2.36 · Consider a D-T system $x[n] \to y[n]$ s.t.

$y[n] = \left(\frac{1}{2}\right) y[n-1] + x[n]$

a) Show that if it satisfies initial rest then it is LTI.

Initial rest $\Rightarrow$ $x[n]=0 \ \forall n < n_0 \Rightarrow y[n]=0 \ \forall n < n_0$.

Let $x_1[n]=0 \ \forall n<n_1$

$x_1[n] \to y_1[n] : y_1[n] = \frac{1}{2} y_1[n-1] + x_1[n]$, $\quad y_1[n]=0 \ \forall n<n_1$

$x_2[n]=0 \ \forall n<n_2$

$x_2[n] \to y_2[n] : y_2[n] = \frac{1}{2} y_2[n-1] + x_2[n]$, $\quad y_2[n]=0 \ \forall n<n_2$

Then $\alpha y_1[n] + \beta y_2[n] = \frac{\alpha}{2} y_1[n-1] + \alpha x_1[n] + \frac{\beta}{2} y_2[n-1] + \beta x_2[n]$, $\alpha y_1[n] + \beta y_2[n] = 0 \ \forall n < \min(n_1, n_2) \Rightarrow$

$\Rightarrow \alpha y_1[n] + \beta y_2[n] = \frac{1}{2}(\alpha y_1[n-1] + \beta y_2[n-1]) + \alpha x_1[n] + \beta x_2[n]$, $\alpha y_1[n] + \beta y_2[n] = 0 \ \forall n < \min(n_1, n_2)$

Clearly, $\alpha x_1[n] + \beta x_2[n] \to \alpha y_1[n] + \beta y_2[n] \Rightarrow$ linear

Let $x_3[n] = x_1[n-k] = 0 \ \forall n-k < n_3 \Leftrightarrow n < n_3 + k$

$x_3[n] \to y_3[n] : y_3[n] = \frac{1}{2} y_3[n-1] + x_3[n] = \frac{1}{2} y_3[n-1] + x_1[n-k]$

$y_1[n-k] = \frac{1}{2} y_1[n-k-1] + x_1[n-k]$

As we can observe, $y_3[n] = y_1[n-k] \Rightarrow$ time invariant

b) Show that if instead of satisfying initial rest, it satisfies $y[0]=0$, it is not causal.

Let $x_1[n] = 0 \ \forall n$, $x_2[n] = \begin{cases} 0 & n \leq -1 \\ 1 & n \geq -1 \end{cases}$

$x_1[n] = 0 \to y_1[n] = 0$

Satisfies: $y_1[n] = \frac{1}{2} y_1[n-1] + x_1[n]$ since $0 = \frac{1}{2} \cdot 0 + 0 = 0$

$y_1[0] = 0$

For $y_2[n]$: $y_2[0] = 0 = \frac{1}{2} y[-1] + x_2[0] \Rightarrow$ $y_2[-1] = 2 \cdot (0 - x_2[0]) = 2 \cdot (-1) = -2$

~~not causal, since $x_1[n] = x_2[n] = 0 \ \forall n <$~~

$-2 = y_2[-1] = \frac{1}{2} y_2[-2] + x_2[-1] \Rightarrow y_2[-2] = 2(-2 - x_2[-1]) = 2 \cdot (-2-1) = -6 \neq 0$

Since $x_1[n] = x_2[n] = 0 \ \forall n \leq -2$, but $y_1[-2] = 0 \neq y_2[-2] = -6$, it is not causal