Linear Systems
(2023/2024)

1 2 3 4 5
6 7 8 9 10

Exercise 2.

E2.Problem 1. Fourier Series Coefficient. [a b c d e f]

a) $x[n] = \sin[\pi(n-1)/4]$, $n \in \mathbb{Z}$

$$x[n] = \sin\left(\frac{\pi}{4}n - \frac{\pi}{4}\right) \Rightarrow N = 8 \Leftarrow \sin\left(\frac{8\pi}{4}n - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}n - \frac{\pi}{4}\right)$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{-j\frac{2\pi}{N}kn} = \frac{1}{8}\sum_{n=0}^{7}\sin\left(\frac{\pi}{4}n-\frac{\pi}{4}\right)e^{-j\frac{\pi}{4}kn} = \frac{1}{8}\sum_{n=0}^{7}\frac{e^{j\left(\frac{\pi}{4}n-\frac{\pi}{4}\right)}-e^{-j\left(\frac{\pi}{4}n-\frac{\pi}{4}\right)}}{2j}e^{-j\frac{\pi}{4}kn} =$$

$$= \frac{1}{16j}\left(e^{-j\frac{\pi}{4}}\sum_{n=0}^{7}e^{jn\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)} - e^{j\frac{\pi}{4}}\sum_{n=0}^{7}e^{-jn\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)}\right) = \frac{1}{16j}\left(e^{-j\frac{\pi}{4}}\frac{1-e^{j8\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)}}{1-e^{j\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)}} - e^{j\frac{\pi}{4}}\frac{1-e^{-j8\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)}}{1-e^{-j\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)}}\right) =$$

$$= -j\frac{1}{16}\left(e^{-j\frac{\pi}{4}}\cdot\frac{1-e^{j2\pi k}}{1-e^{j\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)}} - e^{j\frac{\pi}{4}}\frac{1-e^{-j2\pi k}}{1-e^{-j\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)}}\right) = -j\frac{1}{16}\left(e^{-j\frac{\pi}{4}}\frac{1-\cos(2\pi k)-j\sin(2\pi k)}{1-e^{j\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)}} - e^{j\frac{\pi}{4}}\frac{1-\cos(2\pi k)+j\sin(2\pi k)}{1-e^{-j\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)}}\right) =$$

$\frac{1-k}{4} = 2k \Leftrightarrow k-1 = 8k$ $\qquad$ $\frac{1+k}{4} = 2k \Leftrightarrow k+1 = 8k$

$$= \frac{1}{16}\left(e^{-j\frac{\pi}{4}}\begin{cases}\lim\limits_{k\to 1}\dfrac{\sin(2\pi k)}{1-\cos\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)-j\sin\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)} & \text{if } \frac{\pi}{4}-\frac{\pi}{4}k = 2\pi k,\ k\in\mathbb{Z} \\ 0 & \text{otherwise}\end{cases} - e^{j\frac{\pi}{4}}\begin{cases}\lim\limits_{k\to -1}\dfrac{\sin(2\pi k)}{1-\cos\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)+j\sin\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)} & \text{if } \frac{\pi}{4}+\frac{\pi}{4}k = 2\pi k \\ 0 & \text{otherwise}\end{cases}\right) =$$

$$= \begin{cases}\frac{1}{16}\left(e^{-j\frac{\pi}{4}} j\lim\limits_{k\to 1}\dfrac{\overbrace{\sin(2\pi k)}^{2\pi(k-1)+o(k)}}{\underbrace{\sin\left(\frac{\pi}{4}-\frac{\pi}{4}k\right)}_{-\frac{\pi}{4}(k-1)+o(k)}}\right) & \text{if } k = 8k+1,\ k\in\mathbb{Z} \\ \frac{1}{16}\left(e^{j\frac{\pi}{4}} j\lim\limits_{k\to -1}\dfrac{\sin(2\pi k)}{\underbrace{\sin\left(\frac{\pi}{4}+\frac{\pi}{4}k\right)}_{\frac{\pi}{4}(k+1)+o(k)}}\right) & \text{if } k = 8k-1,\ k\in\mathbb{Z} \\ 0 & \text{otherwise}\end{cases} = \begin{cases}\frac{1}{16}je^{-j\frac{\pi}{4}}\lim\limits_{k\to 1}\dfrac{2\pi(k-1)}{-\frac{\pi}{4}(k-1)} & \text{if } k=8k+1,\ k\in\mathbb{Z} \\ \frac{1}{16}je^{j\frac{\pi}{4}}\lim\limits_{k\to -1}\dfrac{2\pi(k+1)}{\frac{\pi}{4}(k+1)} & \text{if } k=8k-1,\ k\in\mathbb{Z} \\ 0 & \text{otherwise}\end{cases} =$$

$$= \boxed{\begin{cases}-j\frac{1}{2}e^{-j\frac{\pi}{4}} & \text{if } k\in\{8k+1 : k\in\mathbb{Z}\} \\ j\frac{1}{2}e^{j\frac{\pi}{4}} & \text{if } k\in\{8k-1 : k\in\mathbb{Z}\} \\ 0 & \text{otherwise}\end{cases}}$$