ON THE SOLVABILITY OF AN ABSTRACT EVOLUTION PROBLEM

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Abstract. This note is devoted to the proof of a fundamental result concerning the solvability of an abstract evolution problem. It corresponds to material included in the appendix of my undergraduate thesis [Bus25]. If these notes are cited, we kindly request that the appendix of the thesis be referenced. The arguments follow the general ideas in [Sho97, Chapter IV], and in particular those of [Sho97, Theorem IV.6.1(b)]. Nevertheless, our aim is to provide a self-contained proof, including all the necessary preliminaries. We begin by recalling several definitions and results concerning accretive operators. Next, we establish a key result on evolution equations in Hilbert spaces. We then include an interlude describing the relationship between pre-Hilbert and semi-inner product spaces, which offers a convenient framework for passing from just a semi-inner product space to a Hilbert space by duality. Finally, we conclude by proving the main result, which relies on all the preceding developments.

1. Prelude: Accretive operators

Let $(H, (\cdot,\cdot)_H)$ be a Hilbert space and let A be a relation on H, that is, A is a subset of $H \times H$. We define the domain, range and inverse of A as

$$\begin{aligned} \mathcal{D}(A) &:= \Big\{ x \in H \ : \quad (x,y) \in A \quad \text{for some } y \in H \Big\}, \\ R(A) &:= \Big\{ y \in H \ : \quad (x,y) \in A \quad \text{for some } x \in H \Big\}, \\ A^{-1} &:= \Big\{ (y,x) \in H \times H \ : \quad (x,y) \in A \Big\}, \end{aligned}$$

respectively. We may view A as a mapping that assigns to each $x \in H$ the subset $A(x) = \{y \in H \ : \ (x,y) \in A\}$. Then A is a function precisely when each set $A(x)$ consists of a single point. In this sense, we refer to A as a relation or, equivalently, as a multivalued operator. Furthermore, we may define the operations

$$\begin{aligned} \lambda A &:= \Big\{ (x, \lambda y) \in H \times H \ : \quad (x,y) \in A \Big\} \qquad \forall\, \lambda \in \mathbb{R}, \quad \text{and} \\ A + B &:= \Big\{ (x, y+z) \in H \times H \ : \quad (x,y) \in A \quad \text{and} \quad (x,z) \in B \Big\}. \end{aligned}$$

We notice that $\mathcal{D}(\lambda A) = \mathcal{D}(A)$ if $\lambda \neq 0$ and $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$. From now on, we denote by I the identity operator.

Definition 1.1. *An operator A on H is said to be accretive if, for all $(x_j, w_j) \in A$, $j \in \{1,2\}$, then*

$$(w_1 - w_2, x_1 - x_2)_H \geq 0\,.$$

Furthermore, A is called m-accretive if, in addition, $R(I + A) = H$.

The following two lemmas provide characterizations of accretive and m-accretive operators. Their proofs are mainly algebraic and are therefore omitted. For further details, see [Sho97, Corollary IV.1.3 and Lemma IV.1.3].

Lemma 1.1. *The following are equivalent:*

(1) A is accretive,

(2) $(I + \alpha A)^{-1}$ is a contraction on $R(I + \alpha A)$ for all $\alpha > 0$.

Lemma 1.2. *The following are equivalent:*

(1) A is accretive and $R(I + \alpha A) = H$ for some $\alpha > 0$,

(2) A is m-accretive,

(3) A is accretive and $R(I + \alpha A) = H$ for all $\alpha > 0$.

Given an m-accretive operator A and $\alpha > 0$, Lemma 1.2 implies that $R(I + \alpha A) = H$, and, consequently, $\mathcal{D}((I + \alpha\, A)^{-1}) = H$. Moreover, for any $y \in H$, the problem of finding $x \in H$ such that

$$(I + \alpha\, A)^{-1}(y) = x$$

admits a unique solution, since $(I + \alpha\, A)^{-1}$ is a contraction on $R(I + \alpha A) = H$ as stated in Lemma 1.1. Therefore, we may define the single-valued operators

$$J_\alpha := (I + \alpha A)^{-1} \qquad \forall\, \alpha > 0\,,$$

which are called the resolvents of A. Observe that $y = J_\alpha(x)$ if and only if $\frac{1}{\alpha}(x - y) \in A(y)$, whence

$$\frac{1}{\alpha}\,(x - J_\alpha(x)) \in A(J_\alpha(x)) \qquad \forall\, x \in \mathrm{H}\,, \quad \forall\, \alpha > 0\,. \tag{1.1}$$

