2.45. a) Show that if the response of an LTI system to $x(t)$ is $y(t)$, then the response of $x'(t) = \cancel{dx}\ \frac{dx(t)}{dt}$ is $y'(t)$. Do this in three ways:

i) Directly from the properties of linearity and time invariance, and $x'(t) = \lim_{h\to0} \frac{x(t)-x(t-h)}{h}$

$$S:\ x(t) \to y(t) = S(y(t))$$

$$x'(t) = \lim_{h\to0}\frac{x(t)-x(t-h)}{h} \to S\left(\lim_{h\to0}\frac{x(t)-x(t-h)}{h}\right) = \lim_{h\to0} S\left(\frac{x(t)-x(t-h)}{h}\right) \underset{\text{Thanks to linearity}}{=} \lim_{h\to0}\frac{S(x(t))-S(x(t-h))}{h} \underset{\text{Thanks to time invariance}}{=}$$

$$= \lim_{h\to0}\frac{y(t)-y(t-h)}{h} = \frac{dy(t)}{dt}$$

ii) By differentiating the convolution integral

Let $y(t) = x(t) * h(t)$, where $h(t)$ is the impulse response. $y(t) = x(t)*h(t) = h(t)*x(t)$

$$\frac{dx(t)}{dt} \to \int_{-\infty}^{\infty} h(\tau)\frac{dx(t-\tau)}{dt}\,d\tau$$

$$\frac{dy(t)}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau =$$

$$= \int_{-\infty}^{\infty} h(\tau)\frac{dx(t-\tau)}{dt}\,d\tau = S\left(\frac{dx(t)}{dt}\right)$$

iii) By examining this system: $x(t) \to \boxed{u_1(t)} \to \boxed{h(t)} \to y(t)$

Both are LTI systems: $x(t) \to \boxed{u_1(t)} \xrightarrow{x'(t)} \boxed{h(t)} \to z(t) = S(x'(t))$

If we switch the cascade order, the final ~~results~~ result stays the same (since differentiation is also an LTI system)

$$x(t) \to \boxed{h(t)} \xrightarrow{y(t)} \boxed{u_1(t)} \to y'(t)$$

b) Prove the following:

i) $y'(t) = x(t) * h'(t)$

By swapping the ~~names~~ names of x and h we get: $y'(t) = h(t) * x'(t)$, which I proved in part a.

ii) $y(t) = \left(\int_{-\infty}^{t} x(\tau)\,d\tau\right) * h'(t) = \int_{-\infty}^{t} (x'(\tau) * h(\tau))\,d\tau = x'(t) * \int_{-\infty}^{t} h(\tau)\,d\tau$

(Let $u_1(t)$: $x(t) * u_1(t) = \frac{dx(t)}{dt}$)

$$\left(\int_{-\infty}^{t} x(\tau)\,d\tau\right) * h'(t) = \left(\int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau\right) * h(t) * u_1(t) = x(t) * u(t) * u_1(t) * h(t) = x(t) * \delta(t) * h(t) = x(t) * h(t)$$

$$\int_{-\infty}^{t} (x'(\tau) * h(\tau))\,d\tau = \int_{-\infty}^{\infty} (x'(\tau) * h(\tau))\cdot u(t-\tau)\,d\tau = (x'(t) * h(t)) * u(t) = x(t) * u_1(t) * u(t) * h(t) = x(t) * \delta(t) * h(t) = x(t) * h(t)$$

$$x'(t) * \int_{-\infty}^{t} h(\tau)\,d\tau = x(t) * u_1(t) * \int_{-\infty}^{\infty} h(\tau)\,u(t-\tau)\,d\tau = x(t) * u_1(t) * h(t) * u(t) = x(t) * \delta(t) * h(t) = x(t) * h(t)$$