Problems and Exercises Collection - Block I: Estimation

Most of the problems and exercises of this collection have been taken from previous years exams. The topics covered by each exercise are shown next to the exercise number according to:

1.1. General view of estimation problems.

1.2. Bayes' Estimation Theory.

1.3. Frequently used Bayes' estimators: MSE, MAP, MAD.

1.4. Maximum Likelihood estimation.

1.5. Estimation with Gaussian likelihoods.

1.6. Constrained estimators. Linear minimum mean square error estimation.

1.7. Bias and Variance of estimators.

1.8. Machine design of estimators.

Notation:

- $\hat{S}_{\text{MSE}}$: Minimum Mean Square Error estimator.
- $\hat{S}_{\text{MAD}}$: Mininimum Mean Absolute Deviation Error estimator.
- $\hat{S}_{\text{MAP}}$: Maximum a posteriori estimator.
- $\hat{S}_{\text{ML}}$: Maximum likelihood estimator.
- $\hat{S}_{\text{LMSE}}$: Linear Minimum Mean Square Error estimator.

Exercise 1 (1.6)

We wish to design a linear minimum mean square error estimator for the estimation of random variable S based on the observation of random variables X_1 and X_2. It is known that:

$$\begin{array}{lll} \mathbb{E}\{S\} = \frac{1}{2} & \mathbb{E}\{X_1\} = 1 & \mathbb{E}\{X_2\} = 0 \\ \mathbb{E}\{SX_1\} = 1 & \mathbb{E}\{SX_2\} = 2 & \mathbb{E}\{X_1X_2\} = \frac{1}{2} \\ \mathbb{E}\{S^2\} = 4 & \mathbb{E}\{X_1^2\} = \frac{3}{2} & \mathbb{E}\{X_2^2\} = 2 \end{array}$$

Obtain the weights of estimator $\hat{S}_{\text{LMSE}} = w_0 + w_1X_1 + w_2X_2$, and calculate its mean square error $\mathbb{E}\left\{(S - \hat{S}_{\text{LMSE}})^2\right\}$.

Solution: $w_0 = \frac{1}{2} \quad w_1 = 0 \quad w_2 = 1$

$\mathbb{E}\left\{(S - \hat{S}_{\text{LMSE}})^2\right\} = \frac{7}{4}$

Exercise 2 (1.2; 1.3; 1.7)

Consider the estimation of a random variable S from another random variable X, where their joint probability density function (pdf) is given by:

$$p_{S,X}(s,x) = \frac{6}{7}(x+s)^2, \quad 0 \leq x, s \leq 1$$

(a) Obtain $p_X(x)$.

(b) Obtain $p_{S|X}(s|x)$.

(c) Calculate the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(d) Calculate the MAP estimator of S given X, $\hat{S}_{\mathrm{MAP}}$.

(e) Indicate which is the bias and the variance of the MAP estimator.

Solution:

(a) $p_X(x) = \frac{2}{7}\left(3x^2 + 3x + 1\right) \quad 0 < x < 1.$

(b) $p_{S|X}(s|x) = \dfrac{(x+s)^2}{x^2 + x + \frac{1}{3}} \quad 0 < s < 1.$

(c) $\hat{S}_{\mathrm{MSE}} = \dfrac{\frac{X^2}{2} + \frac{2X}{3} + \frac{1}{4}}{X^2 + X + \frac{1}{3}}.$

(d) Given that $p_{S|X}(s|x)$ is strictly increasing with s, $\hat{S}_{\mathrm{MAP}} = 1$.

(e) $p_S(s) = \frac{2}{7}\left(3s^2 + 3s + 1\right), \ 0 < s < 1$, thus, $\mathbb{E}\{S\} = \frac{9}{14}$.
The bias is $-\frac{5}{14}$, whereas the variance is 0.

Exercise 3 (1.4; 1.7)

A random variable X follows a unilateral exponential distribution with parameter $a > 0$:

$$p_X(x) = \frac{1}{a}\exp\left(-\frac{x}{a}\right) \quad x > 0$$

As it is known, the mean and variance of X are given by a and a^2, respectively.

(a) Obtain the maximum likelihood estimator of a, $\hat{A}_{\mathrm{ML}}$, based on a set of K independent observations of random variable X, $\left\{X^{(k)}\right\}_{k=1}^{K}$.

(b) Consider now a new estimator based on the previous one, and characterized by expression:

$$\hat{A} = c \cdot \hat{A}_{\mathrm{ML}},$$

where $0 \leq c \leq 1$ is shrinkage constant that allows re-scaling the ML estimator. Find the bias squared, the variance, and the Mean Square Error (MSE) o the new estimator, and represent them all together in the same plot as a function of c.

(c) Find the value of c which minimizes the MSE, c^*, and discuss its evolution as the number of available observations increases. Calculate the MSE of the estimator associated to c^*.

(d) Determine the range of values of c for which the MSE of $\hat{A}$ is smaller than the MSE of the ML estimator, and explain how such range changes as $K \longrightarrow \infty$. Discuss your result.

Solution:

(a) $\hat{A}_{\text{ML}} = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K} X^{(k)}$

(b) $\hat{A} = \dfrac{c}{K}\displaystyle\sum_{k=1}^{K} X^{(k)}$

$\mathbb{E}\left\{\hat{A} - a\right\}^2 = (c-1)^2 a^2, \qquad \text{Var}\left\{\hat{A}\right\} = \dfrac{c^2 a^2}{K}, \qquad \mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = (c-1)^2 a^2 + \dfrac{c^2 a^2}{K}$

(c) $c^* = \dfrac{K}{K+1}, \qquad c^* \longrightarrow 1 \ (K \longrightarrow \infty), \qquad \mathbb{E}\left\{\left(\hat{A} - a\right)^2\right\} = \dfrac{a^2}{K+1} \ (c = c^*)$

(d) The range of values is: $c \in \left[\dfrac{K-1}{K+1}, 1\right]$, which narrows as K increases.

