Problem 1 *Fourier Series Representations*

Determine the Fourier series representations of each of the following signals:

(a) $x(t) = \cos(4t) + \sin(6t)$, $t \in \mathbb{R}$.

(b) $x(t) = \cos\left(\frac{\pi}{4}(t-1)\right)$, $t \in \mathbb{R}$.

(c) $x(t)$ is periodic with period 2 and

$$x(t) = e^{-t}, \quad -1 < t \leq 1.$$

(d) $x(t)$ is periodic with period 4 and

$$x(t) = \frac{t}{2}, \quad -2 < t \leq 2.$$

(e) $x(t)$ is periodic with period 1 and

$$x(t) = \begin{cases} -1, & \text{if } -\frac{1}{2} < t \leq 0 \\ 1, & \text{if } 0 < t \leq \frac{1}{2}. \end{cases}$$

Problem 2 *Periodic Signal*

Consider the signal

$$x(t) = \cos(2\pi t), \quad t \in \mathbb{R}.$$

Since $x(\cdot)$ is periodic with a fundamental period of 1, it is also periodic with a period of N, where N is any positive integer. What are the Fourier series coefficients of $x(\cdot)$ if we regard it as a periodic signal with period 3?

Problem 3 *Properties of the Fourier Series*

Let $x(\cdot)$ be a periodic signal with fundamental period T_0 and Fourier series coefficients $\{a_k\}$. For each of the following signals, derive the Fourier series coefficients and express them as a function of $\{a_k\}$:

(a) $y(t) = x(t - t_0)$, $t \in \mathbb{R}$.

(b) $y(t) = x(-t)$, $t \in \mathbb{R}$.

(c) $y(t) = x(t) + x^*(t)$, $t \in \mathbb{R}$.

(d) $y(t) = x(t) + \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, $t \in \mathbb{R}$.

(e) $y(t) = x(\alpha t)$, $t \in \mathbb{R}$ for $\alpha > 0$.

Problem 4 *Modulation Property of Fourier Series*

Let $x(\cdot)$ and $y(\cdot)$ be continuous-time periodic signals with fundamental period T_0 and Fourier series representations

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \quad \text{and} \quad y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\frac{2\pi}{T_0}kt}.$$

(a) Show that the Fourier series coefficients of the signal $z(t) = x(t)y(t)$, $t \in \mathbb{R}$ with Fourier series representation

$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T_0}kt}$$

is given by the discrete convolution

$$c_k = \sum_{\ell=-\infty}^{\infty} a_\ell b_{k-\ell}.$$

(b) Use this result to compute the Fourier series coefficients of $z(t) = \cos^2(4t)$, $t \in \mathbb{R}$.

Problem 5 *Fourier Transforms*

Compute the Fourier transforms of each of the following signals:

(a) $x(t) = e^{-|t|}$, $t \in \mathbb{R}$.

(b) $x(t) = \begin{cases} t, & \text{if } |t| \leq 2 \\ 0, & \text{if } |t| > 2. \end{cases}$

(c) $x(t) = \operatorname{sinc}(t)$, $t \in \mathbb{R}$.

(d) $x(t) = \operatorname{sinc}^2(2t)$, $t \in \mathbb{R}$.

(e) $x(t) = e^{-t}\,\mathrm{I}\{t \geq 0\}$, $t \in \mathbb{R}$, where $\mathrm{I}\{\cdot\}$ denotes the indicator function, i.e., $\mathrm{I}\{\text{statement}\} = 1$ if the statement is true and $\mathrm{I}\{\text{statement}\} = 0$ otherwise.

(f) $x(t) = \begin{cases} 2\left(1 - \frac{|t|}{2}\right), & \text{if } |t| \leq 2 \\ 0, & \text{otherwise.} \end{cases}$

(g) $x(t) = \mathrm{I}\{0 < t \leq 1\} - \mathrm{I}\{1 < t \leq 2\}$, $t \in \mathbb{R}$.

Problem 6 *Properties of the Fourier Transform of Real-Valued Signals*

Let $x(\cdot)$ be a real-valued signal.

(a) If $x(\cdot)$ is odd, i.e., $x(-t) = -x(t)$, $t \in \mathbb{R}$, show that its Fourier transform $\hat{x}(\cdot)$ is pure imaginary and odd.

(b) If $x(\cdot)$ is even, i.e., $x(-t) = x(t)$, $t \in \mathbb{R}$, show that its Fourier transform $\hat{x}(\cdot)$ is real and even.

Problem 7 *Derivative of Bandlimited Signal*

Let $x(\cdot)$ be a a given signal with Fourier transform

$$\hat{x}(f) = \begin{cases} 1, & \text{if } |f| < 1 \\ 0, & \text{if } |f| > 1. \end{cases}$$

Define the signal $y(\cdot)$ as

$$y(t) = \frac{\mathrm{d}^2}{dt^2} x(t), \quad t \in \mathbb{R}.$$

What is the value of

$$\int_{-\infty}^{\infty} |y(t)|^2 \,\mathrm{d}t?$$

Problem 8 *Inverse Filter*

Let the signal $x(\cdot)$ be given by

$$x(t) = \operatorname{sinc}(8t), \quad t \in \mathbb{R}$$

and consider a filter $h(\cdot)$ with impulse response

$$h(t) = e^{-|t|}, \quad t \in \mathbb{R}.$$

(a) What is the Fourier transform $\hat{y}(\cdot)$ of the filtered signal

$$y(t) = (x * h)(t), \quad t \in \mathbb{R}?$$

(b) Find a filter $g(\cdot)$ of finite energy such that

$$z(t) = (y * g)(t) = x(t), \quad t \in \mathbb{R}.$$

Problem 9 *LTI System 1*

Consider an LTI system whose input and output satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 2y(t) = x(t), \quad t \in \mathbb{R}.$$

(a) Determine the frequency response

$$\hat{h}(f) = \frac{\hat{y}(f)}{\hat{x}(f)}, \quad f \in \mathbb{R}$$

of the system.

(b) If $x(t) = e^{-t}$ for $t \geq 0$ and $x(t) = 0$ for $t < 0$, determine the Fourier transform of the output $y(\cdot)$.

(c) For the input in Part (b), determine the output $y(\cdot)$.

Problem 10 *LTI System 2*

Consider an LTI system whose input and output satisfy

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 4\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = \frac{\mathrm{d}}{\mathrm{d}t}x(t) + 2x(t), \quad t \in \mathbb{R}.$$

(a) Determine the frequency response

$$\hat{h}(f) = \frac{\hat{y}(f)}{\hat{x}(f)}, \quad f \in \mathbb{R}$$

of the system.

(b) If $x(t) = e^{-t}$ for $t \geq 0$ and $x(t) = 0$ for $t < 0$, determine the Fourier transform of the output $y(\cdot)$.

(c) For the input in Part (b), determine the output $y(\cdot)$.