# Detection Theory

## A Modern Theory of Detection and Estimation.
## Block-2: Analytical Detection

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# Example: Tipping riders

You usually order dinner to your favourite restaurant. The ordering app lets you fix a time slot to get your food delivered. They always serve within this range because if they serve off-range you get your order free. Since the pandemia the service is contactless: the riders leave the packet at your door, ring and rush to serve the next order in time. So you don't know who brought the food.

Since you have had a **long term relationship with them** you know your area is served by 3 riders with different quality of service:

- Early: delivers most frequently at the beginning of the time slot.
- Flatty: delivers indistinctly at any point in the time slot.
- L'80: delivers most frequently by the end of the slot.

The app lets you tip the riders according to their service, and you have made up a tipping policy oriented to getting the best possible service

# Tipping riders: Fixed Tipping policy

Your tipping policy reflects you value of your long lasting relationship with the riders, that is you don't tip based on each independent order but on a long term good quality. Therefore you tip (through the app, as the service is contactless)

- Anytime you think that the rider was Rider Early you'd like to tip 20%
- Anytime you think that the rider was Rider Flatty you'd like to tip 15%
- Anytime you think that the rider was Rider L'80 you'd like to tip 10%

You want to design a **tipping strategy** that helps you maximize the goal of your tipping policy: rewarding each rider with what you consider they deserve. The tipping strategy depends on the **observation of the deliver instant**. Since the time slot for deliver is variable, you consider a normalized slot so $x \in [0, 1]$.
At the core of your tipping strategy lies a **decider** $d = \phi(x)$ that takes $x$ as input and outputs the exact percentage of your tip depending on its guess about the correct rider.
So you define a **cost policy** to evaluate the quality of your decisions.

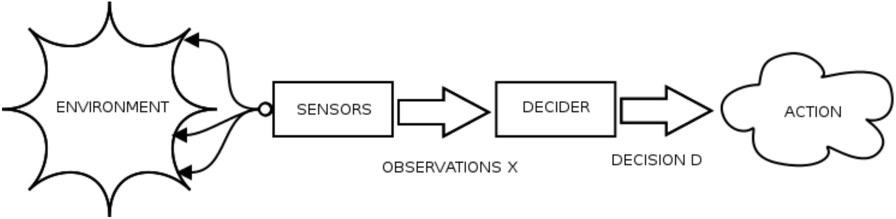
# Examples of decision problems

## Decision problems

Choose one of out several **hypothesis** or possible ways of explaining the observations

- Digital Communications: $H_0$ a zero was sent. $H_1$ a one was sent. $\mathbf{X}$ is the received signal
- Radar. $H_0$ no target. $H_1$ target. $\mathbf{X}$ is the received pulse
- Medical diagnosis. $H_0$ healthy. $H_1$ patient. $\mathbf{X}$ is the outcome of the test.
- Spam filtering. $H_0$ regular mail. $H_1$ spam. $\mathbf{X}$ is the email text.
- OCR. A hypothesis for each character. $\mathbf{X}$ is the written character
- Speech recognition. A hypothesis for each phoneme. $\mathbf{X}$ is the audio clip.
- News classification. A hypothesis for each newspaper section. $\mathbf{X}$ is a text.
- Image classification. A hypothesis for each class. $\mathbf{X}$ are the pixels in the image.

# Elements of a detection problem



- **Hypothesis:** Discrete random variables that represent the several options that explain observations. **Disjoint, exhaustive and finite**.
- **Observations:** Random vector that contains the information recorded by sensors. Statistically related to the hypothesis.
- **Detector:** Implements the **discriminant function** $D = \phi(\mathbf{X})$. Mathematical function of the observation that assigns each observation to a decision.
- **Decision:** Discrete random variable $D$. Deterministic given the observation, i.e., each observation always leads to the same decision
- **Decision region:** part of the input space formed by observations that lead to the same decision

# Decision regions

- **Each observation always leads to the same decision**.
- Input space partitioned into **categories:** regions formed by observations that lead to the same decision
- Decision region for category $d$:

$$\mathcal{X}_d = \{\mathbf{x} \in \mathcal{X} | \phi(\mathbf{x}) = d\}$$

- **Decision boundaries:** separation between regions.
- Every decider **induces a partition** of the input space into decision regions $\mathcal{X} = \bigcup_{d=0}^{M-1} \mathcal{X}_d$
  - This partition completely characterizes the decision function
  - It's equivalent to design a decision function or to design the partition of the input space

# Design of decision functions

- **Analytical methods:** Problem is defined in terms of a complete statistical characterization of the involved random variables. This lecture and the next one
- **Machine learning:** Problem defined in terms of a set of labelled examples: observation and right decision. In two lectures time.

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# Statistical modeling of a decision problem

- **Likelihood** of each hypothesis $p_{\mathbf{X}|H}(X = x|H = h)$. Generation of observations under each hypothesis. In a case with $L$ hypothesis: $p_{\mathbf{X}|H}(x|H = 0)$, $p_{\mathbf{X}|H}(x|H = 1)$, $\ldots$, $p_{\mathbf{X}|H}(x|H = L - 1)$.
- **Prior probability of each hypothesis** $P_H(H = h)$. Note $H$ is a discrete random variable. $\sum_{h=0}^{L-1} P_H(h) = 1$.
- **Prior distribution of the observations** $p_{\mathbf{X}}(\mathbf{x})$.
- **Joint distribution of observations and hypothesis** $p_{\mathbf{X},H}(x, h) = p_{\mathbf{X}|H}(x|h)P_H(h)$
- **Posterior of each hypothesis** $P_{H|\mathbf{X}}(h|X = x)$.

# Tipping riders: math modelling of the problem (I)

The **observation**, $x$, is the precise moment of the time slot you got your dine delivered.

You choose to model this as a multiclass decision problem with **3 hypotheses** with the following **likelihoods**:

$$\begin{array}{lllll} H=0: & \text{Flatty delivered} & p_{X|H}(x|0) = & 1 & 0<x<1 \\ H=1: & \text{Early delivered} & p_{X|H}(x|1) = & 2(1-x) & 0<x<1 \\ H=2: & \text{L'80 delivered} & p_{X|H}(x|2) = & 2x & 0<x<1 \end{array}$$

The prior probabilities (according to your past experiences/beliefs) are $P_H(0) = 0.4$ y $P_H(1) = P_H(2) = 0.3$

# Risk

**Evaluation of the performance** of a decider.

Decisions involve **costs**

## Costs

- Quantification of the consequences of each decision.
- $c(D,H) \in \mathbb{R}$ assigns a penalty $c_{dh}$, with $c_{dh} > c_{hh} \geq 0$, $\forall d \neq h$, to the fact of deciding $D = d$ when the right hypothesis was $H = h$
- We usually call **cost policy** to the set of the $c_{dh}$

## Risk of a decider $\phi(\mathbf{x})$

Risk: Expected cost

$$\begin{aligned} r_\phi =& \mathbb{E}\{c(D,H)\} = \sum_{d=0}^{M-1}\sum_{h=0}^{L-1} c_{dh} P_{D,H}\{D=d, H=h\} \\ &= \sum_{d=0}^{M-1}\sum_{h=0}^{L-1} c_{dh} P_H(h) P_{D|H}(d|h) = \sum_{d=0}^{M-1}\sum_{h=0}^{L-1} c_{dh} P_H(h) \int_{\mathcal{X}_d} p_{\mathbf{x}|H}(\mathbf{x}|h) d\mathbf{x} \end{aligned}$$

