Linear Networks Analysis and Synthesis

Lab 1 Session 2 - Relay activation/deactivation transient analysis

Academic year 2023-2024

Alonso Herreros Copete
NIA: 100493990

Preparatory Homework

In this exercise we study the transient effect of the activation and deactivation of the coil present in a relay. Let's think of a microcontroller that activates/deactivates this relay by means of one of its output pins, as shown in Figure 3.

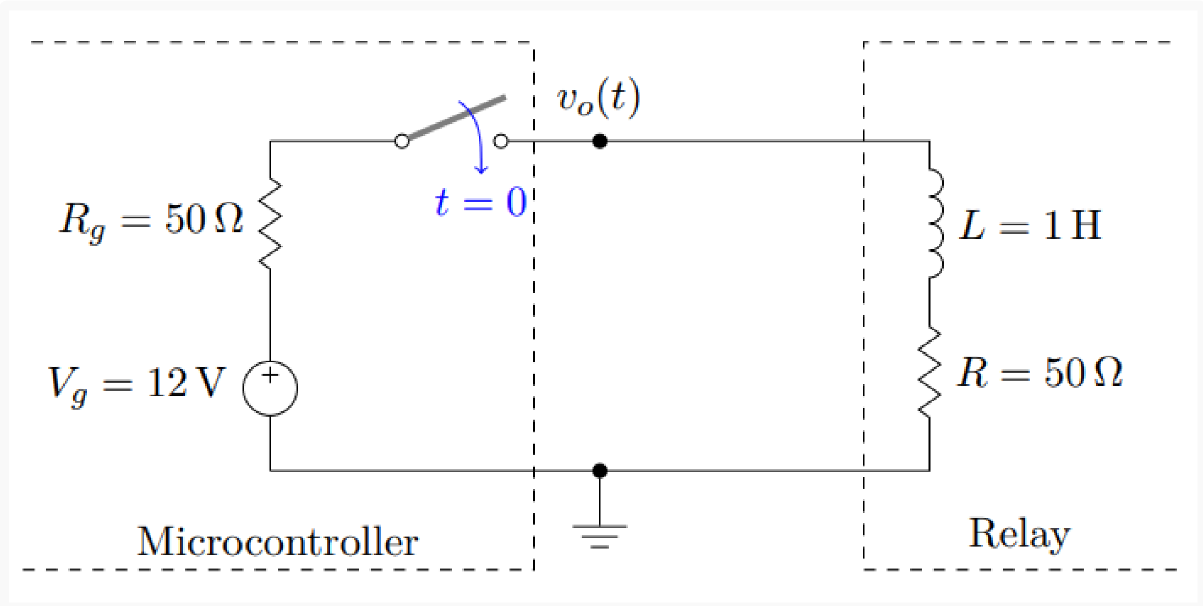


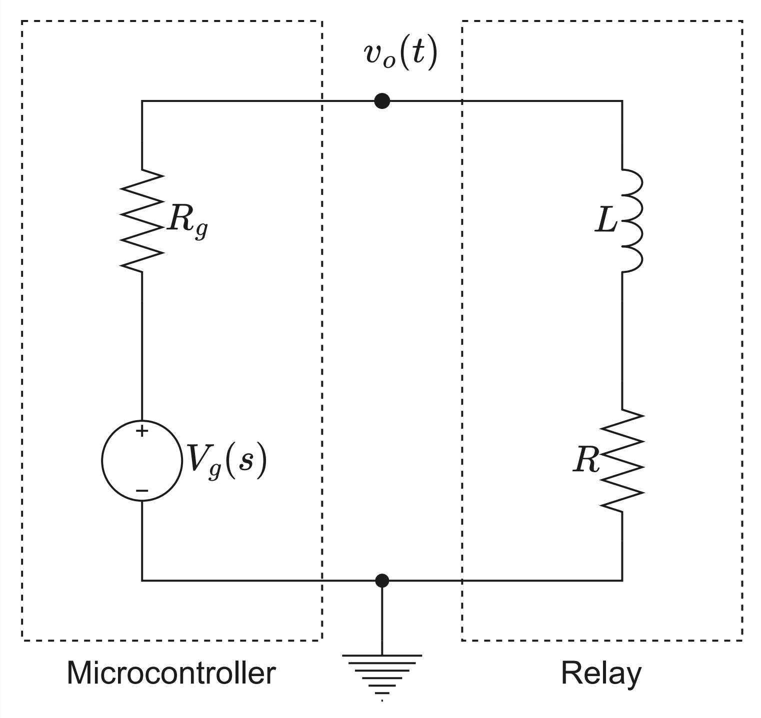
Figure 3: Microcontroller activates the relay at t = 0.