Exercise 4 (1.5)

We have access to the two following observations for estimating a random variable S:

$$X_1 = S + N_1$$
$$X_2 = \alpha S + N_2$$

where α is a known constant, and S, N_1, and N_2 are independent Gaussian random variables, with zero mean and variances v_s, v_n, and v_n, respectively.

(a) Obtain the minimum mean square error estimator of S given X_1 and X_2, $\hat{S}_1$ and $\hat{S}_2$, respectively.

(b) Calculate the mean square error of each of the estimators from the previous section. Which of the two provides a smaller MSE? Justify your answer for the different values of parameter α.

(c) Obtain the minimum mean square error estimator of S based on the joint observation of variables X_1 and X_2, i.e., as a function of the observation vector $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $\hat{S}_{\text{MMSE}}$.

Solution:

(a) $\hat{S}_1 = \dfrac{v_s}{v_s + v_n} X_1$ and $\hat{S}_2 = \dfrac{\alpha v_s}{\alpha^2 v_s + v_n} X_2$.

(b) $\mathbb{E}\{E_1^2\} = \dfrac{v_s v_n}{v_s + v_n}$ and $\mathbb{E}\{E_2^2\} = \dfrac{v_s v_n}{\alpha^2 v_s + v_n}$. For $|\alpha| > 1$ the mean square error of $\hat{S}_2$ is smaller than that of $\hat{S}_1$.

(c) $\hat{S}_{\text{MMSE}} = \left[\dfrac{1}{1 + \alpha^2 + v_n/v_s}, \dfrac{\alpha}{1 + \alpha^2 + v_n/v_s}\right] \mathbf{X}$

Exercise 5 (1.6)

The joint p.d.f. of random variables X and S is given by

$$p_{X,S}(x, s) = \begin{cases} x + s & 0 \leq x \leq 1 \text{ and } 0 \leq s \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the linear minimum mean square error estimator of S given X, $\hat{S}_{\text{LMSE}} = w_0 + w_1 X$.

Solution: $\hat{S}_{\text{LMSE}} = \frac{7}{11} - \frac{X}{11}$

Exercise 6 (1.2; 1.3; 1.4; 1.7)

We want to estimate the value of a positive random variable S using a random observation X, which is related with S via

$$X = R/S$$

R being a r.v. independent of S with p.d.f.

$$p_R(r) = \exp(-r), \quad r > 0$$

(a) Obtain the likelihood of S, $p_{X|S}(x|s)$.

(b) Find the maximum likelihood estimator of S given X, $\hat{S}_{\text{ML}}$.

Knowing also that the p.d.f. of S is $p_S(s) = \exp(-s), \quad s > 0$, obtain:

(c) The joint p.d.f. of S and X, $p_{S,X}(s, x)$, and the *a posteriori* distribution of S, $p_{S|X}(s|x)$.

(d) The maximum *a posteriori* estimator of S given X, $\hat{S}_{\text{MAP}}$.

(e) The minimum mean square error estimator of S given X, $\hat{S}_{\text{MSE}}$.

(f) The bias of estimators $\hat{S}_{\text{MAP}}$ and $\hat{S}_{\text{MSE}}$.

Solution:

(a) $p_{X|S}(x|s) = s\, \exp(-xs), \quad x > 0.$

(b) $\hat{S}_{\text{ML}} = \frac{1}{X}.$

(c) $p_{X,S}(x, s) = s\, \exp\left(-s(x+1)\right), \quad x, s > 0;$

$p_{S|X} = (x+1)^2 s\, \exp\left(-s(x+1)\right), \quad s > 0.$

(d) $\hat{S}_{\text{MAP}} = \frac{1}{X+1}.$

(e) $\hat{S}_{\text{MSE}} = \frac{2}{X+1}.$

(f) $\mathbb{E}\{S - \hat{S}_{\text{MAP}}\} = \frac{1}{2}; \quad \mathbb{E}\{S - \hat{S}_{\text{MSE}}\} = 0.$

Exercise 7 (1.6; 1.8)

We wish to build an estimator for random variable S with the following analytical shape:

$$\hat{S} = w_0 + wX^3$$

(a) Let us define r.v. $Y = X^3$. Indicate which statistics are sufficient to determine the weights of the estimation model.

(b) An analyst wants to adjust the previous model, but he does not have statistical information about the problem. Therefore, he recurs to sample estimations of the sufficient statistics, based on a set of available labelled pairs of the involved random variables:

$$\{X^{(k)}, S^{(k)}\}_{k=1}^{4} = \{(-1, -0.55), (0, 0.5), (1, 1.57), (2, 8.7)\}$$

Determine the weights w_0 and w that the analyst would obtain.

Solution:

(a) $\mathbb{E}\{X\}$, $\mathbb{E}\{Y\}$, v_y and v_{sy} (or any other set from which these can be obtained).

(b) $w = 1.0256$ and $w_0 = 0.5038$.

Exercise 8 (1.2; 1.3; 1.4)

Random variables S and X are jointly distributed according to

$$p_{S,X}(s,x) = \alpha s x^2, \qquad 0 < s < 1 - x, \quad 0 < x < 1$$

α being a parameter that needs to be determined.

(a) Stablish the expressions for the marginal probability density functions $p_X(x)$ and $p_S(s)$.

(b) Obtain the MAP estimator of S given X, $\hat{S}_{\mathrm{MAP}}(X)$.

(c) Obtain the ML estimator of S given X, $\hat{S}_{\mathrm{ML}}(X)$.

(d) Obtain the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}(X)$.

(e) Compare the previous estimators according to the mean square errors given X in which they incur.

