# MICROWAVE CIRCUIT ANALYSIS: SCATTERING PARAMETERS

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# Learning goals

- Describe
  - New tool for analysis of generic microwave circuits that will be valid for any kind of microwave circuit
  - This tool avoids the application of the complex Maxwell equations
  - The circuit as an entity where only the power at its ports will be considered
  - S-parameters as the circuit parameters that relate the power transfer between its ports
  - The properties of these S-parameters for lossless and isotropic networks
  - The concepts of matched and uncoupled port

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- Definition of generalized voltages and currents: concept of impedance.
- Generic concept of microwave circuits.
- Description of junctions of a single waveguide:
- Description of multiple waveguide junctions
    - Description of a junction using generalized voltages and currents
    - Description of a junction using power waves: scattering matrix and S-parameters
        - Properties: symmetry, shift in reference planes.
        - Physical meaning of scattering parameters: transmission and reflection, matching and uncoupled ports

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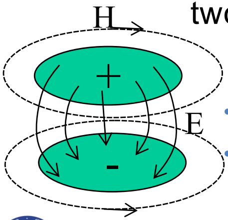
# Concepts of generalized voltage and current I

- Problems and contradictions:
  - Direct measurement of voltages and currents is not possible with microwaves.
  - However, definition of voltages and currents at terminals is useful:
    - For structures that support TEM modes there are unambiguous definitions of voltage and current waveforms along the distance.
    - Structures that do not support pure TEM modes there are not univocal definitions. Voltage is defined as the integral of transversal E-fields between two points and current as circulation of H-fields.

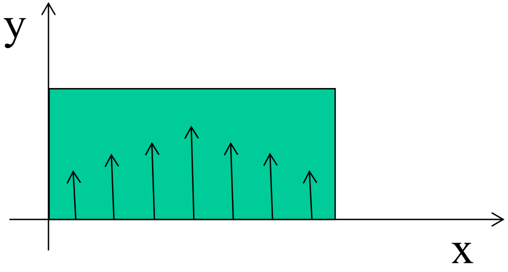
$$V = \int_{+}^{-} \vec{E} \cdot d\vec{l} \, ; I = \oint_{C^+} \vec{H} \cdot d\vec{l}$$




- For TM modes the integral is zero (to prove as an exercise).
- For TE modes the resulting value depends on the path of integration

# Concepts of generalized voltage and current II

- Example for a rectangular waveguide and $TE_{10}$ mode: depends on the position x.




$$E_{y,10} = -\frac{j\omega\mu a}{\pi} P sen \frac{\pi}{a} x \cdot \exp(-j \cdot \beta \cdot z)$$

$$H_{x,10} = \frac{\gamma a}{\pi} P \cdot sen \frac{\pi}{a} x \cdot \exp(-j \cdot \beta \cdot z)$$

$$V = -\frac{j\omega\mu a}{\pi} P sen \frac{\pi}{a} x \cdot \exp(-j \cdot \beta \cdot z) \cdot \int_y dy$$

# Concepts of generalized voltage and current III

- Previous voltage depends on the position x and the integration path. There is not a single voltage.
- Conclusions from waveguide theory:
  - Transversal fields are involved in transmitted power.
  - For a lossless waveguide total transmitted power is superposition of transmitted power by each mode.
  - Transversal fields vary along the longitudinal component as an exponential.
  - Transversal E and H fields are related through the modal impedance.

# Concepts of generalized voltage and current IV

- Suppose a waveguide that supports **only** one propagating mode:

$$\left.\begin{aligned}\vec{E}_t(x,y,z)&=\vec{e}_t(x,y)\cdot\left[A^+\cdot e^{-j\beta z}+A^-\cdot e^{j\beta z}\right]\\ \vec{H}_t(x,y,z)&=\vec{h}_t(x,y)\cdot\left[A^+\cdot e^{-j\beta z}-A^-\cdot e^{j\beta z}\right]\end{aligned}\right\}\Rightarrow \vec{h}_t(x,y)=\frac{\hat{z}\times\vec{e}_t(x,y)}{Z_{wave}}$$

- Equivalent voltage and current are defined as those **complex numbers** associated to the mode so that the product of voltage and current is twice the transmitted power.

$$\left.\begin{aligned}V&=V^+\cdot e^{-j\beta z}+V^-\cdot e^{j\beta z}\\ I&=I^+\cdot e^{-j\beta z}-I^-\cdot e^{j\beta z}\end{aligned}\right\}\Rightarrow\begin{cases}V^+=K_1\cdot C^+;V^-=K_1\cdot C^-\\ I^+=K_2\cdot C^+;I^-=-K_2\cdot C^-\end{cases}$$

# Concepts of generalized voltage and current V

- Applying the definition of generalized voltages and currents

$$\frac{1}{2}V^{+}\cdot\left(I^{+}\right)^{*}=\frac{\left|C^{+}\right|^{2}}{2}\int_{S}\vec{e}\times\vec{h}\cdot\hat{z}ds$$

$$K_{1}\cdot K_{2}=\int_{S}\vec{e}\times\vec{h}\cdot\hat{z}ds$$

- An equivalent characteristic impedance can be defined as:

$$Z_{C}=\frac{V^{+}}{I^{+}}=\frac{V^{-}}{I^{-}}=\frac{K_{1}}{K_{2}}=\begin{cases}1\\ Z_{wave}\end{cases}$$

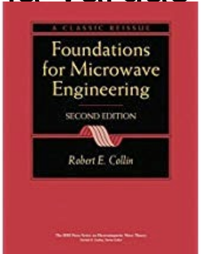
- This way, a waveguide is represented as an equivalent transmission line.

