2.22. For each pair of waveforms, use the convolution integral to find the response $y(t)$ of the LTI system with ~~ip~~ impulse ~~respones~~ response $h(t)$ and input $x(t)$. Sketch the results.

a) $\left.\begin{aligned} x(t) &= e^{-\alpha t}\, u(t) \\ h(t) &= e^{-\beta t}\, u(t) \end{aligned}\right\}$ (both cases, $\alpha \neq \beta$ and $\alpha = \beta$)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau}\, u(\tau)\, e^{-\beta(t-\tau)}\, u(t-\tau)\, d\tau = \int_{0}^{\max(t,0)} e^{-\alpha\tau - \beta t + \beta\tau}\, d\tau =$$

$$= e^{-\beta t}\, u(t) \cdot \int_0^t e^{+\tau(\beta-\alpha)}\, d\tau = e^{-\beta t}\, u(t) \cdot \begin{cases} \int_0^t e^{\tau(\beta-\alpha)}\, d\tau & \text{if } \alpha \neq \beta \\ \int_0^t d\tau & \text{if } \alpha = \beta \end{cases} = e^{-\beta t}\, u(t) \cdot \begin{cases} \left(\dfrac{e^{\tau(\beta-\alpha)}}{\beta-\alpha}\right)\Big|_{\tau=0}^{\tau=t} & \text{if } \alpha \neq \beta \\ t & \text{if } \alpha = \beta \end{cases} =$$

$$= e^{-\beta t}\, u(t) \cdot \begin{cases} \dfrac{e^{t(\beta-\alpha)} - 1}{\beta-\alpha} & \text{if } \alpha \neq \beta \\ t & \text{if } \alpha = \beta \end{cases} = \boxed{\begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{\beta-\alpha} \cdot u(t) & \text{if } \alpha \neq \beta \\ t e^{-\beta t} \cdot u(t) & \text{if } \alpha = \beta \end{cases}}$$

(*a)

b) $x(t) = u(t) - 2u(t-2) + u(t-5)$

$h(t) = e^{2t}\, u(1-t)$

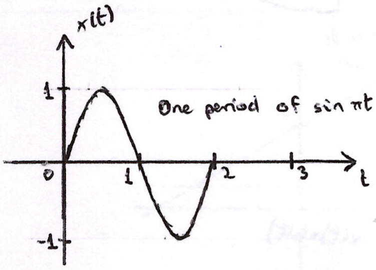
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} (u(t) - 2u(t-2) + u(t-5)) \cdot e^{2(t-\tau)} \cdot u(1-(t-\tau))\, d\tau =$$

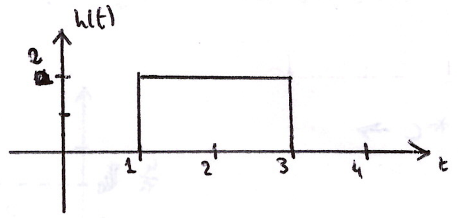
$$= \int_{\max(0,t-1)}^{\max(2,t-1)} e^{2(t-\tau)}\, d\tau - \int_{\max(2,t-1)}^{\max(5,t-1)} e^{2(t-\tau)}\, d\tau = \left(\frac{e^{2t-2\tau}}{-2}\right)\Bigg|_{\tau=\max(0,t-1)}^{\max(2,t-1)} - \left(\frac{e^{2t-2\tau}}{-2}\right)\Bigg|_{\tau=\max(2,t-1)}^{\max(5,t-1)} =$$

$$= \frac{1}{2}\left(-e^{2t-2\max(2,t-1)} + e^{2t-2\max(0,t-1)} + e^{2t-2\max(5,t-1)} - e^{2t-2\max(2,t-1)}\right) = \boxed{\frac{1}{2}\begin{cases} e^{2t}(-2e^{-4} + e^{-10} + 1), & t < 1 \\ e^{2t}(e^{-10} - 2e^{-4}) + e^{2}, & 1 \le t < 3 \\ e^{2t-10} - e^{2}, & 3 \le t < 5 \\ 0, & 5 < t \end{cases}}$$

(*b)

c)





$x(t) = \sin \pi t \cdot u(t) \cdot u(2-t)$; $h(t) = 2u(t-1) \cdot u(3-t)$

$$x(t) * h(t) = \int_{-\infty}^{\infty} \sin \pi\tau \cdot \underbrace{u(\tau)}_{\tau \ge 0} \cdot \underbrace{u(2-\tau)}_{\tau \le 2} \cdot 2\underbrace{u(t-\tau-1)}_{\tau \le t-1} \cdot \underbrace{u(3-t+\tau)}_{\tau \ge t-3}\, d\tau = \begin{cases} 2\int_{\max(0,t-3)}^{\min(2,t-1)} \sin \pi\tau\, d\tau & \text{if } t-3 \le 2 \text{ \& } t-1 \ge 0 \\ 0 & \text{otherwise} \end{cases} =$$

$$= \begin{cases} -\frac{2}{\pi} (\cos \pi\tau)\Big|_{\tau=\max(0,t-3)}^{\min(2,t-1)} & \text{if } t \le 5 \text{ \& } t \ge 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} -\frac{2}{\pi}(\cos(\pi \min(2,t-1)) - \cos(\pi \max(0,t-3))) & \text{if } 1 \le t \le 5 \\ 0 & \text{otherwise} \end{cases} =$$

$$= \begin{cases} -\frac{2}{\pi} \cdot (\cos(\pi t - \pi) - \cos(0)) & \text{if } 1 \le t \le 3 \\ -\frac{2}{\pi} \cdot (\cos 2\pi - \cos(\pi t - 3\pi)) & \text{if } 3 < t \le 5 \\ 0 & \text{otherwise} \end{cases} = \boxed{\frac{2}{\pi}\begin{cases} (\cos(\pi t) + 1) & \text{if } 1 \le t \le 3 \\ -(\cos(\pi t) + 1) & \text{if } 3 \le t \le 5 \\ 0 & \text{otherwise} \end{cases}}$$

(*c)