Analytic Estimation

Modern Theory of Detection and Estimation. Block-1: Estimation

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Estimation problem definition

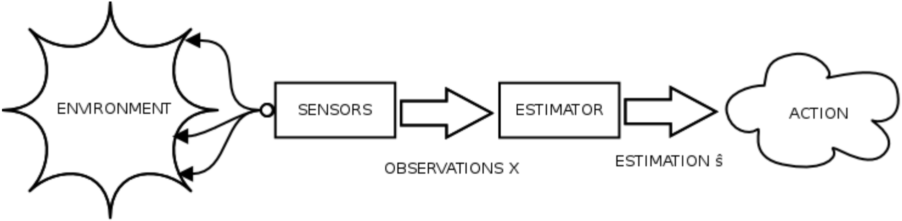
Goal in a estimation problem

You receive a scalar observation (x) or **a vector observation** ($\mathbf{x}$), and your task is to find out (**estimate**) the value of a real number s. The variable you want to find out can be either deterministic or random.

Examples:

- Estimate the number of stocks of a certain instrument that will be traded tomorrow.
- Estimate the daily energy consumption in a building as a function of the weather, season of year, day of week.
- Estimate the price of a house given the surface, number of rooms, size of the yard, distance to the centre of the town, etc.
- Guess the foot length of people based on their height.

Estimation problem setup



- Sensors sample information about the environment and construct observations $\mathbf{x}^k$, $k = 1, \ldots, l$.
- Estimator constructs estimated variable $\hat{s}$ as a function of the observations. There **must be** a statistical relationship between the observations and the variable we want to estimate, s.
- Then we use the estimation $\hat{s}$ instead of s to take actions. Remember we **never** have access to s (that's why we need to estimate it!!)

Notation

- $\mathbf{X}$: **observations**. Random in the analytical case setup. Could be scalars or vectors.
- S: **desired output**. This is the variable we want to estimate. It can be random (S) or deterministic (s). It can be a vector or an scalar, most of the time this course it will be an scalar.
- $\hat{S}$: **estimator**. $\hat{S} = f(\mathbf{X})$ is our approximation to the value of S given the observation. In the analytic estimation case $\hat{S}$ is random since it is a deterministic function ($f()$) of a random variable ($\mathbf{X}$), therefore its pdf can be obtained applying random variable transformation to the pdf of the observations.
- $p_{X,S}(\mathbf{x}, s)$: **statistical relationship between observations and estimated variable**. We can estimate S using $\mathbf{X}$ as input because they are related. We express this relationship as a joint pdf.

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Random variable vs. Deterministic parameter

The variable we want to estimate can be either random S, or deterministic s. Examples:

- Deterministic: I've thought the mean of a Gaussian distribution m. I draw K samples from the distribution and ask you to estimate m. Note that m is unknown but does not follow any prior distribution $p(m)$; m is a fixed [unknown] constant.
- Random: I pick at random one student from the class, I tell you his/her foot length ($X = x$) and you have to guess his/her height ($S = s$). S is random because I can pick any student in the class.

Analytic estimation vs. machine estimation

- **Machine estimation or regression**: the problem is defined in terms of a set of examples, $\{\mathbf{x}^k, s^k\}_{k=1}^l$. We use the examples to fit an estimation model, for example a linear regressor:

  $$\hat{s} = f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + w_0$$

  we learn $\mathbf{w}$ and w_0 with the training set $\{\mathbf{x}^k, s^k\}_{k=1}^l$.
  Notice in the machine estimation setting the training observations $\{\mathbf{x}^k, s^k\}_{k=1}^l$ are not random (we do know them). The test observations can be considered random.
- **Analytic estimation**: the problem is defined in terms of a complete statistical characterisation: there exists a joint pdf $p_{\mathbf{X},S}(\mathbf{x}, s)$ and we know how to calculate it for any pair $(\mathbf{x}, s)$. In this setting $\mathbf{X}$ is random.
- **Semi-analytic estimation**: The problem is defined in terms of a data set, as in the regression case. We do not use this data to fit a regression model but to **learn the statistics**, i.e., estimate $p_{\mathbf{X},S}(\mathbf{x}, s)$. Once we have an estimation of the joint pdf (or any other useful pdf) we can apply analytic estimation.