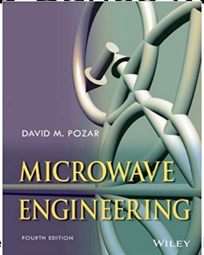
# Concepts of generalized voltage and current VI

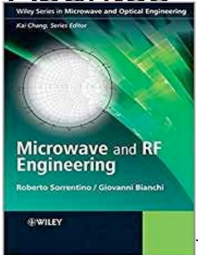
- Considerations:
  - When the waveguide supports N modes it is equivalent to N lines.
  - Therefore the number of physical terminals of the waveguide is lower than the number of mathematical terminals used to represent it.
  - A discontinuity generates N modes. If the waveguide is correctly dimensioned all of them but one will be evanescent modes and will not have any effect at some distance.

# References


- R. Collin, “Foundations For Microwave Engineering”, 2nd Edition, Wiley 2000, or any previous edition: chapter 4.
- D. Pozar, “*Microwave Engineering*”, 4th Edition 2011 (or any previous edition), chapter 4.
- R. Sorrentino, “*Microwave and RF Engineering*”, Wiley 2010, chapter 4.
- D. Segovia Vargas and Carlos Martín; Apuntes

# Index

# Generic Concept of Microwave Circuits: Junctions of a Single Waveguide I

- Definition: a one-port network is a circuit where energy can leave or enter only through one terminal that is prolongation of a waveguide or transmission line.
- Definition: terminal plane of a waveguide is an arbitrary straight section where all higher order modes (not the fundamental one) are evanescent and can be considered completely attenuated. It is also called **reference plane**.
- Computing the Poynting vector that crosses the terminal plane:

$$\frac{1}{2}\oint_{\Sigma} \vec{E}\times\vec{H}^{*}\cdot\hat{z}\cdot dS = P_{loss} + 2jw\cdot\left(W_m - W_e\right)$$

# Generic Concept of Microwave Circuits: Junctions of a Single Waveguide

- The previously defined generalized voltage and current are used (V and I, complex numbers defined on vector spaces $\mathcal{V}$ and $\mathcal{J}$).
- These V and I are unique for each E and H field distribution.
  - **V and I in transmission lines are physical magnitudes**
  - **V and I in waveguides are models**

$$\frac{1}{2} V \cdot I^* = P_{loss} + 2jw \cdot \left( W_m - W_e \right)$$

- The map between vector $\mathcal{V}$ and $\mathcal{J}$ spaces is biunivocal and linear, and therefore there is a complex number that relates each values V and I.

$$Z_{in} = \frac{V}{I} = \frac{\frac{1}{2} V \cdot I^*}{\frac{1}{2} I \cdot I^*} = \frac{P_{loss} + 2jw \cdot \left( W_m - W_e \right)}{\frac{1}{2} I \cdot I^*} = R + jX$$

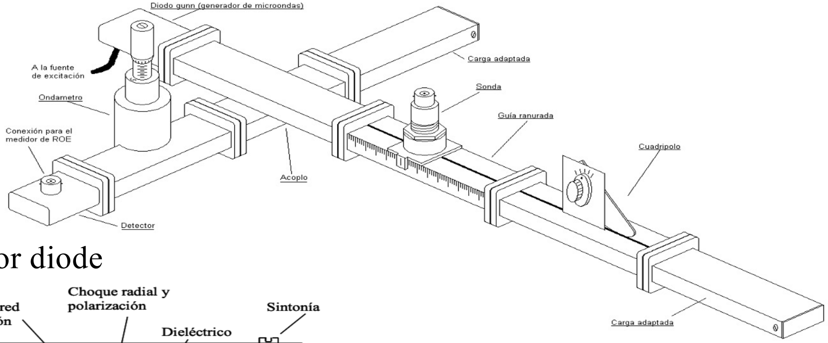
# Physical Conditions for Existance of Equivalent Impedances and Admittances I

- Since the map between $\mathcal{V}$ and $\mathcal{I}$ is bijective and linear, $Y=Z^{-1}=G+jB$ exists. This description using Z and Y is valid as long as a dominant mode is present.
- Passive termination:
  - Lossy ($P_{loss}>0$): $Re(Z)=R>0$, $Re(Y)=G>0$
    - If $W_E=W_H$ there are resonance conditions.
  - Lossless ($P_{loss}=0$):
    - The real part is 0 and the partial derivative of the reactance with respect to the frequency is positive. Therefore the reactance has positive slope and poles and zeros are alternated.
    - The impedance is pure imaginary.
    - If $W_E=W_H$ there are resonance conditions.

