GUIDED PROPAGATION FORMULAS

WAVEGUIDES

SOLVING WAVE EQUATIONS

Separation of variables in E_z

$$E_z = F_e(\xi_1, \xi_2) A_e(z)$$

Separation constant

$$\gamma = \sqrt{\gamma_0^2 - \gamma_c^2} = \sqrt{\omega^2 \mu \epsilon - \gamma_c^2}$$

Where $\gamma_0^2 < 0$ for PEC materials.

TM Fields / phasors

$$egin{aligned} ec{E_z} &= E_z \hat{z} = F_e(\xi_1, \xi_2) e^{-\gamma z} \hat{z} \ ec{H}_t &= -rac{j\omega\epsilon}{\gamma_c^2}
abla_t E_z imes \hat{z} \ ec{E}_t &= rac{\gamma_c^2}{\gamma_c^2}
abla_t E_z \end{aligned}$$

TE Fields / phasors

$$egin{aligned} ec{H_z} &= H_z \hat{z} = F_h(\xi_1, \xi_2) e^{-\gamma z} \hat{z} \ ec{E_t} &= -rac{j\omega\epsilon}{\gamma_c^2}
abla_t H_z imes \hat{z} \ ec{H_t} &= rac{\gamma_z}{\gamma_z^2}
abla_t H_z \end{aligned}$$

Mode impedances

$$egin{aligned} Z_{TM} &= rac{\hat{z} imes ec{E}_t}{ec{H}_t} = rac{\gamma}{j\omega\epsilon} \ Z_{TE} &= rac{\hat{z} imes ec{E}_t}{ec{H}_t} = rac{j\omega\mu}{\gamma} \ Z_{TEM} &= rac{\hat{z} imes ec{E}_t}{ec{H}_t} = \eta_0 \end{aligned}$$

Power flow

$$P_m = rac{1}{2} \Re \left[rac{1}{Z_m}
ight] \iint_{\Sigma} |ec{E}_t|^2 d\sigma$$

PEC BOUNDARY CONDITIONS

Cutoff frequency

$$f_c = rac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{-\gamma_c^2}$$

Propagation constant

$$\gamma = \pm \gamma_0 \sqrt{1 - \left(rac{f_c}{f}
ight)^2}$$

Where $\gamma_0=j\omega\sqrt{\mu\epsilon}$

Mode impedances

$$egin{aligned} Z_{TE} &= rac{\pm \eta}{\sqrt{1-\left(rac{f_c}{f}
ight)^2}} \ Z_{TM} &= \pm \eta \sqrt{1-\left(rac{f_c}{f}
ight)^2} \end{aligned}$$

Using + for $f > f_c$ and - for $f < f_c$.

LOSSY DIELECTRICS

Propagation constant

$$\gamma = \sqrt{-\omega^2 \mu (\epsilon' - j \epsilon'') - \gamma_c^2} = lpha + j eta$$

Cutoff frequency: same as PEC, where $\alpha=\beta$

Good Dielectric (low loss)

$$\gamma = \sqrt{-\omega^2 \mu_0 \epsilon_0 \epsilon_r (1-j an\delta) - \gamma_c^2} \ an\delta << \Longrightarrow egin{cases} lpha_d pprox rac{k_0^2 an\delta}{2eta} & (ext{TE}, ext{TM}) \ lpha_d pprox rac{k_0 an\delta}{2} & (ext{TEM}) \end{cases}$$

REAL CONDUCTORS

$$lphapproxlpha_g+lpha_c \ etapproxeta_g$$

Where $lpha_q$ and eta_q are PEC values

Joule Effect attenuation constant

$$lpha_c = rac{W_c}{2W_t} = rac{rac{1}{2}\oint_Crac{1}{\sigma\delta}ec{H}(0)\cdotec{H}(0)^*dl}{\Re\left[\iint_sec{E}_t(0) imesec{H}_t(0)^*\cdotec{ds}
ight]}$$

Rectangular Waveguide TE_{10}

$$lpha_c = rac{R_s}{a^3beta_g k\eta}\left(2b\pi^2 + a^3k^2
ight)$$

Where R_s is the surface resistance:

$$R_s = \sqrt{rac{\omega \mu}{2\sigma}} = \Re\left[rac{1+j}{\sigma \delta_s}
ight]$$

Rectangular waveguide a imes b

$$\gamma_c^2 = -k_x^2 - k_y^2$$

Rectangular TE_{mn} modes

$$k_x=rac{m\pi}{a} \ k_y=rac{n\pi}{b}$$

Propagation constant

$$\gamma_{mn} = \sqrt{-\omega^2 \mu \epsilon + \left(rac{m\pi}{a}
ight)^2 + \left(rac{n\pi}{b}
ight)^2}$$

Cutoff frequency

$$f_{c;mn} = rac{c}{2\sqrt{\epsilon_{n}}}\sqrt{\left(rac{m}{a}
ight)^{2}+\left(rac{n}{b}
ight)^{2}}$$

