Propagation and standing wave in transmission lines

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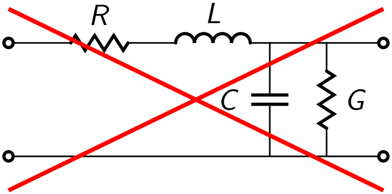
Transmission line

Coaxial cable, twin-lead line, ...

L: total inductance
C: total capacitance
R: total series resistance
G: total parallel conductance

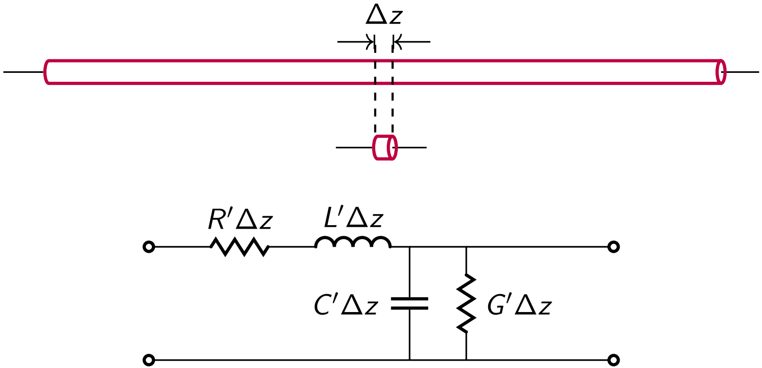


Primary parameters

- L': **inductance** per unit length — H/m
- C': **capacitance** per unit length — F/m
- R': series **resistance** per unit length — Ω/m
- G': parallel **conductance** per unit length — S/m

Transmission line

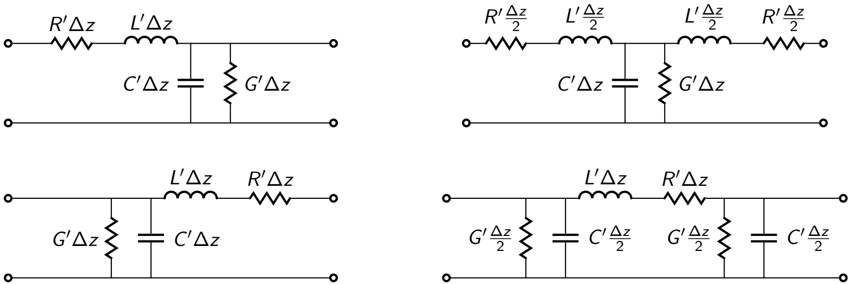
Model of a short section



- $L'\Delta z$: total inductance
- $R'\Delta z$: total series resistance
- $C'\Delta z$: total capacitance
- $G'\Delta z$: total parallel conductance

Transmission line

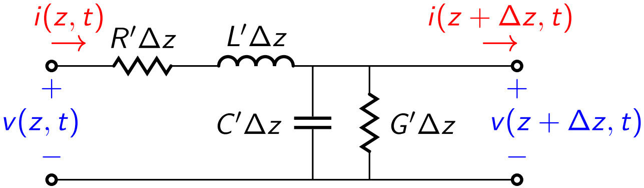
Alternative circuit models for a short section



- $L'\Delta z$: total inductance
- $R'\Delta z$: total series resistance
- $C'\Delta z$: total capacitance
- $G'\Delta z$: total parallel conductance

Circuit analysis of the short section model

Kirchoff's laws



$$v(z,t) - v(z+\Delta z,t) = R'\Delta z\, i(z,t) \qquad + L'\Delta z\, \frac{\partial i(z,t)}{\partial t}$$

$$i(z,t) - i(z+\Delta z,t) = G'\Delta z\, v(z+\Delta z,t) + C'\Delta z\, \frac{\partial v(z+\Delta z,t)}{\partial t}$$

Telegrapher's equations

Circuit differential equations

$$\begin{cases} v(z,t) - v(z+\Delta z,t) = R'\Delta z\, i(z,t) \qquad\quad + L'\Delta z\,\dfrac{\partial i(z,t)}{\partial t} \\ i(z,t) - i(z+\Delta z,t) = G'\Delta z\, v(z+\Delta z,t) + C'\Delta z\,\dfrac{\partial v(z+\Delta z,t)}{\partial t} \end{cases}$$

