E3. Problem 2. Sampling a Periodic Signal

Consider the periodic signal $x(t) = \frac{1}{2}\cos(3t) + 2\sin(4t)$, $t \in \mathbb{R}$ with samples $x[n] = x(nT_s)$, $n \in \mathbb{Z}$

a) Determine the fundamental period T_0 of $x(\cdot)$. If $T_s = T_0/N$, for odd int N, what is the smallest N s.t. ~~x~~ $x(\cdot)$ can be recovered perfectly from $x[\cdot]$?

b) For $N=11$, plot the imaginary and real parts of the F.S. coefficients of $x[\cdot]$. Is there aliasing?

c) For $N=5$, plot the real and im. parts of the F.S. coefficients of $x[\cdot]$. If you reconstruct $x(\cdot)$ from these, what signal do you obtain?

a) $\cos(3t)$ has period $T_A = \frac{2\pi}{3}$
$\sin(4t)$ has period $T_B = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\Rightarrow T_0 = \text{m.c.m.}\left(\frac{2\pi}{3}, \frac{\pi}{2}\right) = \boxed{2\pi = T_0}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j\frac{2\pi}{T_0}kt}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{2}\cos(3t) + 2\sin(4t)\right)e^{-j\frac{2\pi}{2\pi}kt}\,dt =$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\frac{e^{j3t}+e^{-j3t}}{2}e^{-jkt}\,dt + \frac{1}{\pi j}\int_0^{2\pi}\frac{e^{j4t}-e^{-j4t}}{2}e^{-jkt}\,dt =$$

$$= \frac{1}{8\pi}\left(\int_0^{2\pi} e^{j(3-k)t}\,dt + \int_0^{2\pi} e^{-j(3+k)t}\,dt\right) + \frac{1}{2\pi j}\left(\int_0^{2\pi} e^{j(4-k)t}\,dt - \int_0^{2\pi} e^{-j(4+k)t}\,dt\right) =$$

$$= \frac{1}{8\pi}\left(\frac{e^{j(3-k)t}}{j(3-k)} + \frac{e^{-j(3+k)t}}{-j(3+k)}\right)\Bigg|_{t=0}^{2\pi} + \frac{1}{2\pi j}\left(\frac{e^{j(4-k)t}}{j(4-k)} - \frac{e^{-j(4+k)t}}{-j(4+k)}\right)\Bigg|_{t=0}^{2\pi} =$$

$$= \frac{1}{8\pi}\left(\frac{\overbrace{e^{j2\pi(3-k)}}^{1+j\sin(2\pi(3-k))}-1}{j(3-k)} + \frac{\overbrace{e^{-j2\pi(3+k)}}^{1-j\sin(2\pi(3+k))}-1}{-j(3+k)}\right) + \frac{1}{2\pi j}\left(\frac{e^{j2\pi(4-k)}-1}{j(4-k)} - \frac{e^{-j(4+k)2\pi}-1}{-j(4+k)}\right) =$$

$$= \frac{1}{8\pi}\left(\frac{\sin(2\pi(3-k))}{\frac{2\pi}{2\pi}(3-k)} + \frac{\sin(2\pi(3+k))}{\frac{2\pi}{2\pi}(3+k)}\right) + \frac{1}{2\pi}\left(\frac{\sin(2\pi(4-k))}{\frac{2\pi}{2\pi}(4-k)} - \frac{\sin(2\pi(4+k))}{\frac{2\pi}{2\pi}(4+k)}\right) =$$

$$= \frac{1}{4}\left(\underbrace{\operatorname{sinc}(\overbrace{2(3-k)}^{\in\mathbb{Z}})}_{\delta[2(3-k)]} + \operatorname{sinc}(2(3+k)) + \operatorname{sinc}(2(4-k)) - \operatorname{sinc}(2(4+k))\right) = \cancel{\tfrac{1}{4}(\delta[k-3]+\delta[k+3])}$$

$$= \frac{1}{4}\left(\delta[k-3] + \delta[k+3]\right) + \delta[k-4] - \delta[k+4]\,)$$

$a_k = 0 \;\forall |k| > 4 = k_0$; $T_s = \frac{T_0}{N}$ with $N > 2k_0 = 8 \Rightarrow N \geq 9 \Rightarrow \boxed{N_{min} = 9}$