E3. Problem 9. Analog System that Processes a DT Signal

Consider a system that processes $x[\cdot]$ with a filter $h(\cdot)$ s.t. $\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=x(t)$, $t\in\mathbb{R}$. Out of $x[\cdot]$, $x(t)$ is created with $W=\frac{1}{2T_s}$ using $\sum_{n=-\infty}^{\infty}x[n]\,\mathrm{sinc}\left(\frac{t}{T_s}-n\right)$, $t\in\mathbb{R}$. The signal is filtered by $h(\cdot)$ and the output $y(\cdot)$ is sampled with T_s to produce $y[n]=y(nT_s)$, $n\in\mathbb{Z}$. Determine $\hat{h}(\cdot)$ of the overall system $x[\cdot]\mapsto y[\cdot]$

$$\frac{d^2}{dt^2}y(t)+4\frac{d}{dt}y(t)+3y(t)=x(t) \Rightarrow (j2\pi f)^2\hat{y}_c(f)+4(j2\pi f)\hat{y}_c(f)+3\hat{y}_c(f)=\hat{x}_c(f)=(-4\pi^2f^2+j8\pi f+3)\hat{y}_c(f)$$

$$x(t)=\sum_{n=-\infty}^{\infty}x[n]\,\mathrm{sinc}\left(\frac{t}{T_s}-n\right)=(x*\mathrm{sinc})\left(\frac{t}{T_s}\right) \Rightarrow \hat{x}_c(f)=T_s\Big(\hat{x}_s(T_s\cdot f)\cdot \underbrace{\mathrm{rect}(T_s\cdot f)}_{\begin{cases}1, & |T_s\cdot f|<\frac{1}{2} \Leftrightarrow |f|<\frac{1}{2T_s}\ \text{(always)}\\ 0, & \text{otherwise}\end{cases}}\Big) \Rightarrow$$

$$\Rightarrow \hat{x}_c(f)=T_s\hat{x}_s(fT_s) \Rightarrow \hat{x}_s(f)=\frac{1}{T_s}\hat{x}_c\left(\frac{f}{T_s}\right)$$

sampling: w/ Nyquist ok

$$\hat{y}_s(f)=\frac{1}{T_s}\hat{y}_c\left(\frac{f}{T_s}\right)=\frac{1}{T_s}\frac{\hat{x}_c\left(\frac{f}{T_s}\right)}{-4\pi^2\left(\frac{f}{T_s}\right)^2+j8\pi\frac{f}{T_s}+3}=\frac{\hat{x}_s(f)}{-4\pi^2\frac{f^2}{T_s^2}+j8\pi\frac{f}{T_s}+3}$$

$$\boxed{\hat{h}_s(f)=\frac{\hat{y}_s(f)}{\hat{x}_s(f)}=\frac{1}{-4\pi^2\frac{f^2}{T_s^2}+j8\pi\frac{f}{T_s}+3}, \quad |f|\le\frac{1}{2}}$$

discrete h