Linear Systems
(2023-24)

Exercise 1

1 2 3 4 5
6 7 8 9 10

Problem 1. Fourier Series Representation

[a b c d e]

Determine the Fourier Series representations of these signals:

a) $x(t) = \cos(4t) + \sin(6t)$, $t \in \mathbb{R}$

- Period:

~~$x(t) = x(t+T) \iff \cos(4t) + \sin(6t) = \cos(4(t+T)) + \sin(6(t+T)) \iff$~~

$\iff$ $x(t)$ is a linear combination of: ~~sum of:~~

$$\left.\begin{array}{l} \cdot\ \cos(4t),\ \text{periodic with } T_0 = \frac{\pi}{2} \\ \cdot\ \sin(6t),\ \text{periodic with } T_0 = \frac{\pi}{3} \end{array}\right\} x(t) \text{ is periodic with } T_0 = \pi$$

- $$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j\frac{2\pi}{T_0}kt}\, dt = \frac{1}{\pi}\int_0^{\pi} (\cos 4t + \sin 6t)\, e^{-j2kt}\, dt =$$

$$\cancel{= \frac{1}{\pi}\left(\int_0^{\pi} Re(e^{6t})\, e^{-2jkt}\, dt + \int_0^{\pi} Im(e^{6t})\, e^{-j2kt}\, dt\right)}$$

$$= \frac{1}{\pi}\left(\frac{1}{2}\int_0^{\pi} (e^{j4t} + e^{-j4t})\, e^{-j2kt}\, dt + \frac{1}{2j}\int_0^{\pi} (e^{j6t} - e^{-j6t})\, e^{-j2kt}\, dt\right) =$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} e^{2jt(2-k)}\, dt + \int_0^{\pi} e^{-2jt(2+k)}\, dt - j\int_0^{\pi} e^{2jt(3-k)}\, dt + j\int_0^{\pi} e^{-2jt(3+k)}\, dt\right) =$$

$$= \frac{1}{2\pi}\left(\frac{e^{2\pi j(2-k)} - e^0}{2j(2-k)} + \frac{e^{-2\pi j(2+k)} - e^0}{-2j(2+k)} - j\,\frac{e^{2\pi j(3-k)} - e^0}{2j(3-k)} + j\,\frac{e^{-2\pi j(3+k)} - e^0}{-2j(3+k)}\right) =$$

- at $k=-3$: $$a_{-3} = \frac{1}{2\pi}\left(0 + 0 - 0 + j\lim_{k\to -3}\frac{e^{-2\pi j(3+k)} - 1}{-2j(3+k)}\right) = \frac{-1}{4\pi}\lim_{k\to -3}\frac{\cos(2\pi(3+k)) - j\sin(2\pi(3+k)) - 1}{3+k} =$$

$$= \left[u = 3+k \Rightarrow k = u-3\right] = \frac{-1}{4\pi}\lim_{u\to 0}\frac{\cos(2\pi u) - 1 - j\sin(2\pi u)}{u} = \frac{j}{4\pi}\lim_{u\to 0} 2\pi\,\mathrm{sinc}(2u) = \frac{j}{2}$$

skip

- at $k=-2$: $$a_{-2} = \frac{1}{2\pi}\left(0 - \frac{e^{-2\pi j(2+k)} - e^0}{2j(2+k)} - 0 + 0\right) = \frac{j}{4\pi}\cdot\frac{\cos(2\pi(2+k)) - j\sin(2\pi(2+k)) - 1}{2+k} =$$

$$= \frac{-j\cdot j}{4\pi}\cdot\frac{2\pi\sin(2\pi(2+k))}{2\pi(2+k)} = \frac{1}{2}\,\mathrm{sinc}(2\pi(2+k)) = \frac{1}{2}\,\delta[2+k]$$