# Tipping riders: math modelling of the problem (II)

The possible **decisions** output by the decider are

$$\phi(x) = \begin{cases} d = 0 & \text{tip } 15\% \\ d = 1 & \text{tip } 20\% \\ d = 2 & \text{tip } 10\% \end{cases}$$

The last ingredient you need to design your decider is to stablish a cost policy that captures your perception of the impact of the consequences of the decisions. As a first step, you keep it simple $c_{hh} = 0, \ h = 0, 1, 2$ y $c_{dh} = 1, \ d \neq h$

# Tipping riders: evaluate the performance of a generic decider

Compute the risk of the decider $\phi(x)$:

$$\phi(x) = \begin{cases} 1, & x < 0.5 \\ 2, & x > 0.5 \end{cases}$$

## Solution

$$r_\phi = \quad 0.4 \cdot 0.5 + 0.4 \cdot 0.5 + 0.3 \cdot 0.25 + 0.3 \cdot 0.25 = 0.55$$

# Conditional Risk

Evaluate the quality of a decision given the observation

$$\mathbb{E}\{c(d,H)|\mathbf{x}\} = \sum_{h=0}^{L-1} c_{dh} P_{H|\mathbf{X}}(h|\mathbf{x})$$

The conditional risk relates to the risk or overall risk

$$r_\phi = \mathbb{E}\{c(D,H)\} = \int \mathbb{E}\{c(d,H)|\mathbf{x}\} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

# Example continued

Conditional risk for each decision:

$$\mathbb{E}\{c(d,H)|x\} = c_{d0}P_{H|X}(0|x) + c_{d1}P_{H|X}(1|x) + c_{d2}P_{H|X}(2|x)$$

After the application of the Bayes rule and some math we arrive at

- If $d = 0$:

$$\begin{aligned}\mathbb{E}\{c(0,H)|x\} &= c_{00}P_{H|X}(0|x) + c_{01}P_{H|X}(1|x) + c_{02}P_{H|X}(2|x)\\ &= 0\cdot 0.4 + 1\cdot 0.6(1-x) + 1\cdot 0.6x = 0.6\end{aligned}$$

- If $d = 1$:

$$\begin{aligned}\mathbb{E}\{c(1,H)|x\} &= c_{10}P_{H|X}(0|x) + c_{11}P_{H|X}(1|x) + c_{12}P_{H|X}(2|x)\\ &= 1\cdot 0.4 + 0\cdot 0.6(1-x) + 1\cdot 0.6x = 0.4 + 0.6x\end{aligned}$$

- If $d = 2$:

$$\begin{aligned}\mathbb{E}\{c(2,H)|x\} &= c_{20}P_{H|X}(0|x) + c_{21}P_{H|X}(1|x) + c_{22}P_{H|X}(2|x)\\ &= 1\cdot 0.4 + 1\cdot 0.6(1-x) + 0\cdot 0.6x = 1 - 0.6x\end{aligned}$$

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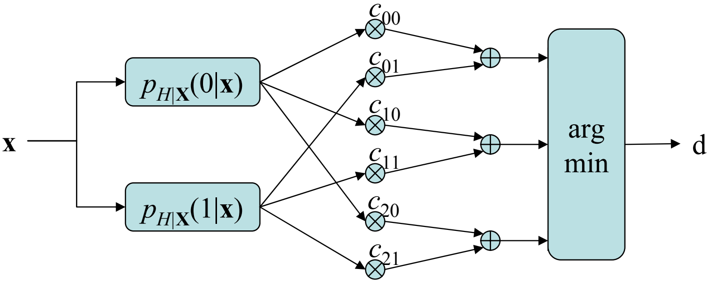
# Bayesian Decision Theory

## Minimum Expected Risk

**Bayesian Decider: choose the decision of minimum risk**

$$\phi^*(\mathbf{x}) = \arg\min_d \sum_{h=0}^{L-1} c_{dh} P_{H|\mathbf{X}}(h|\mathbf{x})$$

For instance, in a case with two hypotheses and three possible decisions



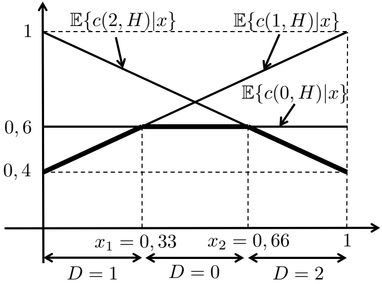
x
$p_{H|\mathbf{X}}(0|\mathbf{x})$
$p_{H|\mathbf{X}}(1|\mathbf{x})$
$c_{00}$
$c_{01}$
$c_{10}$
$c_{11}$
$c_{20}$
$c_{21}$
arg min
d

# Example: Bayesian decider of the previous case

We had the expected cost per decision

$$\begin{aligned} \mathbb{E}\{c(0,H)|x\} &= \quad 0.6 \\ \mathbb{E}\{c(1,H)|x\} &= \quad 0.4+0.6x \\ \mathbb{E}\{c(2,H)|x\} &= \quad 1-0.6x \end{aligned}$$

analyze, for each observation $x$, which term is the minimum:



Computing $x_1$:
$0.4 + 0.6x_1 = 0.6$
$x_1 = 0.33$

Computing $x_2$:
$1 - 0.6x_2 = 0.6$
$x_2 = 0.66$

Solution:

$$\phi(x) = \begin{cases} 1, & 0 < x < 0.33 \\ 0, & 0.33 < x < 0.66 \\ 2, & 0.66 < x < 1 \end{cases}$$

# Bayesian Decider with likelihoods and priors

Applying Bayes' Rule

$$P_{H|\mathbf{X}}(h|\mathbf{x}) = \frac{p_{\mathbf{X}|H}(\mathbf{x}|h)P_H(h)}{p_{\mathbf{X}}(\mathbf{x})}$$

$$\phi^*(\mathbf{x}) = \arg\min_d \sum_{h=0}^{L-1} c_{dh} \frac{p_{\mathbf{X}|H}(\mathbf{x}|h)P_H(h)}{p_{\mathbf{X}}(\mathbf{x})}$$

since the denominator does not depend on the decision

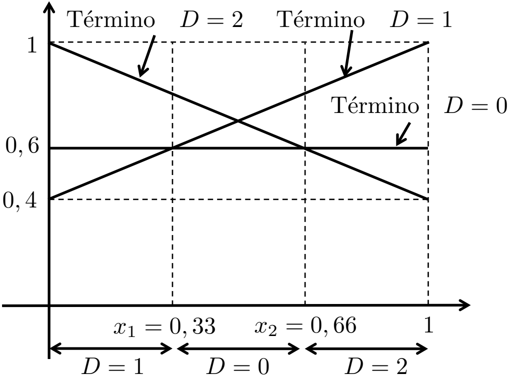
$$\phi^*(\mathbf{x}) = \arg\min_d \sum_{h=0}^{L-1} c_{dh} p_{\mathbf{X}|H}(\mathbf{x}|h)P_H(h)$$

# Example continued

- If $D = 0$:
  $\sum_{h=0}^{2} c_{0h} p_{X|H}(x|h) P_H(h) = 0 \cdot 1 \cdot 0.4 + 1 \cdot 2(1-x) \cdot 0.3 + 1 \cdot 2x \cdot 0.3 = 0.6.$
- If $D = 1$:
  $\sum_{h=0}^{2} c_{1h} p_{X|H}(x|h) P_H(h) = 1 \cdot 1 \cdot 0.4 + 0 \cdot 2(1-x) \cdot 0.3 + 1 \cdot 2x \cdot 0.3 = 0.4 + 0.6x.$
- If $D = 2$:
  $\sum_{h=0}^{2} c_{2h} p_{X|H}(x|h) P_H(h) = 1 \cdot 1 \cdot 0.4 + 1 \cdot 2(1-x) \cdot 0.3 + 0 \cdot 2x \cdot 0.3 = 1 - 0.6x.$

Minimizing the cost for every $x$ (after plotting each term as a function of $x$)

# Example continued



Computing $x_1$:
$0,4 + 0,6x_1 = 0,6$
$x_1 = 0,33$

Computing $x_2$:
$1 - 0,6x_2 = 0,6$
$x_2 = 0,66$

Solution:

$$\phi(x) = \begin{cases} 1, & 0 < x < 0.33 \\ 0, & 0.33 < x < 0.66 \\ 2, & 0.66 < x < 1 \end{cases}$$

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# MAP Decision

What if we have no access to the cost policy?