Solution:

(a) $p_X(x) = 30x^2(1-x)^2, \qquad 0 < x < 1,$

$p_S(s) = 20s(1-s)^3, \qquad 0 < s < 1,$

(b) $\hat{S}_{\mathrm{MAP}}(X) = 1 - X$

(c) $\hat{S}_{\mathrm{ML}}(X) = 1 - X$

(d) $\hat{S}_{\mathrm{MSE}}(X) = \frac{2}{3}(1 - X)$

(e) $\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MAP}}(X)\right)^2 \mid x\right\} = \mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{ML}}(X)\right)^2 \mid x\right\} = \frac{1}{6}(1-x)^2$

$\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MSE}}(X)\right)^2 \mid x\right\} = \frac{1}{18}(1-x)^2$

Exercise 9 (1.2; 1.3; 1.4; 1.7)

Consider the estimation of a r.v. S from another random variable X. The joint p.d.f. of the two variables is given by:

$$p_{X,S}(x,s) = \begin{cases} 6x, & 0 \le x \le s, \quad 0 \le s \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MSE}}$.

(b) Obtain the maximum likelihood estimator of S given X, $\hat{S}_{\mathrm{ML}}$.

(c) Find the probability density function of the previous estimators, $p_{\hat{S}_{\mathrm{MSE}}}(\hat{s})$ and $p_{\hat{S}_{\mathrm{ML}}}(\hat{s})$, and provide a plot of them.

(d) Find the mean and the variance of the error of both estimators.

Solution:

(a) $\hat{S}_{\mathrm{MSE}}(X) = \frac{1}{2}(1 + X)$

(b) $\hat{S}_{\mathrm{ML}}(X) = X$

(c) $p_{\hat{S}_{\mathrm{MSE}}}(\hat{s}) = 24(2\hat{s} - 1)(1 - \hat{s}), \qquad \frac{1}{2} \leq \hat{s} \leq 1$

$p_{\hat{S}_{\mathrm{ML}}}(\hat{s}) = 6\hat{s}(1 - \hat{s}), \qquad 0 \leq \hat{s} \leq 1$

(d) $\mathbb{E}\{S - \hat{S}_{\mathrm{ML}}\} = \dfrac{1}{4}, \qquad \mathbb{E}\{S - \hat{S}_{\mathrm{MSE}}\} = 0$

$\mathrm{Var}\{S - \hat{S}_{\mathrm{ML}}\} = \dfrac{13}{80}, \qquad Var\{S - \hat{S}_{\mathrm{MSE}}\} = \dfrac{1}{40}$

Exercise 10 (1.6)

Consider the design of a linear minimum mean square estimator of random variable S based on the observation of random variable X_1. The following statistical information is known:

$$\begin{array}{ll} \mathbb{E}\{X_1\} = 0 & \mathbb{E}\{S\} = 1 \\ \mathbb{E}\{X_1^2\} = 1 & \mathbb{E}\{X_1 S\} = 2 \end{array}$$

(a) Which of the two following designs will incur in a smaller MSE?

$$\begin{aligned} \hat{S}_a &= w_{0a} + w_{1a} X_1 \\ \hat{S}_b &= w_{1b} X_1 \end{aligned}$$

(b) If we have access to a second random variable X_2 satisfying

$$\begin{array}{ll} \mathbb{E}\{X_2\} = 1 & \mathbb{E}\{X_2^2\} = 2 \\ \mathbb{E}\{X_1 X_2\} = \frac{1}{2} & \mathbb{E}\{S X_2\} = 2 \end{array}$$

justify if estimator $\hat{S}_c = w_{0c} + w_{1c} X_1 + w_{2c} X_2$ has a smaller mean quadratic error than the estimators considered in Section (a).

Solution:

(a) $w_{0,a}$ is different from 0; therefore, the MSE of $\hat{S}_a$ is smaller than the MSE of $\hat{S}_b$.

(b) The optimal weights of $\hat{S}_c$ are

$$w_{1,c} = 2 \qquad w_{2,c} = 0$$

Since $\hat{S}_a = \hat{S}_c$ both estimators incur in the same MSE, which is smaller than that of $\hat{S}_b$.

Exercise 11 (1.2; 1.3; 1.7)

The joint p.d.f. of two random variables S and X is:

$$p_{S,X}(s, x) = \begin{cases} 6s, & 0 < s < x \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find:

(a) The minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{MMSE}}$.

(b) The bias of such estimator.

Solution:

(a) $\hat{S}_{\text{MMSE}} = \frac{2}{3}X$.

(b) The estimator is unbiased.

Exercise 12 (1.2; 1.3)

The joint p.d.f. of two random variables S and X is given by:

$$p_{S,X}(s,x) = \alpha, \quad -1 < x < 1, \quad 0 < s < |x|$$

(a) Obtain the marginal p.d.f. of X, $p_X(x)$, specifying the value of α.

(b) Find the estimators of S based on variable X that minimize the mean square error (MSE), ($\overline{C}_{\text{MSE}} = \mathbb{E}\left\{(S-\hat{S})^2\right\}$) and mean absolute deviation (MAD) ($\overline{C}_{\text{MAD}} = \mathbb{E}\left\{|S-\hat{S}|\right\}$), $\hat{S}_{\text{MMSE}}$ and $\hat{S}_{\text{MAD}}$, respectively.

(c) If the estimators analytical shape is constrained to be quadratic in the observations, obtain the expressions of the optimal estimators with respect to previously considered costs: MSE and MAD, i.e., $\hat{S}_{q,\text{MMSE}} = w_1 X^2$ and $\hat{S}_{q,\text{MAD}} = w_2 X^2$, respectively.

