Exercise 2

Linear Systems 2023/2024

Problem 1 *Fourier Series Coefficients*

Determine the Fourier series coefficients of each of the following signals:

(a) $x[n] = \sin[\pi(n-1)/4]$, $n \in \mathbb{Z}$.

(b) $x[n] = \cos\left(\frac{11\pi n}{4} - \frac{\pi}{3}\right)$, $n \in \mathbb{Z}$.

(c) $x[\cdot]$ is periodic with period 6 and

$$x[n] = \left(\frac{1}{2}\right)^{|n|}, \quad n = -2, \ldots, 3.$$

(d) $x[\cdot]$ is periodic with period 4 and

$$x[n] = 1 - \sin\left(\frac{\pi n}{4}\right), \quad n = 0, \ldots, 3.$$

(e) $x[\cdot]$ is periodic with period 7 and

$$x[n] = \begin{cases} 1, & n = 0, \ldots, 4 \\ 0, & n = 5, 6. \end{cases}$$

(f) $x[\cdot]$ is periodic with period 6 and

$$x[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2 \\ 0, & n = 3, \ldots, 5. \end{cases}$$

Problem 2 *Properties of the Fourier Series*

Let $x[\cdot]$ be a periodic sequence with period N and Fourier series representation

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n}.$$

For each of the following signals, derive the Fourier series coefficients and express them as a function of $\{a_k\}$:

(a) $y[n] = x[n - n_0]$, $n \in \mathbb{Z}$.

(b) $y[n] = x[n] - x[n-1]$, $n \in \mathbb{Z}$.

(c) $y[n] = x[n] + x^*[-n]$, $n \in \mathbb{Z}$.

(d) $y[n] = (-1)^n x[n]$, $n \in \mathbb{Z}$ and N is even.

(e) $y[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{otherwise.} \end{cases}$

Problem 3 *The Fourier Series of Real Periodic Signals*

Let $x[\cdot]$ be a real-valued, periodic signal with period N and Fourier series coefficients $\{a_k\}$.

(a) Let the Cartesian form of $\{a_k\}$ be given by

$$a_k = b_k + jc_k, \quad k \in \mathbb{Z}.$$

Show that $a_{-k} = a_k^*$, $k \in \mathbb{Z}$. What is the relation between b_k and b_{-k}? What is the relation between c_k and c_{-k}?

(b) Suppose that N is even. Show that $a_{N/2}$ is real-valued.

(c) Suppose that N is even, and let the polar form of $\{a_k\}$ be given by

$$a_k = A_k e^{j\theta_k}, \quad k \in \mathbb{Z}.$$

Show that

$$x[n] = A_0 + A_{N/2}(-1)^n + 2\sum_{k=1}^{N/2-1} A_k \cos\left(\frac{2\pi kn}{N} + \theta_k\right).$$

Problem 4 *Modulation Property of Fourier Series*

Let $x[\cdot]$ and $y[\cdot]$ be discrete-time periodic signals with Fourier series representations

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n} \quad \text{and} \quad y[n] = \sum_{k=\langle N\rangle} b_k e^{jk\frac{2\pi}{N}n}.$$

(a) Show that the Fourier series coefficients of the signal $z[n] = x[n]y[n]$, $n \in \mathbb{Z}$ with Fourier series representation

$$z[n] = \sum_{k=\langle N\rangle} c_k e^{jk\frac{2\pi}{N}n}$$

is given by the discrete convolution

$$c_k = \sum_{\ell=\langle N\rangle} a_\ell b_{k-\ell}.$$

(b) Use this result to compute the Fourier series coefficients of $z[n] = x[n]y[n]$, $n \in \mathbb{Z}$, where $x[n] = \cos(\pi n/3)$, $n \in \mathbb{Z}$ and $y[\cdot]$ is periodic with period 12 and

$$y[n] = \begin{cases} 1, & \text{if } n = -3, \dots, 3 \\ 0, & \text{if } n = -6, \dots, -4 \text{ or } n = 4, \dots, 6. \end{cases}$$

Problem 5 *Fourier Transforms I*

Compute the Fourier transforms of each of the following signals:

(a) $x[n] = \left(\frac{1}{2}\right)^{|n|}$, $n \in \mathbb{Z}$.

(b) $x[n] = \begin{cases} 2 - \frac{|n|}{2}, & |n| = 0, \dots, 4 \\ 0, & |n| = 5, 6, \dots \end{cases}$

(c) $x[n] = \mathrm{I}\{0 \le n \le 3\}$, $n \in \mathbb{Z}$, where $\mathrm{I}\{\cdot\}$ denotes the indicator function, i.e., $\mathrm{I}\{\text{statement}\} = 1$ if the statement is true and $\mathrm{I}\{\text{statement}\} = 0$ otherwise.

