E3. Problem 4. Energy of Sampled Signal

Consider $x(\cdot)$ with W bandwidth and samples $x[\cdot]$ at T_s seconds sample interval: $x[n] = x(nT_s)$, $n \in \mathbb{Z}$. Assume $T_s = \frac{1}{2W}$. What is the relation between the energy of ~~the~~ the samples, $E_s = \sum_{n=-\infty}^{\infty} |x[n]|^2$, and the signal, $E_c = \int_{-\infty}^{\infty} |x(t)|^2\, dt$?

No aliasing

$$E_s = \sum_{n=-\infty}^{\infty} |x[n]|^2 \overset{\text{P's rel.}}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{x}_s(f)|^2\, df = \frac{1}{2}\int_{-1}^{1} |\hat{x}_s(f)|^2\, df$$

$$E_c = \int_{-\infty}^{\infty} |x(t)|^2\, dt \overset{\text{P's rel.}}{=} \int_{-\infty}^{\infty} |\hat{x}_c(f)|^2\, df \underbrace{=}_{\hat{x}_c(f)=0\ \forall |f|>W} \int_{-W}^{W} |\hat{x}_c(f)|^2\, df$$

~~No aliasing => F.S. coefficients a_n are the same~~

~~$E_s = \sum_{n=-\infty}^{\infty} |x[n]|^2$~~

$$\hat{x}_s(f) = \frac{1}{T_s}\hat{x}_c\left(\frac{f}{T_s}\right) \Rightarrow E_s = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{1}{T_s}\hat{x}_c\left(\frac{f}{T_s}\right)\right|^2 = \left[\begin{array}{l} f' = \frac{f}{T_s} \Rightarrow f = f' \cdot T_s \Rightarrow df = d(f'T_s) = T_s df' \\ f \to \frac{1}{2} \Rightarrow f' \to \underbrace{\frac{1}{2T_s}}_{W} ;\ f \to -\frac{1}{2} \Rightarrow f' \to \underbrace{\frac{-1}{2T_s}}_{-W} \end{array}\right] =$$

$$= \int_{\frac{-1}{2T_s}}^{\frac{1}{2T_s}} \left|\frac{1}{T_s}\right|^2 |\hat{x}_c(f')|^2\, T_s\, df' = \frac{1}{T_s}\int_{-W}^{W} |\hat{x}_c(f')|^2\, df' = \boxed{\frac{1}{T_s} E_c = E_s}$$