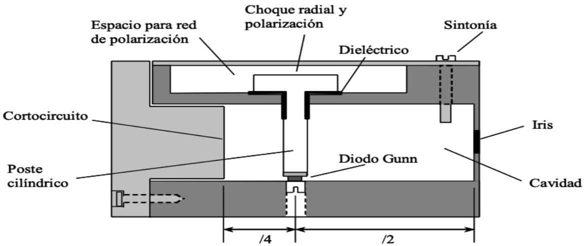
# Physical Conditions for Existance of Equivalent Impedances and Admittances I

- Even and odd properties of input impedance with respect to the frequency (proof at Collin, pp. 232):
  - Real part is an even function of frequency.
  - Imaginary part is an odd function of frequency.
  - (This helps to establish auxiliary functions useful to define resistances or reactances)

# Model of Microwaves Workbench and Detail of One-Port Circuits

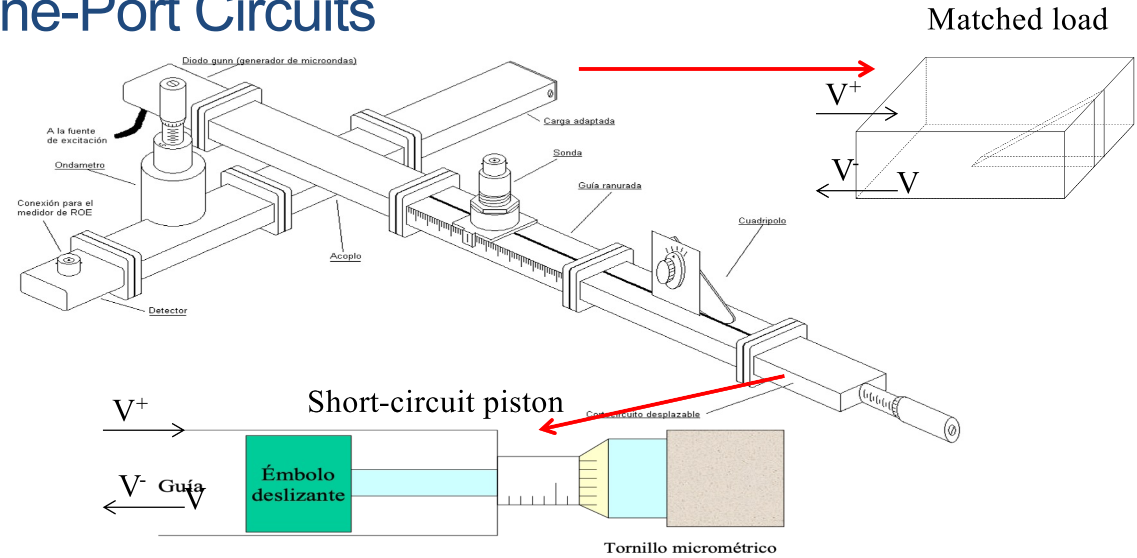



Detector diode




(a)

# Model of Microwaves Workbench and Detail of One-Port Circuits



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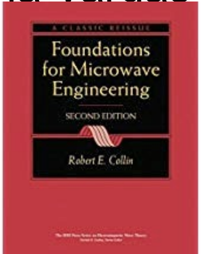


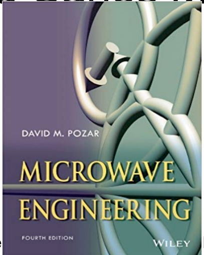


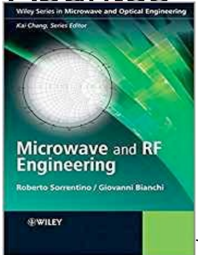

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# Waveform description of a termination I

- Concepts of generalized voltages and currents allow to associate incident and reflected voltage and current waveforms at the reference plane of a termination.

$$V = V^{+} \cdot e^{-j\beta z} + V^{-} \cdot e^{j\beta z} = V_{inc} + V_{ref}$$

$$I = \frac{1}{Z_0}\left[V^{+} \cdot e^{-j\beta z} - V^{-} \cdot e^{j\beta z}\right] = I_{inc} + I_{ref}$$

- Two complex waves *a* and *b* are introduced so that their squared magnitudes are the incident and reflected powers at the reference plane

$$\left.\begin{array}{l} V_{inc} = a \cdot g; I_{inc} = \dfrac{a \cdot g}{Z_0} \left(g, Z_0 : real\right) \\ P_{inc} = \dfrac{1}{2} V_{inc} \cdot I_{inc}^{*} = \dfrac{1}{2} a \cdot g \dfrac{a^{*} \cdot g}{Z_0^{*}} = a \cdot a^{*} \end{array}\right\} \Rightarrow g = \sqrt{2Z_0}$$

- These waves *a* and *b* are called power waveforms.

# Waveform description of a termination II

- Generalized voltages and currents can be expressed in terms of the power waves:

$$\left.\begin{aligned} V &= \sqrt{2Z_0}\cdot(a+b) \\ I &= \frac{\sqrt{2}}{\sqrt{Z_0}}\cdot(a-b) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} a &= \frac{V + Z_0\cdot I}{\sqrt{8Z_0}} \\ b &= \frac{V - Z_0\cdot I}{\sqrt{8Z_0}} \end{aligned}\right.$$

# Waveform description of a termination III

- Definition of reflection coefficient at the reference plane (quotient of tangent components of fields):

$$\Gamma = \frac{V_{ref}}{V_{inc}} = \left.\frac{b}{a}\right|_{si\ plano\ de\ referencia\ en\ -z} = \frac{V^-}{V^+} \cdot e^{-2j\beta z}$$

- Definition of impedance in terms of generalized voltages and currents

$$Z = \frac{V}{I} = \frac{\sqrt{2Z_0}\cdot(a+b)}{\frac{\sqrt{2}}{\sqrt{Z_0}}\cdot(a-b)} = Z_0 \cdot \frac{1+\Gamma}{1-\Gamma}$$

  - Expression that allows the generalization of results for transmission lines to waveguides since the function is biunivocal and linear

