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An improved Helmholtz coil and analysis of its magnetic field homogeneity

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We attempt to improve the Helmholtz coil by adding a third coil with the same radius connected in series. In this three-coil array, the distance from the outside coils to the central coil is 0.760 times radius of the coils, and the ratio of the number of turns, central to the outside, is exactly 59 to 111. Numerical results demonstrate that the uniform region of magnetic field in the improved Helmholtz coil is larger than that of the Helmholtz coil and of Maxwell's tricoil. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1471352]

I. INTRODUCTION

In instruments and systems that require uniform field, it is of significance to obtain a larger uniform region of magnetic field. General analysis of the magnetic field in Helmhotz coil and its homogeneity had been presented.[1–4] However, it is noticed that the uniform region of magnetic field in the Helmhotz coil is not large enough for some applications. To obtain a large uniform region, many works have been done.[1,5] In this article, we attempt to improve the Helmholtz coil by adding a third coil with the same radius connected in series. In this three-coil array, the distance from the outside circles to the central circle is 0.760 times radius of the circles, and the ratio of number of turns, central to the outside, is exactly 59 to 111. The numerical results demonstrate that the uniform region of magnetic field in the improved Helmholtz coil is not only larger than that of the Helmholtz coil, but also Maxwell's tricoil, and the homogeneity of the magnetic field in the improved Helmholtz coil is better than that in the Helmholtz coil and Maxwell's tricoil. At the end of the article, we analyze the reason why the homogeneity of the magnetic field in the improved Helmholtz coil is better than that in the Helmholtz coil and Maxwell's tricoil, and propose a method to further raise the homogeneity of the magnetic field. Because the improved Helmholtz coil can be designed conveniently, we hope that the improved Helmholtz coil proposed will be of value in practical applications where the large uniform region of the magnetic field is desired.

Consider an axial symmetric magnetic field. Let the origin O lie in the middle of the symmetric axis (taken as the z axis). By symmetry, it is evident that the axial field $B(z)$ (magnetic field intensity on the z axis) is an even function. Expanding $B(z)$ in the Taylor series, we can write

$$B(z)=B(0)+\tfrac{1}{2}B^{(2)}(0)z^2+\tfrac{1}{24}B^{(4)}(0)z^4+\tfrac{1}{720}B^{(6)}(0)z^6+\cdots, \tag{1}$$

where $B^{(2)}(0)$, $B^{(4)}(0)$, and $B^{(6)}(0)$ represent values of the second, fourth, and sixth derivatives of $B(z)$ at the center O, respectively.

From expression (1), it can be seen that the magnetic field on the axis is very close to homogeneous magnetic field if the condition $B^{(2)}(0)=0$ is satisfied. (That is the key point for the magnetic field homogeneity of the Helmholtz coil, see below.) It is obvious that the uniform region of the magnetic field will be enlarged if the conditions $B^{(2)}(0)=0$ and $B^{(4)}(0)=0$ are satisfied. In fact, the magnetic fields of Maxwell's tricoil and the improved Helmholtz coil just satisfy the conditions, so the uniform regions of magnetic fields of Maxwell's tricoil and the improved Helmholtz coil are both larger than that of the Helmholtz coil.[6]

For the Helmholtz coil, let N be the number of turns of each coil, I be electric current through the coils connected in series, and R be the radius of the circles (as well as the distance between the two circles), then the magnetic field intensity at any point on the axis of the coils is

$$B(z)=\frac{1}{2}\mu_0NIR^2\left\{\left[R^2+\left(z+\frac{R}{2}\right)^2\right]^{-3/2}+\left[R^2+\left(z-\frac{R}{2}\right)^2\right]^{-3/2}\right\}. \tag{2}$$

At the center O, the magnetic field intensity is

$$B(0)=\frac{16}{5\sqrt{5}}\,\frac{1}{2}\,\frac{\mu_0NI}{R}. \tag{3}$$

By expressions (2) and (3), the relative difference between magnetic field intensity at any point on the axis and that at the center O (this represents a measure of uniformity of magnetic field) is

$$\varepsilon=\frac{B(z)-B(0)}{B(0)}=\frac{5\sqrt{5}}{16}\left\{\left[1+\left(\frac{1}{2}+\xi\right)^2\right]^{-3/2}+\left[1+\left(\frac{1}{2}-\xi\right)^2\right]^{-3/2}\right\}-1, \tag{4}$$

where $\xi=z/R$.

