2.63. A $ 100,000 mortgage is to be retired by equal monthly payments of D dollars, with monthly compounded interest of 12% on the unpaid balance. After the first month, the total debt is

$\$100,000 + \left(\frac{0.12}{12}\right) \cdot \$100,000 = \$101,000$.

Determine D such that after a specified time the mortgage is paid in full.

a) To set up the problem, let $y[n]$ be the unpaid balance after the n-th monthly payment, assuming the principal is borrowed in month 0 and the first payment is made in month 1. Show that $y[n]$ satisfies $y[n] - \gamma\, y[n-1] = -D \quad n \geq 1$, $y[0] = \$100,000$, where $\gamma$ is a constant. Determine $\gamma$.

+12% yearly $\Rightarrow$ +1% monthly (as per the example provided)

Debt on month n is equal to debt on month n-1 plus interest, minus paid amount:

$y[n] = \gamma\, y[n-1] - D$, $n \geq 1 \Rightarrow y[n] - \gamma\, y[n-1] = -D$, $n \geq 1$

From the example, we have that $\boxed{\gamma = 100\% + 1\% = 1.01}$

b) Solve the difference equation to determine $y[n]$ for $n \geq 0$

Let $y_h[n] = A\gamma^n$: $y_h[n] - \gamma\, y_h[n-1] = 0 \Leftrightarrow A\gamma^n - \gamma A\gamma^{n-1} = 0 \Leftrightarrow 0 = 0$

Let $y_p[n] = Y$: $y_p[n] - \gamma\, y_p[n-1] = -D \;\forall n \geq 1 \Leftrightarrow Y - \gamma Y = -D \;\forall n \geq 1 \Leftrightarrow Y = \frac{-D}{1-\gamma} = \frac{-D}{1-1.01} = 100D$ $\Rightarrow$

$y[0] = 100\,000 = y_h[0] + y_p[n] = A\gamma^0 + Y \Rightarrow A = y[0] - Y = 100\,000 - D$

$\Rightarrow \boxed{y[n] = (100,000 - 100D)\,\gamma^n + 100D = (100,000 - 100D)\cdot 1.01^n + 100D \quad \forall\, n \geq 0}$

Solving the first equation manually: $y[1] - \gamma\, y[0] = -D \Rightarrow$

$\Rightarrow y[1] = 1.01 \cdot 100,000 - D = 101,000 - D$

The obtained expression yields $y[1] = (100,000 - 100D) \cdot 1.01 + 100D =$

$= 101,000 - 101D + 100D = 101,000 - D$

Nice!

c) What is the total payment to the bank over the 30-year period?

Period of 30 years $\Rightarrow$ after $30 \cdot 12$ months the debt is paid in full.

$y[360] = 0 = (100,000 - 100D) \cdot 1.01^{360} + 100D =$

$= 100,000 \cdot 1.01^{360} - 100D \cdot 1.01^{360} + 100D \Rightarrow$

$\Rightarrow D\,(100 - 100 \cdot 1.01^{360}) = -100,000 \cdot 1.01^{360} \Rightarrow D = \frac{100,000 \cdot 1.01^{360}}{100 \cdot (1.01^{360} - 1)} = \$1028.61$

$\boxed{\text{Total paid} = 360 \cdot D = \$370,300.53} \approx 3.7 \cdot y[0] = 370\% \cdot y[0]$

c) Why do banks make loans?

Gosh, I wonder why... not to mention the late payment fees.