- Assume a decision corresponds to each hypothesis (number of decisions equal to number of hypotheses).
- Assume cost policy given by

$$c_{dh} = \left[ \begin{array}{ll} 1, & \text{si } d \neq h \\ 0, & \text{si } d = h \end{array} \right] = 1 - \delta_{d-h}$$

MAP decider is Bayesian only if costs are $c_{dh} = 0$ if $d = h$ and $c_{dh} = 1$ if $d \neq h$

# MAP Decider

## Risk equal to probability of error

$$r_\phi = \sum_{d \neq h} P\{D = d, H = h\} = P\{D \neq H\}$$

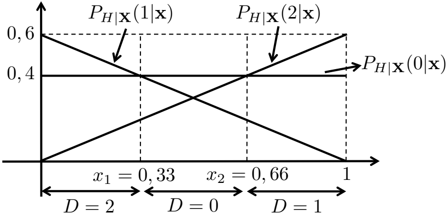
## Minimum risk means maximum posterior probability

Risk of deciding $D = d$: $\mathbb{E}\{C(d, H)|\mathbf{x}\} = \sum_{h \neq d} P_{H|\mathbf{X}}(h|\mathbf{x}) = 1 - P_{H|\mathbf{X}}(d|\mathbf{x})$

$$\phi_{\mathrm{MAP}}(\mathbf{x}) = \arg\max_h P_{H|\mathbf{X}}(h|\mathbf{x})$$

# Example continued

$$P_{H|X}(0|x) = 0.4$$
$$P_{H|X}(1|x) = 0.6(1-x)$$
$$P_{H|X}(2|x) = 0.6x$$



Solution:

$$\phi_{\text{MAP}}(x) = \begin{cases} 1, & 0 < x < 0.33 \\ 0, & 0.33 < x < 0.66 \\ 2, & 0.66 < x < 1 \end{cases}$$

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# ML Decider

What if we have no access to the cost policy nor the prior probabilities of the hypotheses?

## ML rule

$$\phi^*_{\mathrm{ML}}(\mathbf{x}) = \arg\max_h p_{\mathbf{X}|H}(\mathbf{x}|h)$$

## ML and MAP are sometimes equivalent

MAP and ML deciders are equivalent when the hypotheses are **equiprobable**: $P_H(h) = 1/L \ \forall h$

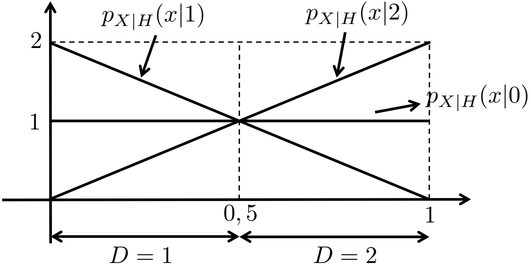
## ML is Bayesian?

ML is Bayesian when

- Hypotheses are equiprobable
- Cost policy is a minimum error probability case

# Example continued

$$\begin{aligned} p_{X|H}(x|0) &= 1 & 0 < x < 1 \\ p_{X|H}(x|1) &= 2(1-x) & 0 < x < 1 \\ p_{X|H}(x|2) &= 2x & 0 < x < 1 \end{aligned}$$



Decision regions:

$$\phi(x) = \begin{cases} 1, & 0 < x < 0.5 \\ 2, & 0.5 < x < 1 \end{cases}$$

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# Binary decision setup

- Two Possible Decisions $D = \{0, 1\}$
- Two Possible Hypotheses $H = \{0, 1\}$
- Correct decision (hit): $\{D = 0, H = 0\}$ or $\{D = 1, H = 1\}$
- Wrong decision (error): $\{D = 1, H = 0\}$ or $\{D = 0, H = 1\}$
- Special names for some joint events:
  - Detection: $\{D = 1, H = 1\}$
  - Missing target: $\{D = 0, H = 1\}$
  - False Alarm: $\{D = 1, H = 0\}$
- Design Bayesian binary deciders using a minimum risk criterion.
- Plus some alternatives based in other criteria different from risk minimization.

# Cost policy and risk

In binary detection the cost policy is defined by 4 constants: $c_{00}$, $c_{11}$, $c_{01}$, $c_{10}$. Costs are positive. The cost of an error must be larger than the cost of the corresponding correct decision: $c_{10} > c_{00}$ and $c_{01} > c_{00}$.

The risk of a binary decider is given by:

$$\begin{aligned} r_\phi =& c_{00} P\{D=0, H=0\} + c_{01} P\{D=0, H=1\} \\ &+ c_{10} P\{D=1, H=0\} + c_{11} P\{D=1, H=1\} \\ =& c_{00} P_H(0) P_{D|H}(0|0) + c_{01} P_H(1) P_{D|H}(0|1) \\ &+ c_{10} P_H(0) P_{D|H}(1|0) + c_{11} P_H(1) P_{D|H}(1|1) \end{aligned}$$

# Probability of False Alarm and probability of Missing Target

## Probability of False Alarm

$$P_{FA} = P_{D|H}(1|0) = \int_{\mathcal{X}_1} p_{\mathbf{x}|H}(\mathbf{x}|0)d\mathbf{x}$$

## Probability of Missing Target

$$P_M = P_{D|H}(0|1) = \int_{\mathcal{X}_0} p_{\mathbf{x}|H}(\mathbf{x}|1)d\mathbf{x}$$

# Risk in terms of $P_{FA}$ and $P_M$

$$
\begin{aligned}
r_\phi =& c_{00}P_H(0)P_{D|H}(0|0) + c_{01}P_H(1)P_{D|H}(0|1) \\
& + c_{10}P_H(0)P_{D|H}(1|0) + c_{11}P_H(1)P_{D|H}(1|1) \\
=& c_{00}P_H(0)(1 - P_{D|H}(1|0)) + c_{01}P_H(1)P_{D|H}(0|1) \\
& + c_{10}P_H(0)P_{D|H}(1|0) + c_{11}P_H(1)(1 - P_{D|H}(0|1)) \\
=& c_{00}P_H(0)(1 - P_{\mathrm{FA}}) + c_{01}P_H(1)P_{\mathrm{M}} + c_{10}P_H(0)P_{\mathrm{FA}} + c_{11}P_H(1)(1 - P_{\mathrm{M}}) \\
=& (c_{01} - c_{11})P_H(1)P_{\mathrm{M}} + (c_{10} - c_{00})P_H(0)P_{\mathrm{FA}} + (c_{00}P_H(0) + c_{11}P_H(1))
\end{aligned}
$$

Risk is sum of three components:

- $(c_{00}P_H(0) + c_{11}P_H(1))$ minimum risk corresponding to an ideal detector $P_M = 0$ and $P_{FA} = 0$ and hits with probability 1.
- $(c_{01} - c_{11})P_H(1)P_M$ contribution of missing targets to the risk
- $(c_{10} - c_{00})P_H(0)P_{FA}$ contribution of false alarms to the risk.

