E4. Problem 3. Partial Fraction Decomposition

~~a) $\hat{x}(z) = 1 - \frac{1-3z}{1-3z+2z^2}$~~

Consider a left-sided sequence $x[\cdot]$ with $\hat{x}(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-z^{-1})}$

a) Rewrite $\hat{x}(z)$ as a ratio of polynomials in z (instead of z^{-1})

b) Using PFD, express $\hat{x}(\cdot)$ as a sum of fractions, where each fraction represents a pole.

c) Determine $x[\cdot]$

a) $$\hat{x}(z) = \frac{1}{(1-\frac{1}{2z})(1-\frac{1}{z})} = \frac{1}{(\frac{2z-1}{2z})(\frac{z-1}{z})} = \frac{1}{\frac{2z^2-2z-z+1}{2z^2}} = \boxed{\frac{2z^2}{2z^2-3z+1}} = 1 - \frac{1-3z}{2z^2-3z+1}$$

b) $$\hat{x}(z) = \frac{1}{(1-\frac{1}{2z})(1-\frac{1}{z})} = \frac{2z^2}{(2z-1)(z-1)} = \frac{2z^2-3z+1-(-3z+1)}{(2z-1)(z-1)} = 1 + \frac{3z-1}{(2z-1)(z-1)} =$$

~~PFD $1-3z = A(z-1)+B(2z-1) = Az-A+2Bz-B = (A+2B)z+(-A-B)$~~

~~$\begin{cases} A+2B = -3 \\ -A-B = 1 \end{cases} \Rightarrow \begin{cases} B = -2 \\ -A-B = 1 \end{cases} \Rightarrow -A-(-2) = 1 \Rightarrow A = +1$~~

PFD

$$= \left[\begin{array}{l} 3z-1 = A(z-1)+B(2z-1) = Az-A+2Bz-B = (A+2B)z+(-A-B) \\ \begin{cases} A+2B = 3 \\ -A-B = -1 \end{cases} \Rightarrow^{+(II)} \begin{cases} 0+B = 2 \\ -A-B = -1 \end{cases} \Rightarrow -A-2 = -1 \Rightarrow A = -1 \end{array} \right] = \boxed{1 - \frac{1}{2z-1} + \frac{2}{z-1}}$$

c) ~~$x[n] = \frac{1}{2\pi j} \oint_c \hat{x}(z) z^{n-1} dz = \frac{1}{2\pi j} \oint_c \left(1 - \frac{1}{2z-1} + \frac{2}{z-1}\right) z^{n-1} dz =$~~

~~$\left[\begin{array}{l} 2z-1=0 \Leftrightarrow z=\frac{1}{2} \\ z-1=0 \Leftrightarrow z=1 \\ \text{left-sided} \Rightarrow \text{contains } 0 \\ \text{ROC: } z < \frac{1}{2} \\ C: \frac{1}{4}e^{i\theta},\ 0 \le \theta \le 2\pi \end{array} \right]$~~

~~$= \frac{1}{2\pi j} \int_0^{2\pi} \left(1 - \frac{1}{\frac{2}{4}e^{i\theta}-1} + \frac{2}{\frac{1}{4}e^{i\theta}-1}\right)\left(\frac{1}{4}e^{i\theta}\right)^{n-1} d\frac{1}{4}e^{i\theta}$~~ ~~$\frac{1}{4}ie^{i\theta}d\theta$~~

~~$= \frac{1}{32\pi} \int_0^{2\pi} \left(1 - \frac{2}{e^{i\theta}-1} + \frac{8}{e^{i\theta}-4}\right) e^{i\theta n} d\theta =$~~

$$x[n] = \mathcal{Z}^{-1}\left(1 - \frac{1}{2z-1} + \frac{2}{z-1}\right) = \mathcal{Z}^{-1}(1) - \mathcal{Z}^{-1}\left(\frac{1}{2z-1}\right) + \mathcal{Z}^{-1}\left(\frac{2}{z-1}\right) =$$

$$= \delta[n] - \left(\frac{1}{2}\right)^n \mathcal{Z}^{-1}\left(\frac{1}{z-1}\right) + 2\,\mathcal{Z}^{-1}\left(\frac{1}{z-1}\right) =$$

$$= \delta[n] + \left(\frac{1}{2}\right)^n \left(\mathcal{Z}^{-1}\left(\frac{1}{1-z^{-1}}\right)\right)\Big|_{n=-n} - 2\left(\mathcal{Z}^{-1}\left(\frac{1}{1-z^{-1}}\right)\right)\Big|_{n=-n} = \delta[n] + \left(\frac{1}{2}\right)^n u[-n] - 2u[-n] =$$

$$= \left[\text{at } n=0:\ \delta[n] + \left(\tfrac{1}{2}\right)^n u[-n] - 2u[-n] = 1 + \left(\tfrac{1}{2}\right)^0 - 2 = 0\right] = \boxed{\left(\left(\frac{1}{2}\right)^n - 2\right) u[-n-1]}$$