2.52 Consider a discrete-time LTI system with unit sample response $h[n]=(n+1)\alpha^n u[n]$, where $|\alpha|<1$. Show that the step response is $s[n]=\left(\frac{1}{(\alpha-1)^2}-\frac{\alpha}{(\alpha-1)^2}\alpha^n+\frac{\alpha}{\alpha-1}(n+1)\alpha^n\right)u[n]$

$$s[n]=u[n]*h[n]=h[n]*u[n]=\sum_{k=-\infty}^{\infty}(k+1)\alpha^k u[k]u[n-k]=\sum_{k=0}^{n}(k+1)\alpha^k=\frac{d}{d\alpha}\overbrace{\sum_{k=0}^{n+1}\alpha^k}^{\text{if } n\geq 0}=$$

$$=\frac{d}{d\alpha}\left(\frac{\overbrace{1-\alpha^{n+2}}^{\text{if } n\geq 0}}{1-\alpha}\right)=\begin{cases}\dfrac{(-(n+2)\alpha^{n+1})(1-\alpha)-(1-\alpha^{n+2})\cdot(-1)}{(1-\alpha)^2} & \text{if } n\geq 0\\ 0 & \text{otherwise}\end{cases}$$

$$=\left(-\frac{(n+2)\alpha^{n+1}}{1-\alpha}+\frac{1-\alpha^{n+2}}{(1-\alpha)^2}\right)u[n]=\left(\frac{1}{(1-\alpha)^2}-\frac{\alpha^2\alpha^n}{(1-\alpha)^2}-\frac{(1-\alpha)(n+2)\alpha\cdot\alpha^n}{(1-\alpha)^2}\right)u[n]=$$

$$=\left(\frac{1}{(1-\alpha)^2}-\frac{(n+1)\alpha\alpha^n+\alpha\alpha^n}{(1-\alpha)}-\frac{\alpha\alpha\alpha^n}{(1-\alpha)^2}\right)\cdot u[n]=\left(\frac{1}{(1-\alpha^2)}-\frac{(n+1)\alpha\alpha^n}{1-\alpha}+\frac{(1-\alpha)\alpha\alpha^n+\alpha\alpha\alpha^n}{(1-\alpha)^2}\right)u[n]=$$

$$=\left(\frac{1}{(1-\alpha)^2}+\frac{\alpha}{\alpha-1}(n+1)\alpha^n-\frac{\alpha\alpha^n}{(1-\alpha)^2}\right)u[n]=\left(\frac{1}{(1-\alpha)^2}-\frac{\alpha}{(1-\alpha)^2}\alpha^n+\frac{\alpha}{\alpha-1}(n+1)\alpha^n\right)u[n]$$