E1. Problem 4. Modulation of Fourier Series

Let $x(\cdot)$ and $y(\cdot)$ be continuous-time signals with fundamental period T_0 and Fourier series representations $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{+j\frac{2\pi}{T_0}kt}$ and $y(t) = \sum_{k=-\infty}^{\infty} b_k e^{+j\frac{2\pi}{T_0}kt}$

a) Show that the Fourier series coefficients of $z(t) = x(t)\,y(t)$, $t \in \mathbb{R}$, with $z(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T_0}kt}$

are given by the discrete-time convolution $c_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}$

$$c_k = \frac{1}{T_0}\int_0^{T_0} z(t) e^{-j\frac{2\pi}{T_0}kt}\,dt = \frac{1}{T_0}\int_0^{T_0} \sum_{n=-\infty}^{\infty}\left(a_n e^{j\frac{2\pi}{T_0}nt}\right)\sum_{m=-\infty}^{\infty}\left(b_m e^{j\frac{2\pi}{T_0}mt}\right) e^{-j\frac{2\pi}{T_0}kt}\,dt = \frac{1}{T_0}\int_0^{T_0}\sum_n\left(a_n e^{j\frac{2\pi}{T_0}nt}\sum_m\left(b_m e^{j\frac{2\pi}{T_0}mt}\right)\right)e^{-j\frac{2\pi}{T_0}kt}\,dt =$$

$$= \frac{1}{T_0}\sum_n\left(a_n\int_0^{T_0} e^{j\frac{2\pi}{T_0}nt}\sum_m\left(b_m e^{j\frac{2\pi}{T_0}mt}\right)e^{-j\frac{2\pi}{T_0}kt}\,dt\right) = \frac{1}{T_0}\sum_n\left(a_n\sum_m\left(b_m\int_0^{T_0} e^{j\frac{2\pi}{T_0}t(n+m-k)}\,dt\right)\right) =$$

$$= \frac{1}{T_0}\sum_n\left(a_n\sum_m\left(b_m\left(\frac{e^{j\frac{2\pi}{T_0}t(n+m-k)}}{j\frac{2\pi}{T_0}(n+m-k)}\right)\Bigg|_{t=0}^{T_0}\right)\right) = \sum_n\left(a_n\sum_m\left(b_m\frac{e^{j2\pi(n+m-k)}-1}{j2\pi(n+m-k)}\right)\right) =$$

$$= \sum_n\left(a_n\sum_m\left(b_m\frac{\overbrace{\cos(2\pi(n+m-k))}^{1}+j\sin(2\pi(n+m-k))-1}{j2\pi(n+m-k)}\right)\right) = \sum_n\left(a_n\sum_m\left(b_m\,\mathrm{sinc}(2(n+m-k))\right)\right) =$$

$$= \sum_n\left(a_n\overbrace{\sum_m\left(b_m\,\delta[n+m-k]\right)}^{=b_m|_{m=k-n}}\right) = \sum_n a_n b_{k-n} \quad \blacksquare \text{ q.e.d.}$$

b) Use this result to compute the Fourier series coefficients of $z(t) = \cos^2(4t)$, $t \in \mathbb{R}$

Let $x(t) = y(t) = \cos(4t) \Rightarrow T_0 = \frac{\pi}{2}$

$$a_k = \frac{1}{\frac{\pi}{2}}\int_0^{\pi/2}\cos(4t)\,e^{-j\frac{2\pi}{\pi/2}kt}\,dt = \frac{2}{\pi}\int_0^{\pi/2}\frac{e^{j4t}+e^{-j4t}}{2}e^{-j4kt}\,dt = \frac{1}{\pi}\left(\int_0^{\pi/2} e^{j4t(1-k)}\,dt + \int_0^{\pi/2} e^{-j4t(1+k)}\,dt\right) =$$

$$= \frac{1}{\pi}\left(\frac{e^{j4t(1-k)}}{j4(1-k)} + \frac{e^{-j4t(1+k)}}{-j4(1+k)}\right)\Bigg|_{t=0}^{\pi/2} = \frac{1}{2}\left(-j\frac{e^{j2\pi(1-k)}-1}{2\pi(1-k)} + j\frac{e^{-j2\pi(1+k)}-1}{2\pi(1+k)}\right) =$$

E's form + simpl.

$$= \frac{1}{2}\left(\mathrm{sinc}(2(1-k)) + \mathrm{sinc}(2(1+k))\right) = \frac{1}{2}\left(\delta[1-k]+\delta[1+k]\right)$$

$$c_k = a_k * a_k = \frac{1}{2}\left(\delta[1-k]+\delta[1+k]\right) * \frac{1}{2}\left(\delta[1-k]+\delta[1+k]\right) = \frac{1}{4}\left(\delta[1-k]*(\delta[1-k]+\delta[1+k])\right.$$

$$= \frac{1}{4}\left(\delta[k-2]+\delta[k]+\delta[k]+\delta[k+2]\right) = \boxed{\frac{1}{2}\delta[k] + \frac{1}{4}\left(\delta[k-2]+\delta[k+2]\right)}$$