# Waveform description of a termination IV

- Physical conditions:
  - Passive termination:
    - Non-dissipative: $|\Gamma| = 1$
    - Resonance: $W_H = W_E$ y $\Gamma\, real$
  - Matched load: $\Gamma = 0$ (complex number *b* is 0 for every *a*)
  - Short-circuit: $\Gamma = -1$

$$1 - \Gamma \cdot \Gamma^* = \left.\frac{P}{a \cdot a^*}\right|_{P>0} \Rightarrow |\Gamma| \leq 1$$

$$\mathrm{Im}(\Gamma) = \omega \cdot \frac{W_H - W_E}{a \cdot a^*}$$

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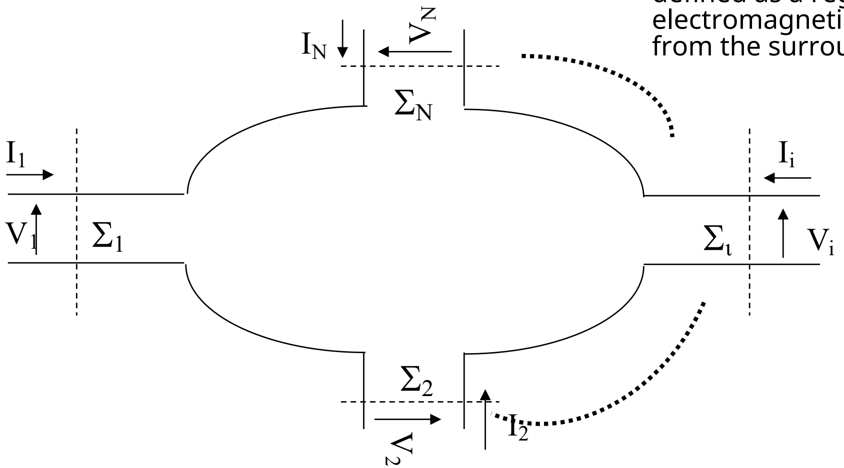
# Waveguide Junctions

A microwave circuit can be defined as a region electromagnetically isolated from the surrounding space.




$$\sum_{n=1}^{N}\frac{1}{2}\oint_{\Sigma_n}\vec{E}\times\vec{H}^{*}\cdot\hat{z}\cdot dS=\sum_{n=1}^{N}\frac{1}{2}V_n\cdot I_n^{*}=P_{loss}+2jw\cdot\left(W_m-W_e\right)$$

# Waveguide Junctions II

- Definition: closed metallic structure that totally encloses a connected region electromagnetically isolated from the outside. The only energy transfer is done through reference planes that are cross sections of waveguides located where only one mode is supported and propagating.
  - If there are several modes, the waveguide itself is modelled as a waveguide junction.
- The junction will be characterized in terms of energy:
  - There is energy flux only at the reference planes.
  - At the reference planes there is a single dominant mode.

# Waveguide Junctions III

- The junction will be characterized in terms of energy:.
  - At each reference plane a generalized voltage and current are defined.
    - Each voltage/current is a complex number defined on vector space $\mathcal{V}/\mathcal{I}$.
    - The junction is defined by means of vectors of complex numbers of dimension N so that each pair of vectors V-I unambiguously defines a pair of vectors E-H.
    - Vectors V and I are not independent, and they are related by a bilinear function between vector spaces $\mathcal{V}^n$ and $\mathcal{I}^n$ (complex vectors of dimension N).
    - Since they are finite-dimensional vector spaces mapped by a bilinear function, there is only one regular matrix that relates vectors V and I.
    - All the statements above are true if conductivity is different from zero at one or more frequency points.

* $V_{N×1} I_{N×1} \in \mathcal V^N \mathcal I^N$ ⟹ Each phase of vectors define univocally a pair of E-H
* $V_{N×1}$ and $I_{N×1}$ are not independent and are related by a bilinear function ⟹ there will be a regular Z ⟹ Y = Z^{-1}

# Waveguide Junctions: Impedance and Admittance Matrices

- The matrix that relates V and I is called **impedance matrix** Z. Since the function is regular, the matrix that relates I and V is the **admittance matrix** Y.
- Impedance and admittance matrices of a homogeneous and isotropic junction are reciprocal.
  - This theorem implies that σ, ε and μ are scalar magnitudes or symmetrical tensors.
- The proof of this theorem is based on the Lorentz reciprocity theorem.

# Waveguide Junctions: Impedance and Admittance Matrices

- The proof of this theorem is based on the Lorentz reciprocity theorem:
  - If ($E_a$, $H_a$) and ($E_b$, $H_b$) are two different solutions of Maxwell's equations for a microwave circuit corresponding to two different sources with the same frequency and mode, it will be verified that

$$\nabla\left[\left(\vec{E}_a \times \vec{H}_b\right) - \left(\vec{E}_b \times \vec{H}_a\right)\right] = 0$$

$$\nabla\left[\left(\vec{E}_a \times \vec{H}_b\right) - \left(\vec{E}_b \times \vec{H}_a\right)\right] =$$

$$\vec{H}_b \nabla \times \vec{E}_a - \vec{E}_a \nabla \times \vec{H}_b - \vec{H}_a \nabla \times \vec{E}_b + \vec{E}_b \nabla \times \vec{H}_a =$$

$$-\vec{H}_b \cdot jw\mu \cdot \vec{H}_a - \vec{E}_a \cdot (\sigma + jw\varepsilon) \cdot \vec{E}_b + \vec{H}_a \cdot jw\mu \cdot \vec{H}_b + \vec{E}_a \cdot (\sigma + jw\varepsilon) \cdot \vec{E}_b$$

