Exercise 3

Linear Systems 2023/2024

Problem 1 *Sampling a Sinc-Square*

Consider the signal
$$x(t) = \operatorname{sinc}^2(t/2), \quad t \in \mathbb{R}$$
with samples
$$x[n] = x(nT_s), \quad n \in \mathbb{Z}.$$

(a) How small must T_s be such that we can perfectly recover $x(\cdot)$ from $x[\cdot]$?

(b) For $T_s = 1$, plot the Fourier transform of $x[\cdot]$. Is there aliasing?

(c) For $T_s = 2$, plot the Fourier transform of $x[\cdot]$. Is there aliasing?

Problem 2 *Sampling a Periodic Signal*

Consider the periodic signal
$$x(t) = \frac{1}{2}\cos(3t) + 2\sin(4t), \quad t \in \mathbb{R}$$
with samples
$$x[n] = x(nT_s), \quad n \in \mathbb{Z}.$$

(a) Determine the fundamental period T_0 of $x(\cdot)$. If we assume that $T_s = T_0/N$ for some odd integer N, what is the smallest value of N such that $x(\cdot)$ can be recovered perfectly from $x[\cdot]$?

(b) For $N = 11$, plot the real and imaginary parts of the Fourier series coefficients of $x[\cdot]$. Is there aliasing?

(c) For $N = 5$, plot the real and imaginary parts of the Fourier series coefficients of $x[\cdot]$. If you reconstruct the continuous-time signal from these samples, what signal do you obtain?

Problem 3 *Product of Two Bandlimited Signals*

Let $x_1(\cdot)$ and $x_2(\cdot)$ be two bandlimited signals, the former with bandwidth W_1 and the latter with bandwidth W_2. Determine the largest sampling period T_s such that
$$y(t) = x_1(t)x_2(t), \quad t \in \mathbb{R}$$
can be recovered perfectly from its samples $y[n] = y(nT_s)$, $n \in \mathbb{Z}$.

Problem 4 *Energy of Sampled Signal*

Consider a signal $x(\cdot)$ of bandwidth W with samples $x[\cdot]$ that are spaced T_s seconds apart, i.e.,
$$x[n] = x(nT_s), \quad n \in \mathbb{Z}.$$
Assume that T_s satisfies the Nyquist criterion, i.e., $T_s \leq \frac{1}{2W}$. What is the relation between the energy of the samples
$$E_s = \sum_{n=-\infty}^{\infty} |x[n]|^2$$
and the energy of the signal
$$E_c = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t?$$

Problem 5 *Upsampling*

Consider two continuous-time signals $x_1(\cdot)$ and $x_2(\cdot)$ of bandwidth $W_1 = 3\text{kHz}$ and $W_2 = 5\text{kHz}$, respectively. Let the discrete-time signals $x_i[\cdot]$, $i = 1, 2$ be obtained by sampling $x_i(\cdot)$ at sampling period $T_{s,i} = \frac{1}{2W_i}$. The discrete-time signals $x_1[\cdot]$ and $x_2[\cdot]$ are processed by a digital signal processor. To this end, both signals need to operate at the same sampling period.

(a) By what factors need the two signals be upsampled such that both signals operate at the same sampling period?

(b) Sketch the Fourier transforms of the continuous-time signals $x_i(\cdot)$, $i = 1, 2$, the corresponding discrete-time signals, and the upsampled signals.

Problem 6 *Echo echo echo*

Consider the continuous-time signal

$$x(t) = s(t) + \frac{1}{8}s(t - T_d), \quad t \in \mathbb{Z}$$

for some $T_d \in \mathbb{R}$ and a signal $s(\cdot)$ that is bandlimited to $\frac{1}{2T_s}$ for some T_s. Let $x[\cdot]$ denote the sampled discrete-time signal, i.e.,

$$x[n] = x(nT_s), \quad n \in \mathbb{Z}.$$

We wish to find the discrete-time filter $h[\cdot]$ such that

$$x[n] = \sum_{k=-\infty}^{\infty} s(kT_s)h[n-k], \quad n \in \mathbb{Z}.$$

(a) Find $h[\cdot]$ when $T_d = T_s$.

(b) Find $h[\cdot]$ when $T_d = \frac{T_s}{2}$.

Problem 7 *Fourier Series Coefficients of Sampled Periodic Signal*

Let the continuous-time signal $x(\cdot)$ be periodic with period $\frac{1}{10}$ seconds and Fourier series coefficients

$$a_k = \left(\frac{1}{2}\right)^{|k|}, \quad k \in \mathbb{Z}.$$

This signal is filtered by an ideal lowpass filter of cutoff frequency 105Hz, i.e.,

$$\hat{h}(f) = \begin{cases} 1, & |f| \leq 105 \\ 0, & |f| > 105 \end{cases}$$

and then sampled at sampling period $T_s = \frac{1}{210}$ seconds.

(a) Show that $x[n] = x(nT_s)$, $n \in \mathbb{Z}$ is a periodic sequence and determine its period.

(b) Determine the Fourier series coefficients of $x[\cdot]$.

Hint: You may use that the Fourier series coefficients of the result of filtering $x(\cdot)$ with $h(\cdot)$ are given by

$$b_k = a_k \hat{h}\left(\frac{k}{T_0}\right), \quad k \in \mathbb{Z}.$$

Problem 8 *Digital System That Processes a Continuous-Time Signal*

Consider a system that processes continuous-time signals using a digital filter $h[\cdot]$. The digital filter is linear and causal, and its input $x[\cdot]$ and output $y[\cdot]$ satisfy

$$y[n] = \frac{1}{2}y[n-1] + x[n], \quad n \in \mathbb{Z}.$$

The continuous-time input signal $x(\cdot)$ is sampled at sampling period T_s, fed to the digital filter, and then lowpass-filtered to reconstruct the continuous-time output signal $y(\cdot)$. If $x(\cdot)$ is bandlimited to $\frac{1}{2T_s}$, then this system is equivalent to a continuous-time LTI system. Determine the frequency response $\hat{h}(\cdot)$ of the equivalent overall system with input $x(\cdot)$ and output $y(\cdot)$.

Problem 9 *Analog System That Processes a Discrete-Time Signal*

Consider a system that processes a discrete-time signal $x[\cdot]$ using a continuous-time filter $h(\cdot)$. Specifically, the filter input and output are related via the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t) + 4\frac{\mathrm{d}}{\mathrm{d}t}y(t) + 3y(t) = x(t), \quad t \in \mathbb{R}.$$

Out of the discrete-time signal $x[\cdot]$, a continuous-time signal $x(\cdot)$ of bandwidth $\frac{1}{2T_s}$ is created by means of the reconstruction formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \operatorname{sinc}\left(\frac{t}{T_s} - n\right), \quad t \in \mathbb{R}.$$

This signal is then filtered by $h(\cdot)$, and the filter output $y(\cdot)$ is sampled at sampling period T_s to create the sequence of samples

$$y[n] = y(nT_s), \quad n \in \mathbb{Z}.$$

Determine the Fourier transform $\hat{h}(\cdot)$ of the equivalent overall system with input $x[\cdot]$ and output $y[\cdot]$.