2.25. Let ~~x~~ $y[n] = x[n] * h[n]$, where $x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$ and $h[n] = \left(\frac{1}{4}\right)^n u[n+3]$.

a) Determine $y[n]$ without using the distributive property of convolution

$$x[n] = \begin{cases} 3^n & \text{if } n \le -1 \\ \left(\frac{1}{3}\right)^n & \text{if } n \ge 0 \end{cases}$$

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k] = \sum_{k=-\infty}^{\infty} \left(\begin{cases} 3^k & \text{if } k \le -1 \\ \left(\frac{1}{3}\right)^k & \text{if } k \ge 0 \end{cases} \right) \cdot \left(\begin{cases} \left(\frac{1}{4}\right)^{n-k} & \text{if } \overbrace{n-k+3 \ge 0}^{k \le n+3} \\ 0 & \text{otherwise} \end{cases} \right) =$$

$$= \sum_{k=0}^{n+3} \left(\frac{1}{3}\right)^k \left(\frac{1}{4}\right)^{n-k} + \sum_{k=-\infty}^{\min(-1,n+3)} 3^k \left(\frac{1}{4}\right)^{n-k} = \left(\frac{1}{4}\right)^n \sum_{k=0}^{n+3} \left(\frac{4}{3}\right)^k + \left(\frac{1}{4}\right)^n \sum_{k=\max(1,-n-3)}^{+\infty} \left(\frac{1}{12}\right)^k =$$

$$= \begin{cases} \left(\frac{1}{4}\right)^n \cdot \frac{1-\left(\frac{4}{3}\right)^{n+4}}{1-\frac{4}{3}} & \text{if } n+3 \ge 0 \\ 0 & \text{if } n+3 < 0 \end{cases} + \left(\frac{1}{4}\right)^n \sum_{k=0}^{\infty} \left(\frac{1}{12}\right)^k - \left(\frac{1}{4}\right)^n \sum_{k=0}^{\max(1,-n-3)-1} \left(\frac{1}{12}\right)^k =$$

$$= \begin{cases} -3 \cdot \left(\frac{1}{4^n} - \frac{4^{n+4}}{4^n \cdot 3^{n+4}}\right) & \text{if } n \ge -3 \\ 0 & \text{if } n < -3 \end{cases} + \frac{\left(\frac{1}{4}\right)^n}{1-\frac{1}{12}} - \left(\frac{1}{4}\right)^n \cdot \frac{1-\left(\frac{1}{12}\right)^{\max(1,-n-3)}}{1-\frac{1}{12}} =$$

$$= \begin{cases} 3\left(\frac{3^{n+4}-4^{n+4}}{4^n \cdot 3^{n+4}}\right) & \text{if } n \ge -3 \\ 0 & \text{if } n < -3 \end{cases} + \left(\frac{12}{11}\right)\left(\frac{1}{4}\right)^n - \left(\frac{12}{11}\right)\left(\frac{1}{4}\right)^n \cdot \left(1 - \left(\frac{1}{12}\right)^{\max(1,-n-3)}\right) =$$

$$= \begin{cases} +3\left(\frac{4^4}{3^4} \cdot \frac{1}{3^n} - \frac{1}{4^n}\right) & \text{if } n \ge -3 \\ 0 & \text{if } n < -3 \end{cases} + \left(\frac{12}{11}\right)\left(\frac{1}{4}\right)^n \begin{cases} \left(\frac{1}{12}\right)^{-n-3} & \text{if } \overbrace{-n-3 > 1}^{n < -4} \\ \left(\frac{1}{12}\right)^1 & \text{if } -n-3 \le 1 \end{cases} =$$

$$= \frac{1}{4^n} \cdot \begin{cases} \frac{12^{n+4}}{11} & , n \le -5 \\ \frac{1}{11} & , n = -4 \\ \frac{1}{11} + 3\left(\left(\frac{4}{3}\right)^{n+4} - 1\right) & , n \ge -3 \end{cases} = \boxed{\frac{1}{4^n} \begin{cases} \frac{12^{n+4}}{11} & n \le -4 \\ \frac{1}{11} + 3\left(\left(\frac{4}{3}\right)^{n+4} - 1\right) & n \ge -3 \end{cases}}$$