# Physical Conditions for Definition of Impedance and Admittance Matrices

- Since Z and Y are symmetric matrices:

$$\frac{1}{2}V^T \cdot I^*\Big|_{escalar} = \frac{1}{2}I^H \cdot V = \frac{1}{2}I^H \cdot Z \cdot I = P + 2jw \cdot \left(W_H - W_E\right)$$

- The conjugate of the previous expression is:

$$\frac{1}{2}I^H \cdot Z^H \cdot I = P - 2jw \cdot \left(W_H - W_E\right)$$

- Adding and subtracting both equations, it results:

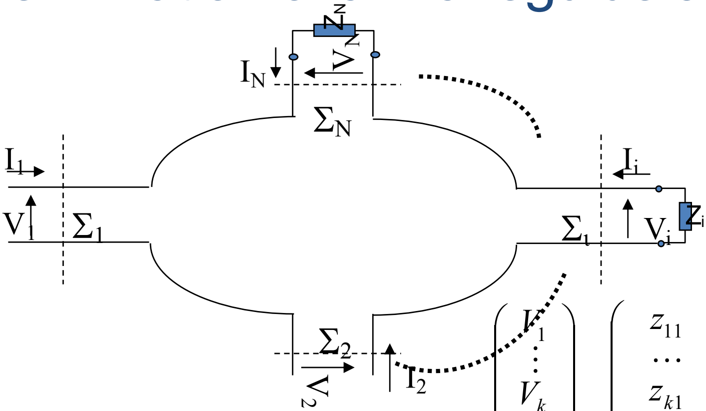
$$\frac{1}{2}I^H \cdot \left(Z + Z^H\right) \cdot I = 2P = I^H \cdot \mathrm{Re}(Z) \cdot I$$

$$\frac{1}{2}I^H \cdot \left(Z - Z^H\right) \cdot I = 4jw\left(W_H - W_E\right) = j \cdot I^H \cdot \mathrm{Im}(Z) \cdot I$$

# Physical Conditions for Definition of Impedance and Admittance Matrices (II)

- If the junction is passive P≥0, thus: $I^H \cdot \mathrm{Re}(Z) \cdot I \geq 0 \Rightarrow \mathrm{Re}(Z)\, semidefinida\ positiva$
- If the junctions is non-dissipative P=0 and it verifies: $I^H \cdot \mathrm{Re}(Z) \cdot I = 0 \Rightarrow \mathrm{Re}(Z) = 0$
  - For a non-dissipative termination the impedance matrix is pure imaginary.
- If $\left(W_H \overset{>}{<} W_E\right) \Rightarrow \begin{pmatrix} \mathrm{Im}(Z) \\ -\mathrm{Im}(Z) \end{pmatrix} definida\ positiva$
- If $\left(W_H = W_E\right) \Rightarrow \mathrm{Im}(Z) = 0$ : resonance condition

# Termination of a Waveguide Junction with Dipoles




$$V_{N\times 1} = Z_{N\times N} \cdot I_{N\times 1} \Rightarrow$$

The new degenerate junction has a matrix $Z$ ($Z_{de}$) that can be written in terms of the matrix $Z$ of the non-degenerate junction

$$\begin{pmatrix} V_1 \\ \vdots \\ V_k \\ V_{k+1} \\ \vdots \\ V_N \end{pmatrix} = \left(\begin{array}{ccc|ccc} z_{11} & \cdots & z_{1k} & \cdots & \cdots & z_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{k1} & \cdots & z_{kk} & \cdots & \cdots & z_{kN} \\ \hline z_{(k+1)1} & \cdots & \cdots & z_{(k+1)(k+1)} & \cdots & z_{(k+1)N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{(k+1)N} & \cdots & \cdots & \cdots & \cdots & z_{NN} \end{array}\right) \cdot \begin{pmatrix} I_1 \\ \vdots \\ I_k \\ I_{k+1} \\ \vdots \\ I_N \end{pmatrix}$$

$$\begin{pmatrix} \bar{V}_1 \\ \bar{V}_2 \end{pmatrix} = \begin{pmatrix} \bar{\bar{M}}_{11} & \bar{\bar{M}}_{12} \\ \bar{\bar{M}}_{21} & \bar{\bar{M}}_{22} \end{pmatrix} \cdot \begin{pmatrix} \bar{I}_1 \\ \bar{I}_2 \end{pmatrix} \Rightarrow \bar{V}_2 = -\bar{\bar{Z}}_L \cdot \bar{I}_2$$

$$\bar{V}_1 = \left[\bar{\bar{M}}_{11} - \bar{\bar{M}}_{12} \cdot \left(\bar{\bar{M}}_{22} + \bar{\bar{Z}}_L\right)^{-1} \cdot \bar{\bar{M}}_{21}\right] \cdot \bar{I}_1 = \bar{\bar{Z}}_{de} \cdot \bar{I}_1$$

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# Waveform characterization of a Waveguide Junction