Solution:

(a) $p_X(x) = |x|$, $-1 < x < 1$

(b) $\hat{S}_{\text{MMSE}}(X) = \hat{S}_{\text{MAD}}(X) = |X|/2$

(c) $\hat{S}_{q,\text{MMSE}}(X) = 3X^2/5$; $\hat{S}_{q,\text{MAD}}(X) = 5X^2/8$

Exercise 13 (1.4)

Consider a random variable X with p.d.f.

$$p_X(x) = a \exp\left[-a(x-d)\right] u(x-d)$$

where $a > 0$ and d are two parameters.

Find the maximum likelihood estimators of both parameters, $\hat{A}_{\text{ML}}$ and $\hat{D}_{\text{ML}}$, as a function of K samples of X independently drawn, $\left\{X^{(k)}\right\}_{k=1}^{K}$.

Solution: $\hat{A}_{\text{ML}} = \left[\frac{1}{K}\sum_{k=1}^{K}\left(X^{(k)} - \text{mín}\{X^{(k)}\}\right)\right]^{-1}, \qquad \hat{D}_{\text{ML}} = \text{mín}_k\{X^{(k)}\}$

Exercise 14 (1.2; 1.7)

Random variables S and X have a joint probability density function given by

$$p_{S,X}(s,x) = \begin{cases} 10s, & 0 < s < x^2 \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Consider the estimation of S based on the observation of X, with the objective to minimize the following cost function:

$$c(S,\hat{S}) = S^2\left(S-\hat{S}\right)^2$$

Find:

(a) The optimal estimator of S, $\hat{S}_C$, which minimizes the mean cost given X, $\mathbb{E}\left\{c(S,\hat{S})|x\right\}$.

(b) The analytically constrained estimator $\hat{S}_L = wX$ which minimizes the mean cost $\mathbb{E}\left\{c(S,\hat{S})\right\}$.

(c) The mean cost of both estimators: $\mathbb{E}\left\{c(S,\hat{S}_C)\right\}$ and $\mathbb{E}\left\{c(S,\hat{S}_L)\right\}$.

(d) The bias of both estimators: $\mathbb{E}\left\{S-\hat{S}_C\right\}$ and $\mathbb{E}\left\{S-\hat{S}_L\right\}$.

(e) The variance of both estimators: $\mathrm{Var}\left\{S-\hat{S}_C\right\}$ and $\mathrm{Var}\left\{S-\hat{S}_L\right\}$.

Solution:

(a) $\hat{S}_C = \frac{4}{5}X^2$

(b) $\hat{S}_L = \frac{11}{15}X$

(c) $\mathbb{E}\left\{c(S,\hat{S}_C)\right\} = \frac{1}{195}$ and $\mathbb{E}\left\{c(S,\hat{S}_L)\right\} = \frac{7}{1170}$

(d) $\mathbb{E}\left\{S-\hat{S}_C\right\} = -\frac{2}{21}$ and $\mathbb{E}\left\{S-\hat{S}_L\right\} = -0.1349$

(e) $\mathrm{Var}\left\{S-\hat{S}_C\right\} = 0.03163$ and $\mathrm{Var}\left\{S-\hat{S}_L\right\} = 0.0326$

Exercise 15 (1.2; 1.3; 1.6)

Random variables S and X are characterized by the following joint distribution:

$$p_{S,X}(s,x) = c, \quad 0 < s < 1, \;\; s < x < 2s$$

with c a constant.

(a) Plot the support of the p.d.f., and use it to calculate the value of c.

(b) Give the expressions for the marginal p.d.f. of the random variables: $p_S(s)$ and $p_X(x)$.

(c) Find the minimum mean square error estimator of S based on the observation of X, $\hat{S}_{\mathrm{MSE}}(X)$. Plot the estimator on the same plot as the support of $p_{S,X}(s,x)$, and discuss whether it would had been possible to obtain the estimator without analytical derivations.

(d) Calculate the mean square error $\mathbb{E}\left\{\left(S-\hat{S}_{\mathrm{MSE}}(X)\right)^2\right\}$ incurred by the previous estimator.

(e) Now, find the linear minimum mean square error estimator of S given X, $\hat{S}_{\mathrm{LMSE}}(X)$. Again, plot the estimator together with the support of $p_{S,X}(s,x)$. Discuss your result.

(f) Obtain the mean square error $\mathbb{E}\left\{\left(S-\hat{S}_{\mathrm{LMSE}}(X)\right)^2\right\}$ of the linear estimator, and compare it with $\mathbb{E}\left\{\left(S-\hat{S}_{\mathrm{MSE}}(X)\right)^2\right\}$.

(g) It is perceived (e.g., visualizing several samples of (X,S)) that there exist different statistical behaviors for $0 < X < 1$ and $1 < X < 2$. What would occur if, based on this, different optimal linear estimators where designed for each of the intervals ($\hat{S}_{A,\mathrm{LMSE}}(X)$ y $\hat{S}_{B,\mathrm{LMSE}}(X)$, respectively)? Verify analytically the proposed solution.

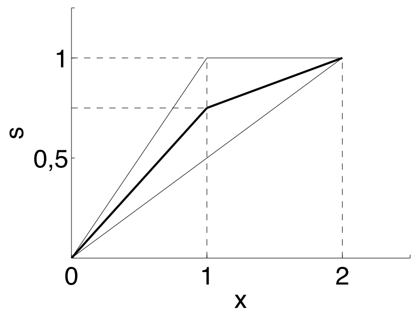
Solution:

(a) Since the area of the support of $p_{S,X}(s,x)$ is $1/2$, $c = 2$.

