

- $L' \equiv \mu K$: inductance p.u.l. (H/m)
- $C' \equiv \frac{\varepsilon'}{K}$: capacitance p.u.l. (F/m)
- $G' \equiv \frac{\omega\varepsilon''}{K}$: shunt conductance p.u.l. (S/m)
- R': series resistance p.u.l. (Ω/m)



There are other possible lumped-element models for a short section of waveguide supporting a TEM mode, as the symmetrical T shown here. Prove that it is a valid model for a lossless waveguide by finding the impedance Z and admittance Y in terms of the primary line constants.