E1. Problem 5. Fourier Transforms [a b c d e f g]

Compute the Fourier Transforms of these signals:

a) $x(t) = e^{-|t|}$, $t \in \mathbb{R}$

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} e^{-|t|} e^{-j2\pi f t}\, dt = \int_{-\infty}^{0} e^{\overbrace{t-j2\pi f t}^{t(1-j2\pi f)}}\, dt + \int_{0}^{\infty} e^{\overbrace{-t-j2\pi f t}^{-t(1+j2\pi f)}}\, dt =$$

$$= \frac{e^{t(1-j2\pi f)}}{(1-j2\pi f)}\Bigg|_{t=-\infty}^{0} + \frac{e^{-t(1+j2\pi f)}}{-(1+j2\pi f)}\Bigg|_{t=0}^{\infty} = \frac{e^{0}-e^{-\infty}}{1-j2\pi f} + \frac{e^{-\infty}-e^{0}}{-(1+j2\pi f)} = \frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} =$$

$$= \frac{1+2j\pi f + 1 - 2j\pi f}{1+4\pi^2 f^2} = \boxed{\frac{2}{1+4\pi^2 f^2}}$$

b) $x(t) = \begin{cases} t & \text{if } |t| \leq 2 \\ 0 & \text{otherwise} \end{cases}$

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = \int_{-2}^{2} t\, e^{-j2\pi f t}\, dt = \left[\begin{matrix} u = t & \Rightarrow du = dt \\ dv = e^{-j2\pi f t}\, dt & \Rightarrow v = \int e^{-j2\pi f t}\, dt = \underbrace{\frac{-1}{j2\pi f} e^{-j2\pi f t}}_{\frac{j}{2\pi f} e^{-j2\pi f t}} \end{matrix}\right] =$$

$$= t \cdot \frac{j\, e^{-j2\pi f t}}{2\pi f}\Bigg|_{t=-2}^{2} - \int_{-2}^{2} \frac{j}{2\pi f} e^{-j2\pi f t}\, dt = \frac{j}{2\pi f}\left(t\, e^{-j2\pi f t}\Big|_{t=-2}^{2} - \frac{e^{-j2\pi f t}}{-j2\pi f}\Bigg|_{t=-2}^{2} \right) =$$

$$= \frac{(2\pi f j t + 1)\, e^{-j2\pi f t}}{4\pi^2 f^2}\Bigg|_{t=-2}^{2} = \frac{(4j\pi f + 1)\, e^{-4j\pi f} - (-4j\pi f + 1)\, e^{4j\pi f}}{4\pi^2 f^2} =$$

$$= \frac{+4j\pi f \cos(4\pi f) + \cos(4\pi f) - i\sin(4\pi f) + 4j\pi f\cos(4\pi f) + 4\pi f \sin(4\pi f) - \cos(4\pi f) - i\sin(4\pi f)}{4\pi^2 f^2} =$$

$$= \frac{+j4\pi f\cos(4\pi f) - j\sin(4\pi f)}{2\pi^2 f^2} = \boxed{+j\,\frac{4\pi f\cos(4\pi f) - \sin(4\pi f)}{2\pi^2 f^2}}$$