··· E1. Problem 5.

e) $x(t) = e^{-t}\, I\{t \geq 0\}$, $t \in \mathbb{R}$, where $I\{\cdot\}$ is the idicator function: $I\{\text{statement}\} = \begin{cases} 1 & \text{if statement} \\ 0 & \text{otherwise} \end{cases}$

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} e^{-t} I\{t \geq 0\}\, e^{-j2\pi f t}\, dt = \int_{0}^{\infty} e^{-t - j2\pi f t}\, dt = \left(\frac{e^{-t(1+j2\pi f t)}}{-(1+j2\pi f)} \right)\Bigg|_{t=0}^{\infty} = \cancel{\lim} - \frac{e^{-\infty} - e^{0}}{1+j2\pi f} =$$

$$= \boxed{\frac{1}{1+j2\pi f}}$$

f) $x(t) = \begin{cases} \overbrace{2\left(1 - \frac{|t|}{2}\right)}^{2-|t|} & \text{if } |t| \leq 2 \\ 0 & \text{otherwise} \end{cases}$

$$\int t\, e^{-j2\pi f t} = \begin{bmatrix} u = t \Rightarrow du = dt \\ dv = e^{-j2\pi f t} dt \Rightarrow v = \frac{e^{-j2\pi f t}}{-j2\pi f} \end{bmatrix} = t\,\frac{e^{-j2\pi f t}}{-j2\pi f} - \int \frac{e^{-j2\pi f t}}{-j2\pi f}\, dt = t\,\frac{e^{-j2\pi f t}}{-j2\pi f} - \frac{e^{-j2\pi f t}}{(-j2\pi f)^2}$$

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-2}^{2} 2\left(1 - \frac{|t|}{2}\right) e^{-j2\pi f t}\, dt = \int_{-2}^{0} \overbrace{2\left(1 + \frac{t}{2}\right)}^{2+t} e^{-j2\pi f t}\, dt + \int_{0}^{2} \overbrace{2\left(1 - \frac{t}{2}\right)}^{2-t} e^{-j2\pi f t}\, dt =$$

$$= 2\int_{-2}^{0} e^{-j2\pi f t}\, dt + \int_{-2}^{0} t\, e^{-j2\pi f t}\, dt + 2\int_{0}^{2} e^{-j2\pi f t}\, dt - \int_{0}^{2} t\, e^{-j2\pi f t}\, dt =$$

$$= 2\left(\frac{e^{-j2\pi f t}}{-j2\pi f} \right)\Bigg|_{t=-2}^{0} + \left(t\,\frac{e^{-j2\pi f t}}{-j2\pi f} - \frac{e^{-j2\pi f t}}{(-j2\pi f)^2} \right)\Bigg|_{t=-2}^{0} + 2\left(\frac{e^{-j2\pi f t}}{-j2\pi f} \right)\Bigg|_{t=0}^{2} - \left(t\,\frac{e^{-j2\pi f t}}{-j2\pi f} - \frac{e^{-j2\pi f t}}{(j2\pi f)^2} \right)\Bigg|_{t=0}^{2} =$$

$$= 2\,\frac{e^{0} - e^{j4\pi f}}{-j2\pi f} + \left(-(-2)\cdot \frac{e^{j4\pi f}}{-j2\pi f} - \frac{e^{0} - e^{j4\pi f}}{-2^2\pi^2 f^2} \right) + 2\,\frac{e^{-j4\pi f} - e^{0}}{-j2\pi f} - \left(2\,\frac{e^{-j4\pi f}}{-j2\pi f} - \frac{e^{-j4\pi f} - e^{0}}{-(2\pi f)^2} \right) =$$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x \Rightarrow 1 - \cos x = 2\sin^2 \frac{x}{2}$

$$= 2\,\frac{1 - e^{j4\pi f} + e^{j4\pi f} + e^{-j4\pi f} - 1 - e^{-j4\pi f}}{-j2\pi f} + \frac{1 - e^{j4\pi f} - e^{-j4\pi f} + 1}{4\pi^2 f^2} = \boxed{\frac{1 - \cos(4\pi f)}{2\pi^2 f^2}} =$$

$$= \frac{2\sin^2(2\pi f)}{2\pi^2 f^2} = \boxed{4\,\mathrm{sinc}(2f)}$$

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