- The formulation for a dipole from 4.9 can be extended to a waveguide junction:

$$\left.\begin{array}{l} V_{N\times1} = H_{N\times N}\cdot\left(A_{N\times1}+B_{N\times1}\right) \\ I_{N\times1} = K_{N\times N}\cdot\left(A_{N\times1}-B_{N\times1}\right)\end{array}\right\} con \left(\begin{array}{l} H_{N\times N} = diag\left(\sqrt{2Z_{0n}}\right) \\ K_{N\times N} = diag\left(\dfrac{\sqrt{2}}{\sqrt{Z_{0n}}}\right)\end{array}\right) \Rightarrow \qquad \boxed{(1)}$$

$$\left\{\begin{array}{l} A_{N\times1} = F_{N\times N}\cdot\left(V_{N\times1}+G_{N\times N}\cdot I_{N\times1}\right) \\ B_{N\times1} = F_{N\times N}\cdot\left(V_{N\times1}-G_{N\times N}\cdot I_{N\times1}\right)\end{array}\right\} con \left(\begin{array}{c} F_{N\times N} = diag\left(\left(\sqrt{8Z_{0n}}\right)^{-1}\right) \\ G_{N\times N} = diag\left(Z_{0n}\right)\end{array}\right)$$

- Physical meaning (“from the circuit point of view”):
  - A: incident power waves at each network port (they enter the circuit)
  - B: reflected power waves at each network port (they leave the circuit)

If B (reflected waves) is solved in terms of A (incident waves):

# Waveform characterization of a Waveguide Junction

- If B (reflected waves) is solved in terms of A (incident waves):

$$A = F \cdot (V + G \cdot I) = F \cdot (Z + G) \cdot I \Rightarrow I = (Z + G)^{-1} \cdot F^{-1} \cdot A$$

$$B = F \cdot (Z - G) \cdot I = F \cdot (Z - G) \cdot (Z + G)^{-1} \cdot F^{-1} \cdot A$$

$$B_{N\times 1} = S_{N\times N} \cdot A_{N\times 1};$$

$$\boxed{S = F \cdot (Z - G) \cdot (Z + G)^{-1} \cdot F^{-1}} \qquad (2)$$

- S is the **scattering matrix** that depends on the junction and the reference planes.
- If all the reference impedances are equal to the characteristic impedance, matrix S coefficients are the (non-generalized) S-parameters and:

$$S = (Z - \Delta) \cdot (Z + \Delta)^{-1}$$

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# Scattering Matrix Symmetry

- The S-matrix of a passive, linear, isotropic network is symmetrical: $S=S^T$

$$S = S^T$$

$$F \cdot (Z-G) \cdot (Z+G)^{-1} \cdot F^{-1} = \left[F \cdot (Z-G) \cdot (Z+G)^{-1} \cdot F^{-1}\right]^T$$

$$F \cdot (Z-G) \cdot (Z+G)^{-1} \cdot F^{-1} = \left(F^{-1}\right)^T \cdot \left((Z+G)^{-1}\right)^T \cdot \left((Z-G)\right)^T \cdot F^T$$

  - Matrices *F* and *G* are diagonal, matrix *Z* is symmetrical

$$F \cdot (Z-G) \cdot (Z+G)^{-1} \cdot F^{-1} = F^{-1} \cdot (Z+G)^{-1} \cdot (Z-G) \cdot F$$

  - Interchanging the inverses between terms, it results:

$$(Z+G) \cdot F \cdot F \cdot (Z-G) = (Z-G) \cdot F \cdot F \cdot (Z+G)$$

  - And computing and simplifying the following expression is obtained,

$$2Z \cdot F \cdot F \cdot G = 2G \cdot F \cdot F \cdot Z$$

  - And taken into account the fact that *F* and *G* are diagonal: $S = S^T$

# Physical Conditions for the Existence of S-matrix

- From equation (1), if the characteristic impedance is unique, it can be written:

$$\left.\begin{aligned} V &= \sqrt{2\cdot Z_0}\cdot(\Delta+S)\cdot A = H\cdot(\Delta+S)\cdot A \\ I &= \sqrt{\frac{2}{Z_0}}\cdot(\Delta-S)\cdot A = K\cdot(\Delta-S)\cdot A \end{aligned}\right\} \Rightarrow \begin{cases} V^H = A^H\cdot(\Delta+S^H)\cdot H \\ I^H = A^H\cdot(\Delta-S^H)\cdot K \end{cases}$$

- Computing the power and its conjugate:

$$\left.\begin{aligned} \frac{1}{2}I^H\cdot V &= A^H\cdot(\Delta-S^H\cdot S-S^H+S)\cdot A = P+2jw(W_H-W_E) \\ \frac{1}{2}V^H\cdot I &= A^H\cdot(\Delta-S^H\cdot S-S+S^H)\cdot A = P-2jw(W_H-W_E) \end{aligned}\right\}$$

- Adding and subtracting each term it results:

$$\left.\begin{aligned} A^H\cdot(\Delta-S^H\cdot S)\cdot A &= P \\ A^H\cdot(S-S^H)\cdot A &= 2jw(W_H-W_E) \end{aligned}\right\}$$

# Physical Conditions for the Existence of S-matrix

- Consequences:
  - Passive termination: $P \geq 0; \Rightarrow \left(\Delta - S^H \cdot S\right) semidefinida\_positiva$
    - Non-dissipative:

$$P = 0; \Rightarrow \left(\Delta - S^H \cdot S\right) = 0 \Rightarrow \Delta = S^H \cdot S \Rightarrow \ 'unitaria$$