(b) $p_S(s) = 2s,\ 0 < s < 1; \qquad p_X(x) = \begin{cases} x & ,0 < x < 1 \\ 2-x & ,1 < x < 2 \end{cases}$

(c)

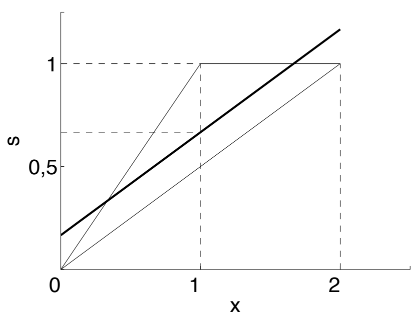
$$\hat{S}_{\mathrm{MSE}}(X) = \begin{cases} \dfrac{3X}{4}, & 0 < X < 1 \\ \dfrac{1}{2}\left(\dfrac{X}{2}+1\right), & 1 < X < 2 \end{cases}$$



Since for every value X we have a uniform *a posteriori distribution* $p_{S|X}(s|x)$, the MSE estimator is given as the average between the minimum and maximum values of S (for each X).

(d) $\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MSE}}(X)\right)^2\right\} = \dfrac{1}{96}$

(e) $\hat{S}_{\mathrm{LMSE}}(X) = \dfrac{X}{2} + \dfrac{1}{6}$



(f) $\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{LMSE}}(X)\right)^2\right\} = \dfrac{11}{24}$, which is larger than $\mathbb{E}\left\{\left(S - \hat{S}_{\mathrm{MSE}}(X)\right)^2\right\}$

(g) $\hat{S}_{A,\mathrm{LMSE}}(X) = \dfrac{3X}{4}$ and $\hat{S}_{B,\mathrm{LMSE}}(X) = \dfrac{1}{2}\left(\dfrac{X}{2}+1\right)$. When jointly considered, these estimators compose $\hat{S}_{\mathrm{MSE}}(X)$.
$p_A(s,x)$ and $p_B(s,x)$ are uniform, and now the linear estimators will also be optimal.

Exercise 16 (1.5; 1.7)

Consider the estimation of a random vector $\mathbf{S}$ from a statistically related observation vector $\mathbf{X}$:

$$\mathbf{X} = H\mathbf{S} + \mathbf{R}$$

H being a known matrix, $\mathbf{R}$ a noise vector with distribution $G(\mathbf{0}, v_r I)$, and $\mathbf{S}$ the random vector to be estimated, whose distribution is $G(\mathbb{E}\{\mathbf{S}\}, V_s)$. It is also known that $\mathbf{S}$ and $\mathbf{R}$ are independent random vectors:

(a) Find the ML estimator of $\mathbf{S}$, $\hat{\mathbf{S}}_{\mathrm{ML}}$.

(b) Is the ML estimator unbiased? Justify your answer.

(c) As it is known, the MSE estimator of $\mathbf{S}$ is given by:

$$\hat{\mathbf{S}}_{\mathrm{MSE}} = \left(H^T H + v_r V_s^{-1}\right)^{-1} H^T \mathbf{X}$$

Obtain the bias of $\hat{\mathbf{S}}_{\mathrm{MSE}}$ and indicate under which conditions such bias vanishes.

Solution:

(a) $\hat{\mathbf{S}}_{\mathrm{ML}} = \left(H^T H\right)^{-1} H^T \mathbf{X}$

(b) The estimator is unbiased.

(c) $\mathbb{E}\left\{\hat{\mathbf{S}}_{\mathrm{MSE}} - \mathbf{S}\right\} = \left(H^T H + v_r V_s^{-1}\right)^{-1} H^T H \mathbb{E}\{\mathbf{S}\} - \mathbb{E}\{\mathbf{S}\}$. The bias goes to zero as the noise power decreases towards 0.

Exercise 17 (1.4; 1.7)

We have access to a set of K samples, $\left\{X^{(k)}\right\}_{k=1}^{K}$, independently drawn from a random variable X with p.d.f.

$$p_X(x) = \frac{1}{bx^2} \exp\left(-\frac{1}{bx}\right) u(x)$$

with $b > 0$ a constant.

(a) Find the ML estimator of b as a function of the available samples, $\hat{B}_{\mathrm{ML}}$.

(b) Verify that random variable $Y = 1/X$ is characterized by a unilateral exponential p.d.f. $p_Y(y)$, and obtain the value of the mean of such distribution.

(c) Considering your answers to the previous sections, is $\hat{B}_{\mathrm{ML}}$ an unbiased estimator?

Solution:

(a) $\hat{B}_{\mathrm{ML}} = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{X^{(k)}}$

(b) $p_Y(y) = \frac{1}{b} \exp\left(-\frac{y}{b}\right) u(y)$

(c) The estimator is unbiased.

Exercise 18 (1.2)

Consider the family of cost functions given by

$$c(S, \hat{S}) = \frac{1}{N+1} \hat{S}^{N+1} + \frac{1}{N(N+1)} S^{N+1} - \frac{1}{N} S \hat{S}^N$$

where N is a non-negative and odd integer.

(a) Assuming that

$$p_{S,X}(s, x) = \frac{1}{\lambda x} \exp\left(-\frac{s}{x} - \frac{x}{\lambda}\right) u(s) u(x) \quad \lambda > 0$$

find the minimum cost estimator of S given X.

(b) Obtain the minimum mean cost.

(c) Determine the coefficient w that minimizes the mean cost of an estimator with analytical shape

$$\hat{S}_L = wX^m$$

m being a positive integer.

Hint: $\int_0^\infty x^N \exp(-x)\, dx = N!$

Solution:

(a) $\hat{S} = X$

(b) $\mathbb{E}\left\{c(S, \hat{S})\right\} = ((N+1)! - 1)\,(N-1)!\lambda^{N+1}$

(c) $w = \dfrac{(Nm+1)!}{(Nm+m)!\lambda^{m-1}}$

Exercise 19 (1.4; 1.7)

An order-N Erlang probability density is characterized by the following expresion:

$$p_X(x) = \frac{a^N x^{N-1} \exp(-ax)}{(N-1)!} \quad x > 0, \quad a > 0$$

Assume that N is known. Considering that the mean of the distribution is given by $m = N/a$, obtain:

(a) The ML estimator of the mean using K independent observations of the variable, $\hat{M}_{\text{ML}}$.