Notice that as long as $p(\mathbf{x}|H = 0)$ and $p(\mathbf{x}|H = 1)$ present some overlapping the decider will incur in errors.

# Discriminant function

Every binary decision can be expressed as comparing a function of the observations with a threshold $\eta$

## Discriminant function

$$g(\mathbf{x}) \underset{D=0}{\overset{D=1}{\gtrless}} \eta$$

Since $\mathbf{X}$ is a random variable, $g(\mathbf{X})$ is a random variable itself. We call $\Lambda = g(\mathbf{X})$

## $P_{\text{FA}}$ and $P_M$ using $\Lambda$

$$P_{\text{FA}} = P_{D|H}(1|0) = P\{\Lambda > \eta | H = 0\} = \int_{\eta}^{\infty} p_{\Lambda|H}(\lambda|0) d\lambda$$

$$P_M = P_{D|H}(0|1) = P\{\Lambda > \eta | H = 1\} = \int_{-\infty}^{\eta} p_{\Lambda|H}(\lambda|1) d\lambda$$

Notice that even when $\mathbf{X}$ could be a random vector, $\Lambda$ is always a **scalar**.

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# Bayesian binary deciders

Design a binary decider means **find a discriminant function** $g(\mathbf{X})$ and a **threshold** $\eta$ that define a decision rule of minimum risk

$$c_{10} P_{H|\mathbf{X}}(0|\mathbf{x}) + c_{11} P_{H|\mathbf{X}}(1|\mathbf{x}) \underset{D=1}{\overset{D=0}{\gtrless}} c_{00} P_{H|\mathbf{X}}(0|\mathbf{x}) + c_{01} P_{H|\mathbf{X}}(1|\mathbf{x})$$

Grouping terms

$$(c_{10} - c_{00}) P_{H|\mathbf{X}}(0|\mathbf{x}) \underset{D=1}{\overset{D=0}{\gtrless}} (c_{01} - c_{11}) P_{H|\mathbf{X}}(1|\mathbf{x})$$

since $c_{10} > c_{00}$ y $c_{01} > c_{11}$,

$$\frac{P_{H|\mathbf{X}}(1|\mathbf{x})}{P_{H|\mathbf{X}}(0|\mathbf{x})} \underset{D=0}{\overset{D=1}{\gtrless}} \frac{c_{10} - c_{00}}{c_{01} - c_{11}}$$

$g(\mathbf{X})$ is the ratio of posterior probabilities, $\eta$ is the ratio of the cost increment.

# Likelihood Ratio Tests

Application of the Bayes' rule to the previous result:

$$\frac{p_{\mathbf{x}|H}(\mathbf{x}|1)}{p_{\mathbf{x}|H}(\mathbf{x}|0)} \underset{D=0}{\overset{D=1}{\gtrless}} \frac{(c_{10}-c_{00})P_H(0)}{(c_{01}-c_{11})P_H(1)}$$

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# MAP decider

- Risk equal to probability of error
- Cost policy given by

$$c_{dh} = \left[ \begin{array}{ll} 1, & \text{if } d \neq h \\ 0, & \text{if } d = h \end{array} \right] = 1 - \delta_{d-h}$$

$$P_{H|\mathbf{x}}(1|\mathbf{x}) \underset{D=0}{\overset{D=1}{\gtrless}} P_{H|\mathbf{x}}(0|\mathbf{x})$$

Using Bayes' rule:

$$\frac{p_{\mathbf{x}|H}(\mathbf{x}|1)}{p_{\mathbf{x}|H}(\mathbf{x}|0)} \underset{D=0}{\overset{D=1}{\gtrless}} \frac{P_H(0)}{P_H(1)}$$

The probability of error is

$$P_e = r_{\phi_{\mathrm{MAP}}} = P\{D \neq H\} = P_H(1)P_{\mathrm{M}} + P_H(0)P_{\mathrm{FA}}$$

# Binary ML Decider

## ML rule

$$\phi^*_{\text{ML}}(\mathbf{x}) = \arg\max_h p_{\mathbf{X}|H}(\mathbf{x}|h)$$

## ML binary case

$$p_{\mathbf{X}|H}(\mathbf{x}|1) \underset{D=0}{\overset{D=1}{\gtrless}} p_{\mathbf{X}|H}(\mathbf{x}|0)$$

Remember

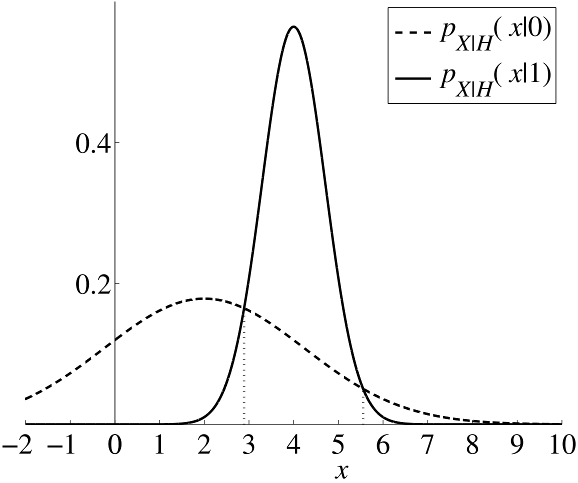
Binary MAP and ML deciders are equivalent when the hypotheses are **equiprobable**: $P(H=0) = P(H=1) = 1/2$

## Binary ML is Bayesian if

- $P(H=0) = P(H=1)$
- Cost policy is a minimum error probability case

# Example Binary decider

Consider a binary decision problem with Gaussian likelihoods with means $m_0 = 2$ and $m_1 = 4$ and variances $v_0 = 5$ and $v_1 = 0.5$



# Example Binary decider

Consider a binary decision problem with Gaussian likelihoods with means $m_0 = 2$ and $m_1 = 4$ and variances $v_0 = 5$ and $v_1 = 0.5$
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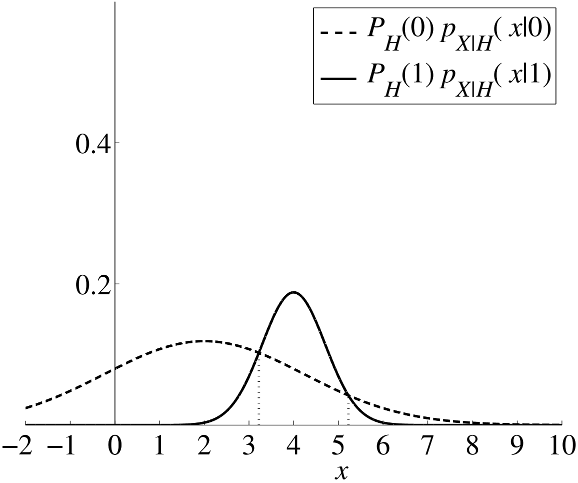

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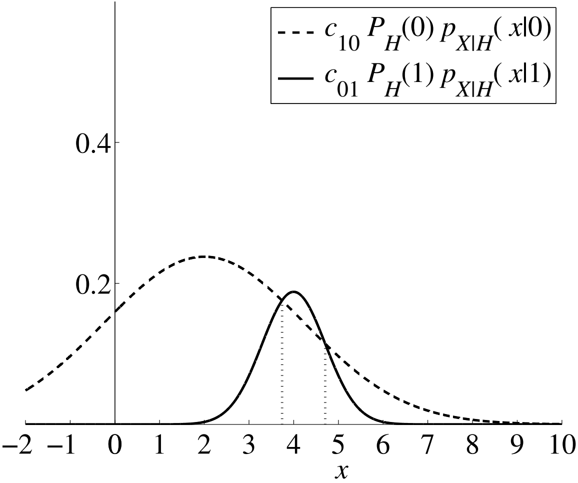