Proposition 1.1. *Let A be an m-accretive operator on H. Then, A is maximal accretive, i.e. if B is accretive and $A \subset B$, then $A = B$.*

Proof. Let $(x, y) \in B$ and define $z := (I + A)^{-1}(y + x)$. Then, $z \in \mathcal{D}(A) \subset \mathcal{D}(B)$ and $y + x \in (I + A)(z) \subset (I + B)(z)$, so $y + x - z \in B(z)$. Using that B is accretive with the pair $(x, y), (z, y + x - z) \in B$, we deduce that

$$0 \le (y - (y + x - z), x - z)_H = (-x + z, x - z)_H = -\|x - z\|_H^2\,,$$

which implies $x = z$. Therefore, $x \in \mathcal{D}(A)$, and, since $y + x \in (I + A)(z)$, it follows that $y \in A(x)$. Thus, $(x, y) \in A$, as desired. □

Hereafter, if S is a subset of a topological space X, we denote by $\overline{S}$ the closure of S in X.

Proposition 1.2. *Let A be an m-accretive operator on H. Then, for all $x \in H$, the set $A(x)$ is closed and convex.*

Proof. Let us define the relation $\widetilde{A} \subset H \times H$ by

$$(x, y) \in \widetilde{A} \quad \text{if and only if} \quad x \in \mathcal{D}(A) \quad \text{and} \quad y \in \overline{A(x)}\,.$$

It is readily verified that $A \subset \widetilde{A}$ and that $\widetilde{A}$ is also accretive. By Proposition 1.1, it follows that $A = \widetilde{A}$. Thus, $A(x) = \overline{A(x)}$ for all $x \in \mathcal{D}(A)$. This proves that $A(x)$ is closed.

On the other hand, to prove that $A(x)$ is convex, we define $A_c \subset H \times H$ by

$$(x, y) \in A_c \quad \text{if and only if} \quad x \in \mathcal{D}(A) \quad \text{and} \quad y \in \operatorname{Conv}(A(x)) \,,$$

where $\operatorname{Conv}(A(x))$ denotes the convex hull of $A(x)$. As before, it is readily verified that $A \subset A_c$ and A_c is also accretive. Hence, $A_c = A$, which implies that for all $x \in \mathcal{D}(A)$, $A(x) = \operatorname{Conv}(A(x))$, that is, $A(x)$ is convex. This completes the proof. □

Proposition 1.3. *Let A be an m-accretive operator on H and let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset A$ be a sequence such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ in H. Then*

(1) If $\liminf\limits_{n\to\infty} (x_n, y_n)_H \le (x, y)_H$, then $(x, y) \in A$.

(2) If $\limsup\limits_{n\to\infty} (x_n, y_n)_H \le (x, y)_H$, then $\lim\limits_{n\to\infty} (x_n, y_n)_H = (x, y)_H$.

Proof.

(1) Consider the relation $\widetilde{A} := A \cup \{(x, y)\}$. Clearly, $A \subset \widetilde{A}$. Let us prove that $\widetilde{A}$ is accretive. Since A is accretive, for all $(u, v) \in A$,

$$0 \le (v - y_n, u - x_n)_H = (v, u)_H - (v, x_n)_H - (y_n, u)_H + (y_n, x_n)_H \,.$$

Then, taking lim inf, using the assumption and the weak convergences,

$$0 \le (v, u)_H - (v, x)_H - (y, u)_H + (y, x)_H = (x - y, u - x)_H \,.$$

Thus, $\widetilde{A}$ is accretive. Therefore, by Proposition 1.1, we conclude that $\widetilde{A} = A$, which means that $(x, y) \in A$.

(2) If the lim sup assumption holds, the lim inf assumption of the previous item also holds, and, thus, $(x, y) \in A$. Then, using that A is accretive,

$$(y - y_n, x - x_n)_H \ge 0 \qquad \forall\, n \in \mathbb{N} \,.$$

Taking lim inf, we obtain that

$$\liminf_{n\to\infty} (x_n, y_n)_H \ge (x, y)_H \,,$$

which along with the lim sup assumption, gives $\lim\limits_{n\to\infty} (x_n, y_n)_H = (x, y)_H$, as desired.

