Linear Networks Analysis and Synthesis

Lab 1 Session 1 - SSS Analysis of a Sallen-Key Circuit

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Preparatory Homework

The circuit in the figure 2 shows a well-known and widely used circuit often referred to as *Sallen-Key*

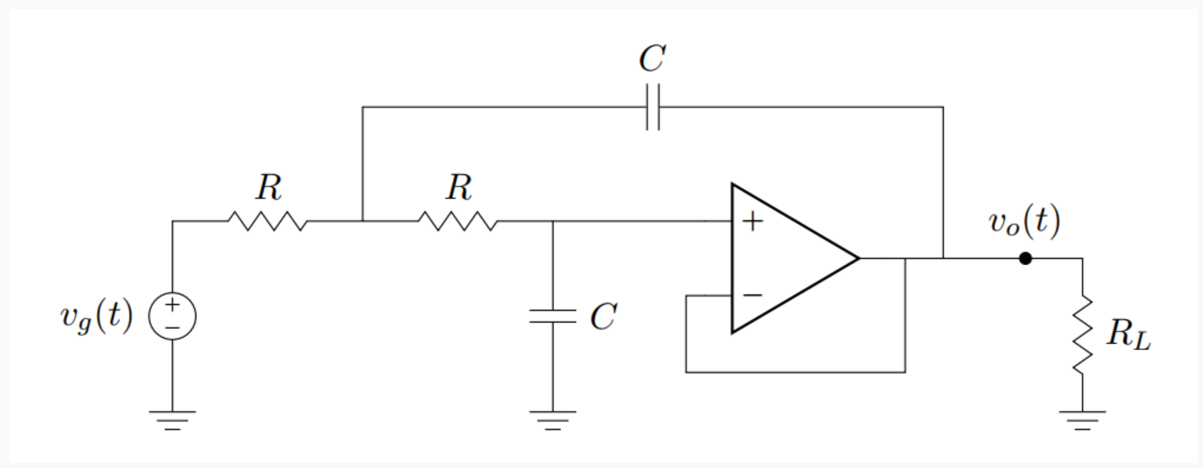
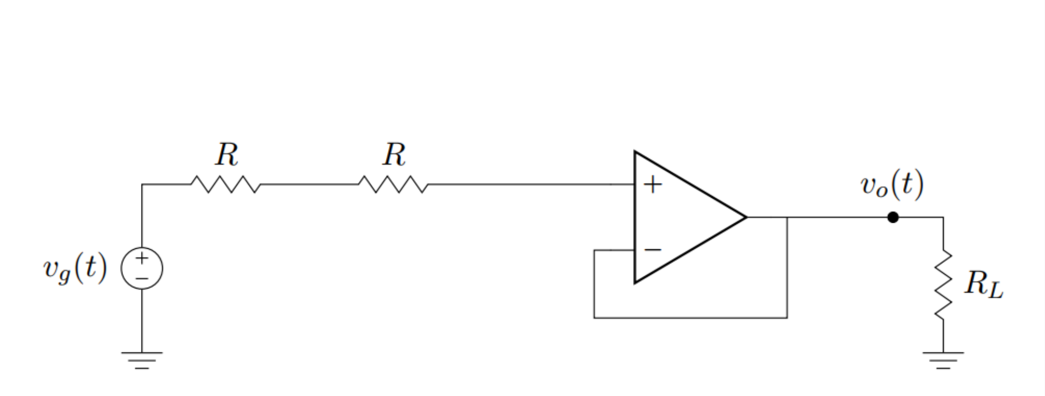


Figure 2: Sallen-Key Circuit

1. In view of the circuit, estimate without calculating the transfer function the value of $V_o(\omega)/V_g(\omega)$ for $\omega = 0$ and $\omega = \infty$. Recall that the impedance of a capacitor depends on the angular frequency ω of the signal.

