✓ E3. Problem 6. Echo echo echo

Consider $x(t) = s(t) + \frac{1}{8} s(t - T_d)$, $t \in \mathbb{R}$ for some $T_d \in \mathbb{R}$ and $s(\cdot)$ bandlimited to $\frac{1}{2T_s}$.
Let $x[\cdot]$ be $x[n] = x(nT_s)$, $n \in \mathbb{Z}$. We want $h[\cdot]$ filter s.t. $x[n] = \sum_{k=-\infty}^{\infty} s(kT_s) h[n-k]$, $n \in \mathbb{Z}$

a) Find $h[\cdot]$ when $T_d = T_s$

b) Find $h[\cdot]$ when $T_d = \frac{T_s}{2}$

a) $\hat{x}_c(f) = \hat{s}_c(f) + \frac{1}{8} e^{-j2\pi f T_d} \hat{s}_c(f)$

Let $s[n] = s(nT_s)$

$$x[n] = \sum_{k=-\infty}^{\infty} s[k]h[n-k] = (s*h)[n] \Rightarrow \hat{x}_s(f) = \hat{s}_s(f) \cdot \hat{h}_s(f) \Rightarrow \hat{h}_s(f) = \frac{\hat{x}_s(f)}{\hat{s}_s(f)}$$

$s(\cdot)$ bandlimited to $\frac{1}{2T_s} \Rightarrow x(\cdot)$ bandlimited to $\frac{1}{2T_s} \Rightarrow$ Nyquist ok $\Rightarrow$ no aliasing

$$x[n] = x(nT_s) = s(nT_s) + \frac{1}{8} s(\underbrace{nT_s - \underbrace{T_d}_{T_s}}_{(n-1)T_s}) = s[n] + \frac{1}{8} s[n-1] \Rightarrow$$

alt

$$\Rightarrow x[n] = (s[n]) * (\delta[n] + \tfrac{1}{8}\delta[n-1]) \Rightarrow$$
$$\Rightarrow h[n] = \delta[n] + \tfrac{1}{8}\delta[n-1]$$

$$\Rightarrow \hat{x}_s(f) = \hat{s}_s(f) + \frac{1}{8} e^{-j2\pi f} \hat{s}_s(f)$$

$$\hat{h}_s(f) = \frac{\hat{s}_s(f) + \frac{1}{8} e^{-j2\pi f} \hat{s}_s(f)}{\hat{s}_s(f)} = 1 + \frac{1}{8} e^{-j2\pi f} \Rightarrow \boxed{h[n] = \delta[n] + \frac{1}{8}\delta[n-1]}$$

b)

~~$x[n] = x(nT_s) = s(nT_s) + \frac{1}{8} s(nT_s - T_d) = s[n] + \frac{1}{8} s[n - \frac{1}{2}]$~~

$$x[n] = x(nT_s) \overset{\text{no aliasing}}{\Rightarrow} \hat{x}_s(f) = \frac{1}{T_s}\hat{x}_c\left(\frac{f}{T_s}\right) = \frac{1}{T_s}\left(\hat{s}_c\left(\frac{f}{T_s}\right) + \frac{1}{8} e^{-j2\pi \frac{f}{T_s} \overset{\frac{T_s}{2}}{T_d}} \hat{s}_c\left(\frac{f}{T_s}\right)\right) =$$

$\hat{s}_s(f) = \frac{1}{T_s}\hat{s}_c\left(\frac{f}{T_s}\right)$

$$= \hat{s}_s(f) + \frac{1}{8} e^{-j\pi f} \hat{s}_s(f)$$

$$\hat{h}_s(f) = \frac{\hat{x}_s(f)}{\hat{s}_s(f)} = \frac{\hat{s}_s(f) + \frac{1}{8} e^{-j\pi f} \hat{s}_s(f)}{\hat{s}_s(f)} = 1 + \frac{1}{8} e^{-j\pi f}$$

$$h_s[n] = \delta[n] + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{8} e^{-j\pi f} e^{j2\pi f n} \, df = \delta[n] + \frac{1}{8} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j\pi f(2n-1)} \, df = \delta[n] + \frac{1}{8} \left. \frac{e^{j\pi(2n-1)f}}{j\pi(2n-1)} \right|_{f=-\frac{1}{2}}^{\frac{1}{2}} =$$

$$= \delta[n] + \frac{1}{8} \frac{e^{j\pi(n-\frac{1}{2})} - e^{-j\pi(n-\frac{1}{2})}}{j\pi(2n-1)} = \delta[n] + \frac{1}{8} \frac{2j\sin(\pi n - \frac{\pi}{2})}{j\pi(2n-1)} = \delta[n] + \frac{1}{8} \frac{-\sin(-\pi n + \frac{\pi}{2})}{\pi(n-\frac{1}{2})} =$$

$$= \delta[n] - \frac{\cos(-\pi n)}{8\pi(n-\frac{1}{2})} = \delta[n] - \frac{\cos(\pi n)}{4\pi(2n-1)} = \boxed{\delta[n] - \frac{(-1)^n}{4\pi(2n-1)} = h_s[n]}$$