E3. Problem 7. F.S. Coefficients of a Sampled Periodic Signal

Let $x(\cdot)$ be periodic with $T = \frac{1}{10}$ seconds, and $a_k = \left(\frac{1}{2}\right)^{|k|}$, $k \in \mathbb{Z}$. The signal is filtered by an ideal lowpass filter $\hat{h}(f) = \begin{cases} 1, & |f| \leq 105 \\ 0, & |f| > 105 \end{cases}$ and sampled at $T_s = \frac{1}{210}$ seconds.

a) Show that $x[n] = x(nT_s)$, $n \in \mathbb{Z}$ is periodic and determine its period.

b) Determine the F.S. coefficients of $x[\cdot]$

*Hint: The F.S. coefficients of the result of filtering are $b_k = a_k \hat{h}\left(\frac{k}{T_0}\right)$, $k \in \mathbb{Z}$

a) $x[n] = x(nT_s)$

$x(\cdot)$ periodic with $T = \frac{1}{10}$ => $x[\cdot]$ periodic with period $N = \frac{T}{T_s} = \frac{\frac{1}{10}}{\frac{1}{210}} = \boxed{21}$

...

$$x(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|k|} e^{j\frac{2\pi}{T}kt} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k} e^{-j20\pi kt} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{k+1} e^{j20\pi(k+1)t} = \frac{1}{1-\frac{1}{2}e^{-j20\pi t}} + \frac{1}{2}e^{j20\pi t} \cdot \frac{1}{1-\frac{1}{2}e^{j20\pi t}} =$$

$$= \frac{\left(1-\frac{1}{2}e^{j20\pi t}\right) + \left(1-\frac{1}{2}e^{-j20\pi t}\right)\left(\frac{1}{2}e^{j20\pi t}\right)}{\left(1-\frac{1}{2}e^{-j20\pi t}\right)\left(1-\frac{1}{2}e^{j20\pi t}\right)} = \frac{1-\frac{1}{2}e^{j20\pi t}+\frac{1}{2}e^{j20\pi t}-\frac{1}{4}}{1-\frac{1}{2}e^{j20\pi t}-\frac{1}{2}e^{-j20\pi t}+\frac{1}{4}} = \frac{\frac{3}{4}}{\frac{5}{4}-\cos(20\pi t)} = \frac{3}{5-4\cos(20\pi t)}$$

$$x[n] = x(nT_s) = \frac{3}{5-4\cos\left(20\pi \frac{1}{210}n\right)} = \frac{3}{5-4\cos\left(\frac{2}{21}\pi n\right)} \overset{?}{=} \underbrace{\frac{3}{5-4\cos\left(\frac{2}{21}\pi(n+\underset{N}{21})\right)}}_{x[n+N]=x[n+21]} = \frac{3}{5-4\cos\left(\frac{2}{21}\pi n+2\pi\right)} = x[n]$$

b) Time scaling doesn't affect F.S. coefficients (only changes period)

Periodic discrete signal => periodic F.S. coefficients, period $N = 21$

Filtered with lowpass => bandlimited at $W = 105$ Hz => $b_k = 0 \ \forall \ |k| >$ $W \cdot T = \frac{105}{10} = 10.5 \Rightarrow b_k = 0 \ \forall |k| \geq 11$

~~Samp~~ Sampled at 210 Hz = 2W => No aliasing

$b_k = \begin{cases} a_k, & |k| \leq 10 \\ 0, & \text{otherwise} \end{cases}$, $|k| \leq 10$ => $\boxed{b_k = a_k = \left(\frac{1}{2}\right)^{|k|}, |k| \leq 10 \quad , \text{periodic } N = 21}$