2.6 SUMMARY

In this chapter, we have developed important representations for LTI systems, both in discrete time and in continuous time. In discrete time we derived a representation of signals as weighted sums of shifted unit impulses, and we then used this to derive the convolution-sum representation for the response of a discrete-time LTI system. In continuous time we derived an analogous representation of continuous-time signals as weighted integrals of shifted unit impulses, and we used this to derive the convolution integral representation for continuous-time LTI systems. These representations are extremely important, as they allow us to compute the response of an LTI system to an arbitrary input in terms of the system's response to a unit impulse. Moreover, in Section 2.3 the convolution sum and integral provided us with a means of analyzing the properties of LTI systems and, in particular, of relating LTI system properties, including causality and stability, to corresponding properties of the unit impulse response. Also, in Section 2.5 we developed an interpretation of the continuous-time unit impulse and other related singularity functions in terms of their behavior under convolution. This interpretation is particularly useful in the analysis of LTI systems.

An important class of continuous-time systems consists of those described by linear constant-coefficient differential equations. Similarly, in discrete time, linear constant-coefficient difference equations play an equally important role. In Section 2.4, we examined simple examples of differential and difference equations and discussed some of the properties of systems described by these types of equations. In particular, systems described by linear constant-coefficient differential and difference equations together with the condition of initial rest are causal and LTI. In subsequent chapters, we will develop additional tools that greatly facilitate our ability to analyze such systems.

Chapter 2 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

Extension problems introduce applications, concepts, or methods beyond those presented in the text.

BASIC PROBLEMS WITH ANSWERS

2.1. Let

$$x[n] = \delta[n] + 2\delta[n-1] - \delta[n-3] \quad \text{and} \quad h[n] = 2\delta[n+1] + 2\delta[n-1].$$

Compute and plot each of the following convolutions:

(a) $y_1[n] = x[n] * h[n]$ **(b)** $y_2[n] = x[n+2] * h[n]$
(c) $y_3[n] = x[n] * h[n+2]$

2.2. Consider the signal

$$h[n] = \left(\frac{1}{2}\right)^{n-1} \{u[n+3] - u[n-10]\}.$$

Express A and B in terms of n so that the following equation holds:

$$h[n-k] = \begin{cases} (\frac{1}{2})^{n-k-1}, & A \leq k \leq B \\ 0, & \text{elsewhere} \end{cases}.$$

2.3. Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$

$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

2.4. Compute and plot $y[n] = x[n] * h[n]$, where

$$x[n] = \begin{cases} 1, & 3 \leq n \leq 8 \\ 0, & \text{otherwise} \end{cases},$$

$$h[n] = \begin{cases} 1, & 4 \leq n \leq 15 \\ 0, & \text{otherwise} \end{cases}.$$

2.5. Let

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 9 \\ 0, & \text{elsewhere} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 1, & 0 \leq n \leq N \\ 0, & \text{elsewhere} \end{cases},$$

where $N \leq 9$ is an integer. Determine the value of N, given that $y[n] = x[n] * h[n]$ and

$$y[4] = 5, \quad y[14] = 0.$$

2.6. Compute and plot the convolution $y[n] = x[n] * h[n]$, where

$$x[n] = \left(\frac{1}{3}\right)^{-n} u[-n-1] \quad \text{and} \quad h[n] = u[n-1].$$

2.7. A linear system S has the relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k]$$

between its input $x[n]$ and its output $y[n]$, where $g[n] = u[n] - u[n-4]$.

(a) Determine $y[n]$ when $x[n] = \delta[n-1]$.
(b) Determine $y[n]$ when $x[n] = \delta[n-2]$.
(c) Is S LTI?
(d) Determine $y[n]$ when $x[n] = u[n]$.

2.8. Determine and sketch the convolution of the following two signals:

$$x(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & \text{elsewhere} \end{cases},$$

$$h(t) = \delta(t+2) + 2\delta(t+1).$$

2.9. Let

$$h(t) = e^{2t}u(-t+4) + e^{-2t}u(t-5).$$

Determine A and B such that

$$h(t-\tau) = \begin{cases} e^{-2(t-\tau)}, & \tau < A \\ 0, & A < \tau < B \\ e^{2(t-\tau)}, & B < \tau \end{cases}.$$

2.10. Suppose that

$$x(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

and $h(t) = x(t/\alpha)$, where $0 < \alpha \le 1$.
(a) Determine and sketch $y(t) = x(t) * h(t)$.
(b) If $dy(t)/dt$ contains only three discontinuities, what is the value of α?

2.11. Let

$$x(t) = u(t-3) - u(t-5) \quad \text{and} \quad h(t) = e^{-3t}u(t).$$

(a) Compute $y(t) = x(t) * h(t)$.
(b) Compute $g(t) = (dx(t)/dt) * h(t)$.
(c) How is $g(t)$ related to $y(t)$?

2.12. Let

$$y(t) = e^{-t}u(t) * \sum_{k=-\infty}^{\infty} \delta(t-3k).$$

Show that $y(t) = Ae^{-t}$ for $0 \le t < 3$, and determine the value of A.

2.13. Consider a discrete-time system S_1 with impulse response

$$h[n] = \left(\frac{1}{5}\right)^n u[n].$$

(a) Find the integer A such that $h[n] - Ah[n-1] = \delta[n]$.
(b) Using the result from part (a), determine the impulse response $g[n]$ of an LTI system S_2 which is the inverse system of S_1.

2.14. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) $h_1(t) = e^{-(1-2j)t}u(t)$ **(b)** $h_2(t) = e^{-t}\cos(2t)u(t)$

2.15. Which of the following impulse responses correspond(s) to stable LTI systems?
(a) $h_1[n] = n\cos(\frac{\pi}{4}n)u[n]$ **(b)** $h_2[n] = 3^n u[-n+10]$

2.16. For each of the following statements, determine whether it is true or false:
(a) If $x[n] = 0$ for $n < N_1$ and $h[n] = 0$ for $n < N_2$, then $x[n] * h[n] = 0$ for $n < N_1 + N_2$.
(b) If $y[n] = x[n] * h[n]$, then $y[n-1] = x[n-1] * h[n-1]$.
(c) If $y(t) = x(t) * h(t)$, then $y(-t) = x(-t) * h(-t)$.
(d) If $x(t) = 0$ for $t > T_1$ and $h(t) = 0$ for $t > T_2$, then $x(t) * h(t) = 0$ for $t > T_1 + T_2$.

2.17. Consider an LTI system whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d}{dt}y(t) + 4y(t) = x(t). \qquad \text{(P2.17–1)}$$

The system also satisfies the condition of initial rest.
(a) If $x(t) = e^{(-1+3j)t}u(t)$, what is $y(t)$?
(b) Note that $\mathcal{R}e\{x(t)\}$ will satisfy eq. (P2.17–1) with $\mathcal{R}e\{y(t)\}$. Determine the output $y(t)$ of the LTI system if

$$x(t) = e^{-t}\cos(3t)u(t).$$

2.18. Consider a causal LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = \frac{1}{4}y[n-1] + x[n].$$

Determine $y[n]$ if $x[n] = \delta[n-1]$.

2.19. Consider the cascade of the following two systems S_1 and S_2, as depicted in Figure P2.19:

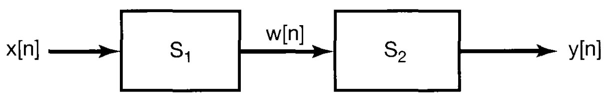


Figure P2.19

S_1 : causal LTI,

$$w[n] = \frac{1}{2}w[n-1] + x[n];$$

S_2 : causal LTI,

$$y[n] = \alpha y[n-1] + \beta w[n].$$

The difference equation relating $x[n]$ and $y[n]$ is:

$$y[n] = -\frac{1}{8}y[n-2] + \frac{3}{4}y[n-1] + x[n].$$

(a) Determine α and β.
(b) Show the impulse response of the cascade connection of S_1 and S_2.

2.20. Evaluate the following integrals:
(a) $\int_{-\infty}^{\infty} u_0(t)\cos(t)\,dt$
(b) $\int_0^5 \sin(2\pi t)\,\delta(t+3)\,dt$
(c) $\int_{-5}^{5} u_1(1-\tau)\cos(2\pi\tau)\,d\tau$

BASIC PROBLEMS

2.21. Compute the convolution $y[n] = x[n] * h[n]$ of the following pairs of signals:

(a) $\left.\begin{aligned} x[n] &= \alpha^n u[n], \\ h[n] &= \beta^n u[n], \end{aligned}\right\}\ \alpha \neq \beta$

(b) $x[n] = h[n] = \alpha^n u[n]$
(c) $x[n] = (-\frac{1}{2})^n u[n-4]$
$h[n] = 4^n u[2-n]$
(d) $x[n]$ and $h[n]$ are as in Figure P2.21.

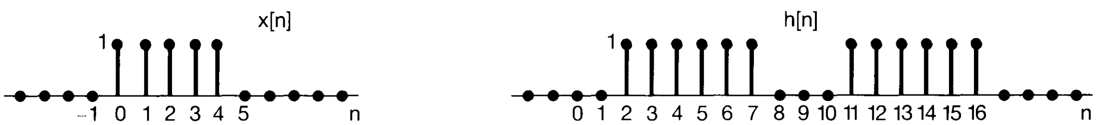


Figure P2.21

2.22. For each of the following pairs of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with impulse response $h(t)$ to the input $x(t)$. Sketch your results.

(a) $\left.\begin{aligned} x(t) &= e^{-\alpha t}u(t) \\ h(t) &= e^{-\beta t}u(t) \end{aligned}\right\}$ (Do this both when $\alpha \neq \beta$ and when $\alpha = \beta$.)

(b) $x(t) = u(t) - 2u(t-2) + u(t-5)$
$h(t) = e^{2t}u(1-t)$
(c) $x(t)$ and $h(t)$ are as in Figure P2.22(a).
(d) $x(t)$ and $h(t)$ are as in Figure P2.22(b).
(e) $x(t)$ and $h(t)$ are as in Figure P2.22(c).