> We can use the fact that capacitors act as open circuits at $\omega = 0$ and as short circuits at $\omega = \infty$ to estimate the behavior of the circuit at these frequencies. The Operational Amplifier is in a *voltage buffer* (also known as *voltage follower*) configuration, which has a gain of 1.
>
> At $\omega = 0$, both capacitors act as **open cirucits**, resulting in the following equivalent circuit:

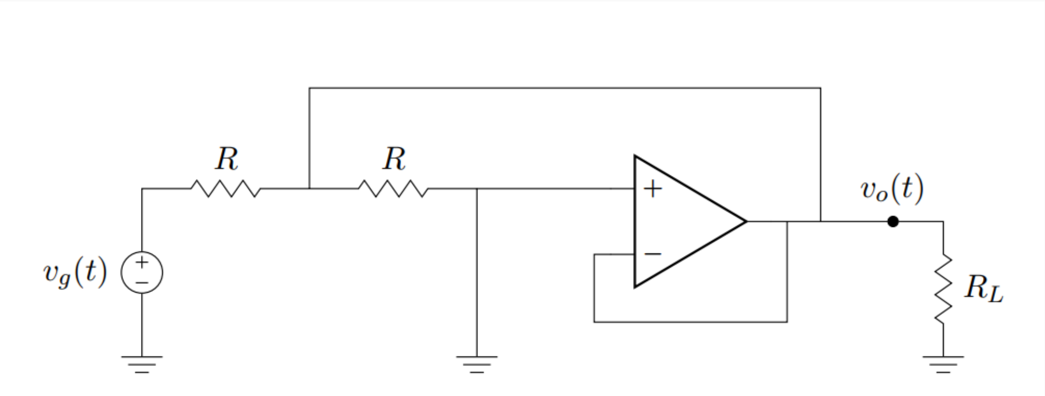


Having an ideal OpAmp, the currents at both input terminals are zero.

$$i_+ = i_- = 0 \Longrightarrow v_+ = v_g$$

From the voltage follower behavior, we can conclude that $v_o(t) = v_g(t)$ when $\omega = 0$, so the transfer function is $V_o(\omega)/V_g(\omega) = 1$ at $\omega = 0$.

At $\omega = \infty$, both capacitors act as **short circuits**, resulting in the following equivalent circuit.



Due to the shorting to ground at the positive OpAmp terminal, the input to the *voltage buffer* will always be 0.

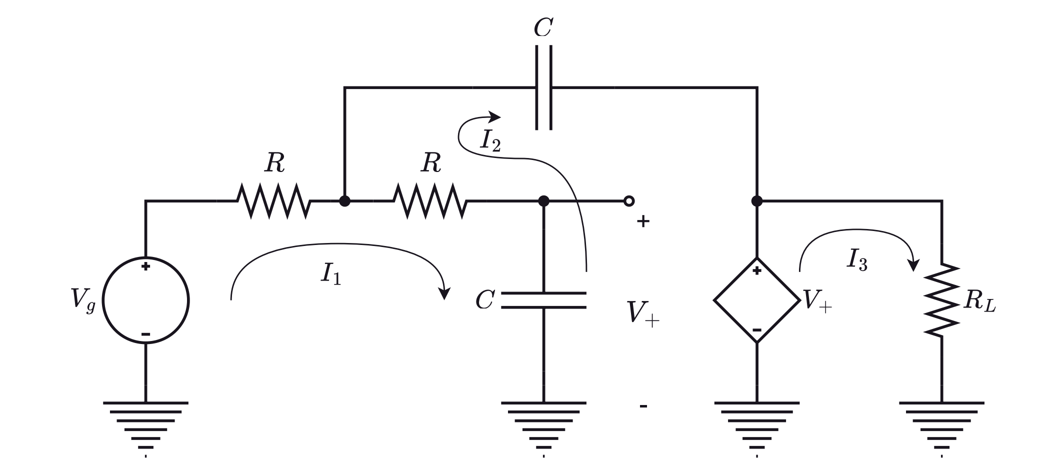
$$v_+ = 0 \Longrightarrow v_o = 0$$

This might seem contradicting with the same output node being connnected to the point between both R resistors, but it's actually compatible: the voltage at this point is also 0, and no current flows through the second R resistor. All current going through the first R resistor is drained through the input of the ideal Operational Amplifier, which has 0 output impedance, keeping the voltage at v_o equal to 0. Therefore, the transfer function is $V_o(\omega)/V_g(\omega) = 0$ at $\omega = \infty$.

