2.31. Consider a system whose input and output are related by $\frac{dy(t)}{dt} + 2y(t) = x(t)$. Assume it satisfies ~~initial~~ final rest. Show that this system is not causal.

(final rest: $x(t) = 0 \ \forall \ t > t_0 \Rightarrow y(t) = 0 \ \forall \ t > t_0$)

Let $x_1(t) = 0$ and $x_2(t) = e^t(u(t) - u(t-1)) = \begin{cases} e^t & \text{if } 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$

- $y_1(t) = 0$
- $y_2(t) = 0 \ \forall \ t > 1 \Leftarrow x_2(t) = 0 \ \forall \ t > 1 \Leftarrow x_2(t) = 0 \ \forall \ t \notin [0,1]$

For $0 \leq t < 1$: $\frac{dy_2(t)}{dt} + 2y_2(t) = e^t \Rightarrow \begin{bmatrix} y_2 = uv \\ y_2' = u'v + uv' \end{bmatrix} \to u'v + uv' + 2uv = e^t \Rightarrow$

$\Rightarrow u'v + u(v' + 2v) = e^t$

- $v' + 2v = 0 \Leftrightarrow$ ~~$v' = -2v$ $v'(t) = -2v(t) \Rightarrow$~~ $\frac{dv(t)}{dt} = -2v(t) \Rightarrow \frac{1}{v(t)} dv(t) = -2\,dt \Rightarrow$

$\Rightarrow \int \frac{dv(t)}{v(t)} = \int -2\,dt \Rightarrow \ln(v(t)) = -2t + c \Rightarrow v(t) = e^{-2t}$

- $\frac{du(t)}{dt} \cdot v(t) = e^t \Rightarrow \frac{du(t)}{dt} e^{-2t} = e^t \Rightarrow \frac{du(t)}{dt} = e^{3t} \Rightarrow \int du(t) = \int e^{3t}\,dt \Rightarrow$

$\Rightarrow u(t) = \frac{1}{3} e^{3t} + c_2$

$y_2 = \left(\frac{1}{3} e^{3t} + c_2\right) \cdot \left(e^{-2t}\right) = \frac{1}{3} e^{t} + c_1 e^{-2t}$

$y_2(1) = 0 = \frac{1}{3} e + c_1 e^{-2} \Rightarrow c_1 e^{-2} = -\frac{1}{3} e \Rightarrow c_1 = -\frac{e^3}{3} \Rightarrow y_2(t) = \frac{1}{3} e^t - \frac{1}{3} e^{-2t+3}$

$y_2(0) = \frac{1}{3} e^0 - \frac{1}{3} e^3 = 1 - \frac{1}{3} e^3$

For $t < 0$:

$\frac{dy_2(t)}{dt} + 2y_2(t) = 0 \Rightarrow \frac{1}{y_2(t)} dy_2(t) = -2\,dt \Rightarrow \int \frac{dy_2(t)}{y_2(t)} = -2 \int dt = \ln(y_2(t)) = -2t + c \Rightarrow y_2(t) = e^{-2t+c} \Rightarrow$

$\Rightarrow y_2(t) = C \cdot e^{-2t}$

$y_2(0) = 1 - \frac{1}{3} e^3 = Ce^{-2 \cdot 0} = Ce^0 = C \Rightarrow y_2(t) = \left(1 - \frac{e^3}{3}\right) e^{-2t}$

Now, let's see $t = -1$: $y_2(-1) = \left(1 - \frac{e^3}{3}\right) e^2 = e^2 - \frac{e^5}{3} \neq 0$

As we can see, both $x_1(t)$ and $x_2(t)$ are $0 \ \forall \ t < 0$, but $y_2(t)$ is not $0 \ \forall \ t < 0$, so the system is not causal.