Moreover, we can get formulas for the off-axis magnetic field as in the following:[6]

$$B_r(r,z)=\frac{\mu_0}{2\pi}\frac{NI(z+R/2)}{r[(R+r)^2+(z+R/2)^2]^{1/2}}\times\left[-K_1+\frac{R^2+r^2+(z+R/2)^2}{(R-r)^2+(z+R/2)^2}E_1\right]+\frac{\mu_0}{2\pi}\frac{NI(z-R/2)}{r[(R+r)^2+(z-R/2)^2]^{1/2}}\times\left[-K_2+\frac{R^2+r^2+(z-R/2)^2}{(R-r)^2+(z-R/2)^2}E_2\right], \tag{5}$$

$$B_z(r,z)=\frac{\mu_0}{2\pi}\frac{NI}{[(R+r)^2+(z+R/2)^2]^{1/2}}\times\left[K_1+\frac{R^2-r^2-(z+R/2)^2}{(R-r)^2+(z+R/2)^2}E_1\right]+\frac{\mu_0}{2\pi}\frac{NI}{[(R+r)^2+(z-R/2)^2]^{1/2}}\times\left[K_2+\frac{R^2-r^2-(z-R/2)^2}{(R-r)^2+(z-R/2)^2}E_2\right], \tag{6}$$

where K_1, K_2 and E_1, E_2 are the first and second complete elliptic integrals with modulus k_1, k_2, respectively, and k_1, k_2 are given, respectively, by

$$k_1^2=\frac{4Rr}{(R+r)^2+(z+R/2)^2}, \tag{7}$$

$$k_2^2=\frac{4Rr}{(R+r)^2+(z-R/2)^2}. \tag{8}$$

By expressions (6) and (3), the relative difference between the z component of the magnetic field intensity at any off-axis point and that at the center O is

$$\varepsilon_z=\frac{B_z(r,z)-B(0)}{B(0)}=\frac{5\sqrt{5}}{16\pi}\frac{1}{[(1+\rho)^2+(\xi+1/2)^2]^{1/2}}\times\left[K_1+\frac{1-\rho^2-(\xi+1/2)^2}{(1-\rho)^2+(\xi+1/2)^2}E_1\right]+\frac{5\sqrt{5}}{16\pi}\frac{1}{[(1+\rho)^2+(\xi-1/2)^2]^{1/2}}\times\left[K_2+\frac{1-\rho^2-(\xi+1/2)^2}{(1-\rho)^2+(\xi-1/2)^2}E_2\right]-1, \tag{9}$$

where $\rho=r/R$.

For Maxwell's tricoil, let R be the radius of the great circle, I be the electric current through the coils connected in series, the radius of each small circle is $\sqrt{4/7}R$, the distance of each small circle from the great circle is $\sqrt{3/7}R$, the numbers of turns of the great circle and each small circle are 64 and 49, respectively; the magnetic field intensity at any point on the axis of the coil is

$$B(z)=\frac{1}{2}\mu_0IR^2\left\{64(R^2+z^2)^{-3/2}+28\left[\frac{4}{7}R^2+\left(z+\sqrt{\frac{3}{7}}R\right)^2\right]^{-3/2}+28\left[\frac{4}{7}R^2+\left(z-\sqrt{\frac{3}{7}}R\right)^2\right]^{-3/2}\right\}. \tag{10}$$

At the center O, the magnetic field intensity is

$$B(0)=60\frac{\mu_0I}{R}. \tag{11}$$

By expressions (10) and (11), the relative difference between magnetic field intensity at any point on the axis and that at the center O is

$$\varepsilon=\frac{B(z)-B(0)}{B(0)}=\frac{1}{30}\left\{16(1+\xi^2)^{-3/2}+7\left[\frac{4}{7}+\left(\xi+\sqrt{\frac{3}{7}}\right)^2\right]^{-3/2}+7\left[\frac{4}{7}+\left(\xi-\sqrt{\frac{3}{7}}\right)^2\right]^{-3/2}\right\}-1. \tag{12}$$

Off-axis magnetic field components can be obtained in a similar way to that for the Helmholtz coil and are omitted for brevity.