(b) The bias of $\hat{M}_{\text{ML}}$.

(c) Is $\hat{M}_{\text{ML}}$ variance-consistent?

Solution:

(a) $\hat{M}_{\text{ML}} = \dfrac{1}{K}\displaystyle\sum_{k=1}^{K} X^{(k)}$

(b) The estimator is unbiased.

(c) $\text{Var}\left\{\hat{M}_{\text{ML}}\right\} = \dfrac{v_x}{K}$; therefore, the estimator is variance-consistent.

Exercise 20 (1.6)

Random vector $X = [X_1, X_2, X_3]^T$ follows a p.d.f. with mean $\mathbf{m} = \mathbf{0}$ and covariance matrix

$$V_{XX} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(a) Obtain the coefficients (w_0, w_1 and w_2) of the linear minimum mean square error estimator of X_3 given X_1 and X_2,

$$\hat{X}_{3,\text{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Calculate the mean square error of the estimator $\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\text{LMSE}}\right)^2\right\}$.

Solution:

(a) $\hat{X}_{3,\text{LMSE}} = -\dfrac{1}{5}X_1 + \dfrac{4}{5}X_2$

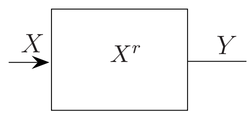
(b) $\mathbb{E}\left\{\left(X_3 - \hat{X}_{3,\text{LMSE}}\right)^2\right\} = \dfrac{8}{5}$

Exercise 21 (1.4)

A random variable X with p.d.f.

$$p_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

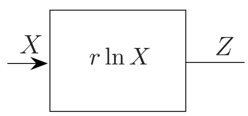
is transformed as indicated in the figure, producing a random observation Y.



$r > 0$

(a) Obtain the maximum likelihood estimator of r, $\hat{R}_{\text{ML}}$, based on K independently drawn observations of Y.

(b) Now, consider the following situation



$r > 0$

and obtain $\hat{R}_{\text{ML}}$ using K independent observations of random variable Z. Discuss your result.

Solution:

(a) $\hat{R}_{\text{ML}} = -\dfrac{1}{K}\displaystyle\sum_{k=1}^{K} \ln Y^{(k)}$. The unknown parameter of the transformation is being identified.

(b) $\hat{R}_{\text{ML}} = -\dfrac{1}{K}\displaystyle\sum_{k=1}^{K} Z^{(k)}$. It is coherent with the previous estimator since $Z = \ln Y$, which is a deterministic (and invertible) transformation of Y.

Exercise 22 (1.4)

An unknown deterministic parameter s, $s > 0$ is measured using two different systems, which provide observations

$$X_i = A_i s + N_i, \quad i = 1, 2$$

where $\{A_i\}, \{N_i\}$, are independent Gaussian random vectors, with means $\mathbb{E}\{A_i\} = 1, \mathbb{E}\{N_i\} = 0$, and variances $\{v_{Ai}\}$, $\{v_{Ni}\}$, respectively $(i = 1, 2)$.

(a) Stablish the expression that defines the ML estimator of s, $\hat{S}_{\text{ML}}$.

(b) Obtain $\hat{S}_{\text{ML}}$ for the particular case $v_{Ai} = 0, i = 1, 2$.

(c) Obtain $\hat{S}_{\text{ML}}$ for the particular case $v_{Ni} = 0, i = 1, 2$.

Solution:

(a) $\hat{S}_{\text{ML}} = \arg\ \text{mín}_s \left\{ \ln\left[(s^2 v_{A1} + v_{N1})(s^2 v_{A2} + v_{N2})\right] + \frac{(s - X_1)^2}{s^2 v_{A1} + v_{N1}} + \frac{(s - X_2)^2}{s^2 v_{A2} + v_{N2}} \right\}$

(b) $\hat{S}_{\text{ML}} = \frac{v_{N2} X_1 + v_{N1} X_2}{v_{N1} + v_{N2}}$

(c) $\hat{S}_{\text{ML}} = \frac{1}{4} \sqrt{\left(\frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}}\right)^2 + 8\left(\frac{X_1^2}{v_{A1}} + \frac{X_2^2}{v_{A2}}\right)} - \left(\frac{X_1}{v_{A1}} + \frac{X_2}{v_{A2}}\right)$

Exercise 23 (1.3; 1.6)

Let X and S be two random variables with joint pdf

$$p_{X,S}(x,s) \begin{cases} \alpha & ; \quad 0 < x < 1,\ 0 < s < 2(1-x) \\ 0 & ; \quad \text{otherwise} \end{cases}$$

with α a constant.

(a) Plot the support of the pdf, and use it to determine the value of α.

(b) Obtain the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) Find the minimum mean square error estimator of S given X, $\widehat{S}_{\text{MSE}}$

(d) Find the linear minimum mean square error estimator of S given X, $\widehat{S}_{\text{LMSE}}$.

Solution:

(a) $\alpha = 1$

(b) $p_{S|X}(s|x) = \frac{1}{2(1-x)}$

(c) $\widehat{S}_{\text{MSE}} = 1 - X$

(d) $\widehat{S}_{\text{LMSE}} = 1 - X$

Exercise 24 (1.3; 1.4; 1.7)

Consider an estimation problem where the goal is to estimate a random variable S using an observation of another random variable X characterized by:

$$X = S + N$$

where the prior pdf of S is

$$p_S(s) = s \exp(-s) \quad s > 0$$

and where N is an additive noise, independent of S, with the following distribution

$$p_N(n) = \exp(-n) \quad n > 0$$

Find:

(a) The maximum likelihood estimator of S, $\widehat{S}_{\text{ML}}$.