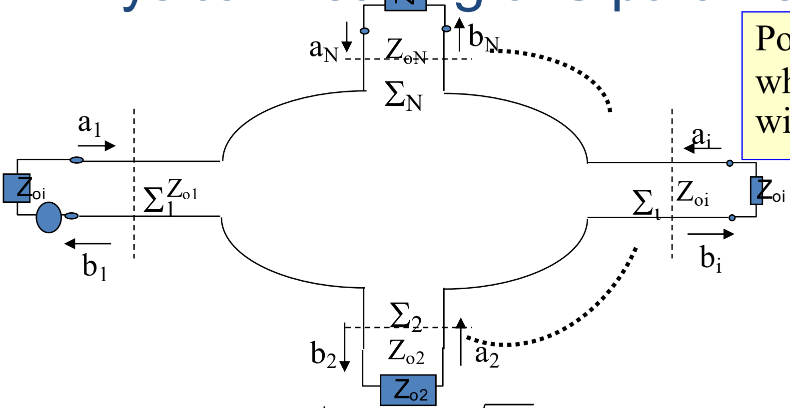
    - Resonance: real scattering matrix $S = S^H$
    - When all the minors of Im(S) are positive, then:

$$W_H > W_E \Rightarrow \{\mathrm{Im}(S)\} definida\_positiva$$

    - When all the minors of –Im(S) are positive, then:

$$W_H > W_E \Rightarrow -\left\{\mathrm{Im}\left(S\right)\right\} definida\_positiva$$

# Physical Meaning of S parameters




Power reflection coefficient at port $i$ when the rest of ports are loaded with the characteristic impedance

$$s_{ii} = \left.\frac{b_i}{a_i}\right|_{Z_{j\neq i}=Z_{oj}} = \frac{V_i^-}{V_i^+}$$

$$\left|s_{ii}\right|^2 = \frac{\left|b_i\right|^2}{\left|a_i\right|^2} = \frac{P_i^-}{P_i^+}$$

$$s_{ki} = \left.\frac{b_k}{a_i}\right|_{Z_{k\neq i}=Z_{ok}} = \frac{\sqrt{Z_{oi}}\cdot V_k^-}{\sqrt{Z_{ok}}\cdot V_i^+}$$

$$\left|s_{ki}\right|^2 = \frac{\left|b_k\right|^2}{\left|a_i\right|^2} = \frac{P_k^-}{P_i^+}$$

Power transmission coefficient from port $i$ to port $k$ when every port except $i$ is loaded with the characteristic impedance

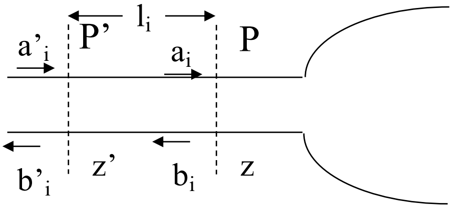
# Definitions of circuit characteristics: matching.

- Hypothesis: all ports of the junction except one are loaded so that there are no reflections: $a_2 = a_3 = \ldots = a_N = 0$
  - The new scattering matrix is reduced to: $b_1 = s_{11} a_1$
- Definition of matched termination: $s_{11} = 0$
- Concept of **matching**: if in waveguide junction $s_{nn} = 0$, the junction is matched from waveguide $n$. If all the diagonal coefficients of the matrix are zero the junction is totally matched.

# Definitions of circuit characteristics: coupling.

- Concept of **uncoupled port**: with the hypothesis above the power transfer to port 2 is given by: $b_2 = s_{21} a_1$.
  - If $s_{21}=0$, port 2 is uncoupled from port 1: there is no power transmission at all from port 1 to port 2.
  - If in a junction of $N$ waveguides there is a subset of $K$ guides that are totally uncoupled from the complementary subset of $N$-$K$ guides, then the junction is said to be degenerate.

# Transformation of the Scattering Matrix by means of Reference Plane Shift




$$\begin{cases} a_i = a \cdot e^{-j\beta_i z} \\ b_i = b \cdot e^{j\beta_i z} \end{cases} \Rightarrow \begin{cases} a'_i = a \cdot e^{-j\beta_i z'} \\ b'_i = b \cdot e^{j\beta_i z'} \end{cases} \xrightarrow{sentido\_de\_onda\_progresiva} z = z' + l_i$$

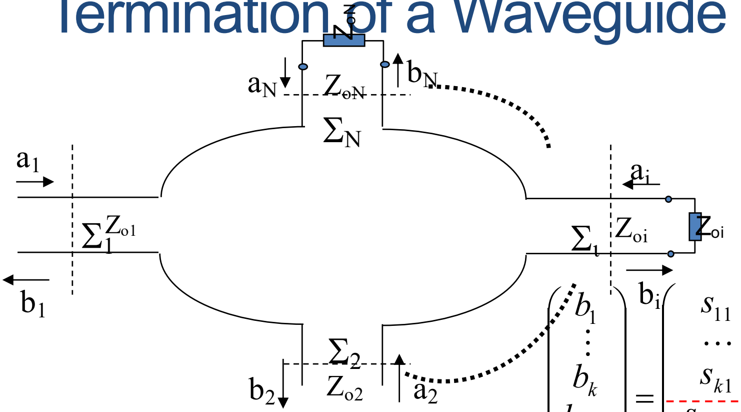
$$\begin{cases} a_i = a'_i \cdot e^{-j\beta_i l_i} \\ b_i = b'_i \cdot e^{j\beta_i l_i} \end{cases} \Rightarrow \begin{cases} A = P \cdot A' \\ B' = P \cdot B \end{cases} \Rightarrow P = diag\left(e^{-j\beta_i l_i}\right)$$