Question 1

Consider that the switch in figure 3 has been open for a long time, and that the micro-controller closes it at an instant we define as $t = 0$. Determine

- (a) $I_L(s)$: current through the coil in the Laplace domain for $t > 0$.

> As the intial state is in reset, the equivalent cirucit in the Laplace domain doesn't require any additional sources.



The equation describing the circuit in Laplace domain for $t > 0$ can be found using mesh analysis, where $V_L(s) = sLI_L(s)$ and $V_g(s) = \mathcal{L}\{V_g\} = \frac{V_g}{s}$

$$\begin{aligned} I_L(s) &= \frac{V_g(s)}{R + R_g + sL} = \frac{V_g}{s(R + R_g + sL)} \\ &= \frac{12}{100s + s^2}\ [\mathrm{A}] \end{aligned}$$

- (b) I_s: stationary value of the current reached after a long time $(t \to \infty)$.

Taking a constant input voltage $V_g(s) = V_g$ and applying the final value theorem:

$$\begin{aligned} I_s &= \lim_{t\to\infty} i_L(t) = \lim_{s\to 0} sI_L(s) \\ &= \lim_{s\to 0} s\frac{V_g}{s(R + R_g + sL)} \\ &= \frac{V_g}{R + R_g} \\ &= 120\ \mathrm{mA} \end{aligned}$$

This means after a long enough time, the current is stabilized and the inductor behaves as a short circuit, leaving only the two resistors contributing with their impedance.

- (c) $i_L(t)$: current through the coil in the time domain for $t > 0$.

We can find the time domain expression for the current through the coil by taking the inverse Laplace of the expression found in the first question. First, we'll use Partial Fraction Decomposition to simplify the expression.

$$I_L(s) = \frac{V_g}{s(R + R_g + sL)} = \frac{A}{s} + \frac{B}{(R + R_g + sL)}$$

> Where
>
> $$\begin{aligned} A &= \lim_{s \to 0} s \frac{V_g}{s(R+R_g+sL)} &= \frac{V_g}{R+R_g} \\ B &= \lim_{s \to -\frac{R+R_g}{L}} \frac{(R+R_g+sL)V_g}{s(R+R_g+sL)} &= -\frac{LV_g}{R+R_g} \end{aligned}$$
>
> Leaving us with this expression.
>
> $$I_L(s) = \frac{V_g}{s(R+R_g)} - \frac{LV_g}{(R+R_g+sL)(R+R_g)}$$
>
> Then, we can adjust the expression to a suitable form and use the inverse Laplace transform of these terms to find the time domain expression for the current.
>
> $$\begin{aligned} i_L(t) &= \mathcal{L}^{-1}\left\{I_L(s)\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{V_g}{s(R+R_g)} - \frac{LV_g}{(R+R_g+sL)(R+R_g)}\right\} \\ &= \frac{V_g}{R+R_g}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(\frac{R+R_g}{L}+s)}\right\}\right) \\ &= \frac{V_g}{R+R_g}\left(1 - e^{-\frac{R+R_g}{L}t}\right) \\ &= 120\left(1 - e^{-100t}\right) \text{ [mA]} \end{aligned}$$

- (d) If the relay is triggered when the current flowing through the coil is 80 % of the final value I_s, how long does it take for the relay to trigger from the time the microcontroller activates it (switch is closed)?