# Detection Theory. Non-bayesian binary detection. Evaluation of classifiers.

A Modern Theory of Detection and Estimation.
Block-2: Analytical Detection

Emilio Parrado-Hernández, emilio.parrado@uc3m.es

October 27, 2022

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   - Example with Gaussian Likelihoods
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   - AUC: Area under the (ROC) curve
   - Sensitivity and Specificity

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   - Neyman-Pearson Detector
   - Minimax classifier

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# LRT rule

Binary deciders are defined by

- Discriminant function $g(\mathbf{x})$
- Threshold $\eta$

## LRT tests

The discriminant function is the likelihood ratio

$$g(\mathbf{x}) = \frac{p_{\mathbf{X}|H}(\mathbf{x}|1)}{p_{\mathbf{X}|H}(\mathbf{x}|0)} = \lambda \underset{D=0}{\overset{D=1}{\gtrless}} \eta$$

## Example

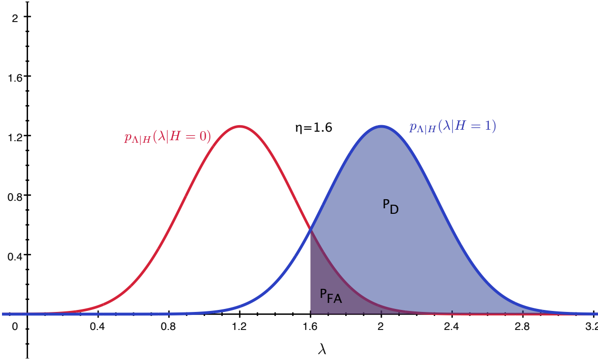
- ML: $\eta = 1$
- MAP: $\eta = \frac{P_H(0)}{P_H(1)}$
- Bayesian: $\eta = \frac{P_H(0)}{P_H(1)} \frac{c_{10}-c_{00}}{c_{01}-c_{11}}$

# LRT decider depending on parameter $\eta$

Every possible value of $\eta \in [-\infty, \infty]$ originates a [potentially] different LRT. The value of $\eta$ determines the performance of each LRT decider:

$$P_{\mathrm{FA}} = \int_{\eta}^{\infty} p_{\Lambda|0}(\lambda|0)d\lambda$$

$$P_D = \int_{\eta}^{\infty} p_{\Lambda|1}(\lambda|1)d\lambda$$



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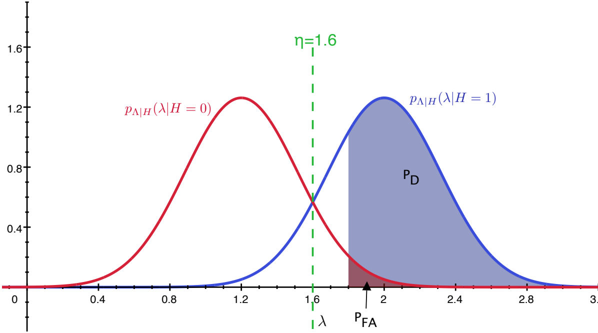

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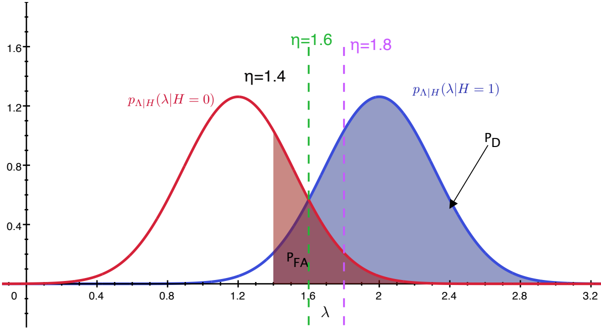
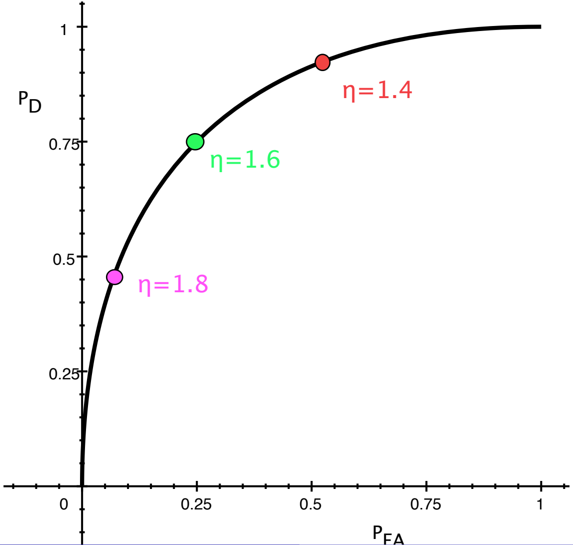


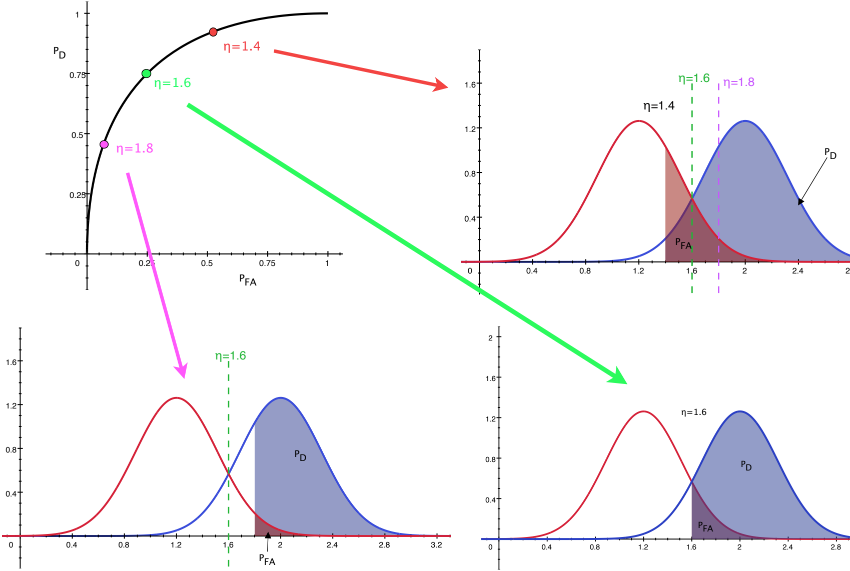
# Receiver Operating Characteristic curve

Assigning values to parameter $\eta$ we can tune a range of deciders with different performances in terms of $P_{\text{FA}}$ and $P_D$.