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Likelihood

We rely on different pdfs that capture the relationship between $\mathbf{x}$ and s.

Likelihood

$$p_{\mathbf{X}|S}(\mathbf{x}|s)$$

It's the key pdf. You will have a likelihood in both random and deterministic variable estimation cases.

The likelihood models the generation of observations for every possible value of the target variable S. If S changes, we get different statistics in the observations. That is why we can guess the value of S.

If s is deterministic there is no point in conditioning in the value of s since it is always the same. Anyway we use $p_{\mathbf{X}|s}(\mathbf{x}|s)$ to unify notations.

S is random

If S is a random variable we have three other pdfs:

- **Prior** or marginal of S: $p_S(s)$ gives information about how are the values of S distributed without access to the observations.
- **Joint distribution** of S and $\mathbf{X}$: $p_{\mathbf{X},S}(\mathbf{x}, s)$
- **Posterior** of S given the observation $\mathbf{X} = \mathbf{x}$: $p_{S|\mathbf{X}}(s|\mathbf{x})$ gives the values of S that concentrate a higher probability density for each particular observed value of $\mathbf{X}$. It is a key pdf in the design of estimators since to know $\mathbf{X} = \mathbf{x}$ **narrows** the uncertainty about S expressed in the prior (I mean $p_{S|\mathbf{X}}(s|\mathbf{x})$ as a function of s is "narrower" than $P_S(s)$).

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Cost Functions

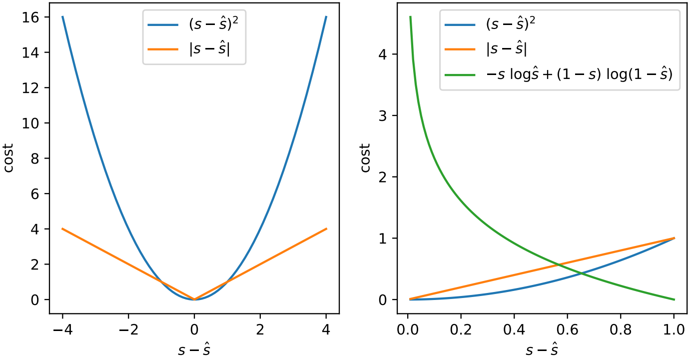
- An estimator is a function of the observations: $\hat{s} = f(\mathbf{x})$.
- The of design an estimator involves an optimisation to pick the best estimation function $f^*()$ from within a family or set of candidates.
- We need a criterion to select this best estimation function from the set of potential candidates. This criterion is the **cost function**.

Cost function

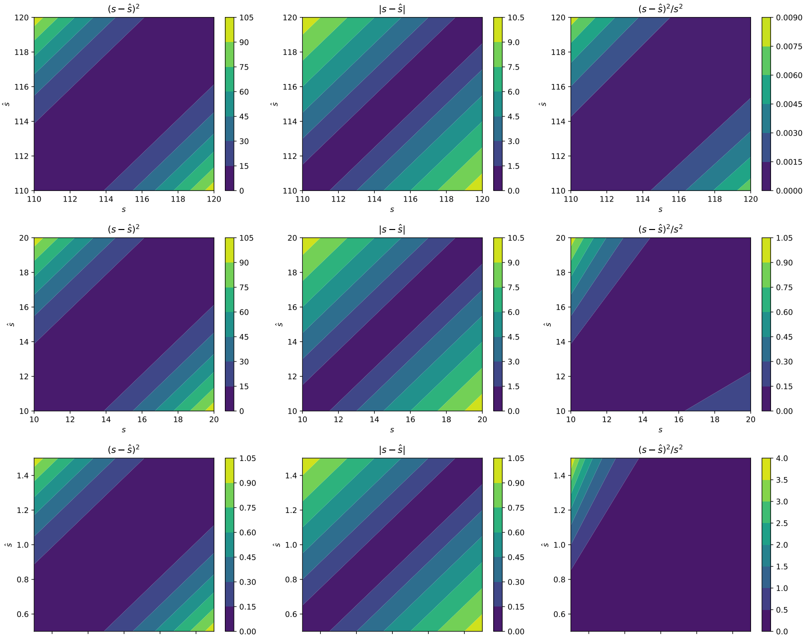
Function of the estimator and of the estimated variable: $C(S, \hat{S})$. It compares both quantities and determines the penalty in which we incur if we approximate $S = s$ with $\hat{S} = \hat{s}$. In most cases exact estimations ($\hat{s} = s$) yield zero cost: $C(s, \hat{s} = s) = 0$. The cost is always positive $C(S, \hat{S}) \geq 0$.

