Bachelor in Mobile and Space Communications Engineering
Bachelor in Telematics Engineering
Bachelor in Sound and Image Engineering
Bachelor in Telecommunication Technologies Engineering

Notation:

- $\widehat{S}_{\text{MMSE}}$: Minimum Mean Square Error estimator.
- $\widehat{S}_{\text{MAD}}$: Mininimum Mean Absolute Deviation Error estimator.
- $\widehat{S}_{\text{MAP}}$: Maximum a posteriori estimator.
- $\widehat{S}_{\text{ML}}$: Maximum likelihood estimator.
- $\widehat{S}_{\text{LMSE}}$: Linear Minimum Mean Square Error estimator.

1. The random variables X and S are related through the conditional distribution:

$$p_{X|S}(x|s) = sx^{s-1}, \qquad 0 \leq x \leq 1, \quad s \geq 0$$

(a) Find the ML estimate of S given X, $\hat{S}_{\text{ML}}$.

(b) Find the ML estimate of S given a set $\{x^{(k)}, k = 0, \ldots, K-1\}$ of independent observations of X.

Solution:

(a)

$$\begin{aligned}
\hat{s}_{\text{ML}} &= \arg \max_s p_{X|S}(x|s) \\
&= \arg \max_s sx^{s-1} \\
&= \arg \max_s [\log(s) + (s-1)\log(x)] \\
&= -\frac{1}{\log(x)}
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{s}_{\text{ML}} &= \arg \max_s \prod_{k=0}^{K-1} p_{X|S}\left(x^{(k)}|s\right) \\
&= \arg \max_s s^K \left(\prod_{k=0}^{K-1} x^{(k)}\right)^{s-1} \\
&= \arg \max_s \left[K \log(s) + (s-1) \sum_{k=0}^{K-1} \log\left(x^{(k)}\right)\right] \\
&= -\frac{K}{\sum_{k=0}^{K-1} \log\left(x^{(k)}\right)}
\end{aligned}$$

2. Consider the estimation of a random variable S from a random variable X, where their joint probability density function is given by:

$$p_{X,S}(x,s) = \begin{cases} 6s & 0 \le s \le x, \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the estimator $\widehat{S}_{\text{MMSE}}$.

(b) Consider the estimation of S based on the observation of X, with the objective to minimize the following cost function:

$$c(S, \widehat{S}) = a^2(\widehat{S} - S)^2$$

Determine the optimal estimator of S, $\widehat{S}$, which minimizes the expected cost of the estimator.

Solution:

(a) $\widehat{S}_{\text{MMSE}} = \dfrac{2X}{3}$

(b) $\widehat{S}^* = \dfrac{2X}{3}$