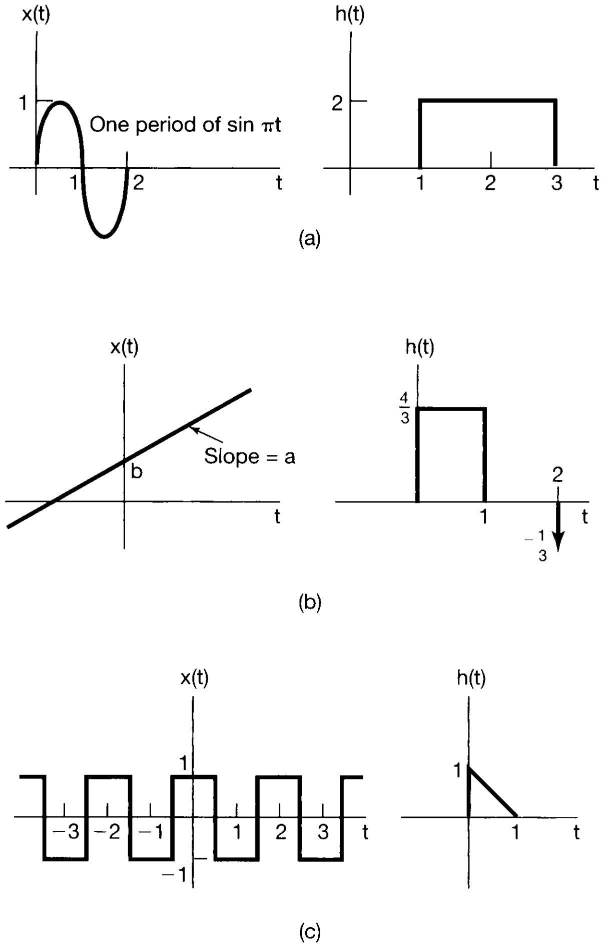
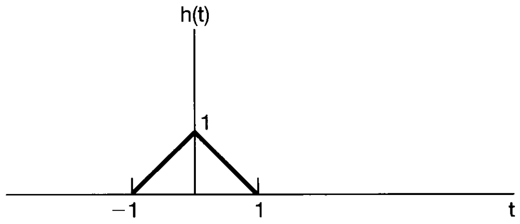


Figure P2.22

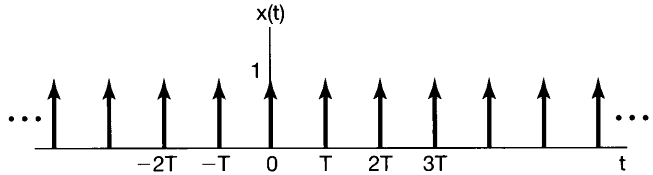
2.23. Let $h(t)$ be the triangular pulse shown in Figure P2.23(a), and let $x(t)$ be the impulse train depicted in Figure P2.23(b). That is,

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT).$$

Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T:
(a) $T = 4$ **(b)** $T = 2$ **(c)** $T = 3/2$ **(d)** $T = 1$



(a)



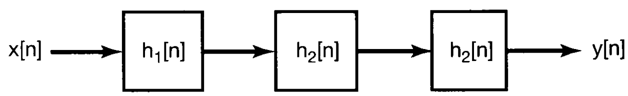
(b)

Figure P2.23

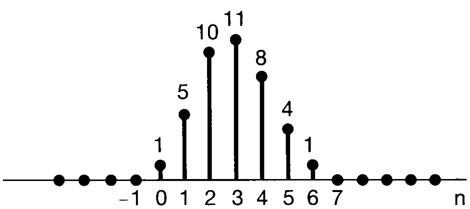
2.24. Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2.24(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2],$$

and the overall impulse response is as shown in Figure P2.24(b).



(a)



(b)

Figure P2.24

(a) Find the impulse response $h_1[n]$.

(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

2.25. Let the signal

$$y[n] = x[n] * h[n],$$

where

$$x[n] = 3^n u[-n-1] + \left(\frac{1}{3}\right)^n u[n]$$

and

$$h[n] = \left(\frac{1}{4}\right)^n u[n+3].$$

(a) Determine $y[n]$ *without* utilizing the distributive property of convolution.
(b) Determine $y[n]$ *utilizing* the distributive property of convolution.

2.26. Consider the evaluation of

$$y[n] = x_1[n] * x_2[n] * x_3[n],$$

where $x_1[n] = (0.5)^n u[n]$, $x_2[n] = u[n+3]$, and $x_3[n] = \delta[n] - \delta[n-1]$.
(a) Evaluate the convolution $x_1[n] * x_2[n]$.
(b) Convolve the result of part (a) with $x_3[n]$ in order to evaluate $y[n]$.
(c) Evaluate the convolution $x_2[n] * x_3[n]$.
(d) Convolve the result of part (c) with $x_1[n]$ in order to evaluate $y[n]$.

2.27. We define the area under a continuous-time signal $v(t)$ as

$$A_v = \int_{-\infty}^{+\infty} v(t)\,dt.$$

Show that if $y(t) = x(t) * h(t)$, then

$$A_y = A_x A_h.$$

2.28. The following are the impulse responses of discrete-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.
(a) $h[n] = (\frac{1}{5})^n u[n]$
(b) $h[n] = (0.8)^n u[n+2]$
(c) $h[n] = (\frac{1}{2})^n u[-n]$
(d) $h[n] = (5)^n u[3-n]$
(e) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[n-1]$
(f) $h[n] = (-\frac{1}{2})^n u[n] + (1.01)^n u[1-n]$
(g) $h[n] = n(\frac{1}{3})^n u[n-1]$

2.29. The following are the impulse responses of continuous-time LTI systems. Determine whether each system is causal and/or stable. Justify your answers.
(a) $h(t) = e^{-4t}u(t-2)$
(b) $h(t) = e^{-6t}u(3-t)$
(c) $h(t) = e^{-2t}u(t+50)$
(d) $h(t) = e^{2t}u(-1-t)$

(e) $h(t) = e^{-6|t|}$
(f) $h(t) = te^{-t}u(t)$
(g) $h(t) = (2e^{-t} - e^{(t-100)/100})u(t)$

2.30. Consider the first-order difference equation

$$y[n] + 2y[n-1] = x[n].$$

Assuming the condition of initial rest (i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), find the impulse response of a system whose input and output are related by this difference equation. You may solve the problem by rearranging the difference equation so as to express $y[n]$ in terms of $y[n-1]$ and $x[n]$ and generating the values of $y[0],\ y[+1],\ y[+2], \ldots$ in that order.

2.31. Consider the LTI system initially at rest and described by the difference equation

$$y[n] + 2y[n-1] = x[n] + 2x[n-2].$$

Find the response of this system to the input depicted in Figure P2.31 by solving the difference equation recursively.

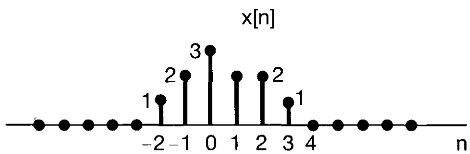


Figure P2.31

2.32. Consider the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n], \qquad \text{(P2.32–1)}$$

and suppose that

$$x[n] = \left(\frac{1}{3}\right)^n u[n]. \qquad \text{(P2.32–2)}$$

Assume that the solution $y[n]$ consists of the sum of a particular solution $y_p[n]$ to eq. (P2.32–1) and a homogeneous solution $y_h[n]$ satisfying the equation

$$y_h[n] - \frac{1}{2}y_h[n-1] = 0.$$

(a) Verify that the homogeneous solution is given by

$$y_h[n] = A\left(\frac{1}{2}\right)^n$$

(b) Let us consider obtaining a particular solution $y_p[n]$ such that

$$y_p[n] - \frac{1}{2}y_p[n-1] = \left(\frac{1}{3}\right)^n u[n].$$

By assuming that $y_p[n]$ is of the form $B(\frac{1}{3})^n$ for $n \geq 0$, and substituting this in the above difference equation, determine the value of B.

(c) Suppose that the LTI system described by eq. (P2.32–1) and initially at rest has as its input the signal specified by eq. (P2.32–2). Since $x[n] = 0$ for $n < 0$, we have that $y[n] = 0$ for $n < 0$. Also, from parts (a) and (b) we have that $y[n]$ has the form

$$y[n] = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$$

for $n \geq 0$. In order to solve for the unknown constant A, we must specify a value for $y[n]$ for some $n \geq 0$. Use the condition of initial rest and eqs. (P2.32–1) and (P2.32–2) to determine $y[0]$. From this value determine the constant A. The result of this calculation yields the solution to the difference equation (P2.32–1) under the condition of initial rest, when the input is given by eq. (P2.32–2).

2.33. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t). \qquad \text{(P2.33–1)}$$

The system also satisfies the condition of initial rest.

(a) (i) Determine the system output $y_1(t)$ when the input is $x_1(t) = e^{3t}u(t)$.

(ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = e^{2t}u(t)$.

(iii) Determine the system output $y_3(t)$ when the input is $x_3(t) = \alpha e^{3t}u(t) + \beta e^{2t}u(t)$, where α and β are real numbers. Show that $y_3(t) = \alpha y_1(t) + \beta y_2(t)$.

(iv) Now let $x_1(t)$ and $x_2(t)$ be arbitrary signals such that

$$x_1(t) = 0, \quad \text{for } t < t_1,$$
$$x_2(t) = 0, \quad \text{for } t < t_2.$$

Letting $y_1(t)$ be the system output for input $x_1(t)$, $y_2(t)$ be the system output for input $x_2(t)$, and $y_3(t)$ be the system output for $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, show that

$$y_3(t) = \alpha y_1(t) + \beta y_2(t).$$

We may therefore conclude that the system under consideration is linear.

(b) (i) Determine the system output $y_1(t)$ when the input is $x_1(t) = Ke^{2t}u(t)$.

(ii) Determine the system output $y_2(t)$ when the input is $x_2(t) = Ke^{2(t-T)}u(t-T)$. Show that $y_2(t) = y_1(t-T)$.

(iii) Now let $x_1(t)$ be an arbitrary signal such that $x_1(t) = 0$ for $t < t_0$. Letting $y_1(t)$ be the system output for input $x_1(t)$ and $y_2(t)$ be the system output for $x_2(t) = x_1(t-T)$, show that

$$y_2(t) = y_1(t-T).$$

We may therefore conclude that the system under consideration is time invariant. In conjunction with the result derived in part (a), we conclude that the given system is LTI. Since this system satisfies the condition of initial rest, it is causal as well.

2.34. The initial rest assumption corresponds to a zero-valued auxiliary condition being imposed at a time determined in accordance with the input signal. In this problem we show that if the auxiliary condition used is nonzero or if it is always applied at a fixed time (regardless of the input signal) the corresponding system cannot be LTI. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation (P2.33–1).

(a) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not linear.

(b) Given the auxiliary condition $y(1) = 1$, use a counterexample to show that the system is not time invariant.

(c) Given the auxiliary condition $y(1) = 1$, show that the system is incrementally linear.

(d) Given the auxiliary condition $y(1) = 0$, show that the system is linear but not time invariant.

(e) Given the auxiliary condition $y(0) + y(4) = 0$, show that the system is linear but not time invariant.

2.35. In the previous problem we saw that application of an auxiliary condition at a fixed time (regardless of the input signal) leads to the corresponding system being not time-invariant. In this problem, we explore the effect of fixed auxiliary conditions on the causality of a system. Consider a system whose input $x(t)$ and output $y(t)$ satisfy the first-order differential equation (P2.33–1). Assume that the auxiliary condition associated with the differential equation is $y(0) = 0$. Determine the output of the system for each of the following two inputs:

(a) $x_1(t) = 0, \text{ for all } t$

(b) $x_2(t) = \begin{cases} 0, & t < -1 \\ 1, & t > -1 \end{cases}$

Observe that if $y_1(t)$ is the output for input $x_1(t)$ and $y_2(t)$ is the output for input $x_2(t)$, then $y_1(t)$ and $y_2(t)$ are not identical for $t < -1$, even though $x_1(t)$ and $x_2(t)$ are identical for $t < -1$. Use this observation as the basis of an argument to conclude that the given system is not causal.

2.36. Consider a discrete-time system whose input $x[n]$ and output $y[n]$ are related by

$$y[n] = \left(\frac{1}{2}\right)y[n-1] + x[n].$$

(a) Show that if this system satisfies the condition of initial rest (i.e., if $x[n] = 0$ for $n < n_0$, then $y[n] = 0$ for $n < n_0$), then it is linear and time invariant.

(b) Show that if this system does not satisfy the condition of initial rest, but instead uses the auxiliary condition $y[0] = 0$, it is not causal. [*Hint:* Use an approach similar to that used in Problem 2.35.]