Examples of cost function

- Quadratic Cost: $c(e) = e^2$.
- Absolute Value of the error: $c(e) = |e|$.
- Relative Square Error: $c(s, \hat{s}) = \frac{(s-\hat{s})^2}{s^2}$
- Log loss: $c(s, \hat{s}) = -s \ln \hat{s} - (1-s) \ln(1-\hat{s})$, for $s \in \{0, 1\}$, and $\hat{s} \in [0, 1]$



Examples of cost function



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Expected Cost

Since we plan to use the estimator a large number of times, we are more interested in the expected cost, that takes into account all the possible values of S and $\hat{S}$.

Expected cost

$$\mathbb{E}\{c(S,\hat{S})\} = \int_{\mathbf{x}}\int_{s} c(s,\hat{s}(\mathbf{x}))\mathbf{p}_{\mathbf{S},\mathbf{X}}(\mathbf{s},\mathbf{x})\mathbf{dsdx}$$

Notice that $\hat{S} = \hat{S}(\mathbf{X})$ is a deterministic function of $\mathbf{X}$, therefore the statistics of $\hat{S}$ can be univocally defined in terms of the statistics of $\mathbf{X}$ (transformation of random variable).

Example

Let X be a noisy observation of S, such that

$$X = S + R$$

with S a random variable with mean 0 and variance 1, and R a Gaussian random variable, independent of S, with mean 0 and variance v. Consider the estimator $\hat{S} = X$, the expected quadratic cost is

Example

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$$\mathbb{E}\{(S - \hat{S})^2\} = \mathbb{E}\{(S - X)^2\} = \mathbb{E}\{R^2\} = v$$

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$$\mathbb{E}\{(S - \hat{S})^2\} = \mathbb{E}\{(S - X)^2\} = \mathbb{E}\{R^2\} = v$$

The expected absolute cost is

$$\begin{aligned}\mathbb{E}\{|S - \hat{S}|\} = \mathbb{E}\{|R|\} &= \int_{-\infty}^{\infty} |r| \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{r^2}{2v}\right) dr \\ &= 2\int_{0}^{\infty} r \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{r^2}{2v}\right) dr = \sqrt{\frac{2v}{\pi}}\end{aligned}$$

Example: Calculation of Expected Quadratic Cost

Two random variables S and X follow a joint pdf

$$p_{S,X}(s,x) = \begin{cases} \frac{1}{x}, & 0 < s < x, \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Consider two estimators $\hat{S}_1 = \frac{1}{2}X$ and $\hat{S}_2 = X$. Which is the best estimator from the point of view of minimising the quadratic cost?

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Consider two estimators $\hat{S}_1 = \frac{1}{2}X$ and $\hat{S}_2 = X$. Which is the best estimator from the point of view of minimising the quadratic cost? To find it out, we'll compute the mean square error of both estimators.

