Guided Propagation Formulas

Waveguides

Solving Wave Equations

Separation of variables in E_z

$$E_z = F_e(\xi_1, \xi_2) A_e(z)$$

Separation constant

$$\gamma = \sqrt{\gamma_0^2 - \gamma_c^2} = \sqrt{\omega^2 \mu \epsilon - \gamma_c^2}$$

Where $\gamma_0^2 < 0$ for PEC materials.

TM Fields / phasors

$$\vec{E}_z = E_z \hat{z} = F_e(\xi_1, \xi_2) e^{-\gamma z} \hat{z}$$
$$\vec{H}_t = -\tfrac{j\omega\epsilon}{\gamma_c^2} \nabla_t E_z \times \hat{z}$$
$$\vec{E}_t = \tfrac{\gamma}{\gamma_c^2} \nabla_t E_z$$

TE Fields / phasors

$$\vec{H}_z = H_z \hat{z} = F_h(\xi_1, \xi_2) e^{-\gamma z} \hat{z}$$
$$\vec{E}_t = -\tfrac{j\omega\epsilon}{\gamma_c^2} \nabla_t H_z \times \hat{z}$$
$$\vec{H}_t = \tfrac{\gamma}{\gamma_c^2} \nabla_t H_z$$

Mode impedances

$$Z_{TM} = \frac{\hat{z} \times \vec{E}_t}{\vec{H}_t} = \frac{\gamma}{j\omega\epsilon}$$
$$Z_{TE} = \frac{\hat{z} \times \vec{E}_t}{\vec{H}_t} = \frac{j\omega\mu}{\gamma}$$
$$Z_{TEM} = \frac{\hat{z} \times \vec{E}_t}{\vec{H}_t} = \eta_0$$

Power flow

$$P_m = \frac{1}{2} \Re \left[\frac{1}{Z_m} \right] \iint_\Sigma |\vec{E}_t|^2 d\sigma$$

PEC Boundary Conditions

Cutoff frequency

$$f_c = \tfrac{1}{2\pi\sqrt{\mu\epsilon}} \sqrt{-\gamma_c^2}$$

Propagation constant

$$\gamma = \pm\gamma_0 \sqrt{1 - \left(\tfrac{f_c}{f}\right)^2}$$

Where $\gamma_0 = j\omega\sqrt{\mu\epsilon}$

Mode impedances

$$Z_{TE} = \frac{\pm\eta}{\sqrt{1 - \left(\tfrac{f_c}{f}\right)^2}}$$
$$Z_{TM} = \pm\eta \sqrt{1 - \left(\tfrac{f_c}{f}\right)^2}$$

Using $+$ for $f > f_c$ and $-$ for $f < f_c$.

Lossy Dielectrics

Propagation constant

$$\gamma = \sqrt{-\omega^2\mu(\epsilon' - j\epsilon'') - \gamma_c^2} = \alpha + j\beta$$

Cutoff frequency: same as PEC, where $\alpha = \beta$

Good Dielectric (low loss)

$$\gamma = \sqrt{-\omega^2\mu_0\epsilon_0\epsilon_r(1 - j\tan\delta) - \gamma_c^2}$$

$$\tan\delta << \implies \begin{cases} \alpha_d \approx \frac{k_0^2\tan\delta}{2\beta} & \text{(TE, TM)} \\ \alpha_d \approx \frac{k_0\tan\delta}{2} & \text{(TEM)} \end{cases}$$

Real Conductors

$$\alpha \approx \alpha_g + \alpha_c$$
$$\beta \approx \beta_g$$

Where α_g and β_g are PEC values

Joule Effect attenuation constant

$$\alpha_c = \frac{W_c}{2W_t} = \frac{\frac{1}{2}\oint_C \frac{1}{\sigma\delta}\vec{H}(0)\cdot\vec{H}(0)^* dl}{\Re\left[\iint_s \vec{E}_t(0)\times\vec{H}_t(0)^*\cdot\vec{ds}\right]}$$

Rectangular Waveguide TE_{10}

$$\alpha_c = \frac{R_s}{a^3 b\beta_g k\eta}\left(2b\pi^2 + a^3k^2\right)$$

Where R_s is the surface resistance:

$$R_s = \sqrt{\frac{\omega\mu}{2\sigma}} = \Re\left[\frac{1+j}{\sigma\delta_s}\right]$$

Rectangular waveguide $a \times b$

$$\gamma_c^2 = -k_x^2 - k_y^2$$

Rectangular TE_{mn} modes

$$k_x = \tfrac{m\pi}{a}$$
$$k_y = \tfrac{n\pi}{b}$$

Propagation constant

$$\gamma_{mn} = \sqrt{-\omega^2\mu\epsilon + \left(\tfrac{m\pi}{a}\right)^2 + \left(\tfrac{n\pi}{b}\right)^2}$$