Similarly, the relative difference between the z component of magnetic field intensity at any off-axis point and that at the center O is

$$\varepsilon_z=\frac{B_z(r,z)-B(0)}{B(0)}=\frac{8}{15\pi}\frac{1}{[(1+\rho)^2+\xi^2]^{1/2}}\left[K_1+\frac{1-\rho^2-\xi^2}{(1-\rho)^2+\xi^2}E_1\right]+\frac{49}{120\pi}\frac{1}{\left[\left(\sqrt{\frac{4}{7}}+\rho\right)^2+\left(\xi+\sqrt{\frac{3}{7}}\right)^2\right]^{1/2}}\times\left[K_2+\frac{\frac{4}{7}-\rho^2-\left(\xi+\sqrt{\frac{3}{7}}\right)^2}{\left(\sqrt{\frac{4}{7}}-\rho\right)^2+\left(\xi+\sqrt{\frac{3}{7}}\right)^2}E_2\right]+\frac{49}{120\pi}\frac{1}{\left[\left(\sqrt{\frac{4}{7}}+\rho\right)^2+\left(\xi-\sqrt{\frac{3}{7}}\right)^2\right]^{1/2}}\times\left[K_3+\frac{\frac{4}{7}-\rho^2-\left(\xi-\sqrt{\frac{3}{7}}\right)^2}{\left(\sqrt{\frac{4}{7}}-\rho\right)^2+\left(\xi-\sqrt{\frac{3}{7}}\right)^2}E_3\right]-1. \tag{13}$$

For the improved Helmholtz coil proposed, let R be the radius of each circle, I be electric current through the coils connected in series, the number of turns of outside coils be equal to N, kN be the number of turns of the central coil (where k is a proportional coefficient to be determined), and a be the distance from the outside circles to the central circle. Then the magnetic field intensity at any point on the axis for the improved Helmholtz coil is

$$B(z)=\frac{\mu_0}{2}NR^2I\{[R^2+(a+z)^2]^{-3/2}+[R^2+(a-z)^2]^{-3/2}+k(R^2+z^2)^{-3/2}\}. \tag{14}$$

From expression (14), we get

$$B^2(0)=\frac{3}{2}\mu_0NR^2I\left[\frac{2(4a^2-R^2)}{(R^2+a^2)^{7/2}}-\frac{k}{R^5}\right], \tag{15}$$

$$B^{(4)}(0)=\frac{1}{2}\mu_0NR^2I\left[\frac{1890a^4}{(R^2+a^2)^{11/2}}-\frac{1260a^2}{(R^2+a^2)^{9/2}}+\frac{90}{(R^2+a^2)^{7/2}}+\frac{45k}{R^7}\right]. \tag{16}$$

To obtain a magnetic field more uniform than that of the Helmholtz coil, one should find suitable values of a, k to satisfy the conditions: $B^{(2)}(0)=0$ and $B^{(4)}(0)=0$. Setting the right-hand side of expressions (15) and (16) to zero simultaneously, we can get the following results through numerical calculations:

$$a/R=0.7601, \quad k=0.5315, \tag{17}$$

i.e., we can let the number of turns of each of the outside circles be $N=111$, and the number of turns of the central coil be 59 (notice that $59/111=0.531\,53$).

By expressions (14) and (17), $B(0)$ can be obtained

$$B(0)=0.7704\frac{111\mu_0I}{R}. \tag{18}$$

The relative difference between magnetic field intensity at any point on the axis and that at the center O is

$$\begin{aligned}\varepsilon&=\frac{B(z)-B(0)}{B(0)}\\&=0.649\{[1+(0.7601+\xi)^2]^{-3/2}\\&\quad+[1+(0.7601-\xi)^2]^{-3/2}+0.5315[1+\xi^2]^{-3/2}\}-1.\end{aligned} \tag{19}$$

The relative difference between the z component of the magnetic field intensity at any off-axis point and that at the center O is

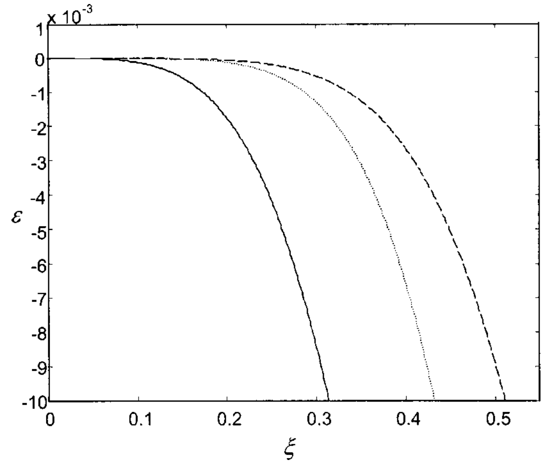

FIG. 1. Dependencies of ε on ξ for the three different kinds of coils: (——) Helmholtz coil, (···) Maxwell's tricoil, and (— — —) an improved Helmholtz coil.