For a general w,

$$\begin{aligned} \mathbb{E}\{(S - wX)^2\} &= \int_0^1 \int_0^x (s - wx)^2 p_{S,X}(s,x) ds dx \\ &= \int_0^1 \int_0^x (s - wx)^2 \frac{1}{x} ds dx \\ &= \int_0^1 \left(\frac{1}{3} - w + w^2 \right) x^2 dx \\ &= \frac{1}{3} \left(\frac{1}{3} - w + w^2 \right) \end{aligned}$$

Example: Calculation of Expected Quadratic Cost (ctd)

If $w = 1/2$

$$\mathbb{E}\{(S - \hat{S}_1)^2\} = \mathbb{E}\{(S - \frac{1}{2}X)^2\} = \frac{1}{3}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{36} \qquad (1)$$

Alternatively, if $w = 1$

$$\mathbb{E}\{(S - \hat{S}_2)^2\} = \mathbb{E}\{(S - X)^2\} = \frac{1}{3}\left(\frac{1}{3} - 1 + 1\right) = \frac{1}{9} \qquad (2)$$

Therefore, from the point of view of the mean square error, $\hat{S}_1$ is better estimator than $\hat{S}_2$.

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Bayesian estimation

Given a cost function and a statistical relationship (joint distribution) between observations and estimated variable, the Bayesian Estimator is the one that minimizes the expected cost.

$$\hat{S}_{\text{Bayes}} = \arg\min_{\hat{S}} \mathbb{E}\{c(S,\hat{S})\} = \arg\min_{\hat{S}} \int_{\mathbf{x}} \int_{s} c(s,\hat{s}) p_{S,\mathbf{X}}(s,\mathbf{x}) ds d\mathbf{x}$$

$$= \arg\min_{\hat{S}} \int_{\mathbf{x}} \int_{s} c(s,\hat{s}) p_{S|\mathbf{X}}(s|\mathbf{x}) ds \; p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} =$$

$$\arg\min_{\hat{S}} \int_{\mathbf{x}} \mathbb{E}\{c(S,\hat{S})|\mathbf{X}\} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

Bayesian estimation

If we look carefully into the last integral, we see that $\mathbb{E}\{c(S, \hat{S})|\mathbf{X}\}$ is multiplied by a positive function $(p_{\mathbf{X}}(\mathbf{x}))$. Therefore, to minimize the expected global cost is to minimise the integral and it is achieved minimising $\mathbb{E}\{c(S, \hat{S})|\mathbf{X}\}$. Thus a first result is

$$\hat{S}_{\text{Bayes}} = \arg\min_{\hat{S}} \mathbb{E}\{c(S, \hat{S})|\mathbf{X}\}$$

In summary, to compute a Bayesian Estimator we need to take two steps:

1. Select a cost function $c(S, \hat{S})$.
2. Minimise $\mathbb{E}\{c(S, \hat{S})|\mathbf{X}\}$

Example: Find Bayesian estimation with Quadratic Cost

Two random variables S and X follow a joint pdf

$$p_{S,X}(s,x) = \begin{cases} \frac{1}{x}, & 0 < s < x, \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the Bayesian estimator that minimizes the quadratic cost

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Find the Bayesian estimator that minimizes the quadratic cost

First, find the posterior

$$p_{S|X}(s|x) = \frac{p_{S,X}(s,x)}{p_X(x)}$$

$$p_{S|X}(s|x) = \begin{cases} \frac{1}{x}, & 0 < s < x, \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Where you should apply that

$$p_X(x) = \int_{(s)} p_{S,X}(s,x)ds = \int_0^x \frac{1}{x}ds = 1$$

Example: Find Bayesian estimation with Quadratic Cost

The expected cost given x:

Example: Find Bayesian estimation with Quadratic Cost

The expected cost given x:

$$\begin{aligned}
\mathbb{E}\{c(S,\hat{S})|\mathbf{X}=\mathbf{x}\} &= \mathbb{E}\{(S-\widehat{s})^2)|\mathbf{X}=\mathbf{x}\} \\
&= \int_{(s)} (s-\widehat{s})^2 p_{S|X}(s|x)ds \\
&= \frac{1}{x}\int_0^x (s-\widehat{s})^2 ds \\
&= \frac{1}{x}\left(\frac{(x-\widehat{s})^3}{3}+\frac{\widehat{s}^3}{3}\right) \\
&= \frac{1}{3}x^2-\widehat{s}x+\widehat{s}^2
\end{aligned}$$

The expected cost given x, as a function of $\widehat{s}$, is a 2nd degree polynomial. We can minimize it by equating its derivative to 0.