The ROC curve is a plot of $P_D$ vs. $P_{\text{FA}}$ indexed by the different values of $\eta$



# Receiver Operating Characteristic curve



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# Example

A binary problem with likelihoods

$$p_{X|H}(x|1) = 2x, \qquad 0 \leq x \leq 1$$
$$p_{X|H}(x|0) = 2(1-x), \qquad 0 \leq x \leq 1$$

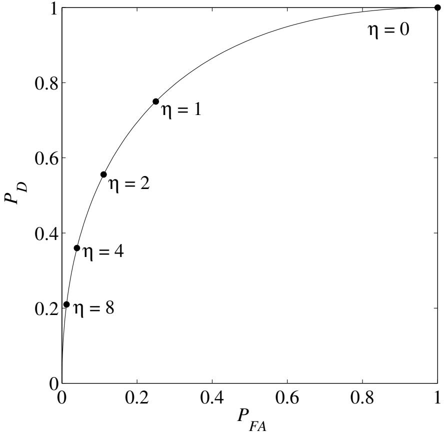
yields the following LRT

$$\frac{x}{1-x} \underset{D=0}{\overset{D=1}{\gtrless}} \eta$$

with performances

$$P_{\text{FA}}(\eta) = \frac{1}{(1+\eta)^2}$$

$$P_{\text{D}} = 1 - \frac{\eta^2}{(1+\eta)^2}$$



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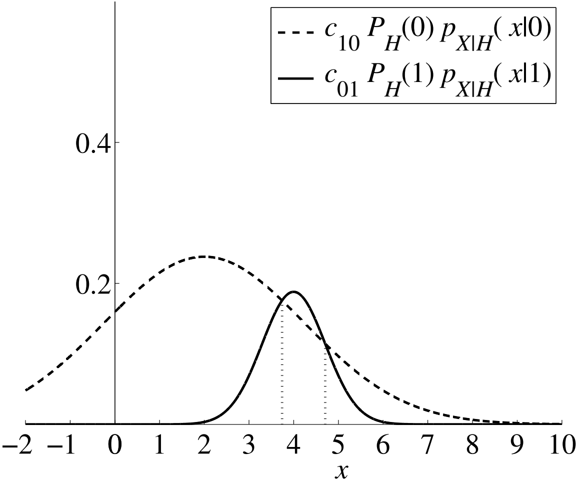
# Another example (from past lecture)

Consider a binary decision problem with Gaussian likelihoods with means $m_0 = 2$ and $m_1 = 4$ and variances $v_0 = 5$ and $v_1 = 0.5$

And Priors $P_H(0) = 2/3$ and $P_H(1) = 1/3$

And costs $c_{10} = 2$, $c_{01} = 1$ and $c_{00} = c_{11} = 0$



# From LRT to g(x)

It is a curve parameterized by $\eta$ in the LRT

$$\frac{p_{\mathbf{x}|H}(\mathbf{x}|1)}{p_{\mathbf{x}|H}(\mathbf{x}|0)} \underset{D=0}{\overset{D=1}{\gtrless}} \eta$$

After some math we have arrived at

$$-9x^2 + 76x \underset{D=0}{\overset{D=1}{\gtrless}} 156 + 10\log(\frac{\eta}{\sqrt{10}})$$

We can rewrite the decider using a new threshold $\eta' = 156 + 10\log(\frac{\eta}{\sqrt{10}})$

$$-9x^2 + 76x \underset{D=0}{\overset{D=1}{\gtrless}} \eta'$$

Notice a quadratic decider involves two thresholds on the observation!!

# Computing $P_{\mathrm{FA}}(\eta)$ and $P_{\mathrm{D}}(\eta)$

1. Selecting $\eta$ determines $\eta'(\eta) = 156 + 10\log(\frac{\eta}{\sqrt{10}})$
2. $\eta'(\eta)$ determines two thresholds on the observation

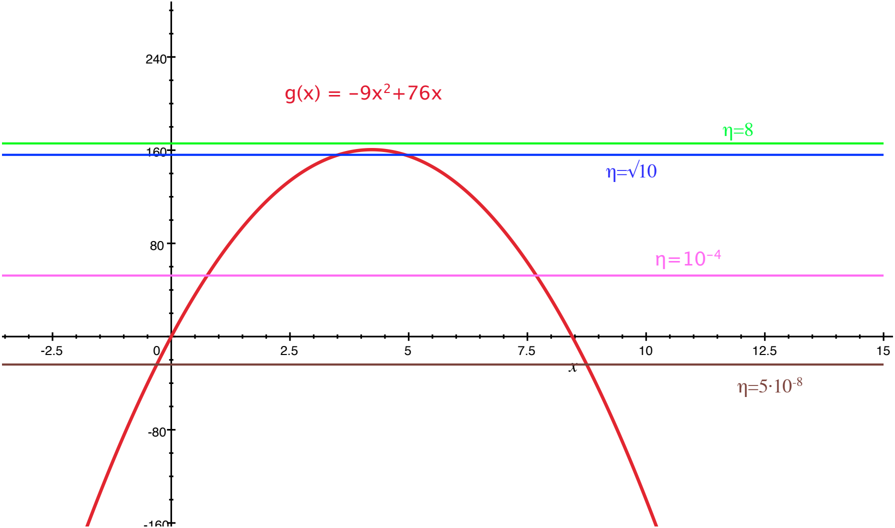
$$x_1(\eta') = \frac{-76 - \sqrt{76^2 + 4(-9)\eta'}}{2(-9)}$$

$$x_2(\eta') = \frac{-76 + \sqrt{76^2 + 4(-9)\eta'}}{2(-9)}$$

3. 
$$P_{\mathrm{FA}}(\eta) = \int_{x_1(\eta'(\eta))}^{x_2(\eta'(\eta))} p_{x|0}(x|0)dx$$

4. 
$$P_{\mathrm{D}}(\eta) = \int_{x_1(\eta'(\eta))}^{x_2(\eta'(\eta))} p_{x|1}(x|1)dx$$

# LRT



$g(x) = -9x^2+76x$

$\eta=8$

$\eta=\sqrt{10}$

$\eta=10^{-4}$

$\eta=5\cdot 10^{-8}$

240

160

80

-80

-160

-2.5

0

2.5

5

7.5

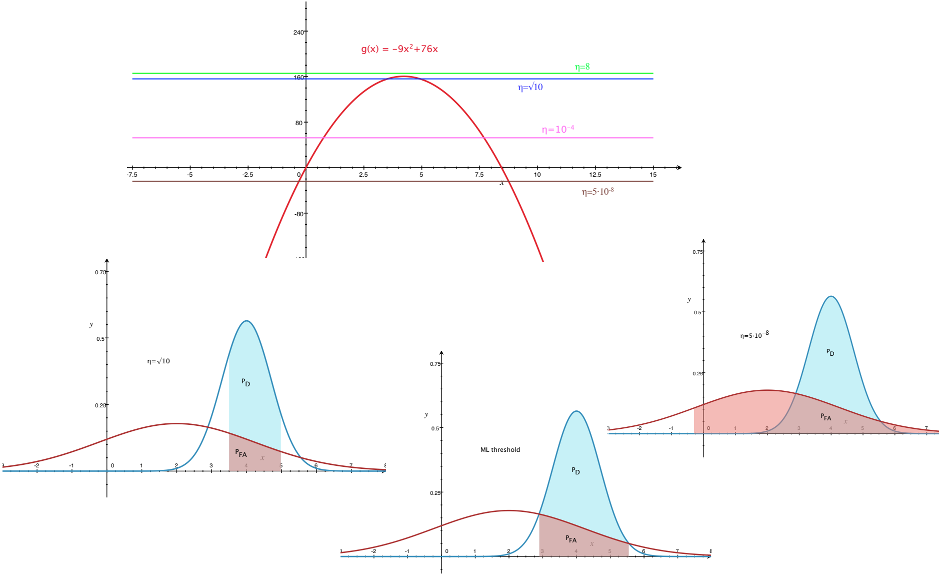
10

12.5

15

$x$

# LRT, $P_{\mathrm{FA}}$ y $P_{\mathrm{D}}$



# Complementary error function erfc$(x)$

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

$$P_{\text{FA}} = \frac{1}{\sqrt{10\pi}} \int_{x_1}^{x_2} \exp\left(-\frac{(x-2)^2}{10}\right) dx$$

$$= \frac{1}{\sqrt{10\pi}} \left( \int_{x_1}^{\infty} \exp\left(-\frac{(x-2)^2}{10}\right) dx - \int_{x_2}^{\infty} \exp\left(-\frac{(x-2)^2}{10}\right) dx \right)$$