$$B' = P \cdot S \cdot A = P \cdot S \cdot P \cdot A' = S' \cdot A'$$

$$S' = P \cdot S \cdot P$$

The movement is outwards direction

# Termination of a Waveguide Junction with Dipole




$$B = S \cdot A \Rightarrow$$

The new degenerate junction has a matrix $S$ ($S_{de}$) that can be written in terms of the matrix $S$ of the non-degenerate junction

$$\begin{pmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \\ \vdots \\ b_N \end{pmatrix} = \left(\begin{array}{ccc|ccc} s_{11} & \cdots & s_{1k} & \cdots & \cdots & s_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{k1} & \cdots & s_{kk} & \cdots & \cdots & s_{kN} \\ \hline s_{(k+1)1} & \cdots & \cdots & s_{(k+1)(k+1)} & \cdots & s_{(k+1)N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{(k+1)N} & \cdots & \cdots & \cdots & \cdots & s_{NN} \end{array}\right) \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_N \end{pmatrix}$$

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \overline{N}_{11} & \overline{N}_{12} \\ \overline{N}_{21} & \overline{N}_{22} \end{pmatrix} \cdot \begin{pmatrix} \overline{A}_1 \\ \overline{A}_2 \end{pmatrix} con \begin{cases} \Gamma_n = \dfrac{a_n}{b_n}; [\Gamma_C]_{(N-k)\times(N-k)} = [\Gamma_C] = diag(\Gamma_n) \\ \overline{A}_2 = [\Gamma_C] \cdot B_2; B_2 = [\Gamma_C]^{-1} \cdot \overline{A}_2 \end{cases}$$

$$B_1 = \boxed{\overline{N}_{11} - \overline{N}_{12} \cdot \left(\overline{N}_{22} - \Gamma_C^{-1}\right)^{-1} \cdot \overline{N}_{21}} \cdot \overline{A}_1 = \overline{S}_{de} \cdot \overline{A}_1$$

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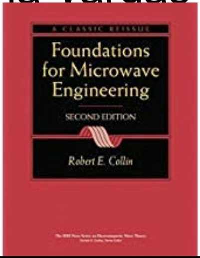
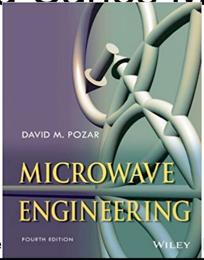
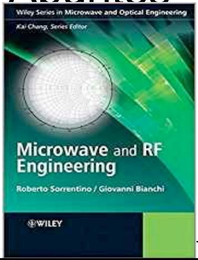


# CONCLUSIONS

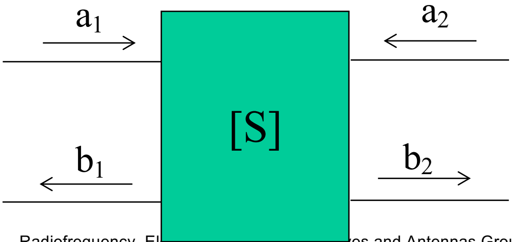
- The definition of scattering parameters is motivated by the need of a set of parameters that clearly relate the magnitudes that are susceptible of measurement in a microwave circuit: power, SWR and reflection.
- Power waveforms are invariant with respect to the length by means of shifting of reference planes.
- S matrix represents in a straightforward way the distribution of power between pairs of ports of the network.
- S parameters are measured with matching conditions at the ports, while Y and Z are measured with short or open circuits.

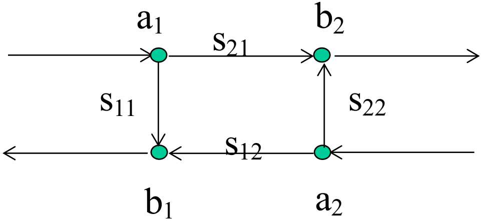
# Appendix

- Description of multiple waveguide junctions
  - Description of a junction using generalized voltages and currents:
    - Vectors of currents and voltages: generalized impedance matrix.
    - Properties and physical conditions of impedance or admittance matrices.
    - Termination of waveguide junction with dipoles.
  - Waveform description of a waveguide junction: scattering matrix.
    - Properties: symmetry, shift in reference planes.
    - Physical meaning of scattering parameters: transmission and reflection.
- **Introduction to graph theory.**
- **Generalized S parameters.**

# Graph Theory I

- Additional technique complimentary to S parameters in order to measure the properties of networks in term of transmitted and reflected powers.
- Elements of a graph:
  - **Node**: each port of a network has two nodes, one associated to an incident wave (*a*) and the other associated to a reflected wave (*b*).
  - **Branch**: direct path between nodes *a* and *b*. Each branch has a coefficient that is a transmission or reflection S parameter.
- Example of two-port network:



# Graph Theory II: rules

- **Series connection**: two branches with a common node and only one input and one output can be merged into a single branch whose coefficient is the product of coefficients.
- **Parallel connection**: two branches with common input and output can be merged into a single branch whose coefficient is the sum of coefficients.
- **Self-loop**: when a branch with coefficient *s* begins and ends on the same node the self-loop can be eliminated by multiplying the other branches that feed the node by *1/(1-s)*.
- **Splitting**: a node can be decomposed into two separate nodes as long as the new graph contains, once and only once, each combination of separated branches connected to the original node.

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