Example: Find Bayesian estimation with Quadratic Cost

The expected cost given x:

$$\mathbb{E}\{c(S,\hat{S})|\mathbf{X}=\mathbf{x}\} = \mathbb{E}\{(S-\widehat{s})^2)|\mathbf{X}=\mathbf{x}\}$$
$$= \frac{1}{3}x^2 - \widehat{s}x + \widehat{s}^2$$

The expected cost given x, as a function of $\widehat{s}$, is a 2nd degree polynomial. We can minimize it by equating its derivative to 0.

$$\frac{d}{d\widehat{s}}\mathbb{E}\{c(S,\hat{S})|\mathbf{X}=\mathbf{x}\} = -x + 2\widehat{s} = 0$$

And the solution is:

$$\widehat{s}_{\text{Bayes}} = \frac{1}{2}x$$

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Minimum Mean Square Error (MMSE) Estimator

This is the Bayesian Estimator when the cost function is the quadratic error: $c(s,\hat{s}) = e^2 = (s-\hat{s})^2$. Therefore the estimator is the solution to the optimisation problem given by

$$\hat{s}_{\mathrm{MMSE}} = \arg\min \hat{s}\mathbb{E}\{c(S,\hat{s})|\mathbf{X}=\mathbf{x}\} = \arg\min \hat{s} \int_s (s-\hat{s})^2 p_{S|\mathbf{X}}(s|\mathbf{x})ds$$

Minimum Mean Square Error (MMSE) Estimator

To minimise, we take the derivative with respect to $\hat{s}$ and make it equal to zero.

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} \int_s (s-\hat{s})^2 p_{S|\mathbf{X}}(s|\mathbf{x}) ds = \int_s \frac{\partial}{\partial \hat{s}} (s-\hat{s})^2 p_{S|\mathbf{X}}(s|\mathbf{x}) ds$$

$$= \int_{-\infty}^{\infty} -2(s-\hat{s}) p_{S|\mathbf{X}}(s|\mathbf{x}) ds$$

In the optimum the above integral is equal to zero:

$$\int_{-\infty}^{\infty} -2(s-\hat{s}_{\text{Bayes}}) p_{S|\mathbf{X}}(s|\mathbf{x}) ds = 0 \Rightarrow$$

$$\hat{s}_{\text{Bayes}} = \mathbb{E}\{s|\mathbf{X}\}$$

The intuition behind this result is that the knowledge of $\mathbf{X}$ determines as optimum estimator the one corresponding to choosing the expected value of S for that particular observed value of $\mathbf{X}$.

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Expected absolute cost

The minimum absolute error estimator corresponds to the cost function $c(e) = |e| = |s - \widehat{s}|$. Therefore:

$$\begin{aligned}\widehat{s}_{\mathrm{MAD}} &= \arg\min_{\widehat{s}} \mathbb{E}\{c(S, \widehat{s})|\mathbf{X} = \mathbf{x}\} \\ &= \arg\min_{\widehat{s}} \int_s |s - \widehat{s}| p_{S|\mathbf{X}}(s|\mathbf{x}) ds\end{aligned}$$

$$\begin{aligned}\mathbb{E}\{|S - \widehat{s})|\mathbf{X} = \mathbf{x}\} &= \int_{-\infty}^{\widehat{s}} (\widehat{s} - s) p_{S|\mathbf{X}}(s|\mathbf{x}) ds + \int_{\widehat{s}}^{\infty} (s - \widehat{s}) p_{S|\mathbf{X}}(s|\mathbf{x}) ds \\ &= \widehat{s}\left[\int_{-\infty}^{\widehat{s}} p_{S|\mathbf{X}}(s|\mathbf{x}) ds - \int_{\widehat{s}}^{\infty} p_{S|\mathbf{X}}(s|\mathbf{x}) ds\right] + \\ &+ \int_{\widehat{s}}^{\infty} s\, p_{S|\mathbf{X}}(s|\mathbf{x}) ds - \int_{-\infty}^{\widehat{s}} s\, p_{S|\mathbf{X}}(s|\mathbf{x}) ds\end{aligned}$$