2.37. Consider a system whose input and output are related by the first-order differential equation (P2.33–1). Assume that the system satisfies the condition of final rest [i. e., if $x(t) = 0$ for $t > t_0$, then $y(t) = 0$ for $t > t_0$]. Show that this system is *not* causal. [*Hint:* Consider two inputs to the system, $x_1(t) = 0$ and $x_2(t) = e^t(u(t) - u(t-1))$, which result in outputs $y_1(t)$ and $y_2(t)$, respectively. Then show that $y_1(t) \neq y_2(t)$ for $t < 0$.]

2.38. Draw block diagram representations for causal LTI systems described by the following difference equations:

(a) $y[n] = \frac{1}{3}y[n-1] + \frac{1}{2}x[n]$

(b) $y[n] = \frac{1}{3}y[n-1] + x[n-1]$

2.39. Draw block diagram representations for causal LTI systems described by the following differential equations:

(a) $y(t) = -(\frac{1}{2})\,dy(t)/dt + 4x(t)$

(b) $dy(t)/dt + 3y(t) = x(t)$

ADVANCED PROBLEMS

2.40. **(a)** Consider an LTI system with input and output related through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)}x(\tau - 2)\,d\tau.$$

What is the impulse response $h(t)$ for this system?

(b) Determine the response of the system when the input $x(t)$ is as shown in Figure P2.40.

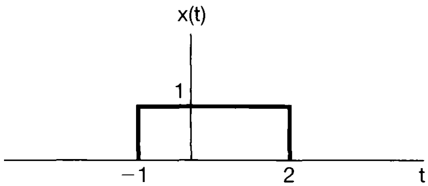


Figure P2.40

2.41. Consider the signal

$$x[n] = \alpha^n u[n].$$

(a) Sketch the signal $g[n] = x[n] - \alpha x[n-1]$.

(b) Use the result of part (a) in conjunction with properties of convolution in order to determine a sequence $h[n]$ such that

$$x[n] * h[n] = \left(\frac{1}{2}\right)^n \{u[n+2] - u[n-2]\}.$$

2.42. Suppose that the signal

$$x(t) = u(t + 0.5) - u(t - 0.5)$$

is convolved with the signal

$$h(t) = e^{j\omega_0 t}.$$

(a) Determine a value of ω_0 which ensures that

$$y(0) = 0,$$

where $y(t) = x(t) * h(t)$.

(b) Is your answer to the previous part unique?

2.43. One of the important properties of convolution, in both continuous and discrete time, is the associativity property. In this problem, we will check and illustrate this property.

(a) Prove the equality

$$[x(t) * h(t)] * g(t) = x(t) * [h(t) * g(t)] \qquad \text{(P2.43–1)}$$

by showing that both sides of eq. (P2.43–1) equal

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(\tau)h(\sigma)g(t-\tau-\sigma)\,d\tau\,d\sigma.$$

(b) Consider two LTI systems with the unit sample responses $h_1[n]$ and $h_2[n]$ shown in Figure P2.43(a). These two systems are cascaded as shown in Figure P2.43(b). Let $x[n] = u[n]$.

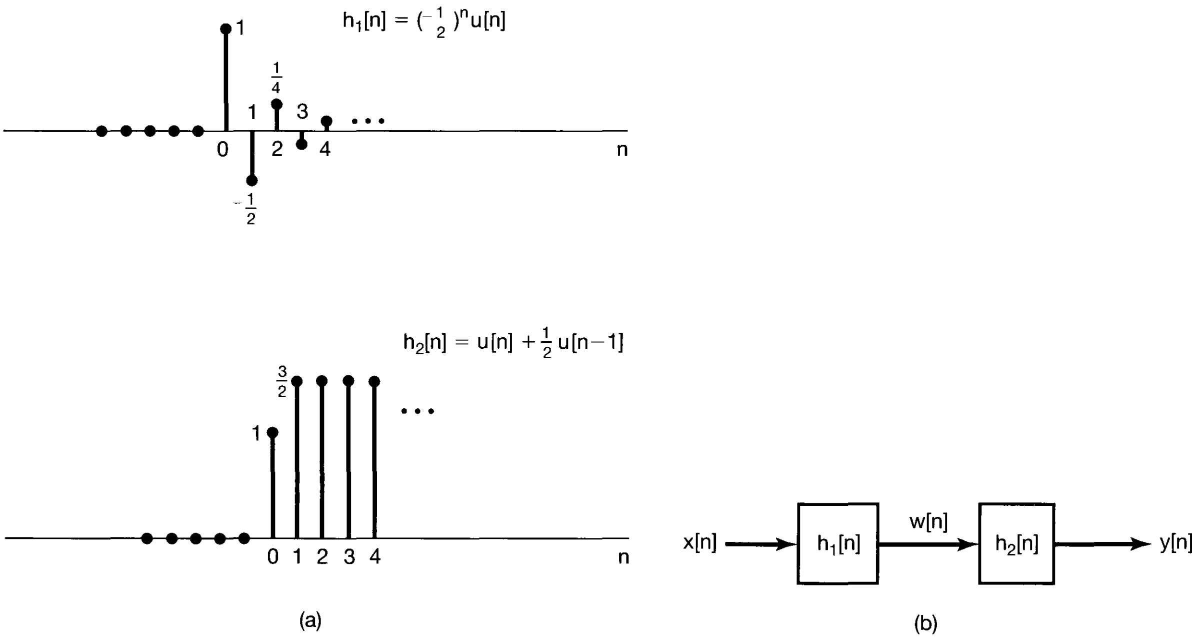


Figure P2.43

(i) Compute $y[n]$ by first computing $w[n] = x[n] * h_1[n]$ and then computing $y[n] = w[n] * h_2[n]$; that is, $y[n] = [x[n] * h_1[n]] * h_2[n]$.

(ii) Now find $y[n]$ by first convolving $h_1[n]$ and $h_2[n]$ to obtain $g[n] = h_1[n] * h_2[n]$ and then convolving $x[n]$ with $g[n]$ to obtain $y[n] = x[n] * [h_1[n] * h_2[n]]$.

The answers to (i) and (ii) should be identical, illustrating the associativity property of discrete-time convolution.

(c) Consider the cascade of two LTI systems as in Figure P2.43(b), where in this case

$$h_1[n] = \sin 8n$$

and

$$h_2[n] = a^n u[n], \quad |a| < 1,$$

and where the input is

$$x[n] = \delta[n] - a\delta[n-1].$$

Determine the output $y[n]$. (*Hint:* The use of the associative and commutative properties of convolution should greatly facilitate the solution.)

2.44. (a) If

$$x(t) = 0, \ |t| > T_1,$$

and

$$h(t) = 0, \ |t| > T_2,$$

then

$$x(t) * h(t) = 0, \ |t| > T_3$$

for some positive number T_3. Express T_3 in terms of T_1 and T_2.

(b) A discrete-time LTI system has input $x[n]$, impulse response $h[n]$, and output $y[n]$. If $h[n]$ is known to be zero everywhere outside the interval $N_0 \le n \le N_1$ and $x[n]$ is known to be zero everywhere outside the interval $N_2 \le n \le N_3$, then the output $y[n]$ is constrained to be zero everywhere, except on some interval $N_4 \le n \le N_5$.

(i) Determine N_4 and N_5 in terms of N_0, N_1, N_2, and N_3.

(ii) If the interval $N_0 \le n \le N_1$ is of length M_h, $N_2 \le n \le N_3$ is of length M_x, and $N_4 \le n \le N_5$ is of length M_y, express M_y in terms of M_h and M_x.

(c) Consider a discrete-time LTI system with the property that if the input $x[n] = 0$ for all $n \ge 10$, then the output $y[n] = 0$ for all $n \ge 15$. What condition must $h[n]$, the impulse response of the system, satisfy for this to be true?

(d) Consider an LTI system with impulse response in Figure P2.44. Over what interval must we know $x(t)$ in order to determine $y(0)$?

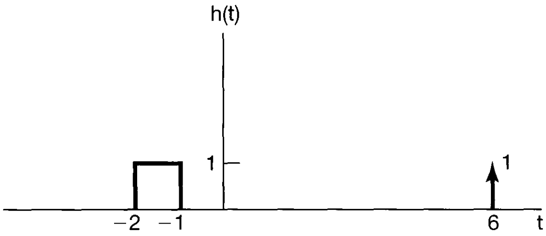


Figure P2.44

2.45. **(a)** Show that if the response of an LTI system to $x(t)$ is the output $y(t)$, then the response of the system to

$$x'(t) = \frac{dx(t)}{dt}$$

is $y'(t)$. Do this problem in three different ways:

(i) Directly from the properties of linearity and time invariance and the fact that

$$x'(t) = \lim_{h \to 0} \frac{x(t) - x(t-h)}{h}.$$

(ii) By differentiating the convolution integral.

(iii) By examining the system in Figure P2.45.

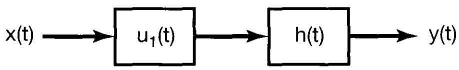


Figure P2.45

(b) Demonstrate the validity of the following relationships:

(i) $y'(t) = x(t) * h'(t)$

(ii) $y(t) = (\int_{-\infty}^{t} x(\tau)\, d\tau) * h'(t) = \int_{-\infty}^{t} [x'(\tau) * h(\tau)] d\tau = x'(t) * (\int_{-\infty}^{t} h(\tau) d\tau)$

[*Hint:* These are easily done using block diagrams as in (iii) of part (a) and the fact that $u_1(t) * u_{-1}(t) = \delta(t)$.]

(c) An LTI system has the response $y(t) = \sin \omega_0 t$ to input $x(t) = e^{-5t} u(t)$. Use the result of part (a) to aid in determining the impulse response of this system.

(d) Let $s(t)$ be the unit step response of a continuous-time LTI system. Use part (b) to deduce that the response $y(t)$ to the input $x(t)$ is

$$y(t) = \int_{-\infty}^{+\infty} x'(\tau) * s(t - \tau)\, d\tau. \qquad \text{(P2.45–1)}$$

Show also that

$$x(t) = \int_{-\infty}^{+\infty} x'(\tau) u(t - \tau)\, d\tau. \qquad \text{(P2.45–2)}$$

(e) Use eq. (P2.45–1) to determine the response of an LTI system with step response

$$s(t) = (e^{-3t} - 2e^{-2t} + 1)u(t)$$

to the input $x(t) = e^t u(t)$.

(f) Let $s[n]$ be the unit step response of a discrete-time LTI system. What are the discrete-time counterparts of eqs. (P2.45–1) and (P2.45–2)?

2.46. Consider an LTI system S and a signal $x(t) = 2e^{-3t}u(t-1)$. If

$$x(t) \longrightarrow y(t)$$

and

$$\frac{dx(t)}{dt} \longrightarrow -3y(t) + e^{-2t}u(t),$$

determine the impulse response $h(t)$ of S.

2.47. We are given a certain linear time-invariant system with impulse response $h_0(t)$. We are told that when the input is $x_0(t)$ the output is $y_0(t)$, which is sketched in Figure P2.47. We are then given the following set of inputs to linear time-invariant systems with the indicated impulse responses:

Input $x(t)$	*Impulse response* $h(t)$
(a) $x(t) = 2x_0(t)$	$h(t) = h_0(t)$
(b) $x(t) = x_0(t) - x_0(t-2)$	$h(t) = h_0(t)$
(c) $x(t) = x_0(t-2)$	$h(t) = h_0(t+1)$
(d) $x(t) = x_0(-t)$	$h(t) = h_0(t)$
(e) $x(t) = x_0(-t)$	$h(t) = h_0(-t)$
(f) $x(t) = x_0'(t)$	$h(t) = h_0'(t)$

[Here $x_0'(t)$ and $h_0'(t)$ denote the first derivatives of $x_0(t)$ and $h_0(t)$, respectively.]