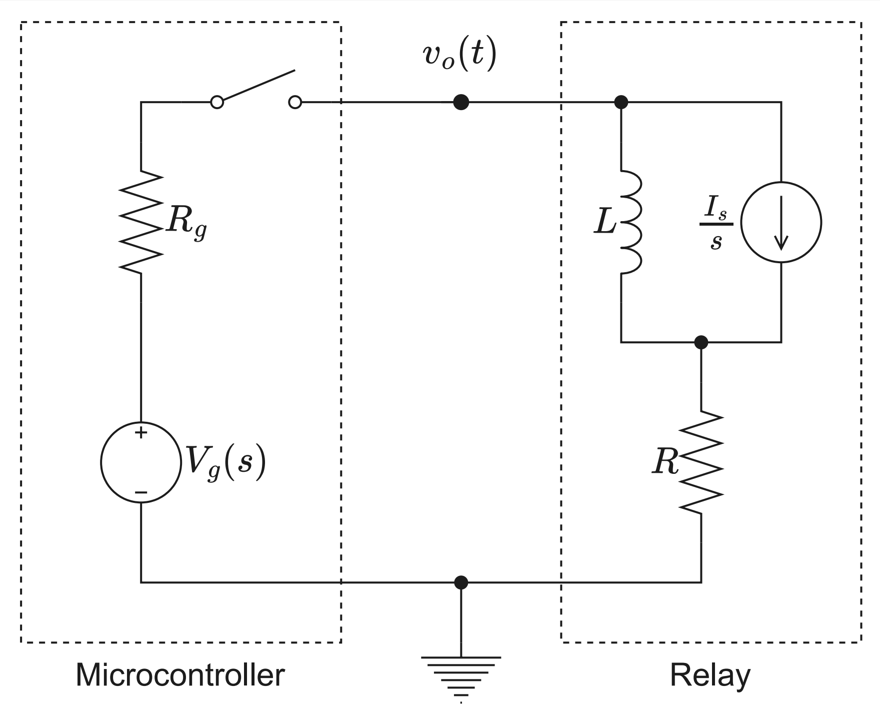
> This is a matter of finding the time t_{80} that it takes for the current i_L to reach $0.8I_s$ in a the studied scenario (switch is closed at $t = 0$). We can use the equations found in the preivous questions for this.
>
> $$\begin{aligned} & & i_L(t_{80}) &= 0.8I_s & \implies \\ \implies & & \frac{V_g}{R+R_g}\left(1 - e^{-\frac{R+R_g}{L}t_{80}}\right) &= 0.8\frac{V_g}{R+R_g} & \implies \\ \implies & & e^{-\frac{R+R_g}{L}t_{80}} &= 1 - 0.8 & \implies \\ \implies & & -\frac{R+R_g}{L}t_{80} &= \ln(0.2) & \implies \\ \implies & & t_{80} &= -\frac{L}{R+R_g}\ln(0.2) \\ & & &= -\frac{1}{50+50}\ln(0.2) \\ & & &= 16.09 \cdot 10^{-3} \text{ [s]} \end{aligned}$$
>
> It would take the relay 16.09 ms to trigger in this scenario.