(b) The joint pdf of X and S, $p_{X,S}(x,s)$, and the posterior pdf of S given X, $p_{S|X}(s|x)$.

(c) The maximum a posteriori estimator of S given X, $\widehat{S}_{\text{MAP}}$.

(d) The minimum mean square error estimator of S given X, $\widehat{S}_{\text{MMSE}}$.

(e) The bias of all previous estimators, $\widehat{S}_{\text{ML}}$, $\widehat{S}_{\text{MAP}}$ and $\widehat{S}_{\text{MMSE}}$.

(f) Which of the previous estimators has a minimum variance? Justify your answer without calculating the variances of the estimators.

Hint: You can use the following expression to solve the exercise:

$$\int_0^\infty x^N \exp(-x)\, dx = N!$$

Solution:

(a) $\widehat{S}_{\text{ML}} = X$

(b) $p_{X,S}(x,s) = s\exp(-x) \quad x > s,\ s > 0$
$p_{S|X}(s|x) = \dfrac{2s}{x^2} \quad 0 < s < x,\ x > 0$

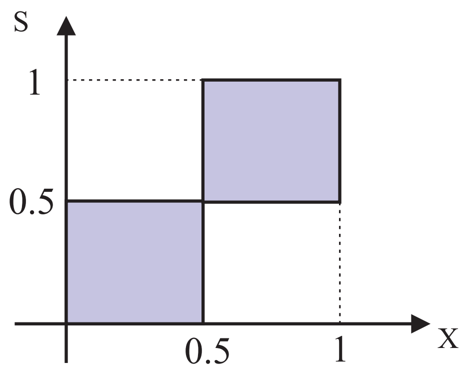
(c) $\widehat{S}_{\text{MAP}} = X$

(d) $\widehat{S}_{\text{MMSE}} = \dfrac{2}{3}X$

(e) $\mathbb{E}\left\{S - \widehat{S}_{\text{ML}}\right\} = \mathbb{E}\left\{S - \widehat{S}_{\text{MAP}}\right\} = -1$
$\mathbb{E}\left\{S - \widehat{S}_{\text{MMSE}}\right\} = 0$
$\text{Var}\left\{\widehat{S}_{\text{MMSE}}\right\} < \text{Var}\left\{\widehat{S}_{\text{MAP}}\right\} = \text{Var}\left\{\widehat{S}_{\text{ML}}\right\}$

Exercise 25 (1.3)

In the plot below, the shaded region shows the domain of a joint distribution of S and X, i.e., the set of points for which $p_{X,S}(x,s) \neq 0$.



Please, provide justified answers to the following questions:

(a) If it is known that $p_{X,S}(x,s)$ is constant in its domain, which is the MSE estimator of S given X? Provide a graphical representation of this estimator.

(b) Is there any $p_{X,S}(x,s)$ with the previous domain for which the MSE estimator of S given X is $\widehat{S}_{\text{MMSE}} = X/2$?

(c) Justify if there exists any $p_{X,S}(x,s)$ with the previous domain, so that $\widehat{S} = 0.5$ is:

- The minimum mean square error estimator of S given X.
- The minimum mean absolute deviation estimator of S given X.
- The maximum *a posteriori* estimator of S given X.

Solution:

(a) $\widehat{S}_{\text{MMSE}} = 0.25$ for $0 < x < 0.5$ and $\widehat{S}_{\text{MMSE}} = 0.75$ for $0.5 < x < 1$

(b) When $0.5 < x < 1$, $p_{S|X}(s|x)$ is non-zero for $0.5 < s < 1$, thus $X/2$ can never be the mean of $p_{S|X}(s|x)$ for that range of X.

(c) $\widehat{S} = 0.5$ cannot be the mean or the median of $p_{S|X}(s|x)$, but it can be its maximum. Therefore, $\widehat{S} = 0.5$ can just be $\widehat{S}_{\text{MAP}}$ (but not $\widehat{S}_{\text{MMSE}}$ or $\widehat{S}_{\text{MAD}}$).

Exercise 26 (1.3; 1.4)

A random variable S follows an exponential pdf

$$p_S(s) = \lambda e^{-\lambda s}, \qquad s > 0$$

with $\lambda > 0$. Consider now a discrete random variable X related to S via a Poisson distribution, i.e.,

$$P_{X|S}(x|s) = \frac{s^x e^{-s}}{x!}, \qquad x = 0, 1, 2, \ldots$$

(a) Determine the ML estimator of S given x.

(b) Assume now that we have access to K independent realizations $\{(x^{(k)}, s^{(k)}), k = 1, \ldots, K\}$ of (X, S). Find the ML estimator of λ based on these observations.

(c) Find the MAP estimation of S for $x = 1$.

Solution:

(a) $\widehat{S}_{\text{ML}} = X$

(b) $\widehat{\lambda}_{\text{ML}} = \dfrac{1}{\frac{1}{K}\sum_{k=1}^{K} s^{(k)}}$

(c) $\widehat{S}_{\text{MAP}} = \dfrac{X}{1+\lambda}$

Exercise 27 (1.5)

We...

Exercise 28 (1.4)

We...

Exercise 29 (1.3)

We...

Exercise 30 (1.4; 1.7)

We...

Exercise 31 (1.5)

We...

A. Additional Problems

We include in this appendix some additional problems taken from the exercises and problems collection used in courses from other degrees. The resolution of these problems should be feasible for students of the new degrees.