Cutoff frequency

$$f_{c;mn} = \frac{c}{2\sqrt{\epsilon_r}}\sqrt{\left(\tfrac{m}{a}\right)^2 + \left(\tfrac{n}{b}\right)^2}$$

Power flow

$$\frac{|\mathrm{E}|^2}{2Z_m}\cdot\frac{ab}{2}$$

Phasors

$$E_x = E_{x0}\cos\left(\tfrac{m\pi x}{a}\right)\sin\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$
$$E_y = E_{y0}\sin\left(\tfrac{m\pi x}{a}\right)\cos\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$

$$H_z = H_{mn}^i\cos\left(\tfrac{m\pi x}{a}\right)\cos\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$

$$H_x = \tfrac{\gamma_{mn}m\pi}{k_c^2 a}H_{mn}^i\sin\left(\tfrac{m\pi x}{a}\right)\cos\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$
$$H_y = \tfrac{\gamma_{mn}n\pi}{k_c^2 b}H_{mn}^i\cos\left(\tfrac{m\pi x}{a}\right)\sin\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$
$$E_x = \tfrac{j\omega\mu n\pi}{k_c^2 b}H_{mn}^i\cos\left(\tfrac{m\pi x}{a}\right)\sin\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$
$$E_y = \tfrac{-j\omega\mu n\pi}{k_c^2 a}H_{mn}^i\sin\left(\tfrac{m\pi x}{a}\right)\cos\left(\tfrac{n\pi y}{b}\right)e^{-\gamma_{mn}z}$$

$$k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Transmission Lines (TEM modes)

Generic TL

- Incident wave (positive): V_0^+
- Reflected wave (negative) V_0^-

Voltage

$$V(z) = V_0 e^{-\gamma_0 z}$$
$$V_0 = \int_1^2 \nabla_t \phi \cdot d\vec{l}$$

Current

$$I(z) = I_0 e^{-\gamma_0 z}$$
$$I_0 = \oint_{C2} \frac{\nabla_t \phi \cdot \hat{n}}{\eta} dl$$

Characteristic Impedance

$$Z_0 = \frac{V_0}{I_0} = \eta \cdot K$$

Where K is a dimensionless factor depending on geometry

Transmitted power

$$P_t = \tfrac{1}{2}\Re(VI^*) = \frac{|V|^2}{2Z_0} = \frac{|I|^2 Z_0}{2}$$

Coaxial Line (a inner, b outer)

Primary Parameters

$$Z_0 = \tfrac{\eta}{2\pi} \ln \tfrac{b}{a}$$
$$\gamma_0 = j\beta = j\omega\sqrt{\mu\epsilon_0\epsilon_r}$$

First HOM f_c (TE_{11})

$$f_c \approx \frac{c}{2\pi\sqrt{\epsilon_r}} \frac{2}{a+b}$$

Circuit Model

ABCD Parameters

$$\begin{pmatrix} a & b \\ c & a \end{pmatrix}$$

$$a = \cosh(\gamma_0 l)$$
$$b = Z_0 \sinh(\gamma_0 l)$$
$$c = \tfrac{1}{Z_0} \sinh(\gamma_0 l)$$

Equivalent Network

Equivalent half-T model (series $Z_s l$, parallel $Y_p l$)

$$Z_s = Z_0 \gamma_0 = j\omega\mu K$$
$$Y_p = \frac{\gamma_0}{Z_0} = \left(\frac{\omega\epsilon''}{K} + j\omega\frac{\epsilon'}{K}\right)$$

Equivalent Lumped Elements

T network with 2 $Ll/2$ inductors in series and Cl capacitor and Gl resistor in parallel

$$L = \mu K = \mu\frac{Z_0}{\eta}$$
$$G = \omega\frac{\epsilon''}{K} = \omega\epsilon''\frac{\eta}{Z_0}$$
$$C = \frac{\epsilon'}{K} = \epsilon'\frac{\eta}{Z_0}$$

Where L, G, C are the **Secondary Parameters**

Parameter relationships

Primary and secondary parameters

$$Z_0 = \sqrt{\frac{j\omega L}{G + j\omega C}} = \sqrt{\frac{Z_s}{Y_p}}$$
$$\gamma_0 = \sqrt{j\omega L(G + j\omega C)} = \sqrt{Z_s Y_p}$$

Low loss dielectric

$$Z_0 \approx \sqrt{\tfrac{L}{C}}$$
$$\gamma_0 \approx G\tfrac{Z_0}{2} + j\omega\sqrt{LC} = \alpha_d + j\beta$$