Divide by Δz and change signs

$$\begin{cases} \dfrac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = -R' i(z,t) - L'\,\dfrac{\partial i(z,t)}{\partial t} \\ \dfrac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G' v(z,t) - C'\,\dfrac{\partial v(z+\Delta z,t)}{\partial t} \end{cases}$$

Limit at $\Delta z \to 0$

$$\begin{cases} \dfrac{\partial v(z,t)}{\partial z} = -R' i(z,t) - L'\,\dfrac{\partial i(z,t)}{\partial t} \\ \dfrac{\partial i(z,t)}{\partial z} = -G' v(z,t) - C'\,\dfrac{\partial v(z,t)}{\partial t} \end{cases}$$

Telegrapher's equations

Time domain t

$$\begin{cases} \dfrac{\partial v(z,t)}{\partial z} = -R' i(z,t) - L' \dfrac{\partial i(z,t)}{\partial t} \\ \dfrac{\partial i(z,t)}{\partial z} = -G' v(z,t) - C' \dfrac{\partial v(z,t)}{\partial t} \end{cases}$$

Frequency domain ω

$$\begin{cases} \dfrac{\mathrm{d}V(z)}{\mathrm{d}z} = -(R' + j\omega L')I(z) \\ \dfrac{\mathrm{d}I(z)}{\mathrm{d}z} = -(G' + j\omega C')V(z) \end{cases}$$

How to solve: differentiate one equation, substitute the other

$$\frac{\mathrm{d}^2V(z)}{\mathrm{d}z^2} = -(R' + j\omega L')\frac{\mathrm{d}I(z)}{\mathrm{d}z} = \overbrace{(R' + j\omega L')(G' + j\omega C')}^{\gamma^2} V(z)$$

Helmholtz wave equations

$$\frac{\mathrm{d}^2V(z)}{\mathrm{d}z^2} - \gamma^2 V(z) = 0 \qquad \frac{\mathrm{d}^2I(z)}{\mathrm{d}z^2} - \gamma^2 I(z) = 0$$

Propagation

Helmholtz equations

$$\frac{d^2 V(z)}{dz^2} - \gamma^2 V(z) = 0$$

$$\frac{d^2 I(z)}{dz^2} - \gamma^2 I(z) = 0$$

General solutions

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$

$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

Propagation *constant*

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$$

- α: attenuation constant Np/m
- β: phase constant rad/m

Characteristic impedance

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$$

But the solutions are not independent: $\frac{\mathrm{d}V(z)}{\mathrm{d}z} = -(R' + j\omega L')I(z)$

Definition: characteristic impedance

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} \triangleq Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

$$\left.\begin{aligned} \gamma &= \sqrt{(R' + j\omega L')(G' + j\omega C')} \\ Z_0 &= \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \end{aligned}\right\} \text{Secondary parameters}$$

Physical interpretation of the solution

Voltage/current waves

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

- Let be $V_0^+ = \left|V_0^+\right| e^{j\phi^+} \qquad V_0^- = \left|V_0^-\right| e^{j\phi^-} \qquad \gamma = \alpha + j\beta$
- Time domain: $v(t, z) = \mathrm{Re}[V(z)e^{j\omega t}]$

Time-domain solution

$$v(z,t) = \underbrace{\left|V_0^+\right| e^{-\alpha z} \cos\left(\omega t - \beta z + \phi^+\right)}_{\text{Incident wave (towards } +z)} + \underbrace{\left|V_0^-\right| e^{\alpha z} \cos\left(\omega t + \beta z + \phi^-\right)}_{\text{Reflected wave (towards } -z)}$$

`https://www.desmos.com/calculator/y3jnr0xqwy`

Lossless (non-dissipative) line

Particular case

$$R = G = 0 \quad \Rightarrow \quad \begin{cases} \gamma = j\beta = j\omega\sqrt{LC} & (\alpha = 0) \\ Z_0 = \sqrt{\dfrac{L}{C}} \end{cases}$$