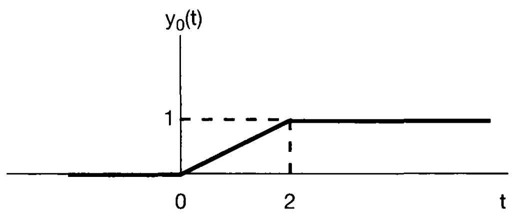


Figure P2.47

In each of these cases, determine whether or not we have enough information to determine the output $y(t)$ when the input is $x(t)$ and the system has impulse response $h(t)$. If it is possible to determine $y(t)$, provide an accurate sketch of it with numerical values clearly indicated on the graph.

2.48. Determine whether each of the following statements concerning LTI systems is true or false. Justify your answers.

(a) If $h(t)$ is the impulse response of an LTI system and $h(t)$ is periodic and nonzero, the system is unstable.

(b) The inverse of a causal LTI system is always causal.

(c) If $|h[n]| \leq K$ for each n, where K is a given number, then the LTI system with $h[n]$ as its impulse response is stable.

(d) If a discrete-time LTI system has an impulse response $h[n]$ of finite duration, the system is stable.

(e) If an LTI system is causal, it is stable.

(f) The cascade of a noncausal LTI system with a causal one is necessarily noncausal.

(g) A continuous-time LTI system is stable if and only if its step response $s(t)$ is absolutely integrable—that is, if and only if

$$\int_{-\infty}^{+\infty} |s(t)|\, dt < \infty.$$

(h) A discrete-time LTI system is causal if and only if its step response $s[n]$ is zero for $n < 0$.

2.49. In the text, we showed that if $h[n]$ is absolutely summable, i.e., if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty,$$

then the LTI system with impulse response $h[n]$ is stable. This means that absolute summability is a *sufficient* condition for stability. In this problem, we shall show that it is also a *necessary* condition. Consider an LTI system with impulse response $h[n]$ that is not absolutely summable; that is,

$$\sum_{k=-\infty}^{+\infty} |h[k]| = \infty.$$

(a) Suppose that the input to this system is

$$x[n] = \begin{cases} 0, & \text{if } h[-n] = 0 \\ \frac{h[-n]}{|h[-n]|}, & \text{if } h[-n] \neq 0 \end{cases}.$$

Does this input signal represent a bounded input? If so, what is the smallest number B such that

$$|x[n]| \leq B \text{ for all } n?$$

(b) Calculate the output at $n = 0$ for this particular choice of input. Does the result prove the contention that absolute summability is a necessary condition for stability?

(c) In a similar fashion, show that a continuous-time LTI system is stable if and only if its impulse response is absolutely integrable.

2.50. Consider the cascade of two systems shown in Figure P2.50. The first system, A, is known to be LTI. The second system, B, is known to be the inverse of system A. Let $y_1(t)$ denote the response of system A to $x_1(t)$, and let $y_2(t)$ denote the response of system A to $x_2(t)$.

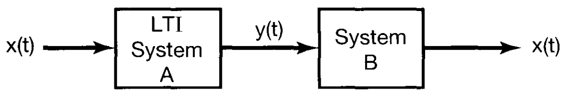


Figure P2.50

(a) What is the response of system B to the input $ay_1(t) + by_2(t)$, where a and b are constants?

(b) What is the response of system B to the input $y_1(t - \tau)$?

2.51. In the text, we saw that the overall input-output relationship of the cascade of two LTI systems does not depend on the order in which they are cascaded. This fact, known as the commutativity property, depends on both the linearity and the time invariance of both systems. In this problem, we illustrate the point.

(a) Consider two discrete-time systems A and B, where system A is an LTI system with unit sample response $h[n] = (1/2)^n u[n]$. System B, on the other hand, is linear but time varying. Specifically, if the input to system B is $w[n]$, its output is

$$z[n] = nw[n].$$

Show that the commutativity property does not hold for these two systems by computing the impulse responses of the cascade combinations in Figures P2.51(a) and P2.51(b), respectively.

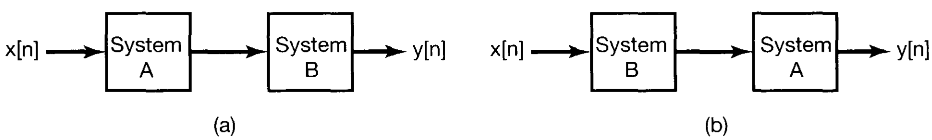


Figure P2.51

(b) Suppose that we replace system B in each of the interconnected systems of Figure P2.51 by the system with the following relationship between its input $w[n]$ and output $z[n]$:

$$z[n] = w[n] + 2.$$

Repeat the calculations of part (a) in this case.

2.52. Consider a discrete-time LTI system with unit sample response

$$h[n] = (n + 1)\alpha^n u[n],$$

where $|\alpha| < 1$. Show that the step response of this system is

$$s[n] = \left[\frac{1}{(\alpha - 1)^2} - \frac{\alpha}{(\alpha - 1)^2}\alpha^n + \frac{\alpha}{(\alpha - 1)}(n + 1)\alpha^n\right]u[n].$$

(*Hint:* Note that

$$\sum_{k=0}^{N}(k + 1)\alpha^k = \frac{d}{d\alpha}\sum_{k=0}^{N+1}\alpha^k.)$$

2.53. **(a)** Consider the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \qquad \text{(P2.53–1)}$$

Show that if s_0 is a solution of the equation

$$p(s) = \sum_{k=0}^{N} a_k s^k = 0, \qquad \text{(P2.53–2)}$$

then $Ae^{s_0 t}$ is a solution of eq. (P2.53–1), where A is an arbitrary complex constant.

(b) The polynomial $p(s)$ in eq. (P2.53–2) can be factored in terms of its roots $s_1, \ldots, s_r$ as

$$p(s) = a_N(s - s_1)^{\sigma_1}(s - s_2)^{\sigma_2}\ldots(s - s_r)^{\sigma_r},$$

where the s_i are the distinct solutions of eq. (P2.53–2) and the σ_i are their *multiplicities*—that is, the number of times each root appears as a solution of the equation. Note that

$$\sigma_1 + \sigma_2 + \ldots + \sigma_r = N.$$

In general, if $\sigma_i > 1$, then not only is $Ae^{s_i t}$ a solution of eq. (P2.53–1), but so is $At^j e^{s_i t}$, as long as j is an integer greater than or equal to zero and less than or equal to $\sigma_i - 1$. To illustrate this, show that if $\sigma_i = 2$, then $Ate^{s_i t}$ is a solution of eq. (P2.53–1). [*Hint:* Show that if s is an arbitrary complex number, then

$$\sum_{k=0}^{N} \frac{d^k(Ate^{st})}{dt^k} = Ap(s)te^{st} + A\frac{dp(s)}{ds}e^{st}.]$$

Thus, the most general solution of eq. (P2.53–1) is

$$\sum_{i=1}^{r}\sum_{j=0}^{\sigma_i-1} A_{ij}t^j e^{s_i t},$$

where the A_{ij} are arbitrary complex constants.

(c) Solve the following homogeneous differential equations with the specified auxiliary conditions:

(i) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0,\ y(0) = 0,\ \ y'(0) = 2$

(ii) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0,\ y(0) = 1,\ \ y'(0) = -1$

(iii) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0,\ y(0) = 0,\ \ y'(0) = 0$

(iv) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = 0,\ y(0) = 1,\ \ y'(0) = 1$

(v) $\frac{d^3y(t)}{dt^3} + \frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - y(t) = 0,\ y(0) = 1,\ \ y'(0) = 1,\ \ y''(0) = -2$

(vi) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 5y(t) = 0,\ y(0) = 1,\ \ y'(0) = 1$

2.54. (a) Consider the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = 0, \tag{P2.54–1}$$

Show that if z_0 is a solution of the equation

$$\sum_{k=0}^{N} a_k z^{-k} = 0, \tag{P2.54–2}$$

then Az_0^n is a solution of eq. (P2.54–1), where A is an arbitrary constant.

(b) As it is more convenient for the moment to work with polynomials that have only nonnegative powers of z, consider the equation obtained by multiplying both sides of eq. (P2.54–2) by z^N:

$$p(z) = \sum_{k=0}^{N} a_k z^{N-k} = 0. \tag{P2.54–3}$$

The polynomial $p(z)$ can be factored as

$$p(z) = a_0(z - z_1)^{\sigma_1} \ldots (z - z_r)^{\sigma_r},$$

where the $z_1, \ldots, z_r$ are the distinct roots of $p(z)$.

Show that if $y[n] = nz^{n-1}$, then

$$\sum_{k=0}^{N} a_k y[n-k] = \frac{dp(z)}{dz} z^{n-N} + (n-N)p(z)z^{n-N-1}.$$

Use this fact to show that if $\sigma_i = 2$, then both Az_i^n and Bnz_i^{n-1} are solutions of eq. (P2.54–1), where A and B are arbitrary complex constants. More generally, one can use this same procedure to show that if $\sigma_i > 1$, then

$$A\frac{n!}{r!(n-r)!}z^{n-r}$$

is a solution of eq. (P2.54–1) for $r = 0, 1, \ldots, \sigma_i - 1$.[7]

(c) Solve the following homogeneous difference equations with the specified auxiliary conditions:

(i) $y[n] + \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 0;\ y[0] = 1,\ y[-1] = -6$

(ii) $y[n] - 2y[n-1] + y[n-2] = 0;\ y[0] = 1,\ y[1] = 0$

(iii) $y[n] - 2y[n-1] + y[n-2] = 0;\ y[0] = 1,\ y[10] = 21$

(iv) $y[n] - \frac{\sqrt{2}}{2}y[n-1] + \frac{1}{4}y[n-2] = 0;\ y[0] = 0,\ y[-1] = 1$

2.55. In the text we described one method for solving linear constant-coefficient difference equations, and another method for doing this was illustrated in Problem 2.30. If the assumption of initial rest is made so that the system described by the difference equation is LTI and causal, then, in principle, we can determine the unit impulse response $h[n]$ using either of these procedures. In Chapter 5, we describe another method that allows us to determine $h[n]$ in a more elegant way. In this problem we describe yet another approach, which basically shows that $h[n]$ can be determined by solving the homogeneous equation with appropriate initial conditions.

(a) Consider the system initially at rest and described by the equation

$$y[n] - \frac{1}{2}y[n-1] = x[n]. \qquad \text{(P2.55–1)}$$

Assuming that $x[n] = \delta[n]$, what is $y[0]$? What equation does $h[n]$ satisfy for $n \geq 1$, and with what auxiliary condition? Solve this equation to obtain a closed-form expression for $h[n]$.

(b) Consider next the LTI system initially at rest and described by the difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1]. \qquad \text{(P2.55–2)}$$

This system is depicted in Figure P2.55(a) as a cascade of two LTI systems that are initially at rest. Because of the properties of LTI systems, we can reverse the order of the systems in the cascade to obtain an alternative representation of the same overall system, as illustrated in Figure P2.55(b). From this fact, use the result of part (a) to determine the impulse response for the system described by eq. (P2.55–2).