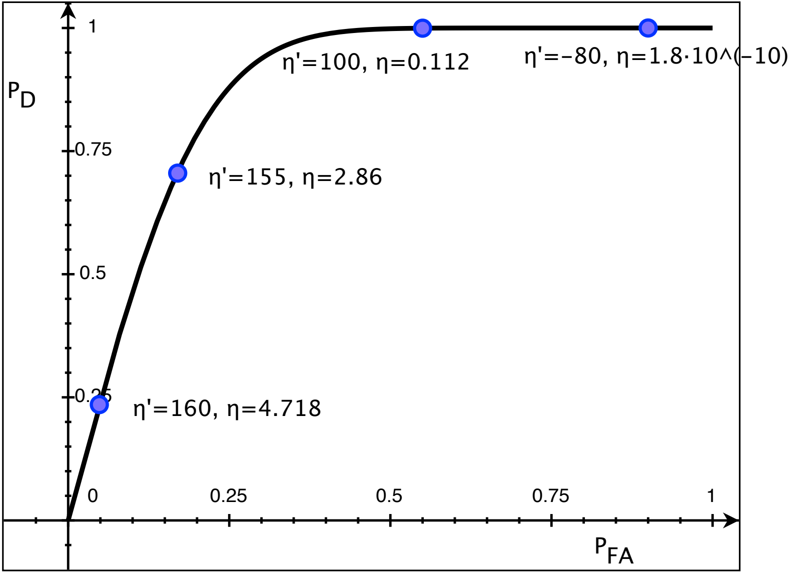
Change of variable $t = \frac{(x-2)}{\sqrt{10}}$

$$P_{\text{FA}} = \frac{1}{\sqrt{10\pi}} \left( \int_{\frac{(x_1-2)}{\sqrt{10}}}^{\infty} \exp(-t^2)\sqrt{10}dt - \int_{\frac{(x_2-2)}{\sqrt{10}}}^{\infty} \exp(-t^2)\sqrt{10}dt \right)$$

$$P_{\text{FA}}(\eta) = 0.5 \left[ \text{erfc}\left(\frac{(x_1(\eta)-2)}{\sqrt{10}}\right) - \text{erfc}\left(\frac{(x_2(\eta)-2)}{\sqrt{10}}\right) \right]$$

Analogously we can compute $P_{\text{D}}(\eta) = 0.5\left[\text{erfc}(x_1(\eta) - 4) - \text{erfc}(x_2(\eta) - 4)\right]$

# ROC curve


1
0.75
0.5
0.25
0
$P_D$
0.25
0.5
0.75
1
$P_{FA}$
η'=100, η=0.112
η'=−80, η=1.8·10^(−10)
η'=155, η=2.86
η'=160, η=4.718

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# ROC of a generic discriminant function

Up to now we have introduced the ROC curve as a means to evaluate the performance of an LRT.

However, in essence, to compute a ROC we only need $P_{\mathrm{FA}}$ and $P_{\mathrm{D}}$. And these two quantities can be calculated for any given binary classifier.

Therefore the ROC curve can be drawn for any binary classifier even if its discriminant function does not arise from an LRT:

- Express the classifier as a discriminant function of the observations $f(\mathbf{x})$ and a threshold $\eta$
- Express $P_{\mathrm{FA}}$ and $P_{\mathrm{D}}$ as functions of $\eta$
- Give values to $\eta$ and draw $P_{\mathrm{FA}}(\eta)$ vs. $P_{\mathrm{D}}(\eta)$

# ROC curves to compare the capabilities of discriminant functions

Each discriminant function $f(\mathbf{x})$ can be represented by its associated ROC curve (the one that can be drawn varying $\eta$ freely).

A perfect classifier would be able to perform in the ROC curve point $(P_{\text{FA}} = 0.0, P_{\text{D}} = 1.0)$. That is, without errors.

A given discriminant function is able implement a perfect classifier as long as its corresponding ROC curve includes the point $(P_{\text{FA}} = 0.0, P_{\text{D}} = 1.0)$.

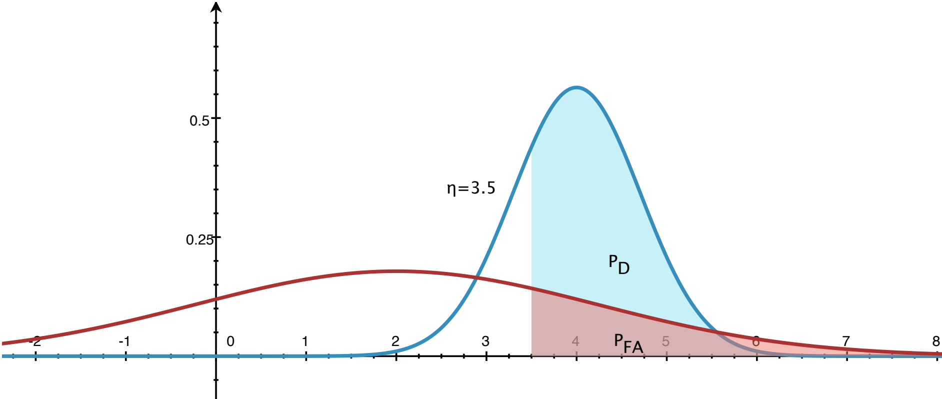
We can use the distance between a ROC curve and point $(P_{\text{FA}} = 0.0, P_{\text{D}} = 1.0)$ as quality measure for a discriminant function (the closer the ROC curve is from $(P_{\text{FA}} = 0.0, P_{\text{D}} = 1.0)$ the better the classifier).

Remember than the ROC curve corresponding to the LRT includes the Bayes classifier. Since the Bayes classifier is optimum, the LRT ROC curve imposes a bound on the performance achievable in a binary problem: there can't be ROC curves between the LRT ROC curve and $(P_{\text{FA}} = 0.0, P_{\text{D}} = 1.0)$

# Example: Single threshold classifier in the previous Gaussian case

Let's analyse the performance of a classifier consisting in a threshold on the observations on the previous problem. The classifier is

$$x \underset{D=0}{\overset{D=1}{\gtrless}} \mu$$



# Computing $P_{\text{FA}}(\eta)$ and $P_{\text{D}}(\eta)$

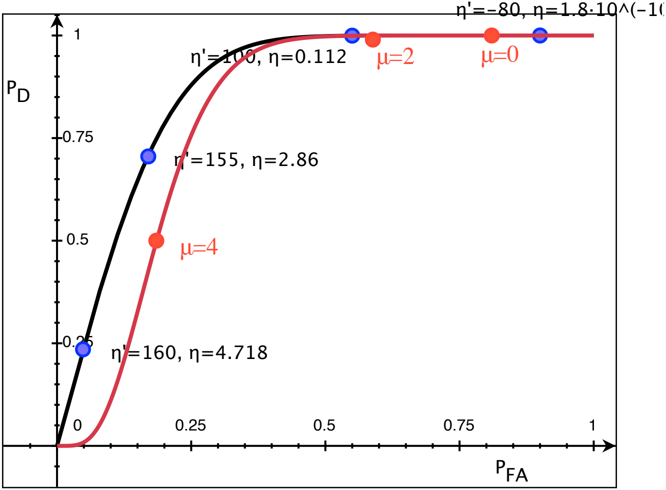
1. $\mu$ determines two decision regions with a single threshold
2. $$P_{\text{FA}}(\mu) = \int_{\mu}^{\infty} p_{x|0}(x|0)dx$$
3. $$P_{\text{D}}(\mu) = \int_{\mu}^{\infty} p_{x|1}(x|1)dx$$