Power flow

$$\frac{|\mathrm{E}|^2}{2Z_m} \cdot \frac{ab}{2}$$

Phasors

$$E_x = E_{x0}\cos\left(rac{m\pi x}{a}
ight)\sin\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ E_y = E_{y0}\sin\left(rac{m\pi x}{a}
ight)\cos\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ H_z = H_{mn}^i\cos\left(rac{m\pi x}{a}
ight)\cos\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ H_x = rac{\gamma_{mn}m\pi}{k_c^2a}H_{mn}^i\sin\left(rac{m\pi x}{a}
ight)\cos\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ H_y = rac{\gamma_{mn}n\pi}{k_c^2b}H_{mn}^i\cos\left(rac{m\pi x}{a}
ight)\sin\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ E_x = rac{j\omega\mu n\pi}{k_c^2b}H_{mn}^i\cos\left(rac{m\pi x}{a}
ight)\sin\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ E_y = rac{-j\omega\mu n\pi}{k_c^2a}H_{mn}^i\sin\left(rac{m\pi x}{a}
ight)\cos\left(rac{n\pi y}{b}
ight)e^{-\gamma_{mn}z} \ k_c = \sqrt{\left(rac{m\pi}{a}
ight)^2 + \left(rac{n\pi}{b}
ight)^2}$$

TRANSMISSION LINES (TEM MODES)

GENERIC TL

- Incident wave (positive): V_0^+
- Reflected wave (negative) V_0^-

Voltage

$$V(z) = V_0 e^{-\gamma_0 z}
onumber \ V_0 = \int_1^2
onumber
onumber \ V_t \phi \cdot dec{l}$$

Current

$$I(z) = I_0 e^{-\gamma_0 z}
onumber \ I_0 = \oint_{C2} rac{
abla_t \phi \cdot \hat{n}}{\eta} dl$$

COAXIAL LINE (a INNER, b OUTER)

Primary Parameters

$$Z_0 = rac{\eta}{2\pi} \ln rac{b}{a} \ \gamma_0 = jeta = j\omega \sqrt{\mu\epsilon_0\epsilon_r}$$

Characteristic Impedance

$$Z_0 = rac{V_0}{I_0} = \eta \cdot K$$

Where K is a dimensionless factor depending on geometry

Transmitted power

$$P_t = rac{1}{2}\Re(VI^*) = rac{|V|^2}{2Z_0} = rac{|I|^2Z_0}{2}$$

CIRCUIT MODEL

ABCD Parameters

$$egin{pmatrix} a & b \ c & a \end{pmatrix} \ a = \cosh(\gamma_0 l) \ b = Z_0 \sinh(\gamma_0 l) \ c = rac{1}{Z_0} \sinh(\gamma_0 l)$$

Equivalent Network

Equivalent half-T model (series $Z_s l$, parallel $Y_p l$)

$$Z_s = Z_0 \gamma_0 = j \omega \mu K \ Y_p = rac{\gamma_0}{Z_0} = \left(rac{\omega \epsilon''}{K} + j \omega rac{\epsilon'}{K}
ight)$$

First HOM f_c (TE_{11})

$$f_c pprox rac{c}{2\pi\sqrt{\epsilon_r}} rac{2}{a+b}$$

Equivalent Lumped Elements

T network with 2 Ll/2 inductors in series and Cl capacitor and Gl resistor in parallel

$$L=\mu K=\mu rac{Z_0}{\eta}$$
 $G=\omega rac{\epsilon''}{K}=\omega \epsilon'' rac{\eta}{Z_0}$ $C=rac{\epsilon'}{K}=\epsilon' rac{\eta}{Z_0}$

Where L, G, C are the **Secondary Parameters**

PARAMETER RELATIONSHIPS

Primary and secondary parameters

$$egin{aligned} Z_0 &= \sqrt{rac{j\omega L}{G+j\omega C}} = \sqrt{rac{Z_s}{Y_p}} \ \gamma_0 &= \sqrt{j\omega L(G+j\omega C)} = \sqrt{Z_s Y_p} \end{aligned}$$

Low loss dielectric

$$Z_0pprox \sqrt{rac{L}{C}} \ \gamma_0pprox Grac{Z_0}{2}+j\omega\sqrt{LC}=lpha_d+jeta$$

PEC and lossless dielectric

$$Z_0pprox \sqrt{rac{L}{C}} \ \gamma_0pprox j\omega\sqrt{LC}=jeta$$

Secondary parameters

$$egin{aligned} L &= \mu rac{\epsilon'}{C} \ G &= \omega rac{\epsilon''}{\epsilon'} C \ C &= rac{Q_t}{V_c} \end{aligned}$$

REAL CONDUCTORS

- TEM mode becomes quasi-TEM
- Add Rl/2 in series with each inductor in T-circuit model