(c) Consider again the system of part (a), with $h[n]$ denoting its impulse response. Show, by verifying that eq. (P2.55–3) satisfies the difference equation (P2.55–1), that the response $y[n]$ to an arbitrary input $x[n]$ is in fact given by the convolution sum

$$y[n] = \sum_{m=-\infty}^{+\infty} h[n-m]x[m]. \qquad \text{(P2.55–3)}$$

[7] Here, we are using factorial notation—that is, $k! = k(k-1)(k-2)\ldots(2)(1)$, where $0!$ is defined to be 1.

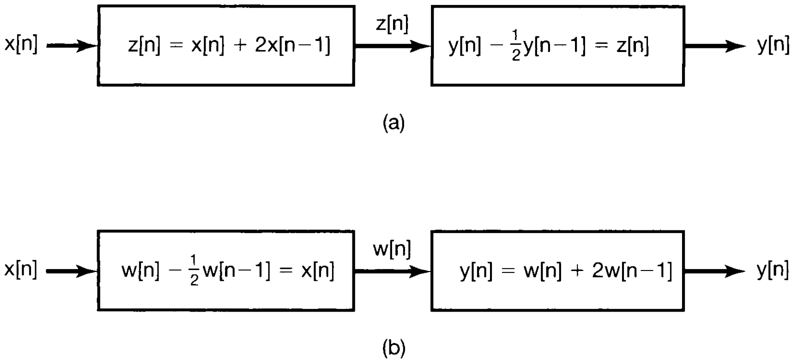


Figure P2.55

(d) Consider the LTI system initially at rest and described by the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = x[n]. \qquad \text{(P2.55–4)}$$

Assuming that $a_0 \neq 0$, what is $y[0]$ if $x[n] = \delta[n]$? Using this result, specify the homogeneous equation and initial conditions that the impulse response of the system must satisfy.

Consider next the causal LTI system described by the difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \qquad \text{(P2.55–5)}$$

Express the impulse response of this system in terms of that for the LTI system described by eq. (P2.55–4).

(e) There is an alternative method for determining the impulse response of the LTI system described by eq. (P2.55–5). Specifically, given the condition of initial rest, i.e., in this case, $y[-N] = y[-N+1] = \ldots = y[-1] = 0$, solve eq. (P2.55–5) recursively when $x[n] = \delta[n]$ in order to determine $y[0], \ldots, y[M]$. What equation does $h[n]$ satisfy for $n \geq M$? What are the appropriate initial conditions for this equation?

(f) Using either of the methods outlined in parts (d) and (e), find the impulse responses of the causal LTI systems described by the following equations:

(i) $y[n] - y[n-2] = x[n]$

(ii) $y[n] - y[n-2] = x[n] + 2x[n-1]$

(iii) $y[n] - y[n-2] = 2x[n] - 3x[n-4]$

(iv) $y[n] - (\sqrt{3}/2)y[n-1] + \frac{1}{4}y[n-2] = x[n]$

2.56. In this problem, we consider a procedure that is the continuous-time counterpart of the technique developed in Problem 2.55. Again, we will see that the problem of determining the impulse response $h(t)$ for $t > 0$ for an LTI system initially at rest and described by a linear constant-coefficient differential equation reduces to the problem of solving the homogeneous equation with appropriate initial conditions.

(a) Consider the LTI system initially at rest and described by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t). \tag{P2.56–1}$$

Suppose that $x(t) = \delta(t)$. In order to determine the value of $y(t)$ *immediately after* the application of the unit impulse, consider integrating eq. (P2.56–1) from $t = 0^-$ to $t = 0^+$ (i.e., from "just before" to "just after" the application of the impulse). This yields

$$y(0^+) - y(0^-) + 2\int_{0^-}^{0^+} y(\tau)\,d\tau = \int_{0^-}^{0^+} \delta(\tau)\,d\tau = 1. \tag{P2.56–2}$$

Since the system is initially at rest and $x(t) = 0$ for $t < 0$, $y(0^-) = 0$. To satisfy eq. (P2.56–2) we must have $y(0^+) = 1$. Thus, since $x(t) = 0$ for $t > 0$, the impulse response of our system is the solution of the homogeneous differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 0$$

with initial condition

$$y(0^+) = 1.$$

Solve this differential equation to obtain the impulse response $h(t)$ for the system. Check your result by showing that

$$y(t) = \int_{-\infty}^{+\infty} h(t-\tau)x(\tau)\,d\tau$$

satisfies eq. (P2.56–1) for any input $x(t)$.

(b) To generalize the preceding argument, consider an LTI system initially at rest and described by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = x(t) \tag{P2.56–3}$$

with $x(t) = \delta(t)$. Assume the condition of initial rest, which, since $x(t) = 0$ for $t < 0$, implies that

$$y(0^-) = \frac{dy}{dt}(0^-) = \ldots = \frac{d^{N-1}y}{dt^{N-1}}(0^-) = 0. \tag{P2.56–4}$$

Integrate both sides of eq. (P2.56–3) once from $t = 0^-$ to $t = 0^+$, and use eq. (P2.56–4) and an argument similar to that used in part (a) to show that the resulting equation is satisfied with

$$y(0^+) = \frac{dy}{dt}(0^+) = \ldots = \frac{d^{N-2}y}{dt^{N-2}}(0^+) = 0 \qquad \text{(P2.56–5}a\text{)}$$

and

$$\frac{d^{N-1}y}{dt^{N-1}}(0^+) = \frac{1}{a_N}. \qquad \text{(P2.56–5}b\text{)}$$

Consequently, the system's impulse response for $t > 0$ can be obtained by solving the homogeneous equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

with initial conditions given by eqs. (P2.56–5).

(c) Consider now the causal LTI system described by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad \text{(P2.56–6)}$$

Express the impulse response of this system in terms of that for the system of part (b). (*Hint:* Examine Figure P2.56.)

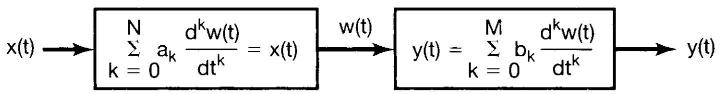


Figure P2.56

(d) Apply the procedures outlined in parts (b) and (c) to find the impulse responses for the LTI systems initially at rest and described by the following differential equations:

(i) $\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$

(ii) $\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = x(t)$

(e) Use the results of parts (b) and (c) to deduce that if $M \geq N$ in eq. (P2.56–6), then the impulse response $h(t)$ will contain singularity terms concentrated at $t = 0$. In particular, $h(t)$ will contain a term of the form

$$\sum_{r=0}^{M-N} \alpha_r u_r(t),$$

where the α_r are constants and the $u_r(t)$ are the singularity functions defined in Section 2.5.

(f) Find the impulse responses of the causal LTI systems described by the following differential equations:

(i) $\frac{dy(t)}{dt} + 2y(t) = 3\frac{dx(t)}{dt} + x(t)$

(ii) $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{d^3x(t)}{dt^3} + 2\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 3x(t)$

2.57. Consider a causal LTI system S whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$y[n] = -ay[n-1] + b_0x[n] + b_1x[n-1].$$

(a) Verify that S may be considered a cascade connection of two causal LTI systems S_1 and S_2 with the following input-output relationship:

$$S_1 : y_1[n] = b_0x_1[n] + b_1x_1[n-1],$$
$$S_2 : y_2[n] = -ay_2[n-1] + x_2[n].$$

(b) Draw a block diagram representation of S_1.
(c) Draw a block diagram representation of S_2.
(d) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.
(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1.
(f) Show that the two unit-delay elements in the block diagram representation of S obtained in part (e) may be collapsed into one unit-delay element. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (d) and (e) are referred to as *Direct Form I* realizations of S.

2.58. Consider a causal LTI system S whose input $x[n]$ and output $y[n]$ are related by the difference equation

$$2y[n] - y[n-1] + y[n-3] = x[n] - 5x[n-4].$$

(a) Verify that S may be considered a cascade connection of two causal LTI systems S_1 and S_2 with the following input-output relationship:

$$S_1 : 2y_1[n] = x_1[n] - 5x_1[n-4],$$
$$S_2 : y_2[n] = \frac{1}{2}y_2[n-1] - \frac{1}{2}y_2[n-3] + x_2[n].$$

(b) Draw a block diagram representation of S_1.
(c) Draw a block diagram representation of S_2.
(d) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.
(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1.
(f) Show that the four delay elements in the block diagram representation of S obtained in part (e) may be collapsed to three. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (d) and (e) are referred to as *Direct Form I* realizations of S.

2.59. Consider a causal LTI system S whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) + b_1 \frac{dx(t)}{dt}.$$

(a) Show that

$$y(t) = A\int_{-\infty}^{t} y(\tau)\,d\tau + Bx(t) + C\int_{-\infty}^{t} x(\tau)\,d\tau,$$

and express the constants A, B, and C in terms of the constants a_0, a_1, b_0, and b_1.

(b) Show that S may be considered a cascade connection of the following two causal LTI systems:

$$S_1 : y_1(t) = Bx_1(t) + C\int_{-\infty}^{t} x(\tau)\,d\tau,$$

$$S_2 : y_2(t) = A\int_{-\infty}^{t} y_2(\tau)\,d\tau + x_2(t).$$

(c) Draw a block diagram representation of S_1.

(d) Draw a block diagram representation of S_2.

(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.

(f) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram of representation S_1.

(g) Show that the two integrators in your answer to part (f) may be collapsed into one. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (e) and (f) are referred to as *Direct Form I* realizations of S.

2.60. Consider a causal LTI system S whose input $x(t)$ and output $y(t)$ are related by the differential equation

$$a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) + b_1 \frac{dx(t)}{dt} + b_2 \frac{d^2 x(t)}{dt^2}.$$

(a) Show that

$$y(t) = A\int_{-\infty}^{t} y(\tau)\,d\tau + B\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} y(\sigma)\,d\sigma\right)d\tau$$
$$+ Cx(t) + D\int_{-\infty}^{t} x(\tau)\,d\tau + E\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} x(\sigma)\,d\sigma\right)d\tau,$$

and express the constants A, B, C, D, and E in terms of the constants a_0, a_1, a_2, b_0, b_1, and b_2.

(b) Show that S may be considered a cascade connection of the following two causal LTI systems:

$$S_1 : y_1(t) = Cx_1(t) + D\int_{-\infty}^{t} x_1(\tau)\,d\tau + E\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} x_1(\sigma)\,d\sigma\right)d\tau,$$

$$S_2 : y_2(t) = A\int_{-\infty}^{t} y_2(\tau)\,d\tau + B\int_{-\infty}^{t}\left(\int_{-\infty}^{\tau} y_2(\sigma)\,d\sigma\right)d\tau + x_2(t).$$

(c) Draw a block diagram representation of S_1.

(d) Draw a block diagram representation of S_2.

(e) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_1 followed by the block diagram representation of S_2.

(f) Draw a block diagram representation of S as a cascade connection of the block diagram representation of S_2 followed by the block diagram representation of S_1.

(g) Show that the four integrators in your answer to part (f) may be collapsed into two. The resulting block diagram is referred to as a *Direct Form II* realization of S, while the block diagrams obtained in parts (e) and (f) are referred to as *Direct Form I* realizations of S.

EXTENSION PROBLEMS

2.61. **(a)** In the circuit shown in Figure P2.61(a), $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered to be the system output.