Minimization of the average cost a posteriori

The Fundamental Theorem of Calculus enables to get the derivative of the average cost a posteriori as:

$$\frac{d\mathbb{E}\{|S-\widehat{s}||\mathbf{X}=\mathbf{x}\}}{d\widehat{s}} =$$

$$\frac{d}{d\widehat{s}}\left(\widehat{s}\left[\int_{-\infty}^{\widehat{s}} p_{S|\mathbf{X}}(s|\mathbf{x})ds - \int_{\widehat{s}}^{\infty} p_{S|\mathbf{X}}(s|\mathbf{x})ds\right] + \int_{\widehat{s}}^{\infty} s\, p_{S|\mathbf{X}}(s|\mathbf{x})ds - \int_{-\infty}^{\widehat{s}} s\, p_{S|\mathbf{X}}(s|\mathbf{x})ds\right)$$

and then make

$$\frac{d\mathbb{E}\{|S-\widehat{s}||\mathbf{X}=\mathbf{x}\}}{d\widehat{s}} = 0$$

MAD estimator

MAD estimator

$$\widehat{s}_{\text{MAD}}(\mathbf{x}) = \text{median}\{S|\mathbf{X} = \mathbf{x}\}$$

Note: It is usually computed with the following expression:

$$\int_{-\infty}^{\widehat{s}_{\text{MAD}}} p_{S|\mathbf{X}}(s|\mathbf{x})ds = \int_{\widehat{s}_{\text{MAD}}}^{\infty} p_{S|\mathbf{X}}(s|\mathbf{x})ds = \frac{1}{2}$$

Example: Find Bayesian estimation with Absolute Error Cost

Two random variables S and X follow a joint pdf

$$p_{S,X}(s,x) = \begin{cases} \frac{1}{x}, & 0 < s < x, \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the Bayesian estimator that minimizes the absolute error cost
From previous example retrieve the posterior

$$p_{S|X}(s|x) = \begin{cases} \frac{1}{x}, & 0 < s < x, \quad 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Example: Find Bayesian estimation with Absolute Error Cost

$$\widehat{s}_{\text{MAD}} \Rightarrow \int_0^{\widehat{s}_{\text{MAD}}} p_{S|\mathbf{x}}(s|\mathbf{x})ds = \frac{1}{2} = \int_{\widehat{s}_{\text{MAD}}}^{x} p_{S|\mathbf{x}}(s|\mathbf{x})ds$$

$$\int_0^{\widehat{s}_{\text{MAD}}} \frac{1}{x}ds = \frac{1}{2} \quad \Rightarrow \quad \widehat{s}_{\text{MAD}}(x) = \frac{x}{2}$$

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MAP estimator

MAP Estimator

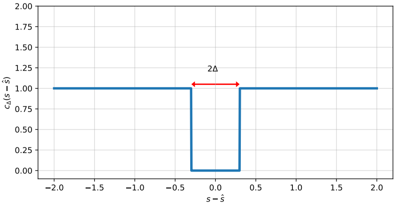
$$\begin{aligned}\widehat{s}_{\mathrm{MAP}}(\mathbf{x}) &= \arg\max_{s} p_{S|\mathbf{X}}(s|\mathbf{x}) \\ &= \arg\max_{s} \ln\left[p_{S|\mathbf{X}}(s|\mathbf{x})\right]\end{aligned}$$

When the distribution a posteiori presents several global maxima, the MAP estimator is not unique!

MAP estimator from a Bayesian point of view

Strictly speaking, the MAP estimator is not Bayesian because it is not minimizing any expected cost. However, if we consider the following cost function:

$$c_\Delta(s - \widehat{s}) = \begin{cases} 1, & \text{if } |s - \widehat{s}| > \Delta \\ 0, & \text{if } |s - \widehat{s}| < \Delta \end{cases}$$



and $\widehat{s}_\Delta$ as the bayesian estimator corresponding to c_Δ:

$$\widehat{s}_{\text{MAP}} = \lim_{\Delta \to 0} \widehat{s}_\Delta$$

Therefore the MAP estimator can be considered a limit case in a family of bayesian estimators.

