Chapter 2 Problems

Extension Problems

61 62 63 64 65 66 67

68 69 70 71 72 73

2.63. A \$ 100,000 mortgage is to be retired by equal monthly payments of D dollars, with monthy compounded interest of 12% on the unpaid balance. After the first month, the total debt is

$\$100,000 + \left(\frac{0.12}{12}\right)\cdot \$100,000 = \cancel{\$100}\ \$101,000.$

Determine D such that after a specified time the mortgage is paid in full.

a) To set up the problem, let $y[n]$ be the unpaid balance after the n-th monthly payment, assuming the principal is borrowed in month 0 and the first payment is made in month 1. Show that $y[n]$ satisfies $y[n] - \gamma y[n-1] = -D \quad n \geq 1$, $y[0] = \$100,000$, where γ is a constant. Determine γ.

+12% ~~yea~~ yearly $\Rightarrow$ +1% monthly (as per the example provided)

Debt on month n is equal to debt on month n-1 plus interest, minus paid amount:

$y[n] = \gamma y[n-1] - D, \ n \geq 1 \Rightarrow y[n] - \gamma y[n-1] = -D, \ n \geq 1$

From the example, we have that $\boxed{\gamma = 100\% + 1\% = 1.01}$

b) Solve the difference equation to determine $y[n]$ for $n \geq 0$

Let $y_h[n] = A\gamma^n$: $y_h[n] - \gamma y_h[n-1] = 0 \Leftrightarrow A\gamma^n - \gamma A \gamma^{n-1} = 0 \Leftrightarrow 0 = 0$

Let $y_p[n] = Y$: $y_p[n] - \gamma y_p[n-1] = -D \ \forall n \geq 1 \Leftrightarrow Y - \gamma Y = -D \ \forall n \geq 1 \Leftrightarrow Y = \frac{-D}{1-\gamma} = \frac{-D}{1-1.01} = 100D$ $\Rightarrow$

$y[0] = 100\,000 = y_h[0] + y_p[n] = A\gamma^0 + Y \Rightarrow A = y[0] - Y = 100\,000 - D$

$\Rightarrow \boxed{y[n] = (100,000 - 100D)\gamma^n + 100D = (100,000 - 100D)\cdot 1.01^n + 100D \quad \forall n \geq 0}$

Solving the first equation manually: ~~y[1] = y[0]~~ $y[1] - \gamma y[0] = -D \Rightarrow$

$\Rightarrow y[1] = 1.01 \cdot 100,000 - D = 101,000 - D$

The obtained expression yields ~~y[n]~~ $y[1] = (100,000 - 100D)\cdot 1.01 + 100D =$

$= 101,000 - 101D + 100D = 101,000 - D$

Nice!

c) What is the total payment to the bank over the 30-year period?

Period of 30 years $\Rightarrow$ after 30·12 months the ~~payment is~~ debt is paid in full.

$y[360] = 0 = (100,000 - 100D)\cdot 1.01^{360} + 100D = \cancel{(100,000 - 100D)\cdot 35.9496 + 100D =}$

$= \cancel{100,000}\ \cancel{100}\ 100,000 \cdot 1.01^{360} - 100D \cdot 1.01^{360} + 100D \Rightarrow$

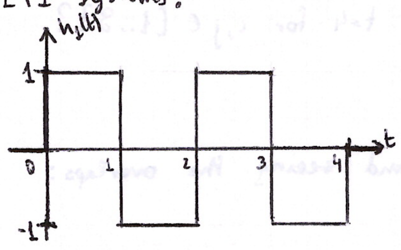
$\Rightarrow D(100 - 100 \cdot 1.01^{360}) = -100,000 \cdot 101^{360} \Rightarrow D = \frac{100,000 \cdot 1.01^{360}}{100 \cdot (1.01^{360} - 1)} = \$1028.61$

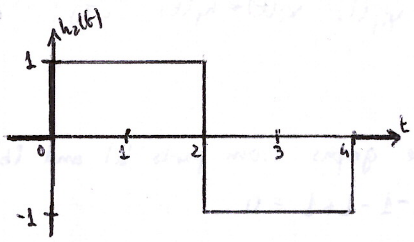
$\boxed{\text{Total paid} = 360 \cdot D = \$370,300.53} \approx 3.7 \cdot y[0] = 370\% \cdot y[0]$

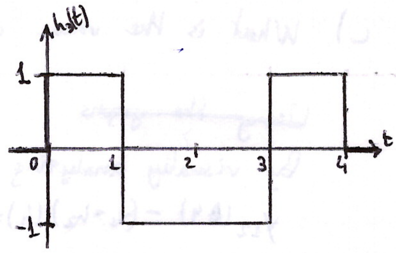
e) Why do banks make loans?

