E1. Problem 10. LTI System 2

Consider an LTI such that $\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=\frac{d}{dt}x(t)+2x(t)$, $t\in\mathbb{R}$

a) Determine the frequency response $\hat{h}(f)=\frac{\hat{y}(f)}{\hat{x}(f)}$, $f\in\mathbb{R}$

$$\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=\frac{d}{dt}x(t)+2x(t) \Rightarrow (j2\pi f)^2\hat{y}(f)+4\cdot j2\pi f\hat{y}(f)+3\hat{y}(f)=j2\pi f\hat{x}(f)+2\cdot\hat{x}(f)\Rightarrow$$

$$\Rightarrow(-4\pi^2f^2+j8\pi f+3)\hat{y}(f)=(j2\pi f+2)\hat{x}(f)\Rightarrow\hat{y}(f)=\frac{j2\pi f+2}{-4\pi^2f^2+j8\pi f+3}\hat{x}(f)$$

$$\hat{h}(f)=\frac{\hat{y}(f)}{\hat{x}(f)}=\boxed{\frac{j2\pi f+2}{-4\pi^2f^2+j8\pi f+3}}=$$

$4x^2+8x+3=(2x+1)(2x+3)$

b) If $x(t)=e^{-t}$ for $t\geq0$ and $x(t)=0$ for $t<0$, determine the F.T. of the output $y(\cdot)$

$$x(t)=e^{-t}\;\circ\!\!-\!\!\bullet\;\frac{1}{1+j2\pi f}=\hat{x}(f)$$

$$\hat{y}(f)=\hat{x}(f)\,\hat{h}(f)=\boxed{\frac{1}{1+j2\pi f}\cdot\frac{j2\pi f+2}{-4\pi^2f^2+j8\pi f+3}}$$

c) For the input in part (b), determine the output $y(\cdot)$

$$\hat{h}(f)=\frac{j2\pi f+2}{-4\pi f^2+j8\pi f+3}=(2+j2\pi f)\frac{1}{(1+j2\pi f)}\frac{1}{(3+j2\pi f)}=$$

$$=\left(1+\frac{1}{1+j2\pi f}\right)\left(\frac{1}{3+j2\pi f}\right)=$$

$$=\underbrace{\frac{1}{3+j2\pi f}}_{\hat{h}_3(f)}+\frac{1}{3+j2\pi f}\frac{1}{1+j2\pi f}=\hat{h}_1(f)+\hat{h}_1(f)\hat{h}_2(f)\;\bullet\!\!-\!\!\circ\;\underbrace{h_1(t)+h_1(t)*h_2(t)}_{h_3(t)}=(e^{-3t}+(e^{-3t})*(e^{-t}))\,I\{t\geq0\}=$$

$$=e^{-3t}I\{t\geq0\}+\int_0^{\infty}e^{-3\tau}e^{-(t-\tau)}I\{t-\tau\geq0\}\,d\tau=e^{-3t}I\{t\geq0\}+\int_0^t e^{-3\tau-t+\tau}\,d\tau=e^{-3t}I\{t\geq0\}+\frac{e^{-2\tau-t}}{-2}\Big|_{\tau=0}^{t}I\{t\geq0\}=$$

$$=I\{t\geq0\}\left(e^{-3t}+\frac{e^{-t}-e^{-3t}}{2}\right)=\frac{e^{-t}+e^{-3t}}{2}I\{t\geq0\}$$

$$y(t)=x(t)*h(t)=\left(e^{-t}I\{t\geq0\}\right)*\left(\frac{e^{-t}+e^{-3t}}{2}I\{t\geq0\}\right)=$$

$$=\frac{1}{2}\int_0^{\infty}(e^{-\tau}+e^{-3\tau})e^{-(t-\tau)}I\{t-\tau\geq0\}\,d\tau=\frac{1}{2}\left(\int_0^t e^{-\tau-t+\tau}d\tau+\int_0^t e^{-3\tau-t+\tau}d\tau\right)=\frac{1}{2}e^{-t}\int_0^t d\tau+\frac{1}{2}\left(\frac{e^{-2\tau-t}}{-2}\right)\Big|_{\tau=0}^{t}=$$

$$=\boxed{\frac{t}{2}e^{-t}+\frac{1}{4}\left(e^{-t}-e^{-3t}\right)}$$