2. Assume that the circuit works in sinusoidal steady state and obtain the transfer function defined by the following ratio:

$$H(\omega) = \frac{V_o(\omega)}{V_g(\omega)}$$

Using the following equivalent circuit, we can perform mesh analysis and come up with three mesh equations to solve for the currents I_1, I_2 and I_3.



$$V_g = I_1\left(2R + \frac{1}{j\omega C}\right) - I_2\left(R + \frac{1}{j\omega C}\right)$$
$$0 = -I_1\left(R + \frac{1}{j\omega C}\right) + I_2\left(R + \frac{2}{j\omega C}\right) + V_+$$
$$V_+ = I_3\left(R_L\right)$$

Using the substitution $V_+ = (I_1 - I_2)\frac{1}{j\omega C}$, we can rewrite the equations as:

$$V_g = I_1\left(2R + \frac{1}{j\omega C}\right) - I_2\left(R + \frac{1}{j\omega C}\right)$$
$$0 = -I_1(R) + I_2\left(R + \frac{1}{j\omega C}\right)$$
$$0 = -I_1\frac{1}{j\omega C} + I_2\frac{1}{j\omega C} + I_3(R_L)$$

This can then be expressed in matrix form:

$$\begin{bmatrix} 2R + \frac{1}{j\omega C} & -R - \frac{1}{j\omega C} & 0 \\ -R & R + \frac{1}{j\omega C} & 0 \\ \frac{-1}{j\omega C} & \frac{1}{j\omega C} & R_L \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} V_g \\ 0 \\ 0 \end{bmatrix}$$

We can solve it using Cramer's rule:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2R + \frac{1}{j\omega C} & -\left(R + \frac{1}{j\omega C}\right) & 0 \\ -R & R + \frac{1}{j\omega C} & 0 \\ \frac{-1}{j\omega C} & \frac{1}{j\omega C} & R_L \end{vmatrix} \\ &= R_L\left(\left(2R + \frac{1}{j\omega C}\right)\left(R + \frac{1}{j\omega C}\right) - R\left(R + \frac{1}{j\omega C}\right)\right) \\ &= R_L\left(R + \frac{1}{j\omega C}\right)^2 \end{aligned}$$

$$I_1 = \frac{\begin{vmatrix} V_g & -R - \frac{1}{j\omega C} & 0 \\ 0 & R + \frac{1}{j\omega C} & 0 \\ 0 & \frac{1}{j\omega C} & R_L \end{vmatrix}}{\det(A)} = \frac{R_L V_g \left(R + \frac{1}{j\omega C}\right)}{R_L \left(R + \frac{1}{j\omega C}\right)^2} = \frac{1}{R + \frac{1}{j\omega C}} V_g$$

$$I_2 = \frac{\begin{vmatrix} 2R + \frac{1}{j\omega C} & V_g & 0 \\ -R & 0 & 0 \\ \frac{-1}{j\omega C} & 0 & R_L \end{vmatrix}}{\det(A)} = \frac{R_L V_g R}{R_L \left(R + \frac{1}{j\omega C}\right)^2} = \frac{R}{\left(R + \frac{1}{j\omega C}\right)^2} V_g$$

$$\begin{aligned} I_3 = \frac{\begin{vmatrix} 2R + \frac{1}{j\omega C} & -\left(R + \frac{1}{j\omega C}\right) & V_g \\ -R & R + \frac{1}{j\omega C} & 0 \\ \frac{-1}{j\omega C} & \frac{1}{j\omega C} & 0 \end{vmatrix}}{\det(A)} &= \frac{V_g \left(-R + R + \frac{1}{j\omega C}\right) \frac{1}{j\omega C}}{R_L \left(R + \frac{1}{j\omega C}\right)^2} \\ &= \frac{1}{R_L \left(j\omega CR + 1\right)^2} V_g \end{aligned}$$

