E1. Problem 9. LTI System 1

Consider an LTI such that $\frac{d}{dt}y(t) + 2y(t) = x(t)$, $t \in \mathbb{R}$

a) Determine the ~~frequency~~ frequency response $\hat{h}(f) = \frac{\hat{y}(f)}{\hat{x}(f)}$, $f \in \mathbb{R}$

$$x(t) = \frac{d}{dt}y(t) + 2y(t) \Rightarrow \hat{x}(t) = j2\pi f\,\hat{y}(f) + 2\hat{y}(f)$$

$$\hat{h}(f) = \frac{\hat{y}(f)}{\hat{x}(f)} = \frac{\hat{y}(f)}{j2\pi f\hat{y}(f) + 2\hat{y}(f)} = \boxed{\frac{1}{2+j2\pi f}}$$

b) If $x(t) = e^{-t}$ for $t \geq 0$ and $x(t) = 0$ for $t < 0$, determine the F.T. of the output.

$$x(t) = e^{-t}\,\mathbb{I}\{t \geq 0\} \circ\!\!-\!\!\bullet \frac{1}{1+j2\pi f} = \hat{x}(f) \qquad \hat{x}(f) = \int_0^\infty e^{-t}e^{-j2\pi ft}\,dt = \left.\frac{e^{-t(1+j2\pi f)}}{-(1+j2\pi f)}\right|_{t=0}^{\infty} = \frac{e^{-\infty} - e^{-0}}{-(1+j2\pi f)} = \frac{1}{1+j2\pi f}$$

$$\hat{y}(f) = \hat{x}(f)\,\hat{h}(f) = \boxed{\frac{1}{1+j2\pi f}\cdot\frac{1}{2+j2\pi f}} = \frac{1}{2-4\pi^2 f^2 + j6\pi f}$$

c) For the input in part (b), determine the output $y(\cdot)$

$$\cancel{h(t) = h(t)} \qquad \hat{h}(f) = \frac{1}{2+j2\pi f} \circ\!\!-\!\!\bullet\; e^{-2t}\,\mathbb{I}\{t \geq 0\} = h(t) \qquad h(t) = \int_{-\infty}^{\infty} \frac{e^{j2\pi ft}}{2+j2\pi f}\,df = \dots$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_0^\infty e^{-\tau}e^{-2(t-\tau)}\mathbb{I}\{t-\tau > 0\}\,d\tau = \int_0^t e^{-\tau-2t+2\tau}\,d\tau =$$

$$= \overbrace{\left.\frac{e^{\tau-2t}}{1}\right|_{\tau=0}^{t}}^{\mathbb{I}\{t\geq 0\}} = \overbrace{e^{t-2t} - e^{-2t}}^{\mathbb{I}\{t\geq 0\}} = \boxed{\left(e^{-t} - e^{-2t}\right)\mathbb{I}\{t \geq 0\}}$$