Question 2

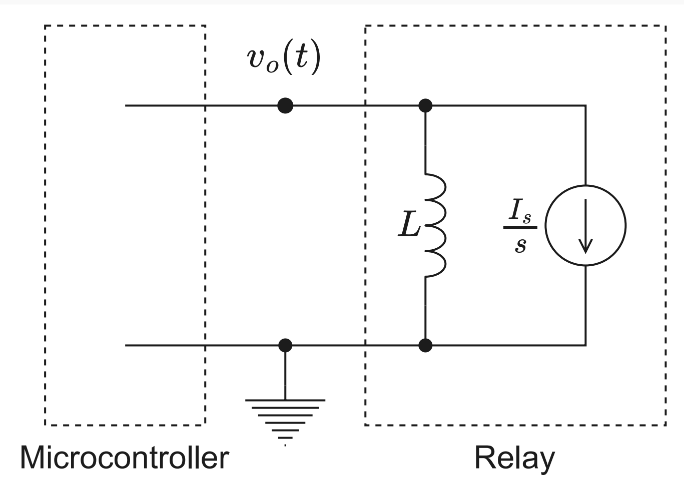
Consider now that the switch in figure 3 has been closed for a long time, and that the microcontroller opens it (puts the output pin in the high-impedance state) at an instant we redefine as $t = 0$.

Under these circumstances, calculate the voltage on the output pin in the Laplace domain, $V_o(s)$, and in the time domain $v_o(t)$. Interpret the result and comment on whether you foresee any problems in the microcontroller.

By using the stabilized current I_s obtained in the previous question, we can find an equivalent circuit in the Laplace domain, adding a current source in parallel with the inductor to account for the initial current flowing trough it, $i_L(0^-) = I_s$.



Due to the high impedance state of the microcontroller pin acting as an open circuit, this can be further simplified:



Using node analysis, we can find the voltage on the output pin, $v_o(t)$

$$
\begin{aligned}
\frac{I_s}{s} &= \frac{0 - v_o(s)}{sL} && \implies \\
\implies \quad v_o(s) &= -I_s L \\
&= -120 \cdot 10^{-3} \text{ [V]}
\end{aligned}
$$

From here, using the inverse Laplace transform, we can find the time domain expression for the voltage on the output pin, $v_o(t)$

$$
\begin{aligned}
v_o(t) &= \mathcal{L}^{-1}\{-I_s L\} \\
&= -I_s L \delta(t) \\
&= -120 \cdot 10^{-3} \delta(t) \text{ [V]}
\end{aligned}
$$

This delta function in our time domain expression tells us that there will be an unbound voltage spike in the opposite direction as soon as the switch is opened. The initial value theorem proves this.

$$
\begin{aligned}
v_o(0^+) = \lim_{t \to 0^+} v_o(t) &= \lim_{s \to \infty} sV_o(s) \\
&= \lim_{s \to \infty} -sI_s L \\
&= -\infty \text{ [V]}
\end{aligned}
$$

This unbound voltage could potentially damage the controller. A capacitor may be added in parallel with the relay in order to limit the effect of this sudden (high-frequency change).

Question 3

To avoid the possible problems mentioned in the previous point, a capacitor is connected in parallel with the relay. $C_p = 200\mu F$, as shown in figure 4. Assuming that the switch has been closed for a long time before opening it at instant $t = 0$, determine:

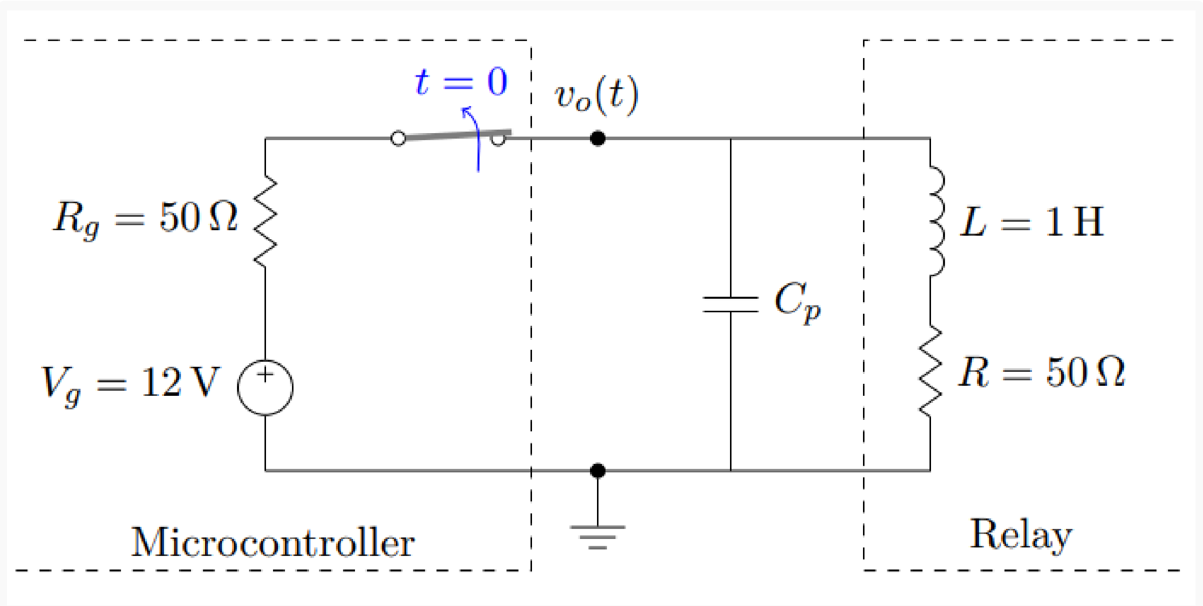


Figure 4: Microcontroller deactivates the relay at t = 0.

- (a) $I_L(s)$: current through the coil in the Laplace domain for $t > 0$.

The first step is to find our new stable *on* state taking the capacitor into account.

Since the circuit is charged with DC for a long time ($\omega = 0$) the **capacitor behaves as an open circuit.**

$$i_C(0^-) = 0$$

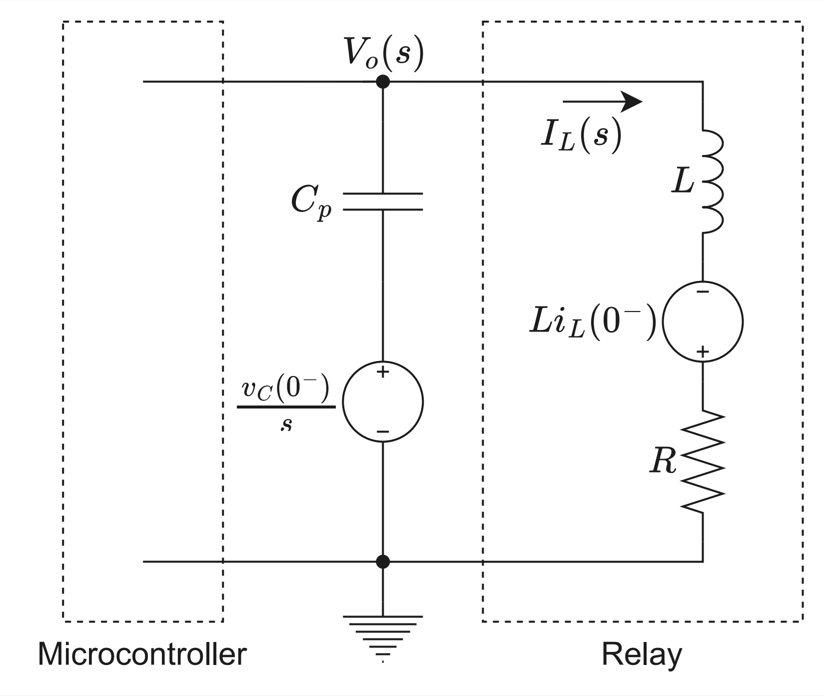
Therefore, the current through the coil after a long time activated **is the same** as I_s in the previous sections.

$$i_L(0^-) = \frac{V_g}{R + R_g} = 120 \text{ mA}$$

Now, we'll find the condition of the capacitor after a long time, $v_C(0^-)$. This can be found by using the previous stable currents.

$$v_C(0^-) = V_g - I_g R_g = V_g - (i_L(0^-) - i_C(0^-))R_g = 6 \text{ V}$$

With this, we can create a new equivalent circuit in the Laplace domain, taking into account the capacitor and using voltage sources as initial conditions in order to make mesh analysis easier.