Finally, we can use the current I_3 to obtain the output voltage, and divide by the input voltage to obtain the transfer function $H(\omega)$:

$$H(\omega) = \frac{V_o(\omega)}{V_g(\omega)} = \frac{I_3 R_L}{V_g} = \frac{1}{(j\omega CR + 1)^2}$$

3. Find the ratio between the square of the amplitudes of the input $v_g(t)$ and the output $v_o(t)$ for each angular frequency ω. That is:

$$|H(\omega)|^2 = \frac{|V_o(\omega)|^2}{|V_g(\omega)|^2}$$

This is a matter of squaring the magnitude of the transfer function obtained above. Using some properties of the magnitudes of products and quotients, it becomes a simple task:

$$\begin{aligned} |H(\omega)|^2 &= \frac{|V_o(\omega)|^2}{|V_g(\omega)|^2} \\ &= \left|\frac{1}{(j\omega CR + 1)^2}\right|^2 \\ &= \frac{|1|}{|j\omega CR + 1|^4} \\ &= \frac{1}{\left(\sqrt{\omega^2 C^2 R^2 + 1}\right)^4} \\ &= \frac{1}{\left(\omega^2 C^2 R^2 + 1\right)^2} \end{aligned}$$

4. Determine, as a function of R and C, the angular frequency ω in which the amplitude of the output signal $v_o(t)$ is 3 dB lower than the amplitude of the input signal $v_g(t)$. That is, to find the angular frequency ω that verify

$$|H(\omega)|^2 = \frac{1}{2}$$

> Using the previous result, we can obtain this by solving for ω in the resulting equation:
>
> $$\begin{aligned} |H(\omega)|^2 = \frac{1}{2} \iff & \frac{1}{\left(\omega^2 C^2 R^2 + 1\right)^2} = \frac{1}{2} & \implies \\ \implies & \omega^2 C^2 R^2 + 1 = \sqrt{2} & \implies \\ \implies & \omega = \sqrt{\frac{\sqrt{2} - 1}{C^2 R^2}} \end{aligned}$$

5. Calculate the current provided by the operational amplifier as a function of V_g, R, C, R_L and ω.

> I will assume that the current *provided* by the Operational Amplifier refers only to the current that flows through the load resistor R_L, and not the the current coming from the top capacitor, which flows through the output terminal of the OpAmp.
>
> This output current is the same as the current I_3 calculated above:
>
> $$I_o\left(V_g, R, C, R_L, \omega\right) = I_3 = \frac{V_g}{R_L\left(j\omega CR + 1\right)^2}$$

6. The datasheet of the operational amplifier to be used specifies that the amplitude of the output current must always be less than 25 mA. If $R_L = 50\Omega$, what is the maximum value that the input signal amplitude can take in order not to exceed this margin when $\omega \approx 0$?

> The previous consideration about the *output current* will be used. However, it's worth noting that the amplifier would sustain greater *input* currents at its *output* due to the currents from the capacitor mentioned above.
>
> Due to the low frequency, we can simplify the equation for the output current I_o and easily calculate the maximum input signal amplitude V_g that to prevent exceeding the 25 mA output limit.
>
> $$I_o = \frac{V_g}{R_L(\underbrace{j\omega CR}_{\approx 0} + 1)^2} \approx \frac{V_g}{R_L}$$
>
> $$\begin{aligned} |I_o| < I_{o;max} \implies & \left|\frac{V_g}{R_L}\right| < I_{o;max} \implies \\ \implies & |V_g| < R_L I_{o;max} = 50\ \Omega \cdot 25\ \text{mA} = 1.25\ \text{V} \end{aligned}$$
>
> The input voltage should not exceed ± 1.25 V to comply with the output limits specified in the datasheet.

NOTE: It is recommended to check with the simulator the results obtained for a particular election of R and C

> Falstad Circuit Link
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> The results were not completely consistent with the simulation, but the theoretical calculations, formulas and equations were checked against multiple sources.