(d) $x[n] = \left(\frac{1}{2}\right)^n \mathrm{I}\{n \ge 0\}$, $n \in \mathbb{Z}$.

(e) $x[n] = n(\mathrm{I}\{n \ge -N\} - \mathrm{I}\{n \ge N+1\})$, $n \in \mathbb{Z}$.

(f) $x[n] = \operatorname{sinc}^2(n/2)$, $n \in \mathbb{Z}$.

Problem 6 *Fourier Transforms II*

Let $\hat{x}(\cdot)$ be the Fourier transform of $x(\cdot)$. Derive expressions in terms of $\hat{x}(\cdot)$ for the Fourier transform of the following signals:

(a) $y[n] = \mathfrak{Re}\{x[n]\}$, $n \in \mathbb{Z}$.

(b) $y[n] = x^*[-n]$, $n \in \mathbb{Z}$.

(c) $y[n] = x[n] - x[-n]$, $n \in \mathbb{Z}$.

(d) $y[n] = 2x[n]x[n-5]$, $n \in \mathbb{Z}$.

Problem 7 *A Discrete-Time System*

A particular discrete-time system has input $x[\cdot]$ and output $y[\cdot]$. The Fourier transforms of these signals are related by the following equation

$$\hat{y}(f) = 2\hat{x}(f) + e^{-j2\pi f}\hat{x}(f) + \frac{\mathrm{d}\hat{x}(f)}{\mathrm{d}f}, \quad -\frac{1}{2} \leq f \leq \frac{1}{2}.$$

(a) Is the system linear?

(b) Is the system time-invariant?

(c) What is $y[\cdot]$ if $x[n] = \delta[n]$, $n \in \mathbb{Z}$?

Problem 8 *LTI System 1*

Consider a discrete-time LTI system with impulse response

$$h[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & \text{if } n = 0, 1, \dots \\ 0, & \text{if } n = -1. -2, \dots \end{cases}$$

Use Fourier transforms to determine the response to each of the following input signals:

- $x[n] = \left(\frac{3}{4}\right)^n \mathrm{I}\{n \geq 0\}$, $n \in \mathbb{Z}$.
- $x[n] = (n+1)\left(\frac{1}{4}\right)^n \mathrm{I}\{n \geq 0\}$, $n \in \mathbb{Z}$.

Problem 9 *LTI System 2*

Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n], \quad n \in \mathbb{Z}.$$

(a) Determine the frequency response

$$\hat{h}(f) = \frac{\hat{y}(f)}{\hat{x}(f)}, \quad -\frac{1}{2} \leq f \leq \frac{1}{2}$$

of the system.

(b) What is the response of this system to the following inputs:

(i) $x[n] = \left(\frac{1}{2}\right)^n \mathrm{I}\{n \geq 0\}$, $n \in \mathbb{Z}$.

(ii) $x[n] = \left(-\frac{1}{2}\right)^n \mathrm{I}\{n \geq 0\}$, $n \in \mathbb{Z}$.

(iii) $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$, $n \in \mathbb{Z}$.

(iv) $x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$, $n \in \mathbb{Z}$.

Problem 10 *Minimum Phase System*

Consider two LTI systems with the following frequency responses:

$$\hat{h}_1(f) = \frac{1 + \frac{1}{2}e^{-j2\pi f}}{1 + \frac{1}{4}e^{-j2\pi f}}, \quad -\frac{1}{2} \leq f \leq \frac{1}{2}$$

$$\hat{h}_2(f) = \frac{\frac{1}{2} + e^{-j2\pi f}}{1 + \frac{1}{4}e^{-j2\pi f}}, \quad -\frac{1}{2} \leq f \leq \frac{1}{2}.$$

(a) Show that both frequency responses have the same magnitude, i.e.,

$$|\hat{h}_1(f)| = |\hat{h}_2(f)|, \quad -1/2 \leq f \leq 1/2$$

but that the phase of $\hat{h}_2(\cdot)$ is larger in magnitude than the phase of $\hat{h}_1(\cdot)$, i.e.,

$$|\angle\hat{h}_1(f)| \leq |\angle\hat{h}_2(f)|, \quad -1/2 \leq f \leq 1/2.$$

(b) Show that

$$\hat{h}_2(f) = \hat{g}(f)\hat{h}_1(f), \quad -\frac{1}{2} \leq f \leq \frac{1}{2}$$

where $\hat{g}(\cdot)$ is an all-pass system, i.e., $|\hat{g}(f)| = 1$, $-1/2 \leq f \leq 1/2$.

The system with frequency response $\hat{h}_1(\cdot)$ is usually referred to as a *minimum-phase system*, and the system with frequency response $\hat{h}_2(\cdot)$ is called a *non-minimum-phase system.*