Gosh, I wonder why... not to mention the late payment ~~penalties~~ fees.

2.66. Let $h_1(t)$, $h_2(t)$, and $h_3(t)$, as sketched, be the impulse responses of three LTI systems.







a) Determine and sketch a choice for $x_1(t)$, a continuous-time signal such that

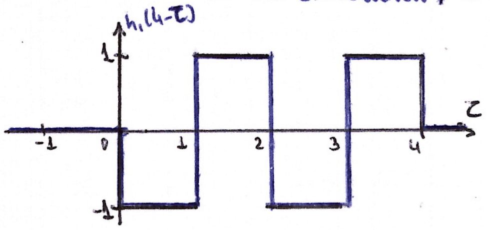
i) $x_1(t)$ is real

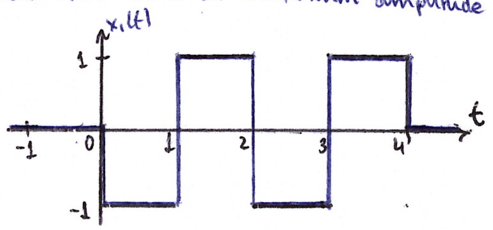
ii) $x_1(t) = 0 \quad \forall \; t<0$

iii) $|x_1(t)| \leq 1 \quad \forall \; t \geq 0$

iv) $y_1(t) = x_1(t) * h(t)$ is as large as possible at $t=4$.

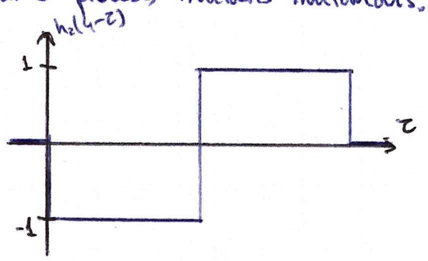
I assume the ~~condition~~ condition is actually $y_1(t) = x_1(t) * h_1(t)$ maximum at $t=4$.
Maximum at $t=4$ means the overlap between ~~x(t) and~~ $x_1(t)$ and $h_1(4-\tau)$ is maximum, so it's a perfect overlap in shape. This conclusion was derived from the graph-manipulation visualization of the convolution, I will build $x_1(t)$ with its maximum amplitude and perfect overlap.

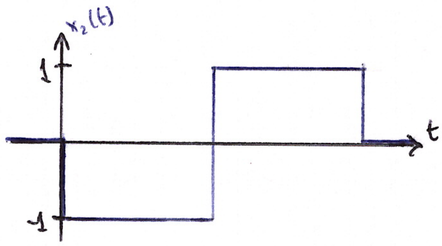


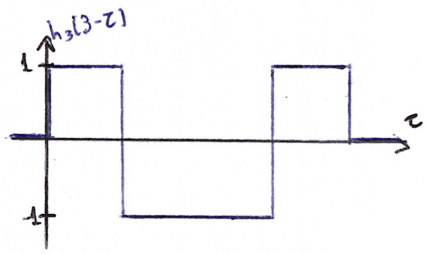


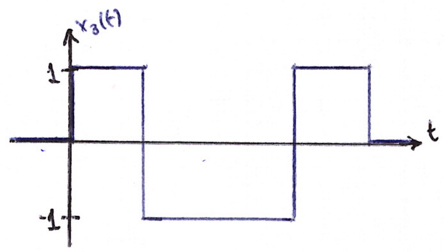
b) Repeat part (a) for $x_2(t)$ and $x_3(t)$.

Same process, mutatis mutandis.









... 2.66. (2)

c) What is the value of $y_{ij}(t) = x_i(t) * h_j(t)$, $i \neq j$ at $t=4$ for $i,j \in [1..3]$?

~~Using the graphs~~

By visually analyzing the graphs from parts (a) and (b) and seeing the overlaps:

$y_{12}(4) = (x_1 * h_2)(4) = 1-1-1+1 = 0$

$y_{13}(4) = (x_1 * h_3)(4) = -1-1+1+1 = 0$

$y_{21}(4) = (x_2 * h_1)(4) = 1-1-1+1 = 0$

$y_{23}(4) = (x_2 * h_3)(4) = -1+1-1+1 = 0$

$y_{31}(4) = (x_3 * h_1)(4) = -1-1+1+1 = 0$

$y_{32}(4) = (x_3 * h_2)(4) = -1+1-1+1 = 0$