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Maximum Likelihood (ML) Estimator

ML estimator

$$\hat{s}_{ML} = \arg\max_s p_{\mathbf{x}|s}(\mathbf{x}|s) = \arg\max_s \ln(p_{\mathbf{x}|s}(\mathbf{x}|s))$$

- No associated cost function, therefore **in general the ML is not a Bayesian estimator**. It could happen that the $\hat{s}_{ML}$ coincides with a Bayesian estimator, but by definition the $\hat{s}_{ML}$ does not optimise a cost function
- $\hat{s}_{ML}$ does not take into account the prior distribution $p_S(s)$. Therefore it can be used when s is a **deterministic parameter**.

Relationship between the MAP estimator and the ML estimator

$$\hat{s}_{MAP} = \arg\max_s p_{S|\mathbf{X}}(s|\mathbf{x}) = \arg\max_s \frac{p_{\mathbf{X}|S}(\mathbf{x}|s)p_S(s)}{p_{\mathbf{X}}(\mathbf{x})}$$

If we don't know $p_S(s)$, we can assume it is **uniform** (no preference for any particular value of s). Therefore

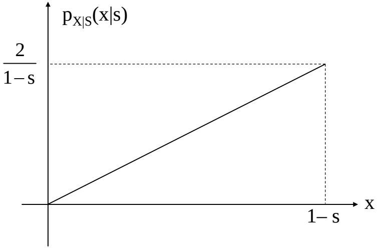
$$\arg\max_s \frac{p_{\mathbf{X}|S}(\mathbf{x}|s)p_S(s)}{p_{\mathbf{X}}(\mathbf{x})} \sim \arg\max_s p_{\mathbf{X}|S}(\mathbf{x}|s) = \hat{s}_{ML}$$

Example: ML estimation of a random variable

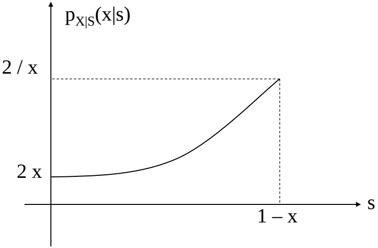
One wishes to estimate the value of a random variable S from the observed value of another variable X, statistically related to S through:

$$p_{X|S}(x|s) = \frac{2x}{(1-s)^2}, \;\; 0 < x < 1-s, \;\; 0 < s < 1$$

We need to maximise $p_{X|S}(x|s)$ with respect to s, (not to x)



$p_{X|S}(x|s)$ as a function of x



$p_{X|S}(x|s)$ as a function of s

Solution: $\hat{s}_{ML} = 1 - X$

Example: Estimation of the parameters of a Gaussian

We have access to a collection of l data samples $\{X^{(k)}\}_{k=1}^{l}$ drawn **independently** from a Gaussian pdf. Construct estimators for the mean m and the variance v of the Gaussian.

- The likelihood of each observation given m and v is

$$p_X(x) = p_{X|m,v}(x|m,v) = \frac{1}{\sqrt{2\pi v}} \exp\left[-\frac{(x-m)^2}{2v}\right]$$

- Since we observe l samples, the **joint** pdf will be the product of the individual pdfs.

$$p_{\{X^{(k)}\}|m,v}(\{x^{(k)}\}|m,v) = \prod_{k=1}^{l} p_{X|m,v}(x^{(k)}|m,v)$$

$$= \frac{1}{(2\pi v)^{l/2}} \prod_{k=1}^{l} \exp\left[-\frac{(x^{(k)}-m)^2}{2v}\right]$$

Example: Estimation of the parameters of a Gaussian

- Taking logs and optimising

$$\hat{m}_{\text{ML}} = \frac{1}{l}\sum_{k=1}^{l} x^{(k)}$$

- Using $\hat{m}_{\text{ML}}$ in the expression for the estimator of the variance

$$\hat{v}_{\text{ML}} = \frac{1}{l}\sum_{k=1}^{l} (x^{(k)} - \hat{m}_{\text{ML}})^2$$