PEC and lossless dielectric

$$Z_0 \approx \sqrt{\tfrac{L}{C}}$$
$$\gamma_0 \approx j\omega\sqrt{LC} = j\beta$$

Secondary parameters

$$L = \mu\tfrac{\epsilon'}{C}$$
$$G = \omega\tfrac{\epsilon''}{\epsilon'}C$$
$$C = \tfrac{Q_t}{V_0}$$

Real Conductors

- TEM mode becomes *quasi-TEM*
- Add $Rl/2$ in series with each inductor in T-circuit model

Parameter relationships

$$\gamma_0 = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{Z_s Y_p}$$
$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{Z_s}{Y_p}}$$

Good dielectric & good conductors

$$\alpha = \tfrac{1}{2}\left(\tfrac{R}{Z_0} + GZ_0\right) = \alpha_c + \alpha_d$$
$$\beta = \omega\sqrt{LC}$$
$$Z_0 = \sqrt{\tfrac{L}{C}}$$

Terminated Transmission Line

- End of TL: $z = 0$
- Load impedance Z_L (known)
- Alias axis $l = -z$

$$Z_L = \frac{V(0)}{I(0)} = \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$$
$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

Reflection

Reflection coefficient (voltage)

$$\Gamma(0) = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- $|\Gamma| \leq 1$ for passive loads and $|V_0^-| \leq |V_0^+|$
- $Z_L = Z_0 \Rightarrow \Gamma(0) = 0$ (no reflected wave)

$$Z_0 = \frac{1 - \Gamma(0)}{1 + \Gamma(0)} Z_L$$

Standing Wave

$$V(z) = V_0^+ \left(e^{-j\beta z} + \Gamma(0) e^{+j\beta z}\right)$$
$$I(z) = \tfrac{V_0^+}{Z_0} \left(e^{-j\beta z} - \Gamma(0) e^{+j\beta z}\right)$$

Generalized Reflection Coefficient

$$\Gamma(z) = \frac{V^-(z)}{V^+(z)} = \Gamma(0) e^{2j\beta z}$$
$$V(z) = V_0^+ e^{-j\beta z} \left(1 + \Gamma(z)\right)$$
$$I(z) = \tfrac{V_0^+}{Z_0} e^{-j\beta z} \left(1 - \Gamma(z)\right)$$

Transmitted Power

$$P_L = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma|^2) = P_i (1 - |\Gamma|^2)$$

- $\Gamma(0) = 0 \Rightarrow P_L = P_i$ (*Matched Load*, all power delivered)
- $|\Gamma(0)| = 1 \Rightarrow P_L = 0$ (*Pure standing wave*, no power delivered)
  - $\Gamma(0) = 1 \Rightarrow Z_L = \infty$ (open)
  - $\Gamma(0) = -1 \Rightarrow Z_L = 0$ (short)
- Rest: $\Gamma(0) = \rho e^{j\theta} = u + jv$

Standing Waves

- Periodicity $\lambda/2$

Voltage magnitude

$$|V(z)| = |V_0^+| \left|1 + \rho e^{\theta - 2\beta l}\right|$$
$$V_{max} = 1 + \rho$$
$$V_{min} = 1 - \rho$$

Standing Wave Ratio

$$SWR = \frac{V_{max}}{V_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + \rho}{1 - \rho}$$

- $SWR = 1$ for matched load
- $SWR = \infty$ for mismatched load (open or short circuit)

Return Loss

$$RL_{dB} = -20 \log |\Gamma|$$

Input (transformed) impedance

$$Z_{in}(l) = \frac{V(l)}{I(l)} = \frac{V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}}{V_0^+ e^{-j\beta z} - V_0^- e^{+j\beta z}} Z_0$$

- Matched load: $Z_{in} = Z_0 \forall l$
- Short termination: $Z_{in}(l) = jZ_0 \tan(\beta l)$
- Open termination: $Z_{in}(l) = -jZ_0 \cot(\beta l)$

Relationship between Z_{in} and $\Gamma(l)$

$$\Gamma(l) = \frac{Z_{in}(l) - Z_0}{Z_{in}(l) + Z_0}$$
$$Z_{in}(l) = Z_0 \frac{1 + \Gamma(l)}{1 - \Gamma(l)}$$

Lossy Transmission Line

- Z_0 is complex
- No stable standing wave

Reflection coefficient (lossy)

$$\Gamma(l) = \Gamma(0)e^{-2\alpha l}e^{2j\beta l}$$

Input impedance (lossy)

$$\begin{aligned} Z_{in}(-l) &= Z_0\frac{Z_L\cosh(\gamma l) + Z_0\sinh(\gamma l)}{Z_0\cosh(\gamma l) + Z_L\sinh(\gamma l)} \\ &= R + jX \end{aligned}$$