Exercise 3.E1 (1.2; 1.3)

Consider an observation

$$X = S + N$$

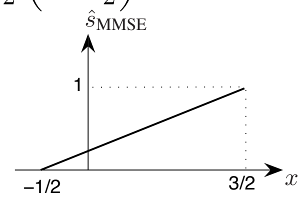
where S is a signal contaminated by additive noise N, and where S and N are independent of each other, and with probability density functions given by:

$$p_S(s) = \begin{cases} 1, & 0 < s < 1 \\ 0, & \text{otherwise} \end{cases} = \Pi(s - 1/2)$$

$$p_N(n) = \begin{cases} 1, & -1/2 < n < 1/2 \\ 0, & \text{otherwise} \end{cases} = \Pi(n)$$

Find the minimum mean square error estimator of S, $\hat{S}_{\text{MMSE}}$. Discuss your result.

Solution: $\hat{S}_{\text{MMSE}} = \dfrac{1}{2}\left(X + \dfrac{1}{2}\right) \qquad (-1/2 < x < 1/2)$



The linear change of the estimator between its minimum and maximum values ($\hat{s}_{\text{MMSE}}(-1/2) = 0, \hat{s}_{\text{MMSE}}(3/2) = 1$) are due to the addition of uniform noise.

Exercise 3.E2 (1.4)

We have access to K samples independently drawn from a random variable X which follows a Laplace distribution $L(m, v)$

$$p_X(x) = \frac{1}{\sqrt{2v}} \exp\left(-\sqrt{\frac{2}{v}}|x - m|\right)$$

Find the joint ML estimators of m, v.

Solution:

$$\hat{M}_{\text{ML}} = \text{med}_K\{X^{(k)}\} \quad \text{(sample median)}$$

$$\hat{V}_{\text{ML}} = \frac{2}{K^2}\left(\sum_k |X^{(k)} - \hat{M}_{\text{ML}}|\right)^2$$

Exercise 4.Q1 (1.5)

Unidimensional random variables S and R are characterized by the following joint distribution.

$$G\left(\mathbf{0}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right),$$

The observable variable is given by $X = S + R$.

(a) Obtain the estimator $\hat{S}_{\text{MSE}}$.

(b) Was this result to be expected? (Consider the existing relationship between $\mathbb{E}\{R|x\}$ and $\mathbb{E}\{S|x\}$).

(c) Obtain the MSE.

Solution:

(a) $\hat{S}_{\text{MSE}} = X/2$

(b) $\mathbb{E}\{R|x\} = \mathbb{E}\{S|x\}$ (since both variables distribute identically given X)
$\mathbb{E}\{X|x\} = x = \mathbb{E}\{S + R|x\} = \mathbb{E}\{S|x\} + \mathbb{E}\{R|x\}$

(c) $\mathbb{E}\left\{\left(S - \hat{S}\right)^2\right\} = \frac{1}{2} - \frac{1}{2}\rho$

Exercise 6.E1 (1.6)

Let S, X_1, and X_2 be three zero-mean random variables satisfying:

- The covariance matrix of X_1 and X_2 is:

$$\mathbf{V}_{xx} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

- the cross-covariance between S and observation vector $X = [X_1, X_2]^T$ is:

$$\mathbf{v}_{sx} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(a) Obtain the coefficients of the linear minimum mean square error estimator

$$\hat{S}_{\text{LMSE}} = w_0 + w_1 X_1 + w_2 X_2$$

(b) Find the expected quadratic value of the estimation error, $\hat{E} = S - \hat{S}_{\text{LMSE}}$.

(c) Explain which is the role of variable X_2, which, as can be seen, is uncorrelated with the variable to be estimated (S).

Solution:

(a) $w_1 = \frac{1}{1-\rho^2}; \qquad w_2 = -\frac{\rho}{1-\rho^2}; \qquad w_0 = 0$

(b) $\mathbb{E}\{\hat{E}^2\} = \mathbb{E}\{S^2\} - \frac{1}{1-\rho^2}$

(c) X_2 is combined with X_1 allowing a better approximation of S.

Exercise 6.E5 (1.6)

We want to design a linear minimum mean square error estimator of a random variable S based on the observation of random variables X_1 and X_2:

$$\hat{S}_{\text{LMSE}}(X_1, X_2) = w_0 + w_1 X_1 + w_2 X_2$$

The means of the random variables are $\mathbb{E}\{S\} = 1$, $\mathbb{E}\{X_1\} = 1$, and $\mathbb{E}\{X_2\} = 0$, whereas the correlations are given by $\mathbb{E}\{S^2\} = 4$, $\mathbb{E}\{X_1^2\} = 3$, $\mathbb{E}\{X_2^2\} = 2$, $\mathbb{E}\{SX_1\} = 2$, $\mathbb{E}\{SX_2\} = 0$, and $\mathbb{E}\{X_1 X_2\} = 1$.

(a) Obtain the optimal coefficients $\{w_i\}, i = 0, 1, 2$ of $\hat{S}_{\text{LMSE}}(X_1, X_2)$.

(b) Check that $v_{SX_2} = 0$. Why can still be $w_2 \neq 0$?

(c) Calculate the mean square error incurred by the application of estimator $\hat{S}_{\text{LMSE}}(X_1, X_2)$.

(d) How does the mean square error changes if the estimator $\hat{S}'_{\text{LMSE}}(X_1) = w_0' + w_1' X_1$, based on the sole observation of X_1, is used instead of $\hat{S}_{\text{LMSE}}(X_1, X_2)$?

Solution:

(a) $w_0 = 1/3, \ w_1 = 2/3, \ w_2 = -1/3$

(b) Combining X_1 and X_2 is better than just using X_1 (using the geometric analogy of the Orthogonality Principle, the projection space spanned by X_1 and X_2 is larger than the one spanned by X_1 alone).

(c) $\mathbb{E}\{E^2\} = 7/3$

(d) ($w_0' = 1/2; \ w_1' = 1/2$). $\mathbb{E}\{E'^2\} = 3$. It increases by 2/3 (confirming our answer to the previous subquestion).