$$\hat{a}(f)=\int_0^\infty e^{-t}e^{-j2\pi ft}\,dt=-\frac{e^{-t(1+j2\pi f t)}}{1+j2\pi f}\Bigg|_0^\infty=-\frac{e^{-\infty}-e^{0}}{1+j2\pi f}=\frac{1}{1+j2\pi f}$$

E1. Problem 8. Inverse Filter

Let $x(\cdot)$ be given by $x(t)=\operatorname{sinc}(8t)$, $t\in\mathbb{R}$ and consider a filter $h(t)=e^{-|t|}$, $t\in\mathbb{R}$

a) What is the F.T. $\hat{y}(\cdot)$ of $y(t)=(x*h)(t)$, $t\in\mathbb{R}$?

$$y(t)=(x*h)(t)\multimap \hat{x}(f)\,\hat{h}(f),\ f\in\mathbb{R}\quad=\cancel{\hat{y}(f)=\operatorname{sinc}(8t)\,e^{-|t|}}\ \underset{\frac{|f|}{8}\le\frac{1}{2}}{}$$

$$\operatorname{sinc}(t)\multimap\Pi(f)$$

$$x(t)=\operatorname{sinc}(8t)\multimap\frac{1}{8}\Pi\left(\frac{f}{8}\right)=\hat{x}(f)=\begin{cases}\frac{1}{8} & \text{if } |f|\le 4\\ 0 & \text{otherwise}\end{cases}$$

$$h(t)=e^{-|t|}=\begin{cases}e^{-t}, & t\ge 0\\ e^{t} & t<0\end{cases}=e^{-t}I\{t\ge0\}+e^{t}I\{t<0\}=e^{-t}I\{t\ge0\}+e^{-(-t)}I\{-(t)\ge0\}=$$

$$=a(t)+a(-t)$$

$$a(t)=e^{-t}I\{t\ge0\}\multimap\frac{1}{1+j2\pi f}=\hat{a}(f)$$

$$h(t)=a(t)+a(-t)\multimap\frac{1}{1+j2\pi f}+\frac{1}{1-j2\pi f}=\hat{h}(f)=\frac{1-j2\pi f+1+j2\pi f}{1-(j2\pi f)^2}=\frac{2}{1+4\pi^2f^2}$$

$$y(t)=(x*h)(t)\Rightarrow\hat{y}(f)=\hat{x}(f)\hat{h}(f)=\begin{cases}\frac{1}{8} & \text{if } |f|\le4\\ 0 & \text{otherwise}\end{cases}\cdot\left(\frac{1}{1+j2\pi f}+\frac{1}{1-j2\pi f}\right)=\begin{cases}\frac{1}{8} & \text{if } |f|\le4\\ 0 & \text{otherwise}\end{cases}\cdot\frac{2}{1+4\pi^2f^2}=$$

$$=\boxed{\begin{cases}\frac{1}{4}\,\frac{1}{1+4\pi^2f^2} & \text{if } |f|\le4\\ 0 & \text{otherwise}\end{cases}}$$

b) Find a finite energy filter $g(\cdot)$ such that $z(t)=(y*g)(t)=x(t)$, $t\in\mathbb{R}$

$$z(t)=(y*g)(t)\Rightarrow\cancel{\hat{z}(f)=\hat{y}(f)\hat{g}(f)=\hat{x}(f)\hat{h}(f)\hat{g}(f)=\hat{x}(f)\Rightarrow\hat{g}(f)=\frac{1}{\hat{h}(f)}}$$

$$\Rightarrow\hat{z}(f)=\hat{y}(f)\hat{g}(f)=\hat{x}(f)\Rightarrow\hat{g}(f)=\frac{\hat{x}(f)}{\hat{y}(f)}=\begin{cases}\dfrac{\frac{1}{8}}{\frac{1}{4}\frac{1}{1+4\pi^2f^2}} & \text{if } |f|\le4\\ 0 & \text{otherwise}\end{cases}=$$

$$=\boxed{\begin{cases}\frac{1+4\pi^2f^2}{2} & \text{if } |f|\le4\\ 0 & \text{otherwise}\end{cases}}$$

$$g(t)=\int_{-\infty}^{\infty}\hat{g}(f)e^{j2\pi ft}\,df=\int_{-4}^{4}\frac{1+4\pi^2f^2}{2}e^{j2\pi ft}\,df=\frac{1}{2}\int_{-4}^{4}e^{j2\pi ft}\,df+2\pi^2\int_{-4}^{4}f^2e^{j2\pi ft}=\frac{1}{2}\left(\frac{e^{j2\pi ft}}{j2\pi t}\right)\Bigg|_{f=-4}^{4}+2\pi^2\left(f^2\frac{e^{j2\pi ft}}{j2\pi t}-2f\frac{e^{j2\pi ft}}{(j2\pi t)^2}+2\frac{e^{j2\pi ft}}{(j2\pi t)^3}\right)\Bigg|_{f=-4}^{4}=$$

$$=\frac{1}{2}\,\frac{e^{j8\pi t}-e^{-j8\pi t}}{j2\pi t}+\left(\frac{16e^{j8\pi t}-16e^{-j8\pi t}}{j2\pi t}-\frac{8e^{j8\pi t}+8e^{-j8\pi t}}{(j2\pi t)^2}+\frac{2e^{j8\pi t}-2e^{-j8\pi t}}{(j2\pi t)^3}\right)2\pi^2=$$

$$=\frac{\sin 8\pi t}{2\pi t}+2\pi^2\left(\frac{32\sin 8\pi t}{2\pi t}-\frac{16\cos 8\pi t}{(j2\pi t)^2}+\frac{4\sin 8\pi t}{j^2(2\pi t)^3}\right)=\boxed{4\operatorname{sinc}8t+256\pi^2\operatorname{sinc}8t+8\frac{\cos 8\pi t}{t^2}-8\frac{\operatorname{sinc}8t}{t^2}}$$