

2.45. a) Show that if the response of an LTI system to  $x(t)$  is  $y(t)$ , then the response of  $x'(t) = \frac{dx(t)}{dt}$  is  $y'(t)$ . Do this in three ways:

i) Directly from the properties of linearity and time invariance, and  $x'(t) = \lim_{h \rightarrow 0} \frac{x(t) - x(t-h)}{h}$

$\uparrow$  S:  $x(t) \rightarrow y(t) = S(y(t))$

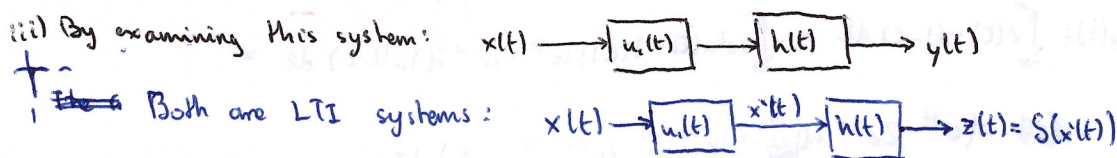
$$x'(t) = \lim_{h \rightarrow 0} \frac{x(t) - x(t-h)}{h} \rightarrow S\left(\lim_{h \rightarrow 0} \frac{x(t) - x(t-h)}{h}\right) = \lim_{h \rightarrow 0} S\left(\frac{x(t) - x(t-h)}{h}\right) \stackrel{\text{Thanks to linearity}}{=} \lim_{h \rightarrow 0} \frac{S(x(t)) - S(x(t-h))}{h} \stackrel{\text{Thanks to time invariance}}{=} \lim_{h \rightarrow 0} \frac{y(t) - y(t-h)}{h} = y'(t)$$

ii) By differentiating the convolution integral

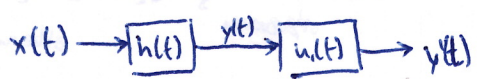
$\uparrow$  let  $y(t) = x(t) * h(t)$ , where  $h(t)$  is the impulse response.  $y(t) = x(t) * h(t) = h(t) * x(t)$

$$\frac{dy(t)}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_{-\infty}^{\infty} h(t-\tau) \frac{dx(\tau)}{d\tau} d\tau = \int_{-\infty}^{\infty} h(\tau) \frac{dx(t-\tau)}{d\tau} d\tau = S\left(\frac{dx(t)}{dt}\right)$$

iii) By examining this system:



If we switch the cascade order, the final result stays the same (since differentiation is also an LTI system)



b) Prove the following:

i)  $y'(t) = x(t) * h'(t)$

By swapping the names of  $x$  and  $h$  we get:  $y(t) = h(t) * x'(t)$ , which I proved in part a.

ii)  $y(t) = \int_{-\infty}^t x(\tau) d\tau * h'(t) = \int_{-\infty}^t (x(\tau) * h(\tau)) d\tau = x(t) * \int_{-\infty}^t h(\tau) d\tau$  (let  $u_1(t) = x(t) * u_2(t) = \frac{dx(t)}{dt}$ )

$\uparrow$   $\left(\int_{-\infty}^t x(\tau) d\tau\right) * h'(t) = \int_{-\infty}^t \left(\int_{-\infty}^{\tau} x(\tau_1) u_1(\tau_1 - \tau) d\tau_1\right) h'(t) d\tau = \int_{-\infty}^t \int_{-\infty}^{\tau} x(\tau_1) u_1(\tau_1 - \tau) h'(t) d\tau_1 d\tau = \int_{-\infty}^t \int_{-\infty}^{\tau} x(\tau_1) u_1(\tau_1 - \tau) h(t) d\tau_1 d\tau = \int_{-\infty}^t x(\tau_1) \left(\int_{-\infty}^{\tau} u_1(\tau_1 - \tau) h(t) d\tau\right) d\tau_1 = \int_{-\infty}^t x(\tau_1) \delta(t) h(t) d\tau_1 = x(t) * h(t)$