Exercise 4

Linear Systems 2023/2024

Problem 1 *z-Transforms*

Determine the z-transforms of the following sequences. Sketch the pole-zero plot and indicate the region of convergence. Indicate whether or not the Fourier transforms of the sequences exists.

(a) $x[n] = \delta[n]$, $n \in \mathbb{Z}$.

(b) $x[n] = \delta[n-1]$, $n \in \mathbb{Z}$.

(c) $x[n] = \delta[n+1]$, $n \in \mathbb{Z}$.

(d) $x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \leq 0 \\ 0, & n > 0. \end{cases}$

(e) $x[n] = \begin{cases} \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n, & n \geq 0 \\ 0, & n < 0. \end{cases}$

(f) $x[n] = \left(\frac{1}{3}\right)^{|n|}$, $n \in \mathbb{Z}$.

(g) $x[n] = \begin{cases} 0, & n < 0 \\ 1, & 0 \leq n \leq 9 \\ 0, & n > 9. \end{cases}$

(h) $x[n] = \begin{cases} 0, & n < 0 \\ \left(\frac{1}{4}\right)^n, & 0 \leq n \leq 10 \\ 0, & n > 10. \end{cases}$

Problem 2 *Inverse z-Transform I*

Determine sequences $x[\cdot]$ corresponding to the following z-transforms $\hat{x}(\cdot)$:

(a) $\hat{x}(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$.

(b) $\hat{x}(z) = \dfrac{1}{1 + \frac{1}{2}z^{-1}}$, $|z| < \frac{1}{2}$.

(c) $\hat{x}(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 + \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$, $|z| > \frac{1}{2}$.

(d) $\hat{x}(z) = \dfrac{1 - \frac{1}{2}z^{-1}}{1 - \frac{1}{4}z^{-1}}$, $|z| > \frac{1}{2}$.

(e) $\hat{x}(z) = \dfrac{1 - az^{-1}}{z^{-1} - a}$, $|z| > \left|\frac{1}{a}\right|$.

Problem 3 *Partial Fraction Decomposition*

Consider a left-sided sequence $x[\cdot]$ with z-transform

$$\hat{x}(z) = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}.$$

(a) Write $\hat{x}(\cdot)$ as a ratio of polynomials in z instead of z^{-1}.

(b) Using the partial fraction decomposition, express $\hat{x}(\cdot)$ as a sum of fractions, where each fraction represents a pole from your answer in part (a).

(c) Determine $x[\cdot]$.

Problem 4 *Inverse z-Transform II*

A right-sided sequence $x[\cdot]$ has the z-transform

$$\hat{x}(z) = \frac{3z^{-10} + z^{-7} - 5z^{-2} + 4z^{-1} + 1}{z^{-10} - 5z^{-7} + z^{-3}}.$$

Determine $x[n]$ for $n < 0$.

Problem 5 *Power-Series Expansion*

By using the power-series expansion

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \quad |w| < 1$$

determine the inverse of each of the following z-transforms:

(a) $\hat{x}(z) = \log(1 - 2z)$, $|z| < \frac{1}{2}$.

(b) $\hat{x}(z) = \log\left(1 - \frac{1}{2}z^{-1}\right)$, $|z| > \frac{1}{2}$.

Problem 6 *Real-Valued Sequences*

Consider a real-valued sequence $x[\cdot]$ with rational z-transform $\hat{x}(\cdot)$.

(a) From the definition of the z-transform, show that

$$\hat{x}(z) = \hat{x}^*(z^*).$$

(b) From your result in part (a), show that if a pole (zero) of $\hat{x}(\cdot)$ occurs at $z = z_0$, then a pole (zero) must also occur at $z = z_0^*$.

(c) Verify the result in part (b) for each of the following sequences:

(i) $x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0. \end{cases}$

(ii) $x[n] = \delta[n] - \frac{1}{2}\delta[n-1] + \frac{1}{4}\delta[n-2]$, $n \in \mathbb{Z}$.

Problem 7 *Time Reversal*

Consider two sequences $x_1[\cdot]$ and $x_2[\cdot]$ with respective z-transforms $\hat{x}_1(\cdot)$ and $\hat{x}_2(\cdot)$, where

$$x_2[n] = x_1[-n], \quad n \in \mathbb{Z}.$$

Show that $\hat{x}_1(z) = \hat{x}_2(1/z)$ and, from this, show that if $\hat{x}_1(\cdot)$ has a pole (zero) at $z = z_0$, then $\hat{x}_2(\cdot)$ has a pole (zero) at $z = 1/z_0$.

Problem 8 *Even Sequences*

Consider an even sequence $x[\cdot]$, i.e., $x[n] = x[-n]$, $n \in \mathbb{Z}$, with rational z-transform $\hat{x}$.

(a) From the definition of the z-transform, show that

$$\hat{x}(z) = \hat{x}\left(\frac{1}{z}\right).$$

(b) From your result in part (a), show that if a pole (zero) of $\hat{x}(\cdot)$ occurs at $z = z_0$, then a pole (zero) must also occur at $z = 1/z_0$.

(c) Verify the result in part (b) for each of the following sequences:

(i) $x[n] = \delta[n-1] + \delta[n+1]$, $n \in \mathbb{Z}$.

(ii) $x[n] = \delta[n-1] - \frac{5}{2}\delta[n] + \delta[n+1]$, $n \in \mathbb{Z}$.

Problem 9 *LTI System I*

Consider an LTI system with input $s[\cdot]$ and output $x[\cdot]$. This system is described by the difference equation

$$x[n] = s[n] - e^{-8\alpha} s[n-8]$$

where $0 < \alpha < 1$.

(a) Find the system function

$$\hat{h}_1(z) = \frac{\hat{x}(z)}{\hat{s}(z)}$$

and plot the poles and zeros in the z-plane. Indicate the region of convergence.

(b) We wish to recover $s[\cdot]$ from $x[\cdot]$ with another LTI system. Find the system function

$$\hat{h}_2(z) = \frac{\hat{y}(z)}{\hat{x}(z)}$$

such that $y[n] = s[n]$, $n \in \mathbb{Z}$. Find all possible regions of convergence for $\hat{h}_2(\cdot)$ and indicate whether the system is causal and stable.

(c) Find all possible choices for the impulse response $h_2[\cdot]$ such that

$$y[n] = (h_2 * x)[n] = s[n], \quad n \in \mathbb{Z}.$$

Problem 10 *LTI System II*

Consider an LTI system with impulse response $h[\cdot]$ and input $x[\cdot]$ given by

$$h[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

and

$$x[n] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the output $y[\cdot]$ by explicitly evaluating the discrete convolution of $x[\cdot]$ and $h[\cdot]$.

(b) Determine the output $y[\cdot]$ by computing the inverse z-transform of the product of the z-transforms of $x[\cdot]$ and $h[\cdot]$.