□

Definition 1.2. *Let A be an m-accretive operator on H. The Yosida approximations of A are defined by*

$$A_\alpha := \frac{1}{\alpha}\,(I - J_\alpha) \qquad \forall\, \alpha > 0 \,.$$

Certainly, from (1.1) it follows that $A_\alpha(x) \in A(J_\alpha(x))$ for all $x \in H$ and $\alpha > 0$. Moreover, using once again the characterization of J_α (1.1), it is straightforward to verify that $y = A_\alpha(x)$ if and only if $y \in A(x - \alpha y)$.

Since A is m-accretive, by Proposition 1.2, $A(x)$ is closed and convex for all $x \in H$. This allows us to define the minimal section operator $A^0 : H \to H$ by

$$A^0(x) = \operatorname{Proj}_{A(x)}(0) = \operatorname*{arg\,min}_{y\in A(x)} \|y\|_H \,.$$

Theorem 1.3. *Let A be an m-accretive operator on H.*

(1) *Each* A_α *is* m*-accretive and Lipschitz with constant* $1/\alpha$.

(2) *For each* $x \in \mathcal{D}(A)$, $\|A_\alpha(x)\|_H$ *converges upward to* $\|A^0(x)\|_H$, $\lim\limits_{\alpha\to 0} A_\alpha(x) = A^0(x)$ *and*
$$\|A_\alpha(x) - A^0(x)\|_H^2 \le \|A^0(x)\|_H^2 - \|A_\alpha(x)\|_H^2 \qquad \forall\, \alpha > 0\,. \tag{1.2}$$

Proof. See [Sho97, Theorem IV.1.1]. □

2. Transitions: The theorem of Kato

An important tool in the development of the theory of abstract evolution equations is the use of Yosida approximations. Later in this section, we shall prove Kato's theorem, which establishes the existence and uniqueness of a solution to the problem
$$\frac{du}{dt}(t) + A(u(t)) \ni \omega\, u(t) + f(t) \qquad \text{for a.e.} \quad t \in (0,T)\,,$$
where A is an m-accretive operator in the Hilbert space H, $\omega \ge 0$, $f : [0,T] \to H$ is absolutely continuous and $u(0) = u_0$. In general, if A is replaced by its Yosida approximation A_α, with $\alpha > 0$, the proof of existence and uniqueness becomes simpler. In this way, we can construct a sequence of solutions u_α corresponding to each approximated problem and then show that $\lim\limits_{\alpha\to 0} u_\alpha$ exists and solves the original problem. Therefore, our first step will be to prove the existence and uniqueness of the solution for each approximation.

Lemma 2.1. *Let* A *be* m*-accretive in the Hilbert space* H *and* $\omega \ge 0$. *For each* $u_0 \in \mathcal{D}(A)$, *absolutely continuous* $f : [0,T] \to H$, *and* $\alpha > 0$, *there is a unique* $u_\alpha \in C^1([0,T], H)$, *such that* $u_\alpha(0) = u_0$ *and*
$$\frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega\, u_\alpha(t) + f(t) \qquad \textit{for a.e.} \quad t \in [0,T]\,. \tag{2.1}$$

Proof. If u_α is a solution of (2.1), then by integrating (2.1) from 0 to $t \in (0,T]$, we obtain
$$u_\alpha(t) = u_0 + \int_0^t \Big\{ - A_\alpha(u_\alpha(s)) + \omega\, u_\alpha(s) + f(s) \Big\}\, ds\,, \tag{2.2}$$
which motivates the definition of the operator $\Phi_\alpha : C([0,T],H) \to C([0,T],H)$ such that for each $v \in C([0,T],H)$,
$$\Phi_\alpha(v)(t) := u_0 + \int_0^t \Big\{ - A_\alpha(v(s)) + \omega\, v(s) + f(s) \Big\}\, ds \qquad \forall\, t \in [0,T]\,.$$
We shall consider the weighted norm $\|\cdot\|_\lambda : C([0,T],H) \to \mathbb{R}$, for some $\lambda > 0$ to be chosen later, defined by
$$\|v\|_\lambda := \sup_{0 \le t \le T} e^{-\lambda t}\, \|v(t)\|_H \qquad \forall\, v \in C([0,T],H)\,,$$
which is equivalent to the usual norm in $C([0,T],H)$. We now proceed to prove that Φ_α is a contractive operator on $C([0,T],H)$ with this norm. Given $v_1, v_2 \in C([0,T],H)$, we have
$$\|\Phi_\alpha(v_1)(t) - \Phi_\alpha(v_2)(t)\|_H \le \int_0^t \Big\{ \|A_\alpha(v_1(s)) - A_\alpha(v_2(s))\|_H + \omega\, \|v_1(s) - v_2(s)\|_H \Big\}\, ds\,,$$