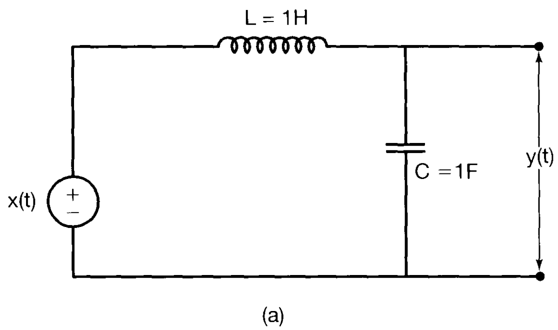


Figure P2.61a

(i) Determine the differential equation relating $x(t)$ and $y(t)$.

(ii) Show that the homogeneous solution of the differential equation from part (i) has the form $K_1 e^{j\omega_1 t} + K_2 e^{j\omega_2 t}$. Specify the values of ω_1 and ω_2.

(iii) Show that, since the voltage and current are restricted to be real, the natural response of the system is sinusoidal.

(b) In the circuit shown in Figure P2.61(b), $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered to be the system output.

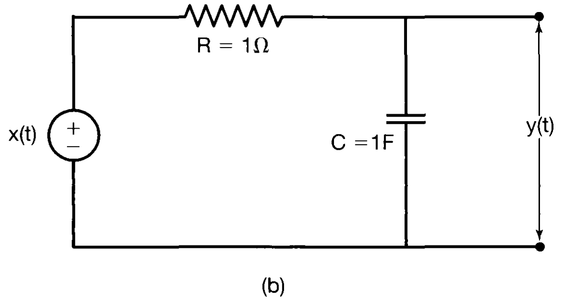


Figure P2.61b

(i) Determine the differential equation relating $x(t)$ and $y(t)$.

(ii) Show that the natural response of this system has the form Ke^{-at}, and specify the value of a.

(c) In the circuit shown in Figure P2.61(c), $x(t)$ is the input voltage. The voltage $y(t)$ across the capacitor is considered to be the system output.

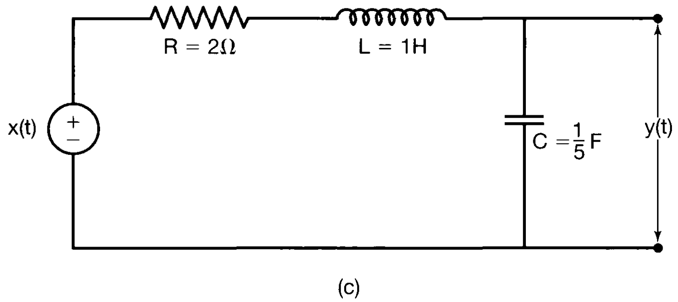


Figure P2.61c

(i) Determine the differential equation relating $x(t)$ and $y(t)$.

(ii) Show that the homogeneous solution of the differential equation from part (i) has the form $e^{-at}\{K_1 e^{j2t} + K_2 e^{-j2t}\}$, and specify the value of a.

(iii) Show that, since the voltage and current are restricted to be real, the natural response of the system is a decaying sinusoid.

2.62. (a) In the mechanical system shown in Figure P2.62(a), the force $x(t)$ applied to the mass represents the input, while the displacement $y(t)$ of the mass represents the output. Determine the differential equation relating $x(t)$ and $y(t)$. Show that the natural response of this system is periodic.

(b) Consider Figure P2.62(b), in which the force $x(t)$ is the input and the velocity $y(t)$ is the output. The mass of the car is m, while the coefficient of kinetic friction is ρ. Show that the natural response of this system decays with increasing time.

(c) In the mechanical system shown in Figure P2.62(c), the force $x(t)$ applied to the mass represents the input, while the displacement $y(t)$ of the mass represents the output.

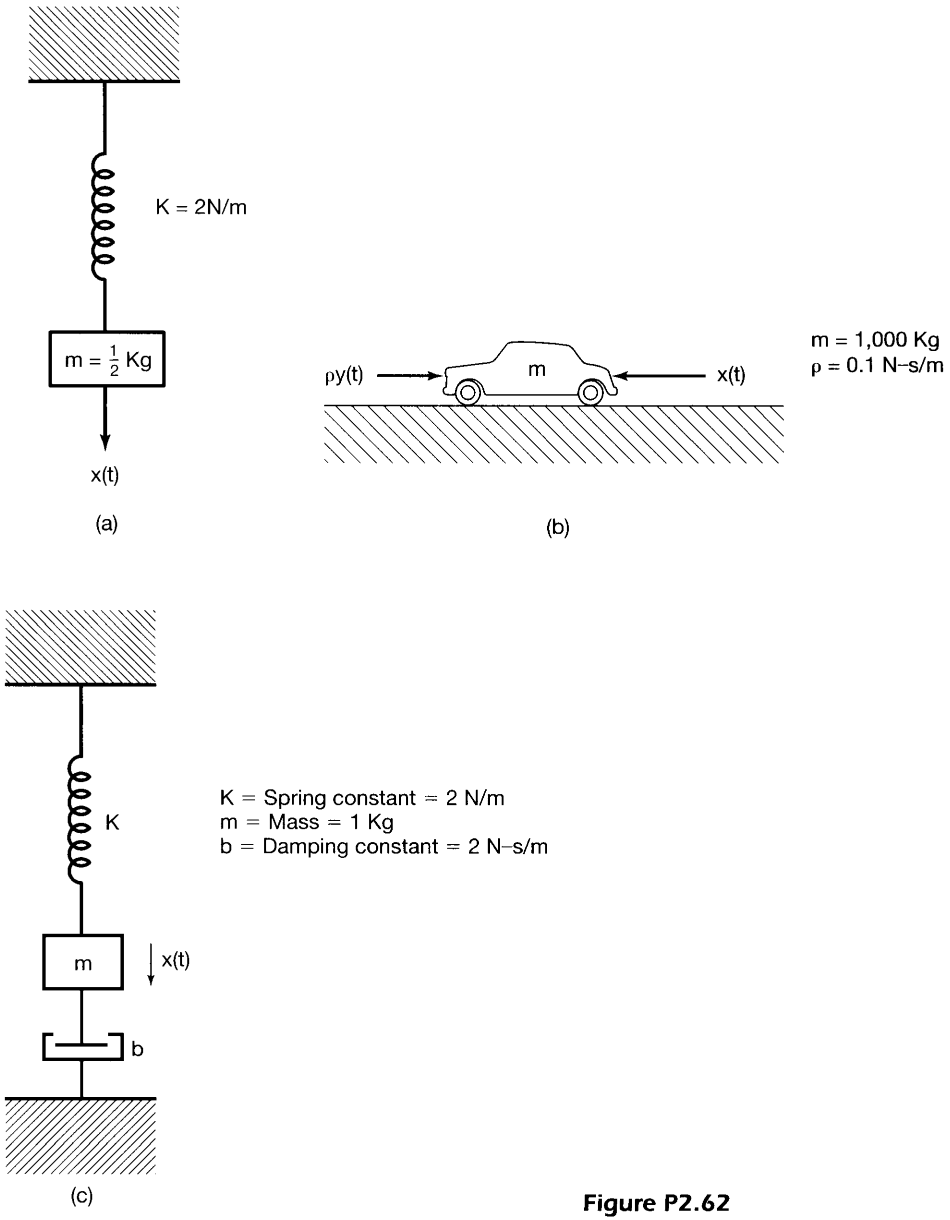


Figure P2.62

(i) Determine the differential equation relating $x(t)$ and $y(t)$.

(ii) Show that the homogeneous solution of the differential equation from part (i) has the form $e^{-at}\{K_1 e^{jt} + K_2 e^{-jt}\}$, and specify the value of a.

(iii) Show that, since the force and displacement are restricted to be real, the natural response of the system is a decaying sinusoid.

2.63. A \$100,000 mortgage is to be retired by *equal* monthly payments of D dollars. Interest, compounded monthly, is charged at the rate of 12% per annum on the unpaid balance; for example, after the first month, the total debt equals

$$\$100{,}000 + \left(\frac{0.12}{12}\right)\$100{,}000 = \$101{,}000.$$

The problem is to determine D such that after a specified time the mortgage is paid in full, leaving a net balance of zero.

(a) To set up the problem, let $y[n]$ denote the unpaid balance after the nth monthly payment. Assume that the principal is borrowed in month 0 and monthly payments begin in month 1. Show that $y[n]$ satisfies the difference equation

$$y[n] - \gamma y[n-1] = -D \quad n \geq 1 \tag{P2.63–1}$$

with initial condition

$$y[0] = \$100{,}000,$$

where γ is a constant. Determine γ.

(b) Solve the difference equation of part (a) to determine

$$y[n] \quad \text{for } n \geq 0.$$

(*Hint:* The particular solution of eq. (P2.63–1) is a constant Y. Find the value of Y, and express $y[n]$ for $n \geq 1$ as the sum of particular and homogeneous solutions. Determine the unknown constant in the homogeneous solution by directly calculating $y[1]$ from eq. (P2.63–1) and comparing it to your solution.)

(c) If the mortgage is to be retired in 30 years after 360 monthly payments of D dollars, determine the appropriate value of D.

(d) What is the total payment to the bank over the 30-year period?

(e) Why do banks make loans?

2.64. One important use of inverse systems is in situations in which one wishes to remove distortions of some type. A good example of this is the problem of removing echoes from acoustic signals. For example, if an auditorium has a perceptible echo, then an initial acoustic impulse will be followed by attenuated versions of the sound at regularly spaced intervals. Consequently, an often-used model for this phenomenon is an LTI system with an impulse response consisting of a train of impulses, i.e.,

$$h(t) = \sum_{k=0}^{\infty} h_k \delta(t - kT). \tag{P2.64–1}$$

Here the echoes occur T seconds apart, and h_k represents the gain factor on the kth echo resulting from an initial acoustic impulse.

(a) Suppose that $x(t)$ represents the original acoustic signal (the music produced by an orchestra, for example) and that $y(t) = x(t) * h(t)$ is the actual signal that is heard if no processing is done to remove the echoes. In order to remove the distortion introduced by the echoes, assume that a microphone is used to sense $y(t)$ and that the resulting signal is transduced into an electrical signal. We will also use $y(t)$ to denote this signal, as it represents the electrical equivalent of the acoustic signal, and we can go from one to the other via acoustic-electrical conversion systems.

The important point to note is that the system with impulse response given by eq. (P2.64–1) is invertible. Therefore, we can find an LTI system with impulse response $g(t)$ such that

$$y(t) * g(t) = x(t),$$

and thus, by processing the electrical signal $y(t)$ in this fashion and then converting back to an acoustic signal, we can remove the troublesome echoes.

The required impulse response $g(t)$ is also an impulse train:

$$g(t) = \sum_{k=0}^{\infty} g_k \delta(t - kT).$$

Determine the algebraic equations that the successive g_k must satisfy, and solve these equations for g_0, g_1, and g_2 in terms of h_k.

(b) Suppose that $h_0 = 1$, $h_1 = 1/2$, and $h_i = 0$ for all $i \geq 2$. What is $g(t)$ in this case?

(c) A good model for the generation of echoes is illustrated in Figure P2.64. Hence, each successive echo represents a fed-back version of $y(t)$, delayed by T seconds and scaled by α. Typically, $0 < \alpha < 1$, as successive echoes are attenuated.

