3 exercises in a trenchcoat, disguised as 1 exercise

2.43. ~~One of the~~

a) Prove that $(x(t)*h(t))*g(t) = x(t)*(h(t)*g(t))$ by proving that both sides equal $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)\,d\tau\,d\sigma$.

$$\cancel{(x(t)*h(t))*g(t) = \left(\int_{-\infty}^{\infty} x(t)h(t-\tau)\,d\tau\right)*g(t) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\right)\cdot g}$$

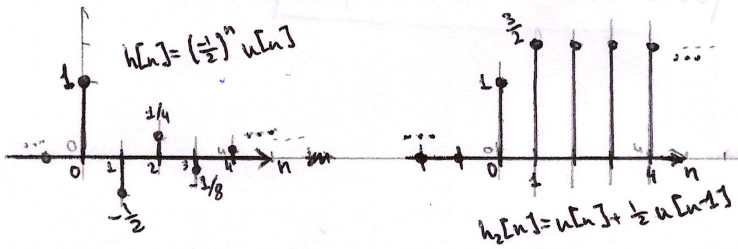
$$(x(t)*h(t))*g(t) = \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\right)*g(t) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)\cdot h(\sigma-\tau)\,d\tau\right)\cdot g(t-\sigma)\,d\sigma = \text{C.V.}\begin{bmatrix}\sigma-\tau=\sigma' \\ \sigma=\sigma'+\tau\end{bmatrix} =$$

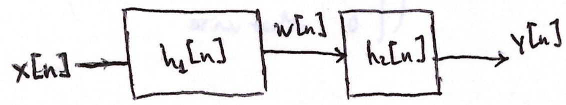
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma')\cdot g(t-\sigma'-\tau)\,d\tau\,d\sigma'$$

$$x(t)*(h(t)*g(t)) = x(t)*\int_{-\infty}^{\infty} h(\tau)g(t-\tau)\,d\tau = \cancel{\int_{-\infty}^{\infty} x(\sigma)\int_{-\infty}^{\infty} h(\tau)g(\sigma-\tau)\,d\tau\,d\sigma}$$

$$= \int_{-\infty}^{\infty} x(\sigma)\int_{-\infty}^{\infty} h(\tau)\cdot g((t-\sigma)-\tau)\,d\tau\,d\sigma = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\sigma)g(t-\sigma-\tau)\,d\tau\,d\sigma = (x(t)*h(t))*g(t)$$

b) Consider two LTI system with impulse responses $h_1[n]$ and $h_2[n]$ shown below. They are cascaded as shown. Let $x[n]=u[n]$





i) Compute $y[n]$ ~~as~~ by computing $w[n]=x[n]*h_1[n]$. $y[n]=(x_1[n]*h_1[n])*h_2[n]$

$n-0 \to n-n$, $n \to 0$, $0 \to n$ } equivalent

$$y[n] = (x[n]*h_1[n])*h_2[n] = \left(\sum_{k=-\infty}^{\infty} x[k]h_1[n-k]\right)*h_2[n] = \left(\sum_{k=-\infty}^{\infty} u[k]\cdot(-\tfrac12)^{n-k}u[n-k]\right)*h_2[n] = \left(\sum_{k=0}^{n}(-\tfrac12)^{n-k}\right)*h_2[n] =$$

$$= \left(\sum_{k=0}^{n}(-\tfrac12)^{n}\right)*h_2[n] = \left(\frac{1-(-\frac12)^{n+1}}{1+\frac12} \text{ if } n\ge0\right)*h_2[n] = \cancel{\sum_{k=-\infty}^{\infty} \cdots (u[n-k]+\tfrac12 u[n-k-1])}$$

3am, almost fell asleep on the desk

Spent 1hr looking at this, I wrote the formula upside down

$$\cancel{\frac32\sum_{k=0}^{n}\frac{1}{1-(-\frac12)^k \cdots} + \frac34\sum_{k=0}^{n-1}\frac{1}{1+\frac12(\cdots)^k}} = \sum_{k=-\infty}^{\infty}\frac{2(1-(-\frac12)^{k+1})}{3}\cdot u[k]\cdot(u[n-k]+\tfrac12 u[n-k-1]) =$$

$$= \sum_{k=0}^{n}\frac23 - \sum_{k=0}^{n}\frac23\cdot(-\tfrac12)(-\tfrac12)^k + \frac12\sum_{k=0}^{n-1}\frac23 - \frac12\sum_{k=0}^{n-1}\frac23(-\tfrac12)(-\tfrac12)^k = \overbrace{\frac23(n+1)+\frac13\cdot\frac{1-(-\frac12)^{n+1}}{1+\frac12}}^{\text{if } n\ge0\text{, else }0} + \overbrace{\frac13(n)+\frac16\cdot\frac{1-(-\frac12)^{n}}{1+\frac12}}^{\text{if } n\ge1\text{, else }0} =$$

$$= \overbrace{\frac{2n}{3}+\frac23+\frac12\cdot\frac23-\frac13\cdot\frac23(-\tfrac12)(-\tfrac12)^n}^{\text{if } n\ge0\text{, else }0} + \underbrace{\overbrace{\frac n3+\frac16\cdot\frac23-\frac16\cdot\frac23(-\tfrac12)^n}^{\text{if } n\ge1\text{, else }0}}_{n=0 \Rightarrow \text{this}=\frac{2}{18}-\frac{2}{18}=0} = \begin{cases} n+\frac23+\frac29+\frac{2}{18}+\frac19(-\frac12)^n-\frac19(-\frac12)^n & \text{if } n\ge0 \\ 0 & \text{otherwise}\end{cases} = \begin{cases} n+1 & \text{if } n\ge0 \\ 0 & \text{otherwise}\end{cases} =$$

$$= \boxed{(n+1)u[n] = y[n]}$$