Using $\text{erfc}(x)$

$$P_{\text{FA}}(\mu) = 0.5\text{erfc}\left(\frac{\mu - 2}{\sqrt{10}}\right)$$

$$P_{\text{D}}(\mu) = 0.5\text{erfc}\left(\mu - 4\right)$$

# ROC for the single threshold classifier



Black: ROC of the LRT. Red: ROC of the single threshold classifier

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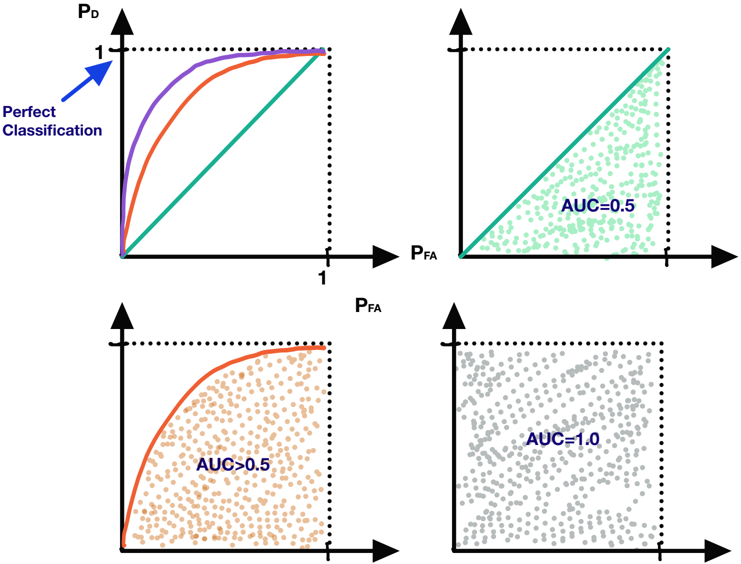
# Area under the ROC curve

We can asses the quality of a binary classifier by evaluating the distance between its ROC curve and the **perfect performance** working point: $(P_{\mathrm{FA}} = 0, P_D = 1)$
However, the quantification of this feature can result tricky.
An alternative is to evaluate the **Area under the ROC curve** as proxy to the quality of a detector:

- It's is bounded between 1 (perfect classification) and 0 (misguided classifier)
- Notice however the worse possible case is when AUC=0.5: A classifier that is *always wrong* is as perfect as a classifier that is *always right*, you just need to negate its output.

# AUC for binary classifiers



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# Sensitivity and Specificity

## Sensitivity

Probability of detecting correctly the presence of a target

$$\text{Sensitivity} \ = P\{D=1|H=1\} \int_{\eta}^{\infty} p_{\Lambda|H}(\lambda|H=1)d\lambda$$

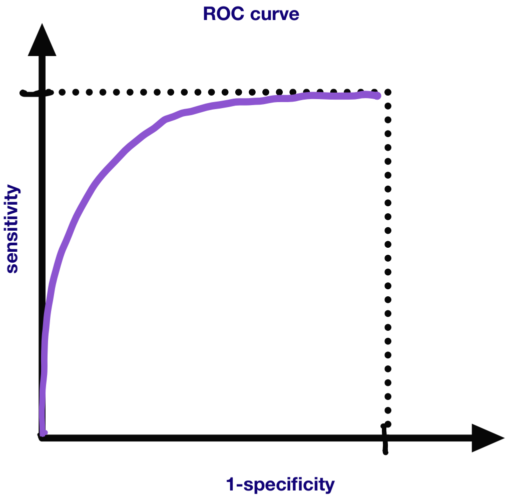
## Specificity

Probability of detecting correctly the absence of a target

$$\text{Specificity} \ = P\{D=0|H=0\} \int_{-\infty}^{\eta} p_{\Lambda|H}(\lambda|H=0)d\lambda$$

# Sensitivity and Specificity and ROC curve

- Sensitivity is the Probability of Detection
- Specificity $= 1 - P_{\text{FA}}$



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# Neyman-Pearson Detectors

In some cases the importance of falses alarms is so great that the probability of false alarm becomes a critical criterion in the design of the classifier

## Neyman-Pearson

$$\phi^* = \arg\max_{\phi}\{P_{\mathrm{D}}\} \text{ subject to } P_{\mathrm{FA}} \leq \alpha$$

- Impose a bound on $P_{\mathrm{FA}}$.
- Maximize $P_{\mathrm{D}}$ but guaranteeing that $P_{\mathrm{FA}}$ stays below the bound.

Procedure to determine an LRT Neyman Pearson Detector:

1. Obtain the LRT as a function of $\eta$
2. Express $P_{\mathrm{FA}}$ as a function of $\eta$
3. Make $P_{\mathrm{FA}} = \alpha$ and solve for $\eta$
4. Plug that value of $\eta$ in the LRT

# Example

Determine the LRT Neyman-Pearson classifier with $P_{\mathrm{FA}} \le \alpha = 0.1$ in a binary problem with likelihoods

$$p_{X|H}(x|1) = 2x, \qquad 0 \le x \le 1$$
$$p_{X|H}(x|0) = 2(1-x), \qquad 0 \le x \le 1$$

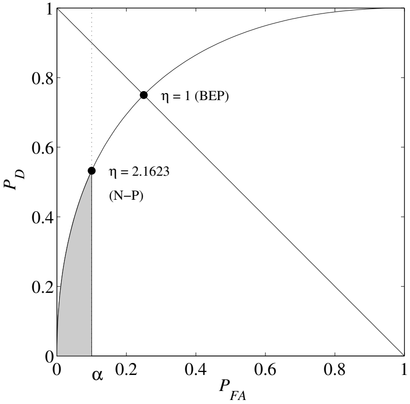
The LRT is $\dfrac{x}{1-x} \underset{D=0}{\overset{D=1}{\gtrless}} \eta$

with $P_{\mathrm{FA}}(\eta) = \dfrac{1}{(1+\eta)^2}$

Making $P_{\mathrm{FA}} = \alpha = 0.1$:

$$\eta = \frac{1}{\sqrt{P_{\mathrm{FA}}}} - 1 \approx 2.1623$$

and $P_{\mathrm{D}} = 1 - \dfrac{\eta^2}{(1+\eta)^2} \approx 0.53$



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# Minimax strategy

Minimax is a broadly used strategy to take decisions. It takes the decision that minimizes the effects of the worst case scenario.
Example: Imagine the following cost policy (no likelihoods available)
(remember $c_{dh}$ is the cost of deciding $d$ when the right hypothesis is $h$)

$$C = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 1 & 5 \\ 6 & 1 & 0 \end{bmatrix}$$

The worst case scenarios for each decision are:

$$\begin{bmatrix} c_{02} = 4 \\ c_{12} = 5 \\ c_{03} = 6 \end{bmatrix}$$

So the minimax decision is $D = 0$. Notice that $D = 2$ would eventually lead to $c = 0$ if $H = 2$, but to $c = 6$ if the worst case $H = 0$ happens

# Minimax classifier

The extension of the minimax strategy to a classification problem defined in terms of $P_{\mathrm{FA}}$ and $P_{\mathrm{D}}$ becomes

### Minimax binary classifier

Choose $\eta$ so that $P_{\mathrm{FA}} = P_{\mathrm{M}} = 1 - P_{\mathrm{D}}$

This point is the intersection of the ROC curve with the line $P_{\mathrm{FA}} = 1 - P_{\mathrm{D}}$. We term it **Break Even Point** (BEP).

It is easy to show that for this classifier $P_e = P_{\mathrm{FA}} = P_{\mathrm{M}}$, what means that $P_e$ in this case is independent of the prior probabilities of the hypotheses. Therefore this classifier is robust to changes in the prior probabilities of the classes.