$$\begin{aligned}\varepsilon_z&=\frac{B_z(r,z)-B(0)}{B(0)}\\&=\frac{0.649}{\pi}\frac{1}{[(1+\rho)^2+(\xi+0.7061)^2]^{1/2}}\\&\quad\times\left[K_1+\frac{1-\rho^2-(\xi+0.7061)^2}{(1-\rho)^2+(\xi+0.7061)^2}E_1\right]\\&\quad+\frac{0.345}{\pi}\frac{1}{[(1+\rho)^2+\xi^2]^{1/2}}\left[K_2+\frac{1-\rho^2-\xi^2}{(1-\rho)^2+\xi^2}E_2\right]\\&\quad+\frac{0.649}{\pi}\frac{1}{[(1+\rho)^2+(\xi-0.7061)^2]^{1/2}}\left[K_3\right.\\&\quad\left.+\frac{1-\rho^2-(\xi-0.7061)^2}{(1-\rho)^2+(\xi-0.7061)^2}E_3\right]-1.\end{aligned} \tag{20}$$

II. COMPARISON OF MAGNETIC FIELD HOMOGENEITY

Comparison among the magnetic fields in the Helmholtz coil, Maxwell's tricoil, and the improved Helmholtz coil are as follows.

A. Comparison of axial uniform regions of magnetic fields

For Helmholtz coil, by expression (4), for $|\varepsilon|\leqslant 1\%$, $|z/R|\leqslant 0.3137$; for $|\varepsilon|\leqslant 0.1\%$, $|z/R|\leqslant 0.1731$.

For Maxwell's tricoil, by expression (12), for $|\varepsilon|\leqslant 1\%$, $|z/R|\leqslant 0.4320$; for $|\varepsilon|\leqslant 0.1\%$, $|z/R|\leqslant 0.2864$.

For the improved Helmholtz coil, by expression (19), for $|\varepsilon|\leqslant 1\%$, $|z/R|\leqslant 0.5104$; for $|\varepsilon|\leqslant 0.1\%$, $|z/R|\leqslant 0.3304$.

It is seen that, for the same value of relative difference ε, the axial uniform region of magnetic field in Maxwell's tricoil is larger than that in the Helmholtz coil, and the axial uniform region of the magnetic field in the improved Helmholtz coil is larger than that in Maxwell's tricoil. Figure 1 shows dependencies of the relative difference ε on ξ for the three cases.

TABLE I. Comparison of the uniform regions of off-axis magnetic fields for three different kinds of coils.

ε_z	Uniform regions of magnetic field	Helmholtz coil	Maxwell's tricoil	Improved Helmholtz coil
$<1\%$	Axial direction	$\|z\|\leqslant 0.3137R$	$\|z\|\leqslant 0.4320R$	$\|z\|\leqslant 0.5104R$
	Radial direction	$r\leqslant 0.2744R$	$r\leqslant 0.2639R$	$r\leqslant 0.3192R$
$<0.1\%$	Axial direction	$\|z\|\leqslant 0.1731R$	$\|z\|\leqslant 0.2864R$	$\|z\|\leqslant 0.3304R$
	Radial direction	$r\leqslant 0.1491R$	$r\leqslant 0.1735R$	$r\leqslant 0.2210R$

B. Comparison of off-axis uniform regions of magnetic fields

By expressions (9), (13), and (20), the off-axis uniform regions of magnetic field for the three different kinds of coils are calculated (see Table I). A simple explanation for Table I is given below.

For the Helmholtz coil, we have obtained that the relative difference of magnetic field intensity on the z axis $|\varepsilon|\leqslant 1\%$ in the region $|z|\leqslant 0.3137R$. If it is required that the relative difference ε_z between the axial component $B_z(r,z)$ and $B(0)$ is also less than 1%, then, by expression (9), it can be seen that we should have $\rho\leqslant 0.2744$, i.e., in the region $|z|<0.3137R$ and $r<0.2744R$, $|\varepsilon_z|\leqslant 1\%$.

By a similar process of reasoning, for Maxwell's tricoil, if $\varepsilon_z\leqslant 1\%$, by expression (13), it can be seen that we should have $\rho\leqslant 0.2639$, i.e., in the region $|z|<0.4320R$ and $r<0.2639R$ $|\varepsilon_z|\leqslant 1\%$.

For the improved Helmholtz coil, if $\varepsilon_z\leqslant 1\%$, by expression (20), it can be seen that we should have $\rho\leqslant 0.3192$, i.e., in the region $|z|<0.5104R$ and $r<0.3192R$ $|\varepsilon_z|\leqslant 1\%$. The case for which $|\varepsilon_z|\leqslant 0.1\%$ can be analyzed in a similar way.

