E1. Problem 6. Properties of Fourier Transform of real-valued signals

Let $x(\cdot)$ be a real-valued signal.

a) If $x(\cdot)$ is odd (ie. $x(-t) = -x(t)$), show that $\hat{x}(\cdot)$ is pure imaginary and odd

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{-\infty}^{0} x(t)\,e^{-j2\pi ft}\,dt + \int_{0}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int_{\infty}^{0} x(-t)\,e^{-j2\pi f(-t)}\,d(-t) + \int_{0}^{-\infty} x(-t)e^{-j2\pi f(-t)}\,d(-t) =$$

$$\cancel{\int x(t)e^{j2\pi ft}\,dt + \int_{0} x(t)e^{-j2\pi ft}\,dt = \int x(t)\left(e^{j2\pi ft} + e^{-j2\pi ft}\right)dt = 2\int x(t)\cos(2\pi ft)\,dt}$$

$$= + \int_{\infty}^{0} x(t)e^{+j2\pi ft}\,dt + \int_{0}^{-\infty} x(t)\,e^{+j2\pi ft}\,dt = \int_{0}^{\infty} x(t)e^{j2\pi ft}\,dt - \int_{-\infty}^{0} x(t)\,e^{j2\pi ft}\,dt = -\int_{-\infty}^{\infty} x(t)\,e^{-j2\pi(-f)t}\,dt = -\hat{x}(-f) \Rightarrow$$

$\Rightarrow \hat{x}(\cdot)$ is odd

$$\hat{x}(f) \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} x(t)\left(\cos(2\pi ft) - j\sin(2\pi ft)\right)dt = \cancel{\int_{-\infty} x(t)\cos(2\pi ft) + \int_{0} x(t)\cos(2\pi ft) - j\int_{-\infty} x(t)\sin}$$

$$= \int_{-\infty}^{0} x(t)\cos(2\pi ft)\,dt + \int_{0}^{\infty} x(t)\cos(2\pi ft)\,dt - j\int_{-\infty}^{0} x(t)\sin(2\pi ft)\,dt - j\int_{0}^{\infty} x(t)\sin(2\pi ft)\,dt =$$

$$= \int_{\infty}^{0} x(-t)\cos(2\pi f(-t))\,d(-t) + \int_{0}^{\infty} x(t)\cos(2\pi ft)\,dt - j\int_{\infty}^{0} x(-t)\sin(2\pi f(-t))\,d(-t) - j\int_{0}^{\infty} x(t)\sin(2\pi ft)\,dt =$$

$$= -\int_{0}^{\infty} x(t)\cos(2\pi ft)\,dt + \int_{0}^{\infty} x(t)\cos(2\pi ft)\,dt - j\int_{0}^{\infty} x(t)\sin(2\pi ft) - j\int_{0}^{\infty} x(t)\sin(2\pi ft)\,dt =$$

$$= -j2\underbrace{\int_{0}^{\infty} x(t)\sin(2\pi ft)\,dt}_{\text{integral of real functions}} \in \{0 + iy\} \Rightarrow \hat{x}(\cdot) \text{ is purely imaginary}$$