With this, we can analyze the circuit using KVL and obtain the expression for $I_L(s)$ for $t > 0$.

> $$\frac{v_C(0^-)}{s} + Li_L(0^-) = I_L(s)\left(\frac{1}{sC_p} + sL + R\right) \implies$$
> $$\implies \quad I_L(s) = \frac{\frac{v_C(0^-)}{s} + Li_L(0^-)}{\frac{1}{sC_p} + sL + R}$$
> $$= \frac{v_C(0^-) + sLi_L(0^-)}{C_p^{-1} + sR + s^2L}$$
> $$= \frac{6 + 0.12s}{5000 + 50s + s^2} \text{ [A]}$$

- (b) $V_o(s)$: voltage on the microcontroller output pin in the Laplace domain for $t > 0$.

> We can find the voltage in the Laplace domain by using the current found in the previous question and the same equivalent circuit.
>
> $$V_o(s) = 0 + \frac{v_C(0^-)}{s} + \frac{I_L(s)}{sC_p}$$
> $$= \frac{v_C(0^-)}{s} + \frac{v_C(0^-) + sLi_L(0^-)}{s\left(1 + sRC_p + s^2LC_p\right)}$$
> $$= \frac{6}{s} + \frac{6 + 0.12s}{s\left(1 + 0.01s + 0.0002s^2\right)}$$

- (c) $v_o(0+)$: voltage to be supported by the output pin of the microcontroller at $t = 0^+$. Has the problem discussed in section 2 been solved?

> As the time point we care the most about is $t = 0^+$, we can find the voltage to be supported by the output pin by using the initial value theorem.
>
> $$v_o(0^+) = \lim_{t\to 0^+} v_o(t) = \lim_{s\to\infty} sV_o(s)$$
> $$= \lim_{s\to\infty} s\left(\frac{v_C(0^-)}{s} + \frac{v_C(0^-) + sLi_L(0^-)}{s\left(1 + sRC_p + s^2LC_p\right)}\right)$$
> $$= v_C(0^-)$$
>
> In this case, the voltage to be supported by the output pin is the same as the initial voltage on the capacitor, unlike the previous scenario where the voltage was unbounded. This means that the power spike problem has been solved by adding the capacitor in parallel with the relay. Instead of an unbounded spike, the voltage will decrease and oscillate as it approaches 0.

- (d) In view of the equation for the current flowing in the coil, what can be concluded about its behavior? What would you change in the circuit to avoid this effect?

> After analyzing the poles and zeros of the transient response (second-order), the system seems to be underdamped, as both poles fall in the left half-plane, with negative real parts and non-zero imaginary parts.
>
> The response is faster than that of a critically damped system, and the overshoot is not too dramatic on the turning-on curve, so there is no risk of damaging the relay. The turn-off curve has a more significant overshoot, but it may not be a problem either.

In order to change this to a critically damped system with no overshoot at all, both poles must be equal, real, and negative. This can be achieved without modifying the maximum current by adjusting the capacitor value, which is present in the polynomial of the denominator of the transient reponse.

$$I_L(s) = \frac{v_C(0^-) + sLi_L(0^-)}{C_p^{-1} + sR + s^2L}$$

In the second-degree polynomial of the denominator to make the discriminant ($B^2 - 4AC$) in the quadratic equation equal to zero.

$$R^2 - 4LC_p^{-1} = 0 \Longleftrightarrow C_p = \frac{4L}{R^2} = 1.6 \text{ mF}$$

However, this would require a big capacitor, and the response speed would be much lower than the one studied in these previous questions.