## E2. Problem 3. Fourier Series of Real Periodic Signals

Let $x[\cdot]$ be a real-valued signal with period $N$ and F.S. coefficients $\{a_n\}$

a) Let $a_k = b_k + j c_k$, $k \in \mathbb{Z}$

Show that $a_{-k} = a^*_k$, $k \in \mathbb{Z}$. What is the relationship between $b_k$ and $b_{-k}$? And $c_k$ and $c_{-k}$?

$$a_k = \sum_{n=\langle N\rangle} x[n] e^{-j\frac{2\pi}{N}kn} = \sum_{n=\langle N\rangle} x[n] \cos(\tfrac{2\pi}{N}kn) - j \sum_{n=\langle N\rangle} x[n] \sin(\tfrac{2\pi}{N}kn) = b_k + j c_k$$

$$a_{-k} = \sum_{n=\langle N\rangle} x[n] \cos(-\tfrac{2\pi}{N}kn) - j \sum_{n=\langle N\rangle} x[n] \sin(-\tfrac{2\pi}{N}kn) = \sum_{n=\langle N\rangle} x[n] \cos(\tfrac{2\pi}{N}kn) + j \sum_{n=\langle N\rangle} x[n] \sin(\tfrac{2\pi}{N}kn) = b_k - j c_k =$$

$$= a^*_k$$

$$\boxed{b_{-k} = \sum_{n=\langle N\rangle} x[n] \cos(\tfrac{2\pi}{N}kn) = \sum_{n=\langle N\rangle} x[n] \cos(\tfrac{2\pi}{N}kn) = b_k}$$

$$\boxed{c_{-k} = -\sum_{n=\langle N\rangle} x[n] \sin(-\tfrac{2\pi}{N}kn) = +\sum_{n=\langle N\rangle} x[n] \sin(\tfrac{2\pi}{N}kn) = -c_k}$$

b) Suppose $N$ is even. Show that $a_{N/2}$ is real-valued.

$$a_{N/2} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j\frac{2\pi}{N}\frac{N}{2}n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-j\pi n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] (\underbrace{\cos \pi n}_{(-1)^n} - j \underbrace{\sin \pi n}_{0}) =$$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} x[n] \cos \pi n \in \mathbb{R}$$

c) Suppose $N$ is even, and let $a_k = A_k e^{j\theta_k}$ $k \in \mathbb{Z}$.

$$a_0 = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] e^{j\frac{2\pi}{N}k\cdot 0} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n] \Rightarrow \theta_0 = 0$$

Show that $x[n] = A_0 + A_{N/2}(-1)^n + 2\sum_{k=1}^{\frac{N}{2}-1} A_k \cos(\frac{2\pi kn}{N} + \theta_k)$

$$x[n] = \sum_{k=\langle N\rangle} A_k e^{j\theta_k} e^{j\frac{2\pi}{N}kn} = A_0 e^{j\theta_0} e^0 + \sum_{k=1}^{N-1} A_k e^{j(\frac{2\pi}{N}kn + \theta_k)} =$$

$\angle(\frac{1}{N}\sum x[n]) = 0$

$a_{N/2} \in \mathbb{R} \Rightarrow \theta_{N/2} = 0$

$$= A_0 + \sum_{k=1}^{N/2-1} A_k e^{j(\frac{2\pi}{N}kn + \theta_k)} + A_{N/2} e^{j\theta_{N/2}} e^{j\frac{2\pi}{N}\frac{N}{2}n} + \sum_{k=\frac{N}{2}+1}^{N-1} A_k e^{j(\frac{2\pi kn}{N} + \theta_k)} =$$

$$= A_0 + A_{N/2} \cdot e^{j\pi k} + \sum_{k=1}^{\frac{N}{2}-1} A_k e^{j(\frac{2\pi}{N}kn + \theta_k)} + \sum_{k=-N+1}^{-\frac{N}{2}-1} A_{-k} e^{j(-\frac{2\pi}{N}kn + \theta_{-k})} = \left[ a_{-k} = a^*_k \Rightarrow A_{-k} = A_k,\ \theta_{-k} = -\theta_k,\ k \to k-N \right] =$$

$$= A_0 + A_{N/2} \cdot (\cos(\pi k) + j\underbrace{\sin(\pi k)}_{0}) + \sum_{k=1}^{\frac{N}{2}-1} A_k e^{j(\frac{2\pi}{N}k + \theta_k)} + \sum_{k=1}^{\frac{N}{2}-1} A_{k-N} e^{j(\frac{2\pi(k-N)n}{N} + \theta_{k-N})} =$$

$$= A_0 + A_{N/2} \cdot (\cos(\pi)) + \sum_{k=1}^{\frac{N}{2}-1} A_k (e^{j(\frac{2\pi}{N}k + \theta_k)} + e^{-j(\frac{2\pi}{N}k + \theta_k)}) =$$

$$= A_0 + A_{N/2} \cdot (-1)^n + 2\sum_{k=1}^{\frac{N}{2}-1} A_k \cos(\tfrac{2\pi}{N}k + \theta_k)$$