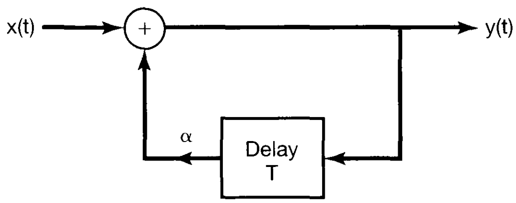


Figure P2.64

(i) What is the impulse response of this system? (Assume initial rest, i.e., $y(t) = 0$ for $t < 0$ if $x(t) = 0$ for $t < 0$.)

(ii) Show that the system is stable if $0 < \alpha < 1$ and unstable if $\alpha > 1$.

(iii) What is $g(t)$ in this case? Construct a realization of the inverse system using adders, coefficient multipliers, and T-second delay elements.

(d) Although we have phrased the preceding discussion in terms of continuous-time systems because of the application we have been considering, the same general ideas hold in discrete time. That is, the LTI system with impulse response

$$h[n] = \sum_{k=0}^{\infty} h_k \delta[n - kN]$$

is invertible and has as its inverse an LTI system with impulse response

$$g[n] = \sum_{k=0}^{\infty} g_k \delta[n - kN].$$

It is not difficult to check that the g_k satisfy the same algebraic equations as in part (a).

Consider now the discrete-time LTI system with impulse response

$$h[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN].$$

This system is *not* invertible. Find two inputs that produce the same output.

2.65. In Problem 1.45, we introduced and examined some of the basic properties of correlation functions for continuous-time signals. The discrete-time counterpart of the correlation function has essentially the same properties as those in continuous time, and both are extremely important in numerous applications (as is discussed in Problems 2.66 and 2.67). In this problem, we introduce the discrete-time correlation function and examine several more of its properties.

Let $x[n]$ and $y[n]$ be two real-valued discrete-time signals. The *autocorrelation functions* $\phi_{xx}[n]$ and $\phi_{yy}[n]$ of $x[n]$ and $y[n]$, respectively, are defined by the expressions

$$\phi_{xx}[n] = \sum_{m=-\infty}^{+\infty} x[m+n]x[m]$$

and

$$\phi_{yy}[n] = \sum_{m=-\infty}^{+\infty} y[m+n]y[m],$$

and the *cross-correlation functions* are given by

$$\phi_{xy}[n] = \sum_{m=-\infty}^{+\infty} x[m+n]y[m]$$

and

$$\phi_{yx}[n] = \sum_{m=-\infty}^{+\infty} y[m+n]x[m].$$

As in continuous time, these functions possess certain symmetry properties. Specifically, $\phi_{xx}[n]$ and $\phi_{yy}[n]$ are even functions, while $\phi_{xy}[n] = \phi_{yx}[-n]$.

(a) Compute the autocorrelation sequences for the signals $x_1[n]$, $x_2[n]$, $x_3[n]$, and $x_4[n]$ depicted in Figure P2.65.

(b) Compute the cross-correlation sequences

$$\phi_{x_i x_j}[n],\ i \neq j,\ \ i, j = 1, 2, 3, 4,$$

for $x_i[n]$, $i = 1, 2, 3, 4$, as shown in Figure P2.65.

(c) Let $x[n]$ be the input to an LTI system with unit sample response $h[n]$, and let the corresponding output be $y[n]$. Find expressions for $\phi_{xy}[n]$ and $\phi_{yy}[n]$ in terms of $\phi_{xx}[n]$ and $h[n]$. Show how $\phi_{xy}[n]$ and $\phi_{yy}[n]$ can be viewed as the output

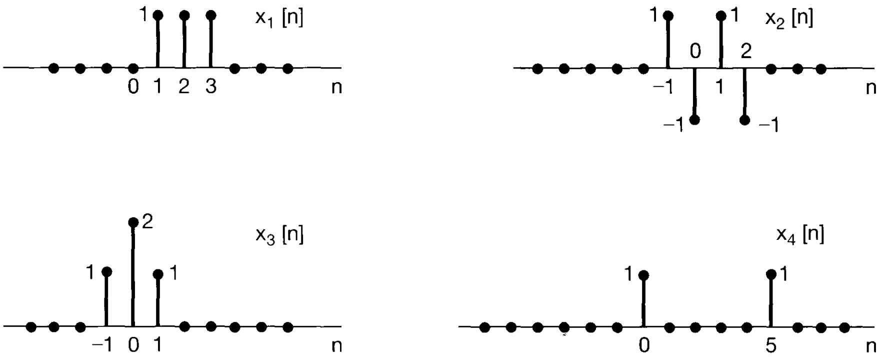


Figure P2.65

of LTI systems with $\phi_{xx}[n]$ as the input. (Do this by explicitly specifying the impulse response of each of the two systems.)

(d) Let $h[n] = x_1[n]$ in Figure P2.65, and let $y[n]$ be the output of the LTI system with impulse response $h[n]$ when the input $x[n]$ also equals $x_1[n]$. Calculate $\phi_{xy}[n]$ and $\phi_{yy}[n]$ using the results of part (c).

2.66. Let $h_1(t)$, $h_2(t)$, and $h_3(t)$, as sketched in Figure P2.66, be the impulse responses of three LTI systems. These three signals are known as *Walsh functions* and are of considerable practical importance because they can be easily generated by digital logic circuitry and because multiplication by each of them can be implemented in a simple fashion by a polarity-reversing switch.

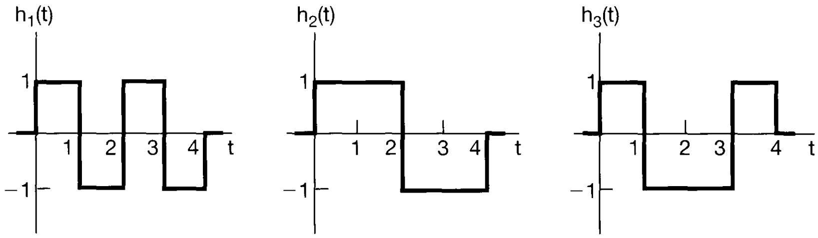


Figure P2.66

(a) Determine and sketch a choice for $x_1(t)$, a continuous-time signal with the following properties:

(i) $x_1(t)$ is real.

(ii) $x_1(t) = 0$ for $t < 0$.

(iii) $|x_1(t)| \leq 1$ for $t \geq 0$.

(iv) $y_1(t) = x_1(t) * h(t)$ is as large as possible at $t = 4$.

(b) Repeat part (a) for $x_2(t)$ and $x_3(t)$ by making $y_2(t) = x_2(t) * h_2(t)$ and $y_3(t) = x_3(t) * h_3(t)$ each as large as possible at $t = 4$.

(c) What is the value of

$$y_{ij}(t) = x_i(t) * h_j(t), \ i \neq j$$

at time $t = 4$ for $i, j = 1, 2, 3$?

The system with impulse response $h_i(t)$ is known as the *matched filter* for the signal $x_i(t)$ because the impulse response is tuned to $x_i(t)$ in order to produce the maximum output signal. In the next problem, we relate the concept of a matched filter to that of the correlation function for continuous-time signals.

2.67. The *cross-correlation function* between two continuous-time real signals $x(t)$ and $y(t)$ is

$$\phi_{xy}(t) = \int_{-\infty}^{+\infty} x(t+\tau)y(\tau)\,d\tau. \qquad \text{(P2.67–1)}$$

The *autocorrelation function* of a signal $x(t)$ is obtained by setting $y(t) = x(t)$ in eq. (P2.67–1):

$$\phi_{xx}(t) = \int_{-\infty}^{+\infty} x(t+\tau)x(\tau)\,d\tau.$$

(a) Compute the autocorrelation function for each of the two signals $x_1(t)$ and $x_2(t)$ depicted in Figure P2.67(a).

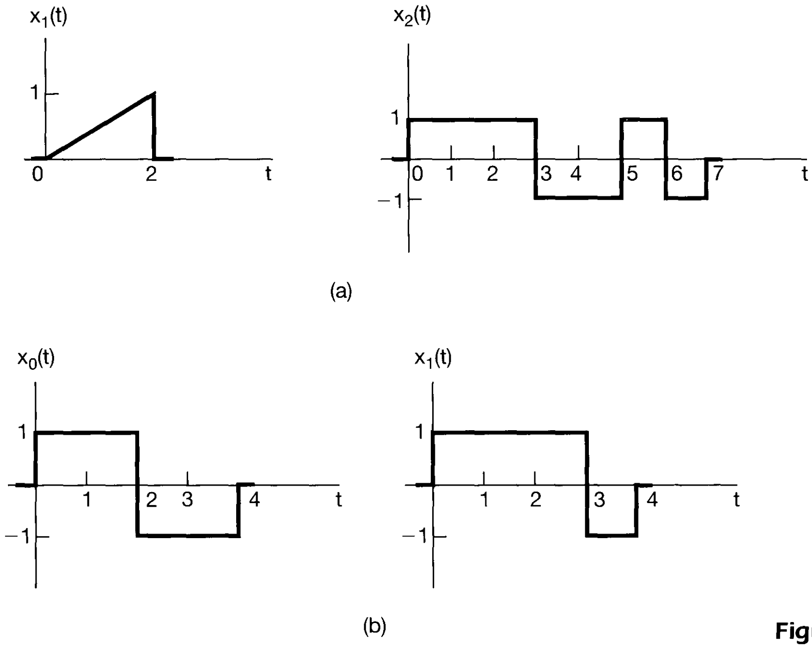


Figure P2.67

(b) Let $x(t)$ be a given signal, and assume that $x(t)$ is of finite duration—i.e., that $x(t) = 0$ for $t < 0$ and $t > T$. Find the impulse response of an LTI system so that $\phi_{xx}(t - T)$ is the output if $x(t)$ is the input.

(c) The system determined in part (b) is a *matched filter* for the signal $x(t)$. That this definition of a matched filter is identical to the one introduced in Problem 2.66 can be seen from the following:

Let $x(t)$ be as in part (b), and let $y(t)$ denote the response to $x(t)$ of an LTI system with real impulse response $h(t)$. Assume that $h(t) = 0$ for $t < 0$ and for $t > T$. Show that the choice for $h(t)$ that maximizes $y(T)$, subject to the constraint that

$$\int_0^T h^2(t)dt = M, \text{ a fixed positive number,} \qquad \text{(P2.67–2)}$$

is a scalar multiple of the impulse response determined in part (b). [*Hint:* Schwartz's inequality states that

$$\int_b^a u(t)v(t)dt \leq \left[\int_a^b u^2(t)dt\right]^{1/2}\left[\int_a^b v^2(t)dt\right]^{1/2}$$

for any two signals $u(t)$ and $v(t)$. Use this to obtain a bound on y(T).]

(d) The constraint given by eq. (P2.67–2) simply provides a scaling to the impulse response, as increasing M merely changes the scalar multiplier mentioned in part (c). Thus, we see that the particular choice for $h(t)$ in parts (b) and (c) is matched to the signal $x(t)$ to produce maximum output. This is an extremely important property in a number of applications, as we will now indicate.

In communication problems, one often wishes to transmit one of a small number of possible pieces of information. For example, if a complex message is encoded into a sequence of binary digits, we can imagine a system that transmits the information bit by bit. Each bit can then be transmitted by sending one signal, say, $x_0(t)$, if the bit is a 0, or a different signal $x_1(t)$ if a 1 is to be communicated. In this case, the receiving system for these signals must be capable of recognizing whether $x_0(t)$ or $x_1(t)$ has been received. Intuitively, what makes sense is to have two systems in the receiver, one tuned to $x_0(t)$ and one tuned to $x_1(t)$, where, by "tuned," we mean that the system gives a large output after the signal to which it is tuned is received. The property of producing a large output when a particular signal is received is exactly what the matched filter possesses.

