2.33. Consider a system whose input $x(t)$ and output $y(t)$ satisfy

$$\frac{dy(t)}{dt} + 2y(t) = x(t),$$ and the condition of initial rest

a)

i) Determine $y_1(t)$ for $x_1(t) = e^{3t}u(t)$

Assuming: $y(t) = y_h(t) + y_p(t)$

$y_h(t) = Ae^{-2t}u(t)$ iff $\frac{d}{dt}Ae^{-2t} + 2Ae^{-2t} = 0 \ \forall t \geq 0 \Leftrightarrow -2Ae^{-2t} + 2Ae^{-2t} = 0 \ \forall t \geq 0 \Leftrightarrow$

$\Leftrightarrow 0 = 0 \ \forall t \geq 0 \Rightarrow y_h(t) = Ae^{-2t}u(t)$ is valid

let $y_p(t) = Be^{3t}u(t) \Rightarrow \frac{d}{dt}Be^{3t} + 2Be^{3t} = e^{3t} \ \forall t \geq 0 \Leftrightarrow 3Be^{3t} + 2Be^{3t} = e^{3t} \ \forall t \geq 0 \Leftrightarrow$

$\Leftrightarrow B = \frac{1}{5} \Rightarrow y_p(t) = \frac{1}{5}e^{3t}u(t)$

~~Initial~~ Initial rest : $y(t) = 0 \ \forall t \leq 0$

$y_1(0) = y_p(0) + y_h(0) = Be^0 + Ae^0 = B + A \Rightarrow A = y(0) - B = 0 - \frac{1}{5} = -\frac{1}{5}$

$$\boxed{y_1(t) = \left(\frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}\right)u(t)}$$

ii) Determine $y_2(t)$ for $x_2(t) = e^{2t}u(t)$

$y_h(t) = Ae^{-2t}$ (checked)

Let $y_p(t) = Be^{2t}u(t)$: $\frac{d}{dt}Be^{2t} + 2Be^{2t} = e^{2t} \ \forall t \geq 0 \Leftrightarrow 2Be^{2t} + 2Be^{2t} = e^{2t} \ \forall t \geq 0 \Leftrightarrow B = \frac{1}{4}$

$y_2(0) = y_p(0) + y_h(0) = Be^0 + Ae^{-0} = A + B \Rightarrow A = y(0) - B = 0 - \frac{1}{4} = -\frac{1}{4} = A$

initial rest

$$\boxed{y_2(t) = \left(\frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}\right)u(t)}$$

iii) Determine $y_3(t)$ for $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$. Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ $(\alpha, \beta \in \mathbb{R})$

- $y_h(t) = Ae^{-2t}$ (checked)
- Let $y_p(t) = B_1e^{3t} + B_2e^{2t}$: $\frac{d}{dt}(B_1e^{3t} + B_2e^{2t}) + 2B_1e^{3t} + 2B_2e^{2t} = \alpha e^{3t} + \beta e^{2t} \ \forall t \geq 0 \Leftrightarrow$

$\Leftrightarrow 3B_1e^{3t} + 2B_1e^{3t} + 2B_2e^{2t} + 2B_2e^{2t} = \alpha e^{3t} + \beta e^{2t} \Leftrightarrow \begin{cases} 5B_1 = \alpha \\ 4B_2 = \beta \end{cases} \Leftrightarrow \begin{cases} B_1 = \frac{1}{5}\alpha \\ B_2 = \frac{1}{4}\beta \end{cases}$

- $y_3(0) = y_h(0) + y_p(0) = Ae^{-0} + B_1e^0 + B_2e^0 = A + B_1 + B_2 \Rightarrow A = y(0) - B_1 - B_2 = 0 - \frac{1}{5}\alpha - \frac{1}{4}\beta = -\left(\frac{1}{5}\alpha + \frac{1}{4}\beta\right)$

$$\boxed{y_3(t) = \left(-\left(\frac{\alpha}{5} + \frac{\beta}{4}\right)e^{-2t} + \frac{\alpha}{5}e^{3t} + \frac{\beta}{4}e^{2t}\right)u(t) = \alpha\left(-\frac{1}{5}e^{-2t} + \frac{1}{5}e^{3t}\right)u(t) + \beta\left(-\frac{1}{4}e^{-2t} + \frac{1}{4}e^{2t}\right)u(t) = \alpha y_1(t) + \beta y_2(t)}$$

iv) Now let $x_1(t)$ and $x_2(t)$ be arbitrary signals such that $x_1(t) = 0 \ \forall t < t_1$, $x_2(t) = 0 \ \forall t < t_2$. Letting $y_1(t)$, ~~be the~~ $y_2(t)$, and $y_3(t)$ be the outputs for $x_1(t)$, $x_2(t)$ and $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ respectively. Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

$\frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t)$, $\quad y_1(t) = x_1(t) = 0 \ \forall t < t_1$ $\quad \cdot\alpha$

$\frac{d}{dt}y_2(t) + 2y_2(t) = x_2(t)$, $\quad y_2(t) = x_2(t) = 0 \ \forall t < t_2$ $\quad \cdot\beta$

$\alpha\frac{d}{dt}y_1(t) + \beta\frac{d}{dt}y_2(t) + \alpha 2y_1(t) + 2\beta y_2(t) = \alpha x_1(t) + \beta x_2$, $\begin{cases} y_1(t) = x_1(t) = 0 \ \forall t < t_1 \\ y_2(t) = x_2(t) = 0 \ \forall t < t_2 \end{cases} \Rightarrow$

$\Rightarrow \frac{d}{dt}[\alpha y_1(t) + \beta y_2(t)] + 2(\alpha y_1(t) + \beta y_2(t)) = \alpha x_1(t) + \beta x_2(t)$, $\begin{cases} y_1(t) = x_1(t) = 0 \ \forall t < t_1 \\ y_2(t) = x_2(t) = 0 \ \forall t < t_2 \end{cases} \Rightarrow$

$\Rightarrow \frac{d}{dt}y_3(t) + 2y_3(t) = x_3(t)$