...E1. Problem 5.

d) $x(t) = \text{sinc}^2(2t)\,,\ t \in \mathbb{R}$

$$x(t) = \text{sinc}^2(2t) = \text{sinc}(2t)\,\text{sinc}(2t) = y(t)\cdot y(t)$$

$$\text{sinc}(t) \quad \circ\!\!-\!\!\bullet \quad \Pi(f)$$

$$y(t) = \text{sinc}(2t) \quad \circ\!\!-\!\!\bullet \quad \tfrac{1}{2}\Pi\left(\tfrac{f}{2}\right) = \hat{y}(t)$$

$$\hat{x}(f) = \hat{y}(f) * \hat{y}(f) = \left(\tfrac{1}{2}\Pi\left(\tfrac{f}{2}\right)\right) * \left(\tfrac{1}{2}\Pi\left(\tfrac{f}{2}\right)\right) = \frac{1}{2}\left(\Pi\left(\tfrac{f}{2}\right) * \Pi\left(\tfrac{f}{2}\right)\right) = \frac{1}{2}\Lambda\left(\frac{f}{2}\right) = \boxed{\begin{cases}\frac{1}{2} - \frac{|f|}{4} & \text{if } |f| \le 2 \\ 0 & \text{otherwise}\end{cases}}$$

$$= \frac{1}{2}\left(\int_{-\infty}^{\infty} \Pi(\tau)\,\Pi\left(\tfrac{f}{2} - \tau\right) d\tau\right) = \frac{1}{2}\int_{-1/2}^{1/2} 1\cdot\Pi\left(\tfrac{f}{2} - \tau\right) d\tau = \frac{1}{2}\int_{-1/2}^{1/2} \begin{cases} 1 & \text{if } \underbrace{\left|\tfrac{f}{2} - \tau\right| \le \tfrac{1}{2}}_{\frac{f-1}{2} \le \tau \le \frac{f+1}{2}} \\ 0 & \text{otherwise}\end{cases} d\tau =$$

$$\frac{f-1}{2} < -\frac{1}{2} \Leftrightarrow f < 0 \qquad \frac{f+1}{2} > \frac{1}{2} \Leftrightarrow f > 0$$

$$= \left[\cancel{I\{-\tfrac{1}{2} \le \tau \le \tfrac{1}{2}\}\, I\{|\tfrac{f}{2} - \tau| \le \tfrac{1}{2}\}}\right] = \left[\begin{cases} -\frac{1}{2} \le \tau \le \frac{1}{2} \\ \wedge \\ \frac{f-1}{2} \le \tau \le \frac{f+1}{2}\end{cases} \Leftrightarrow \begin{cases} -\frac{1}{2} \le \tau < \frac{f+1}{2}\,, & -2 \le f < 0 \\ \vee \\ \frac{f-1}{2} \le \tau \le \frac{1}{2}\,, & 0 \le f \le 2\end{cases}\right] =$$

$$\frac{f+1}{2} > -\frac{1}{2} \Leftrightarrow f > -2 \qquad \frac{f-1}{2} < \frac{1}{2} \Leftrightarrow f < 2$$

$$= \frac{1}{2}\begin{cases} \int_{-1/2}^{\frac{f+1}{2}} 1\,d\tau & \text{if } -2 \le f < 0 \\ \int_{\frac{f-1}{2}}^{1/2} 1\,d\tau & \text{if } 0 \le f \le 2 \\ 0 & \text{otherwise}\end{cases} = \frac{1}{2}\begin{cases} \underbrace{\frac{f+1}{2} + \frac{1}{2}}_{1+\frac{f}{2}} & \text{if } -2 \le f < 0 \\ \underbrace{\frac{1}{2} - \frac{f-1}{2}}_{1-f/2} & \text{if } 0 \le f \le 2 \\ 0 & \text{otherwise}\end{cases} = \frac{1}{2}\begin{cases} 1 - \frac{|f|}{2} & \text{if } |f| \le 2 \\ 0 & \text{otherwise}\end{cases} = \boxed{\begin{cases}\frac{1}{2} - \frac{|f|}{4} & \text{if } |f| \le 2 \\ 0 & \text{otherwise}\end{cases}}$$

e)

...E1. Problem 5.

g) $x(t) = I\{0 < t \le 1\} - I\{1 < t \le 2\}\,,\ t \in \mathbb{R}$

$$x(t) = \begin{cases} 1 & \text{if } 0 < t \le 1 \\ -1 & \text{if } 1 < t \le 2 \\ 0 & \text{otherwise}\end{cases}$$

$$\hat{x}(t) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\,dt = \int_0^1 1 e^{-j2\pi ft}\,dt + \int_1^2 -1 e^{-j2\pi ft}\,dt = \left(\frac{e^{-j2\pi ft}}{-j2\pi f}\right)\Bigg|_{t=0}^{1} - \left(\frac{e^{-j2\pi ft}}{-j2\pi f}\right)\Bigg|_{t=1}^{2} =$$

$$= \frac{e^{-j2\pi f} - e^0}{-j2\pi f} - \frac{e^{-j4\pi f} - e^{-j2\pi f}}{-j2\pi f} = \frac{-e^{-j4\pi f} + 2e^{-j2\pi f} - 1}{-j2\pi f} = \boxed{\frac{(1 - e^{-j2\pi f})^2}{j2\pi f}}$$

$$\cancel{= \frac{\cos(4\pi f) - j\sin(4\pi f) - 2(\cos(2\pi f) - j\sin(2\pi f)) + 1}{j2\pi f}}$$