Review of Statistics

Block 0
Modern Theory of Detection and Estimation.

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1. Review of Probability and Random Variables
   - Probability
   - Random Variables
   - Conditional densities and distributions
   - Transformation of Random Variables

Probability measures

Ω : Sample space of a random experiment: {Set of possible outcomes}

$\omega_i \in \Omega$: outcome of a realization of the random experiment

Probability Measures

Positive measure P satisfying

- $P(A) \geq 0, \forall A \subseteq \Omega$
- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability

$$P(A|B) = \frac{P(A \cap B}{P(B)}, \qquad \text{assuming } P(B) > 0$$

Independence between A and B

$$P(A|B) = P(A)$$

$$P(A \cap B) = P(A) \cdot P(B)$$

Theorems

- Total Probability: $P(A|B)P(B) = P(A, B)$
- Bayes' Rule:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

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Random Variable

Mapping

$$X : \Omega \to \mathbb{R}$$

$$\mathbf{X} : \Omega \to \mathbb{R}^d$$

Characterization with **Probability distribution function** and **Probability density function**

Probability Distributions and densities

Probability Distribution Function

$P_X(x) = \text{Prob.}\{X \leq x\}$

Probability density function (pdf)

If $P_X(x)$ is differentiable: $P_X(x) = \int_{-\infty}^{\infty} p_X(z)dz$

- $p(x) \geq 0$, $\forall x$
- $\int p(x)dx = 1$
- $p(x) = \frac{dP(x)}{dx}$
- $p(x)dx = \text{Prob}\{x \leq X \leq x + dx\}$

Probability mass function

For discrete random variables

$$X : \Omega \to \{x_i\}, \text{contable set}$$

$p_X(x_i) = \text{Prob.}\{X = x_i\}$, role of a pdf
Distribution: $P_X(t) = \sum_{j:x_j<t} p_X(x_j)$

Expectations

$$\mathbb{E}\{g(X)\} = \int_X g(x)p_X(x)dx$$

Mean

$\mathbb{E}(X) = m_X$

Variance

$\{(X - m_X)^2\} = \mathrm{Var}X = \mathbb{E}\{X^2\} - m_X^2$

Discrete random variable

$\mathbb{E}\{X\} = \sum_i x_i p(X = x_i)$

Jointly distributed Random Variables

$$P_{XY}(x,y) = \text{Pob.}\{X \leq x \cap Y \leq y\}$$

Joint density

$$P_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} p(x',y')dx'dy' = \text{Pob.}\{X \leq x + dx \cap Y \leq y + dy\}$$

$$p_{X,Y}(x,y) \geq 0$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x',y')dx'dy' = 1$$

$$p_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} P_{X,Y}(x,y)$$

Marginals

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dy \qquad p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)dx$$

Random Vectors

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_d \end{bmatrix} : \Omega \to \mathbb{R}^d$$

Distribution of a random vector

Use the joint distribution over the components of the vector

$$P_{\mathbf{X}}(\mathbf{x}) = \text{Prob.}\{X_1 \leq x_1, X_2 \leq x_2 \ldots, X_d \leq x_d\}$$

Density of a random vector

Use the joint density over the components of the vector

$$P_{\mathbf{X}}(\mathbf{x}) = \int_{X_1 \leq x_1} \int_{X_2 \leq x_2} \cdots \int_{X_d \leq x_d} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}$$

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Conditional pdf

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

Statistical Independence

$p(x|y) = p(x) \Rightarrow p(x,y) = p(x)p(y)$

Uncorrelated

$\mathbb{E}\{XY\} = \mathbb{E}\{X\}\mathbb{E}\{Y\}$

Watch out!!

- Independence implies uncorrelated
- Uncorrelated DOES NOT imply independence

Theorems

Total Probability

$$p_X(x) = \int_{-\infty}^{\infty} p_{X|Y}(x|Y=y)p_Y(y)dy$$

Bayes' Rule

$$p_{X|Y}(x|Y=y) = \frac{p_{Y|X}(y|X=x)p_X(x)}{p_X(x)} = \frac{p_{Y|X}(y|X=x)p_X(x)}{\int_{-\infty}^{\infty} p_{X|Y}(x|Y=y)p_Y(y)dy}$$

Expectations

Conditional Expectation

$$\mathbb{E}\{g(X,Y)|M\} = \int_X \int_Y g(x,y) p_{X,Y|M}(x,y|M) dy$$

$$\mathbb{E}\{Y|X=x\} = \int_Y y p_{Y|X}(y|X=x) dy$$

Covariance

$$\text{Cov}\{X,Y\} = \mathbb{E}\{(X - \mathbb{E}\{X\})(Y - \mathbb{E}\{Y\})\}$$

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Transformation of Random Variables

Examples in the blackboard