whence using that A_α is Lipschitz with constant $1/\alpha$ (cf. Theorem 1.3) and denoting $L := 1/\alpha + \omega$, we obtain that

$$\begin{aligned}\|\Phi_\alpha(v_1)(t) - \Phi_\alpha(v_2)(t)\|_H &\le L \int_0^t \|v_1(s) - v_2(s)\|_H \, ds \\ &\le L \, \|v_1 - v_2\|_\lambda \int_0^t e^{\lambda s} \, ds = \frac{L}{\lambda} \, \|v_1 - v_2\|_\lambda \, (e^{\lambda t} - 1) \, .\end{aligned}$$

Thus,

$$\|\Phi_\alpha(v_1) - \Phi_\alpha(v_2)\|_\lambda \le \frac{L}{\lambda} \, \|v_1 - v_2\|_\lambda \sup_{0 \le t \le T} (1 - e^{-\lambda t}) \le \frac{L}{\lambda} \, \|v_1 - v_2\|_\lambda \, .$$

Hence, by chosing $\lambda := 2L$, we conclude that Φ_α is a contractive operator in the Banach space $C([0,T], H)$ with the norm $\| \cdot \|_\lambda$. Applying the Banach fixed-point theorem, we deduce that there is a unique $u_\alpha \in C([0,T], H)$ such that $\Phi_\alpha(u_\alpha) = u_\alpha$, which means that (2.2) holds for all $t \in [0,T]$, thereby implying (2.1). Moreover, since every term in the integrand in (2.2) is continuous, it follows that the derivative of u_α is also continuous. Hence, $u_\alpha \in C^1([0,T], H)$. This completes the proof. □

The following lemma, which is a version of the so-called Gronwall inequality, will be used repeatedly in the main result of this section.

Lemma 2.2 (Gronwall inequality)**.** *Let $a, b \in \mathrm{L}^1(0,T)$ with $b(t) \ge 0$ a.e. in $(0,T)$, and let $v : [0,T] \to \mathbb{R}^+$ be an absolutely continuous function that satisfies, for some $0 \le \delta < 1$,*

$$(1 - \delta) \, v'(t) \le a(t) \, v(t) + b(t) \, v^\delta(t) \qquad \text{for a.e.} \quad t \in [0,T] \, .$$

Then,

$$v^{1-\delta}(t) \le v^{1-\delta}(0) \, \eta(0) + \int_0^t \eta(s) \, b(s) \, ds \qquad \forall \, t \in [0,T] \, ,$$

where

$$\eta(s) := \exp\left(\int_s^t a(\xi) \, d\xi \right) \, .$$

Proof. See [Sho97, Lemma IV.4.1]. □

We are now in a position to prove the existence and uniqueness of a solution to the abstract evolution problem introduced at the beginning of this section. This result is originally due to Kato [Kat53].

Theorem 2.3 (Kato)**.** *Let A be m-accretive in the Hilbert space H and $\omega \ge 0$. For each $u_0 \in \mathcal{D}(A)$ and absolutely continuous $f : [0,T] \to H$, there is a unique absolutely continuous $u : [0,T] \to H$, such that $u(0) = u_0$ and*

$$\frac{du}{dt}(t) + A(u(t)) \ni \omega \, u(t) + f(t) \qquad \text{for a.e.} \quad t \in (0,T) \, . \tag{2.3}$$

Furthermore, u is Lipschitz continuous and $u(t) \in \mathcal{D}(A)$ for every $t \ge 0$.

Proof. We first prove uniqueness. Let u_1 and u_2 be two solutions of (2.3). Since A is accretive, it follows that

$$\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|_H^2 \le \omega \, \|u_1(t) - u_2(t)\|_H^2 \qquad \forall \, t \ge 0 \, ,$$

which, by using Gronwall inequality with $\delta = 0$ (cf. Lemma 2.2), yields

$$\|u_1(t) - u_2(t)\|_H \leq e^{\omega\, t}\, \|u_1(0) - u_2(0)\|_H \qquad \forall\, t \geq 0\,.$$

Thus, since $u_1(0) = u_2(0) = u_0$, any solution, if it exists, must be unique.

