~~1.4.4~~

1.4.4.

a) Show that causality for a continuous time-invariant system is equivalent to this statement:

For any time t_0 and input $x(t)$ such that $x(t)=0$ for $t<t_0$, its output $y(t)$ must be 0 $\forall t<t_0$.

The analogous statement works for discrete time.

A causal system is one ~~which~~ where the output at any given time depends only on the present and the past, but not on the ~~fut~~ future.

Since a causal system can't see the future, the outputs of two signals that are identical up to a certain point in time must be the same up to that point in time, because the system can't know if they differ in the future (assuming it's a deterministic system which generates the same output with the same input).

This can be mathematically described as:

For two input signals $x_1(t)$ and $x_2(t)$ and their outputs $y_1(t)$ and $y_2(t)$:

$$x_1(t)=x_2(t)\ \forall t<t_0 \Rightarrow y_1(t)=y_2(t)\ \forall t<t_0$$

Then, let $x_2(t)=0$ constant

$$x_1(t)=0\ \forall t<t_0 \Rightarrow y_1(t)=0\ \forall t<t_0$$

~~Which proves the statement~~ *

This can ~~be~~ be used to prove the statement for discrete time.

b) Find a nonlinear system that satisfies the condition but is not causal

One such system would be the system with output $y(t)=\begin{cases}0 & \text{if } x(t)=0\\ x(t+1)^2 & \text{if } x(t)\neq 0\end{cases}$ for input $x(t)$

It's easy to see that it is nonlinear.

It obviously satisfies the condition: $x(t)=0\ \forall t<t_0 \Rightarrow y(t)=0$ if $0=0$ $\forall t<t_0$

But it is not causal, since it depends on x at $t+1>t$

c) Find a nonlinear system that is causal but does not satisfy the equation.

One such system would be the system with output $y(t)=x(t)+1$

It's easy to see that it is nonlinear.

It doesn't satisfy the condition: $x(t)=0\ \forall t<t_0 \Rightarrow y(t)=0+1=1\ \forall t<t_0$

But it is causal, since it only depends on x at t (it's also memoryless)

* →

$$\text{Linear: } \alpha x_1(t)+\beta x_2(t) \xrightarrow{S} \alpha y_1(t)+\beta y_2(t)$$

Whenever we have $x_1(t)=x_2(t)\ \forall t<t_0 \Leftrightarrow x_1(t)-x_2(t)=0\ \forall t<t_0$

Due to linearity, $x_3(t)=x_1(t)-x_2(t) \xrightarrow{S} y_3(t)=y_1(t)-y_2(t)$

If we know that $x(t)=0\ \forall t<t_0 \Rightarrow y(t)=0\ \forall t<t_0$, then:

$x_1(t)-x_2(t)=0\ \forall t<t_0 \Rightarrow y_1(t)-y_2(t)=0\ \forall t\leq t_0$, and so $y_1(t)=y_2(t)=0\ \forall t\leq t_0$

Therefore, using the statement we can prove a linear system is causal, proving the statement is equivalent to causality.