Furthermore, for $|\varepsilon_z|\leqslant 1\%$ and $|\varepsilon_z|\leqslant 0.1\%$, we can get that $|B_r/B_z|\leqslant 1\%$ and $|B_r/B_z|\leqslant 0.1\%$ for the three different kinds of coils.

Figure 2 shows the dependence of the off-axis relative differences ε_z on ρ when the axial relative differences $\varepsilon=1\%$ for the three different kinds of coils. Figure 3 shows the dependence of the off-axis relative differences ε_z on ρ when the axial relative differences $\varepsilon=0.1\%$ for the three different kinds of coils.

From the above discussions, it is seen that the uniform regions of magnetic field in Maxwell's tricoil and the improved Helmholtz coil are larger than that in Helmholtz's coil when the magnetic field homogeneity is the same. If the uniform regions are regarded as cylinders, the volumes of Maxwell's tricoil and the improved Helmholtz coil are 1.27 times and 2.20 times that of Helmholtz coil, respectively, when $\varepsilon_z=1\%$; the volumes of Maxwell's tricoil and the improved Helmholtz coil are 2.24 times and 4.19 times that of Helmholtz coil, respectively, when $\varepsilon_z=0.1\%$.

Finally, the reason why the homogeneity of magnetic fields in Maxwell's tricoil and the improved Helmholtz coil is better than that in the Helmholtz coil is: $B^{(2)}(0)=0$ for the Helmholtz coil, whereas both $B^{(2)}(0)=0$ and $B^{(4)}(0)=0$ for Maxwell's tricoil and the improved Helmholtz coil. Furthermore, the reason for which the homogeneity of magnetic fields in the improved Helmholtz coil is better than that in Maxwell's tricoil is $B^{(6)}(0)$ of the Helmholtz coil is smaller than $B^{(6)}(0)$ of Maxwell's tricoil.

If we would wish to further raise the magnetic field homogeneity, we may let $B^{(2n)}(0)=0$ $(n=1,2,3,4,5,...)$. These conditions can be satisfied, for example, by using more symmetrically spaced coils with the same radius and an appropriate number of turns and distances to the central.

We hope that the improved Helmholtz coil proposed and the method for further improvements proposed will be of value in practical applications where a large uniform region of magnetic field is desired. The practical realization of the improved Helmholtz coil in apparatus remains to be studied.

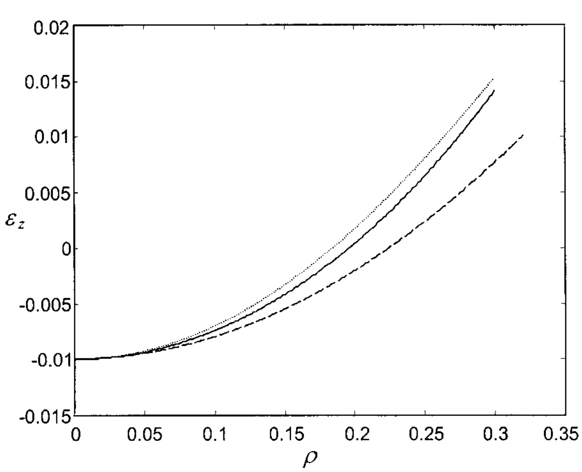

FIG. 2. Dependence of ε_z on ρ when $\varepsilon=1\%$ for the three different kinds of coils. (——) Helmholtz coil, (···) Maxwell's tricoil, and (— — —) an improved Helmholtz coil.

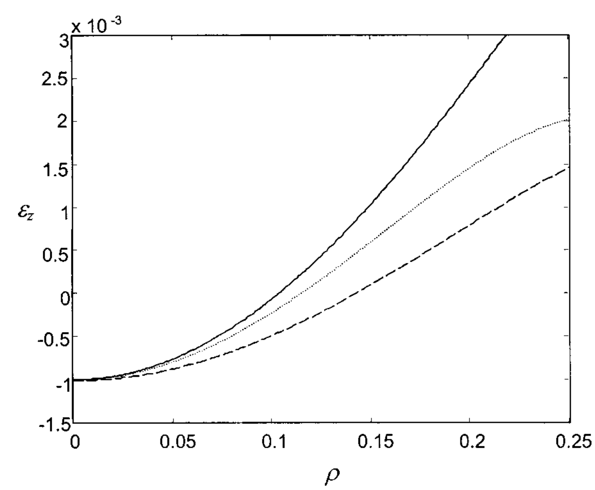

FIG. 3. Dependence of ε_z on ρ when $\varepsilon=0.1\%$ for the three different kinds of coils. (——) Helmholtz coil, (···) Maxwell's tricoil, and (— — —) an improved Helmholtz coil.

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