... 1.44. (2)

d) Show that invertibility for a discrete-time linear system is equivalent to this statement:

The only input that produces $y[n]=0$ for all n is $x[n]=0$ for all n.

The analogous holds for continuous time.

An invertible system is one which allows determining the input that produced any given output. This is only possible if each unique input produces a unique output. If this were not the case, and two different inputs were to produce the ~~same~~ same output, it would be ~~for~~ ~~fundamentally~~ impossible to determine which input signal led to that output, making the system non-invertible.

This can be expressed as:

$$y_1[n]=y_2[n] \ \forall n\in\mathbb{Z} \iff x_1[n]=x_2[n] \ \forall n\in\mathbb{Z}$$

- Linear system: $\alpha x_1[n]+\beta x_2[n] \xrightarrow{S} \alpha y_1[n]+\beta y_2[n]$

~~Where~~

Given $y_1[n]=y_2[n] \ \forall n\in\mathbb{Z} \iff y_1[n]-y_2[n]=0 \ \forall n\in\mathbb{Z}$

We may call this $y_3[n]=0 \ \forall n\in\mathbb{Z}$

If our statement holds, $y_3[n]=0 \ \forall n\in\mathbb{Z} \iff x_3[n]=0 \ \forall n\in\mathbb{Z}$

Due to linearity, $y_3[n]=y_1[n]-y_2[n]$ is the output of $x_3[n]=x_1[n]-x_2[n]$ (where $x_1[n]\to y_1[n]$, $x_2[n]\to y_2[n]$)

$x_3[n]=0 \ \forall n\in\mathbb{Z} \iff x_1[n]-x_2[n]=0 \ \forall n\in\mathbb{Z} \iff x_1[n]=x_2[n] \ \forall n\in\mathbb{Z}$

Therefore, in a linear ~~system~~ system, if the statement holds, it can be concluded that the system is invertible, so the statement is equivalent to invertibility.

e) Find a nonlinear system that satisfies the previous condition but is not invertible.

One such system is the one that maps $x[n]$ to $y[n]=\begin{cases}0 & \text{if } x[n]=0\\ 1 & \text{if } x[n]\neq 0\end{cases}$

This system is clearly nonlinear, and it satisfies the condition: $y[n]=0 \ \forall n\in\mathbb{Z} \iff x[n]=0 \ \forall n\in\mathbb{Z}$

However, it is not invertible, since any signal with the same support will produce the same output. Take $x_1[n]=u[n]$ and $x_2[n]=2u[n]$.

$$y_1[n]=\begin{cases}0 & \text{if } u[n]=0\\ 1 & \text{if } u[n]\neq 0\end{cases} = \begin{cases}0 & \text{if } \begin{cases}1 & \text{if } n\geq 0\\ 0 & \text{if } n<0\end{cases}=0\\ 1 & \text{if } \begin{cases}1 & \text{if } n\geq 0\\ 0 & \text{if } n<0\end{cases}\neq 0\end{cases} = \begin{cases}0 & \text{if } n<0\\ 1 & \text{if } n\geq 0\end{cases}$$

$$y_2[n]=\begin{cases}0 & \text{if } 2u[n]=0\\ 1 & \text{if } 2u[n]\neq 0\end{cases} = \begin{cases}0 & \text{if } u[n]=0\\ 1 & \text{if } u[n]\neq 0\end{cases} = \begin{cases}0 & \text{if } n<0\\ 1 & \text{if } n\geq 0\end{cases} = y_1[n] \Rightarrow \text{non invertible}$$