(5) E2. Problem 1.

e) $x[\cdot]$ periodic w/ $N=7$, $\quad x[n] = \begin{cases} 1, & n = 0,\ldots,4 \\ 0, & n = 5,6 \end{cases}$

$$a_k = \frac{1}{7}\sum_{n=0}^{6} x[n]\, e^{-j\frac{2\pi}{7}kn} = \frac{1}{7}\sum_{n=0}^{4} e^{-j\frac{2\pi}{7}kn} = \frac{1}{7}\frac{1-\left(e^{-j\frac{2\pi}{7}k}\right)^5}{\underbrace{1-e^{-j\frac{2\pi}{7}k}}_{=0 \Leftrightarrow k=7l}} = \frac{1}{7}\frac{\overbrace{1}^{1^2}-\overbrace{e^{-j\frac{10\pi}{7}k}}^{\left(e^{-j\frac{5\pi}{7}k}\right)^2}}{1-e^{-j\frac{2\pi}{7}k}} = \frac{1}{7}\frac{\left(1-e^{-j\frac{5\pi}{7}k}\right)\left(1+e^{-j\frac{5\pi}{7}k}\right)}{\left(1-e^{-j\frac{\pi}{7}k}\right)\left(1+e^{-j\frac{\pi}{7}k}\right)} =$$

$$= \frac{1}{7}\frac{\left(1-e^{-j\frac{5\pi}{7}k}\right)\left(1-e^{j\frac{\pi}{7}k}\right)\left(1+e^{-j\frac{5\pi}{7}k}\right)\left(1-e^{j\frac{5\pi}{7}k}\right)}{\left(1-e^{-j\frac{\pi}{7}k}\right)\left(1-e^{j\frac{5\pi}{7}k}\right)\left(1+e^{-j\frac{\pi}{7}k}\right)\left(1-e^{j\frac{\pi}{7}k}\right)} = \frac{1}{7}\frac{1-e^{-j\frac{5\pi}{7}k}}{1-e^{j\frac{5\pi}{7}k}}\cdot\frac{1-e^{j\frac{\pi}{7}k}}{1-e^{-j\frac{\pi}{7}k}}\cdot\frac{\left(1-e^{j\frac{5\pi}{7}k}+e^{-j\frac{5\pi}{7}k}-1\right)}{\left(1-e^{j\frac{\pi}{7}k}+e^{-j\frac{\pi}{7}k}-1\right)} =$$

$$= \frac{1}{7}\left(-e^{-j\frac{5\pi}{7}k}\right)\left(-e^{j\frac{\pi}{7}k}\right)\cdot\frac{-\sin\left(\frac{5\pi}{7}k\right)}{\underbrace{-\sin\left(\frac{\pi}{7}k\right)}_{=0 \Leftrightarrow k=7l}} = \begin{cases} \frac{1}{7}\cancelto{1}{e^{-j\frac{4\pi}{7}k}}\ \cancelto{5}{\lim_{k\to 0}\frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}}, & k \in \overset{\{\ldots,-7,0,7,\ldots\}}{\cancel{\{-7,\ldots,0,\ldots,7\}}} \\ \frac{1}{7}e^{-j\frac{4\pi}{7}k}\frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}, & \text{otherwise} \end{cases} =$$

$$= \boxed{\begin{cases} \frac{5}{7}, & k \in \{\ldots,-7,0,7,\ldots\} \\ \frac{1}{7}e^{-j\frac{4\pi}{7}k}\frac{\sin\left(\frac{5\pi}{7}k\right)}{\sin\left(\frac{\pi}{7}k\right)}, & \text{otherwise} \end{cases} = a_k}$$