In practice, there is always distortion and interference in the transmission and reception processes. Consequently, we want to maximize the difference between the response of a matched filter to the input to which it is matched and the response of the filter to one of the other signals that can be transmitted. To illustrate this point, consider the two signals $x_0(t)$ and $x_1(t)$ depicted in Figure P2.67(b). Let L_0 denote the matched filter for $x_0(t)$, and let L_1 denote the matched filter for $x_1(t)$.

(i) Sketch the responses of L_0 to $x_0(t)$ and $x_1(t)$. Do the same for L_1.

(ii) Compare the values of these responses at $t = 4$. How might you modify $x_0(t)$ so that the receiver would have an even easier job of distinguishing between $x_0(t)$ and $x_1(t)$ in that the response of L_0 to $x_1(t)$ and L_1 to $x_0(t)$ would both be zero at $t = 4$?

2.68. Another application in which matched filters and correlation functions play an important role is radar systems. The underlying principle of radar is that an electromagnetic pulse transmitted at a target will be reflected by the target and will subsequently return to the sender with a delay proportional to the distance to the target. Ideally, the received signal will simply be a shifted and possibly scaled version of the original transmitted signal.

Let $p(t)$ be the original pulse that is sent out. Show that

$$\phi_{pp}(0) = \max_t \phi_{pp}(t).$$

That is, $\phi_{pp}(0)$ is the largest value taken by $\phi_{pp}(t)$. Use this equation to deduce that, if the waveform that comes back to the sender is

$$x(t) = \alpha p(t - t_0),$$

where α is a positive constant, then

$$\phi_{xp}(t_0) = \max_t \phi_{xp}(t).$$

(*Hint:* Use Schwartz's inequality.)

Thus, the way in which simple radar ranging systems work is based on using a matched filter for the transmitted waveform $p(t)$ and noting the time at which the output of this system reaches its maximum value.

2.69. In Section 2.5, we characterized the unit doublet through the equation

$$x(t) * u_1(t) = \int_{-\infty}^{+\infty} x(t-\tau)u_1(\tau)\,d\tau = x'(t) \qquad \text{(P2.69–1)}$$

for any signal $x(t)$. From this equation, we derived the relationship

$$\int_{-\infty}^{+\infty} g(\tau)u_1(\tau)\,d\tau = -g'(0). \qquad \text{(P2.69–2)}$$

(a) Show that eq. (P2.69–2) is an equivalent characterization of $u_1(t)$ by showing that eq. (P2.69–2) implies eq. (P2.69–1). [*Hint:* Fix t, and define the signal $g(\tau) = x(t-\tau)$.]

Thus, we have seen that characterizing the unit impulse or unit doublet by how it behaves under convolution is equivalent to characterizing how it behaves under integration when multiplied by an arbitrary signal $g(t)$. In fact, as indicated in Section 2.5, the equivalence of these operational definitions holds for all signals and, in particular, for all singularity functions.

(b) Let $f(t)$ be a given signal. Show that

$$f(t)u_1(t) = f(0)u_1(t) - f'(0)\delta(t)$$

by showing that both functions have the same operational definitions.

(c) What is the value of

$$\int_{-\infty}^{\infty} x(\tau)u_2(\tau)\,d\tau?$$

Find an expression for $f(t)u_2(t)$ analogous to that in part (b) for $f(t)u_1(t)$.

2.70. In analogy with continuous-time singularity functions, we can define a set of discrete-time signals. Specifically, let

$$u_{-1}[n] = u[n],$$
$$u_0[n] = \delta[n],$$

and

$$u_1[n] = \delta[n] - \delta[n-1],$$

and define

$$u_k[n] = \underbrace{u_1[n] * u_1[n] * \cdots * u_1[n]}_{k \text{ times}}, \quad k > 0$$

and

$$u_k[n] = \underbrace{u_{-1}[n] * u_{-1}[n] * \cdots * u_{-1}[n]}_{|k| \text{ times}}, \quad k < 0.$$

Note that

$$x[n] * \delta[n] = x[n],$$
$$x[n] * u[n] = \sum_{m=-\infty}^{\infty} x[m],$$

and

$$x[n] * u_1[n] = x[n] - x[n-1],$$

(a) What is

$$\sum_{m=\infty}^{\infty} x[m]u_1[m]?$$

(b) Show that

$$x[n]u_1[n] = x[0]u_1[n] - [x[1] - x[0]]\delta[n-1]$$
$$= x[1]u_1[n] - [x[1] - x[0]]\delta[n].$$

(c) Sketch the signals $u_2[n]$ and $u_3[n]$.
(d) Sketch $u_{-2}[n]$ and $u_{-3}[n]$.
(e) Show that, in general, for $k > 0$,

$$u_k[n] = \frac{(-1)^n k!}{n!(k-n)!}[u[n] - u[n-k-1]]. \tag{P2.70–1}$$

(*Hint:* Use induction. From part (c), it is evident that $u_k[n]$ satisfies eq. (P2.70–1) for $k = 2$ and 3. Then, assuming that eq. (P2.70–1) satisfies $u_k[n]$, write $u_{k+1}[n]$ in terms of $u_k[n]$, and show that the equation also satisfies $u_{k+1}[n]$.)

(f) Show that, in general, for $k > 0$,

$$u_{-k}[n] = \frac{(n+k-1)!}{n!(k-1)!}u[n]. \qquad \text{(P2.70–2)}$$

(*Hint:* Again, use induction. Note that

$$u_{-(k+1)}[n] - u_{-(k+1)}[n-1] = u_{-k}[n]. \qquad \text{(P2.70–3)}$$

Then, assuming that eq. (P2.70–2) is valid for $u_{-k}[n]$, use eq. (P2.70–3) to show that eq. (P2.70–2) is valid for $u_{-(k+1)}[n]$ as well.)

2.71. In this chapter, we have used several properties and ideas that greatly facilitate the analysis of LTI systems. Among these are two that we wish to examine a bit more closely. As we will see, in certain very special cases one must be careful in using these properties, which otherwise hold without qualification.

(a) One of the basic and most important properties of convolution (in both continuous and discrete time) is associativity. That is, if $x(t)$, $h(t)$, and $g(t)$ are three signals, then

$$x(t) * [g(t) * h(t)] = [x(t) * g(t)] * h(t) = [x(t) * h(t)] * g(t). \qquad \text{(P2.71–1)}$$

This relationship holds as long as all three expressions are well defined and finite. As that is usually the case in practice, we will in general use the associativity property without comments or assumptions. However, there are some cases in which it does *not* hold. For example, consider the system depicted in Figure P2.71, with $h(t) = u_1(t)$ and $g(t) = u(t)$. Compute the response of this system to the input

$$x(t) = 1 \text{ for all } t.$$

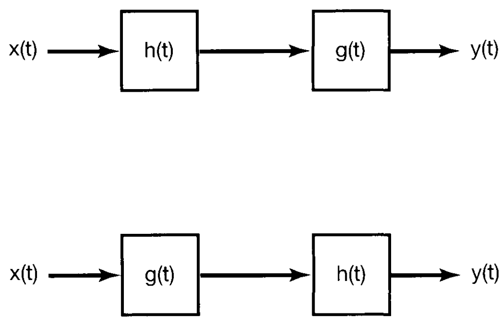


Figure P2.71

Do this in the three different ways suggested by eq. (P2.71–1) and by the figure:

(i) By first convolving the two impulse responses and then convolving the result with $x(t)$.

(ii) By first convolving $x(t)$ with $u_1(t)$ and then convolving the result with $u(t)$.

(iii) By first convolving $x(t)$ with $u(t)$ and then convolving the result with $u_1(t)$.

(b) Repeat part (a) for

$$x(t) = e^{-t}$$

and

$$h(t) = e^{-t}u(t),$$
$$g(t) = u_1(t) + \delta(t).$$

(c) Do the same for

$$x[n] = \left(\frac{1}{2}\right)^n,$$
$$h[n] = \left(\frac{1}{2}\right)^n u[n],$$
$$g[n] = \delta[n] - \frac{1}{2}\delta[n-1].$$

Thus, in general, the associativity property of convolution holds if and only if the three expressions in eq. (P2.71–1) make sense (i.e., if and only if their interpretations in terms of LTI systems are meaningful). For example, in part (a) differentiating a constant and then integrating makes sense, but the process of integrating the constant from $t = -\infty$ and *then* differentiating does not, and it is only in such cases that associativity breaks down.

Closely related to the foregoing discussion is an issue involving inverse systems. Consider the LTI system with impulse response $h(t) = u(t)$. As we saw in part (a), there are inputs—specifically, $x(t) =$ nonzero constant—for which the output of this system is infinite, and thus, it is meaningless to consider the question of inverting such outputs to recover the input. However, if we limit ourselves to inputs that do yield finite outputs, that is, inputs which satisfy

$$\left| \int_{-\infty}^{t} x(\tau)\, d\tau \right| < \infty, \tag{P2.71–2}$$

then the system *is* invertible, and the LTI system with impulse response $u_1(t)$ is its inverse.

(d) Show that the LTI system with impulse response $u_1(t)$ is *not* invertible. (*Hint:* Find two different inputs that both yield zero output for all time.) However, show that the system is invertible if we limit ourselves to inputs that satisfy eq. (P2.71–2). [*Hint:* In Problem 1.44, we showed that an LTI system is invertible if no input other than $x(t) = 0$ yields an output that is zero for all time; are there two inputs $x(t)$ that satisfy eq. (P2.71–2) and that yield identically zero responses when convolved with $u_1(t)$?]

What we have illustrated in this problem is the following:

(1) If $x(t)$, $h(t)$, and $g(t)$ are three signals, and if $x(t) * g(t)$, $x(t) * h(t)$, and $h(t) * g(t)$ are *all* well defined and finite, then the associativity property, eq. (P2.71–1), holds.

(2) Let $h(t)$ be the impulse response of an LTI system, and suppose that the impulse response $g(t)$ of a second system has the property

$$h(t) * g(t) = \delta(t). \qquad \text{(P2.71–3)}$$

Then, from (1), *for* all inputs $x(t)$ for which $x(t) * h(t)$ and $x(t) * g(t)$ are both well defined and finite, the two cascades of systems depicted in Figure P2.71 act as the identity system, and thus, the two LTI systems can be regarded as inverses of one another. For example, if $h(t) = u(t)$ and $g(t) = u_1(t)$, then, as long as we restrict ourselves to inputs satisfying eq. (P2.71–2), we can regard these two systems as inverses.

Therefore, we see that the associativity property of eq. (P2.71–1) and the definition of LTI inverses as given in eq. (P2.71–3) are valid, as long as all convolutions that are involved are finite. As this is certainly the case in any realistic problem, we will in general use these properties without comment or qualification. Note that, although we have phrased most of our discussion in terms of continuous-time signals and systems, the same points can also be made in discrete time [as should be evident from part (c)].

2.72. Let $\delta_\Delta(t)$ denote the rectangular pulse of height $\frac{1}{\Delta}$ for $0 < t \leq \Delta$. Verify that

$$\frac{d}{dt}\delta_\Delta(t) = \frac{1}{\Delta}[\delta(t) - \delta(t - \Delta)].$$

2.73. Show by induction that

$$u_{-k}(t) = \frac{t^{k-1}}{(k-1)!}u(t) \text{ for } k = 1, 2, 3 \ldots$$