2.34. ~~Initial rest correspons~~ The initial rest assumption corresponds to a zero-valued auxiliary condition at a time in ~~acc~~ accordance to the input ~~sig~~ signal. ~~the~~ Now, we'll show that if the condition is nonzero or if it is applied at a time that's independent of the input signal, the system is not LTI. Consider a system of input $x(t)$ and output $y(t)$ such that

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

a) Given $y(1) = 1$, use a counter example to disprove its linearity.

~~Let $y_1(t)$ and $y_2(t)$ be outputs to $x_1(t) + x_2(t)$ such that~~

~~Let $y_1(t)$, $y_2(t)$ and $y_3(t)$ be the outputs of $x_1(t)$, $x_2(t)$ and~~

Let $y_1(t)$ and $y_2(t)$ be the outputs to $x_1(t)$ and $x_2(t) = 2x_1(t)$ respectively

$\cancel{y_1(t) = y_2(t)}$ $y_1(1) = y_2(1) = 1 \neq 2 \cdot y_1(1) = 2 \Rightarrow$ not linear.

b) Given $y(1)=1$, use a counterexample to disprove its time invariance.

Let $x_1(t) = e^t u(t)$

$y_h(t) = Ae^{-2t}u(t)$ (checked in 2.33.a.i)

Let $y_p(t) = Be^t u(t)$: $\frac{d}{dt} Be^t + 2Be^t = e^t \ \forall t \geq 0 \Leftrightarrow B = \frac{1}{3}$

$y_1(1) = y_h(1) + y_p(1) \Rightarrow 1 = Ae^{-2\cdot 1} + Be \Rightarrow A = (1 - \frac{1}{3}e)e^2$

$\Bigg\} \Rightarrow y_1(t) = \left((1+\frac{e}{3})e^{2-2t} + \frac{1}{3}e^t\right)u(t)$

d) $0 = Ae^{-2} + \frac{1}{3}e^1 \Rightarrow A = -\frac{1}{3}e^3$

Now, let $x_2(t) = x_1(t-T) = e^{t-T} u(t-T)$

- $y_h(t) = Ae^{-2t} u(t)$ (checked)

Let $y_p(t) = Be^{t-T}u(t-T)$: $\frac{d}{dt} Be^{t-T} + 2Be^{t-T} = e^{t-T} \ \forall t \geq T \Leftrightarrow B = \frac{1}{3}$

$y_2(1) = y_h(1) + y_p(1) \Rightarrow 1 = Ae^{-2} + Be^{1-T} \ \forall t \geq T$, assuming $T < 1$ $\Rightarrow A = (1 - \frac{1}{3}e^{1-T})e^2$ $\Bigg\} \Rightarrow$

d) $0 = Ae^{-2} + Be^{1-T} \Rightarrow A = (-\frac{1}{3}e^{1-T})e^2 = -\frac{1}{3}e^{3-T}$

$\Rightarrow y_2(t) = (1 - \frac{e^{1-T}}{3})e^{2-2t} u(t) + \frac{1}{3}e^{t-T} \cdot u(t-T) \neq y_1(t-T) = \left((1+\frac{e}{3})e^{2-2(t-T)} + \frac{1}{3}e^{t-T}\right)u(t) \Rightarrow$

$\Rightarrow$ not time invariant

c) Given $y(1)=1$, show that the system is incrementally linear

Let $x_0(t) \longrightarrow y_0(t)$: $\frac{d}{dt} y_0(t) + 2y_0(t) = x_0(t)$, $y_0(1) = 1$

$x_0(t) + x_1(t) \longrightarrow y_0(t) + y_1(t)$: $\frac{d}{dt}(y_0(t) + y_1(t)) + 2(y_0(t) + y_1(t)) = x_0(t) + x_1(t)$, $y_0(1) + y_1(1) = 1$

$x_0(t) + x_2(t) \longrightarrow y_0(t) + y_2(t)$: $\frac{d}{dt}(y_0(t) + y_2(t)) + 2(y_0(t) + y_2(t)) = x_0(t) + x_2(t)$, $y_0(1) + y_2(1) = 1$

Note that $x_1(t) \not\rightarrow y_1(t)$ and $x_2(t) \not\rightarrow y_2(t)$

Then: $\alpha \frac{dy_0(t)}{dt} + \alpha\frac{dy_1(t)}{dt} + 2\alpha y_0(t) + 2\alpha y_1(t) + \beta\frac{dy_0(t)}{dt} + \beta\frac{dy_2(t)}{dt} + 2\beta y_0(t) + 2\beta y_2(t) + \frac{dy_0(t)}{dt} + 2y_0(t) = \alpha x_0(t) + \alpha x_1(t) + \beta x_0(t) + \beta x_2(t) + x_0(t)$,

$\alpha y_0(1) + \alpha y_1(1) + \beta y_0(1) + \beta y_2(1) + y_0(1) = \alpha + \beta + 1$ $\Rightarrow$ by subtracting the first equation scaled by $(\alpha+\beta)$ from both sides

$\Rightarrow \alpha\frac{dy_1(t)}{dt} + 2\alpha y_1(t) + \beta\frac{dy_2(t)}{dt} + 2\beta y_2(t) + \frac{dy_0(t)}{dt} + 2y_0(t) = \alpha x_1(t) + \beta x_2(t) + x_0(t)$,

$\alpha y_1(1) + \beta y_2(1) + y_0(1) = +1 \Rightarrow$

$\Rightarrow \frac{d}{dt}(y_0(t) + \alpha y_1(t) + \beta y_2(t)) + 2(y_0(t) + \alpha y_1(t) + \beta y_2(t)) = x_0(t) + x_1(t) + x_2(t)$, $y_0(1) + y_1(1) + y_2(1) = 1 \Rightarrow$

$\Rightarrow$ the system is incrementally linear, as we can see from the equation.