Parameter relationships

$$egin{aligned} \gamma_0 &= \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{Z_s Y_p} \ Z_0 &= \sqrt{rac{R+j\omega L}{G+j\omega C}} = \sqrt{rac{Z_s}{Y_p}} \end{aligned}$$

Good dielectric & good conductors

$$lpha = rac{1}{2}\left(rac{R}{Z_0} + GZ_0
ight) = lpha_c + lpha_d \ eta = \omega\sqrt{LC} \ Z_0 = \sqrt{rac{L}{C}}$$

TERMINATED TRANSMISSION LINE

- End of TL: z=0
- Load impedance Z_L (known)
- Alias axis l=-z

$egin{aligned} Z_L &= rac{V(0)}{I(0)} = rac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0 \ V_0^- &= rac{Z_L - Z_0}{Z_L + Z_0} V_0^+ \end{aligned}$

REFLECTION

Reflection coefficient (voltage)

$$\Gamma(0) = rac{V_0^-}{V_0^+} = rac{Z_L - Z_0}{Z_L + Z_0}$$

- ullet $|\Gamma| \leq 1$ for passive loads and $|V_0^-| \leq |V_0^+|$
- ullet $Z_L=Z_0\Rightarrow \Gamma(0)=0$ (no reflected wave)

$$Z_0=rac{1-\Gamma(0)}{1+\Gamma(0)}Z_L$$

Standing Wave

$$egin{aligned} V(z) &= V_0^+ \left(e^{-jeta z} + \Gamma(0)e^{+jeta z}
ight) \ I(z) &= rac{V_0^+}{Z_0} \left(e^{-jeta z} - \Gamma(0)e^{+jeta z}
ight) \end{aligned}$$

Generalized Reflection Coefficient

$$egin{aligned} \Gamma(z) &= rac{V^-(z)}{V^+(z)} = \Gamma(0) e^{2jeta z} \ V(z) &= V_0^+ e^{-jeta z} \left(1 + \Gamma(z)
ight) \ I(z) &= rac{V_0^+}{Z_0} e^{-jeta z} \left(1 - \Gamma(z)
ight) \end{aligned}$$

Transmitted Power

$$P_L = rac{|V_0^+|^2}{2Z_0}(1-|\Gamma|^2) = P_i(1-|\Gamma|^2)$$

- $\Gamma(0)=0\Rightarrow P_L=P_i$ (Matched Load, all power delivered)
- $|\Gamma(0)|=1\Rightarrow P_L=0$ (Pure standing wave, no power delivered)

$$\circ \ \Gamma(0) = 1 \Rightarrow Z_L = \infty$$
 (open)

$$\circ \ \Gamma(0) = -1 \Rightarrow Z_L = 0$$
 (short)

• Rest:
$$\Gamma(0) = \rho e^{j\theta} = u + jv$$

STANDING WAVES

• Periodicity $\lambda/2$

Voltage magnitude

$$egin{aligned} |V(z)| &= |V_0^+| \left| 1 +
ho e^{ heta - 2eta l}
ight| \ V_{max} &= 1 +
ho \ V_{min} &= 1 -
ho \end{aligned}$$

Standing Wave Ratio

$$SWR = rac{V_{max}}{V_{min}} = rac{1+|\Gamma|}{1-|\Gamma|} = rac{1+
ho}{1-
ho}$$

- ullet SWR=1 for matched load
- ullet $SWR=\infty$ for mismatched load (open or short circuit)

Return Loss

$$RL_{dB} = -20 \log |\Gamma|$$

Input (transformed) impedance

$$Z_{in}(l) = rac{V(l)}{I(l)} = rac{V_0^+ e^{-jeta z} + V_0^- e^{+jeta z}}{V_0^+ e^{-jeta z} - V_0^- e^{+jeta z}} Z_0$$

- ullet Matched load: $Z_{in}=Z_0 orall l$
- Short termination: $Z_{in}(l) = jZ_0 \tan(\beta l)$
- Open termination: $Z_{in}(l) = -jZ_0\cot(eta l)$

Relationship between Z_{in} and $\Gamma(l)$

$$\Gamma(l) = rac{Z_{in}(l)-Z_0}{Z_{in}(l)+Z_0} \ Z_{in}(l) = Z_0rac{1+\Gamma(l)}{1-\Gamma(l)}$$

LOSSY TRANSMISSION LINE

- ullet Z_0 is complex
- No stable standing wave

Reflection coefficient (lossy)

$$\Gamma(l) = \Gamma(0) e^{-2lpha l} e^{2jeta l}$$

Input impedance (lossy)

$$egin{aligned} Z_{in}(-l) &= Z_0 rac{Z_L \cosh(\gamma l) + Z_0 \sinh(\gamma l)}{Z_0 \cosh(\gamma l) + Z_L \sinh(\gamma l)} \ &= R + j X \end{aligned}$$