Voltage and current waves

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$$

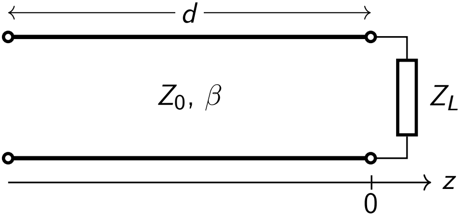
$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z}$$

Time-domain solution

$$v(z,t) = \underbrace{\left|V_0^+\right| \cos\left(\omega t - \beta z + \phi^+\right)}_{\text{Incident wave (towards } +z\text{)}} + \underbrace{\left|V_0^-\right| \cos\left(\omega t + \beta z + \phi^-\right)}_{\text{Reflected wave (towards } -z\text{)}}$$

Terminated line

Boundary condition: impedance



$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$$
$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z}$$

For $z = 0$:

$$\left.\begin{aligned} V(0) &= V_0^+ + V_0^- \\ I(0) &= \frac{V_0^+ - V_0^-}{Z_0} \end{aligned}\right\} \Rightarrow Z_L = \frac{V(0)}{I(0)} = Z_0 \frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}$$

Definition: reflection coefficient

$$\Gamma \triangleq \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Reflection coefficient

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$Z_L = Z_0 \frac{1+\Gamma}{1-\Gamma}$$

Voltage and current waves

$$V(z) = V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z} \right)$$

$$I(z) = \frac{V_0^+}{Z_0} \left(e^{-j\beta z} - \Gamma e^{j\beta z} \right)$$

Particular case: **matched load**

- $V_0^- = 0$
- $\Gamma = 0$
- $Z_L = Z_0$

Average power

$$
\begin{aligned}
P(z) &= \frac{1}{2}\,\mathrm{Re}\left[V(z)I(z)^*\right] \\
&= \frac{1}{2}\frac{\left|V_0^+\right|^2}{Z_0}\,\mathrm{Re}\left[(e^{-j\beta z}+\Gamma e^{j\beta z})(e^{j\beta z}-\Gamma^* e^{-j\beta z})\right] \\
&= \frac{1}{2}\frac{\left|V_0^+\right|^2}{Z_0}\,\mathrm{Re}\left[1-|\Gamma|^2+\Gamma e^{2j\beta z}-\Gamma^* e^{-2j\beta z}\right] \\
&= \frac{1}{2}\frac{\left|V_0^+\right|^2}{Z_0}\left(1-|\Gamma|^2\right)
\end{aligned}
$$

- Independent of z
- Two terms: $P^+ = \dfrac{1}{2}\dfrac{\left|V_0^+\right|^2}{Z_0}$ $\qquad P^+ = \dfrac{1}{2}\dfrac{\left|V_0^-\right|^2}{Z_0} = \dfrac{1}{2}\dfrac{\left|V_0^+\right|^2}{Z_0}|\Gamma|^2$
- $\Gamma = 0$: Maximum power transfer $\qquad |\Gamma| = 1$: no power transfer

Standing wave

Voltage amplitude

Let be $\Gamma = |\Gamma| e^{j\theta}$

$$\begin{aligned}|V(z)| &= \left|V_0^+\right| \left|e^{-j\beta z} + \Gamma e^{j\beta z}\right| = \left|V_0^+\right| \left|1 + \Gamma e^{2j\beta z}\right| \\ &= \left|V_0^+\right| \left|1 + |\Gamma| e^{j(2\beta z + \theta)}\right|\end{aligned}$$

Maxima and minima condition: $|\Gamma| e^{j(2\beta z+\theta)} \in \mathbb{R}$

- $V_{\text{max}} = \left|V_0^+\right| \left|1 + |\Gamma|\right| \quad 2\beta z + \theta = 2n\pi$ rad
- $V_{\text{min}} = \left|V_0^+\right| \left|1 - |\Gamma|\right| \quad 2\beta z + \theta = (2n+1)\pi$ rad
- Distance: $z_{\text{max}} - z_{\text{min}} = \dfrac{\pi}{2\beta} = \dfrac{\lambda}{4}$
- Distance: $z_{\text{max}} - z'_{\text{max}} = \dfrac{\pi}{\beta} = \dfrac{\lambda}{2}$

`https://www.desmos.com/calculator/2pwe46lry0`