... 2.34. (2)

d) Given $y(1)=0$ show that the system is linear but not time invariant.

Let $x_1(t) \longrightarrow y_1(t): \frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t), \quad y_1(1)=0$

$x_2(t) \longrightarrow y_2(t): \frac{d}{dt}y_2(t) + 2y_2(t) = x_2(t), \quad y_2(1)=0$

~~$x_3(t) = \alpha x_1(t) + \beta x_2(t) \longrightarrow y_3(t)$~~

Then: $\alpha \frac{d}{dt} y_1(t) + 2\alpha y_1(t) + \beta \frac{d}{dt} y_2(t) + 2\beta y_2(t) = \alpha x_1(t) + \beta x_2(t), \quad \alpha y_1(1) + \beta y_2(1) = 0 \Rightarrow$

$\Rightarrow \frac{d}{dt}(\alpha y_1(t) + \beta y_2(t)) + 2(\alpha y_1(t) + \beta y_2(t)) = \alpha x_1(t) + \beta x_2(t), \quad \alpha y_1(1) + \beta y_2(1) = 0$

It's easy to see that $\alpha x_1(t) + \beta x_2(t) \longrightarrow \alpha y_1(t) + \beta y_2(t) \Rightarrow$ linear system

Let $x_1(t) = e^t u(t) \longrightarrow y_1(t) \Rightarrow y_1(t) = y_h(t) + y_p(t) = \left(-\frac{1}{3}e^{3-2t} + \frac{1}{3}e^t\right)u(t)$ (solved using the methods we know)

~~Now let $x_2(t) = e^{t-T}u(t-T) \rightarrow$~~

Now let $x_2(t) = x_1(t-T) = e^{t-T}u(t-T) \rightarrow y_2(t) \Rightarrow \left(-\frac{1}{3}e^{3-T-2t} + \left(\frac{1}{3}e^{t-T}\right)\right)u(t-T)$

$y_1(t-T) = \left(-\frac{1}{3}e^{3-2t+2T} + \frac{1}{3}e^{t-T}\right)u(t-T) \neq y_2(t) \Rightarrow$ not time invariant

e) Given $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

Let $x_1(t) \longrightarrow y_1(t): \frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t), \quad y_1(0) + y_1(4) = 0$

$x_2(t) \rightarrow y_2(t): \frac{d}{dt}y_2(t) + 2y_2(t) = x_2(t), \quad y_2(0) + y_2(4) = 0$

Then: $\alpha\frac{d}{dt}y_1(t) + 2\alpha y_1(t) + \beta\frac{d}{dt}y_2(t) + 2\beta y_2(t) = \alpha x_1(t) + \beta x_2(t), \quad \alpha y_1(0) + \alpha y_1(4) + \beta y_2(0) + \beta y_2(4) = 0 + 0 \Rightarrow$

$\Rightarrow \frac{d}{dt}(\alpha y_1(t) + \beta y_2(t)) + 2(\alpha y_1(t) + \beta y_2(t)) = \alpha x_1(t) + \beta x_2(t), \quad \alpha(y_1(0) + y_1(4)) + \beta(y_2(0) + y_2(4)) = 0 \Rightarrow$

$\Rightarrow \alpha x_1(t) + \beta x_2(t) \longrightarrow \alpha y_1(t) + \beta y_2(t) \Rightarrow$ linear

Let $x_1(t) = e^t u(t) \rightarrow y_1(t) : \frac{d}{dt}y_1(t) + 2y_1(t) = x_1(t)$

$y_h(t) = Ae^{-2t}u(t), \quad y_p(t) = Be^t u(t) = \frac{1}{3}e^t u(t) \Rightarrow B = \frac{1}{3}$

$y_1(0) + y_1(4) = 0 = y_h(0) + y_p(0) + y_h(4) + y_p(4) = A\cdot e^0 + \frac{1}{3} + A\cdot e^{-8} + \frac{1}{3}e^{4} \Rightarrow A = \left(-\frac{1}{3} - \frac{1}{3}e^4\right)\cdot\frac{1}{1+e^{-8}}$

$y_1(t) = \left(\left(-\frac{1}{3} - \frac{1}{3}e^4\right)\frac{e^{-2t}}{1+e^{-8}} + \frac{1}{3}e^t\right)u(t)$

Let $x_2(t) = x_1(t-T) = e^{t-T}u(t-T)$

$y_h(t) = Ae^{-2t}u(t-T); \quad y_p(t) = Be^{t-T}u(t-T) = \frac{1}{3}e^{t-T} \Rightarrow B = \frac{1}{3}$

~~$y_1(0) + y_1(4) = 0 = Ae^0$~~

Let's take some $T < 0$ and $T < t$

$y_2(0) + y(4) = 0 = Ae^0 + Be^{0-T} + Ae^{-8} + Be^{4-T} \Rightarrow A = \left(-\frac{e^{-T}}{3} - \frac{e^{4-T}}{3}\right)\frac{1}{1+e^{-8}}$

$y_2(t) = \left(\left(-\frac{e^{-T}}{3} - \frac{e^{4-T}}{3}\right)\frac{e^{-2t}}{1+e^{-8}} + \frac{1}{3}e^{t-T}\right)u(t-T)$

$y_1(t-T) = \left(\left(-\frac{1}{3} - \frac{1}{3}e^4\right)\frac{e^{-2(t-T)}}{1+e^{-8}} + \frac{1}{3}e^{t-T}\right)u(t-T) \neq y_2(t) \Rightarrow$ not time invariant