1.45. It's often important in practice to compute the correlation function $\phi_{hx}(t)$, where $h(t)$ is a fixed given signal ~~but~~ but $x(t)$ may be any of a wide variety of signals. What is done is to design a system S with input $x(t)$ and output $\phi_{hx}(t)$

a) Is S linear? Is it ~~time~~ time invariant? Is it causal? Explain.

$$S: x(t) \longrightarrow \phi_{hx}(t) = \int_{-\infty}^{\infty} h(t+\tau)\cdot x(\tau)\, d\tau$$

- $$x'(t) = \alpha x_1(t) + \beta x_2(t) \longrightarrow \phi_{hx'}(t) = \int_{-\infty}^{\infty} h(t+\tau)\, x'(\tau)\, d\tau = \int_{-\infty}^{\infty} h(t+\tau)\cdot(\alpha x_1(\tau) + \beta x_2(\tau))\, d\tau =$$

$$= \int_{-\infty}^{\infty} h(t+\tau)\cdot\alpha x_1(\tau) + h(t+\tau)\cdot\beta x_2(\tau)\, d\tau = \alpha\int_{-\infty}^{\infty} h(t+\tau)\, x_1(\tau)\, d\tau + \beta\int_{-\infty}^{\infty} h(t+\tau)\, x_2(\tau)\, d\tau$$

$$\alpha\phi_{hx_1}(t) + \beta\phi_{hx_2}(t) = \alpha\int_{-\infty}^{\infty} h(t+\tau)\, x_1(\tau)\, d\tau + \beta\int_{-\infty}^{\infty} h(t+\tau)\, x_2(\tau)\, d\tau = \phi_{hx'}(t) \Rightarrow \boxed{\text{S is linear}}$$

- $$x'(t) = x(t+k) \longrightarrow \phi_{hx'}(t) = \int_{-\infty}^{\infty} h(t+\tau)\, x'(\tau)\, d\tau = \int_{-\infty}^{\infty} h(t+\tau)\, x(\tau+k)\, d\tau$$

$$\phi_{hx'}(t+k) = \int_{-\infty}^{\infty} h(t+k+\tau)\, x(\tau)\, d\tau \neq \phi_{hx'}(t) \Rightarrow \text{~~not time invariant~~} \boxed{\text{S is not time invariant}}$$

- $\phi_{hx}(t)$ depends on x at τ, where τ goes from $-\infty$ to ∞. That is greater than any $t_0 \in \mathbb{R}$, so $\boxed{\text{the system is not causal}}$

b) Do any of your answers change if the output is $\phi_{xh}(t)$?

$$S: x(t) \longrightarrow \phi_{xh}(t) = \int_{-\infty}^{\infty} x(t+\tau)\cdot h(\tau)\, d\tau$$

- $$x'(t) = \alpha x_1(t) + \beta x_2(t) \longrightarrow \phi_{x'h}(t) = \int_{-\infty}^{\infty} x'(t+\tau)\, h(\tau)\, d\tau = \int_{-\infty}^{\infty} (\alpha x_1(t+\tau) + \beta x_2(t+\tau))\, h(\tau)\, d\tau =$$

$$= \alpha\int_{-\infty}^{\infty} x_1(t+\tau)\, h(\tau)\, d\tau + \beta\int_{-\infty}^{\infty} x_2(t+\tau)\, h(\tau)\, d\tau = \alpha\phi_{x_1h}(t) + \beta\phi_{x_2h}(t) \Rightarrow \boxed{\text{S would be linear}}$$

(no change)

- $$x'(t) = x(t+k) \longrightarrow \phi_{x'h}(t) = \int_{-\infty}^{\infty} x'(t+\tau)\, h(\tau)\, d\tau = \int_{-\infty}^{\infty} x(t+\tau+k)\, h(\tau)\, d\tau$$

$$\phi_{xh}(t+k) = \int_{-\infty}^{\infty} x(t+k+\tau)\, h(\tau)\, d\tau = \phi_{x'h}(t) \Rightarrow \boxed{\text{S would be time invariant}}$$

(change)

- $\phi_{xh}(t)$ still depends on x at $\tau\ \forall \tau \in (-\infty, +\infty)$, where ~~where~~ τ may be $> t$ $\forall t \in \mathbb{R}$, so the $\boxed{\text{system is still not causal.}}$