Now, in order to prove existence, for each $\alpha > 0$, we let $u_\alpha \in C^1([0,T], H)$ be the unique solution (cf. Lemma 2.1) to

$$\frac{du_\alpha}{dt}(t) + A_\alpha(u_\alpha(t)) = \omega\, u_\alpha(t) + f(t) \qquad \forall\, t \in [0,T]\,, \tag{2.4}$$

with $u_\alpha(0) = u_0$. We observe that, if $h > 0$, then $u_\alpha(t+h)$ is a solution of (2.4) with $f(t)$ replaced by $f(t+h)$. Using the accretivity of A_α, a few algebraic manipulations yield

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\, \|u_\alpha(t+h) - u_\alpha(t)\|_H^2 &\leq \omega\, \|u_\alpha(t+h) - u_\alpha(t)\|_H^2 \\ &\quad + \|f(t+h) - f(t)\|_H\, \|u_\alpha(t+h) - u_\alpha(t)\|_H\,. \end{aligned}$$

By applying Gronwall inequality with $\delta = 1/2$ (cf. Lemma 2.2) and letting $h \to 0$, we arrive at

$$\big\|u_\alpha'(t)\big\|_H \leq e^{\omega\, t}\, \big\| - A_\alpha(u_0) + \omega\, u_0 + f(0)\big\|_H + \int_0^t e^{\omega\,(t-s)}\, \big\|f'(s)\big\|_H\, ds\,,$$

where the prime indicates differentiation in time. This, together with the fact that $\|A_\alpha(u_0)\|_H \leq \|A^0(u_0)\|_H$ (cf. (1.2)), yields

$$\|u_\alpha'(t)\|_H \leq e^{\omega\, t}\, \|A^0(u_0)\|_H + e^{\omega\, t}\|\omega\, u_0 + f(0)\|_H + \int_0^t e^{\omega\,(t-s)}\, \|f'(s)\|_H\, ds\,. \tag{2.5}$$

This implies that $\{u_\alpha'\}$ is a bounded sequence in $C([0,T], H)$. Moreover, from this fact and employing (2.4), it follows that $\{u_\alpha\}$ and $\{A_\alpha(u_\alpha)\}$ are also bounded in $C([0,T], H)$.

We now proceed to prove that the sequence $\{u_\alpha\}$ is Cauchy in $C([0,T], H)$. To this end, let $\alpha, \beta > 0$, and use (2.4) to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\, \|u_\alpha(t) - u_\beta(t)\|_H^2 \\ &\qquad = \omega\, \|u_\alpha(t) - u_\beta(t)\|_H^2 - \big(A_\alpha(u_\alpha(t)) - A_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\big)_H\,. \end{aligned} \tag{2.6}$$

Writing $u_* = \alpha\, A_*(u_*) + J_*(u_*)$ for $* \in \{\alpha, \beta\}$, and omitting the time dependence for clarity, the last term in (2.6) can be rewritten as

$$\begin{aligned} \big(A_\alpha(u_\alpha) - A_\beta(u_\beta), u_\alpha - u_\beta\big)_H &= \big(A_\alpha(u_\alpha) - A_\beta(u_\beta), \alpha\, A_\alpha(u_\alpha) - \beta\, A_\beta(u_\beta)\big)_H \\ &\quad + \big(A_\alpha(u_\alpha) - A_\beta(u_\beta), J_\alpha(u_\alpha) - J_\beta(u_\beta)\big)_H\,. \end{aligned} \tag{2.7}$$

As mentioned just below of Definition 1.2, we have $A_*(u_*) \in A(J_*(u_*))$ for each $* \in \{\alpha, \beta\}$. This, along with the fact that A is accretive, implies that the last term in (2.7) is non-negative. Thus,

$$\big(A_\alpha(u_\alpha) - A_\beta(u_\beta), u_\alpha - u_\beta\big)_H \geq \big(A_\alpha(u_\alpha) - A_\beta(u_\beta), \alpha\, A_\alpha(u_\alpha) - \beta\, A_\beta(u_\beta)\big)_H\,. \tag{2.8}$$

By performing some algebraic manipulations and using Cauchy–Schwarz and Young's inequalities, we obtain that

$$
\begin{aligned}
&\big(A_\alpha(u_\alpha) - A_\beta(u_\beta),\, u_\alpha - u_\beta\big)_H \\
&\quad \geq \alpha\,\|A_\alpha(u_\alpha)\|_H^2 + \beta\,\|A_\beta(u_\beta)\|_H^2 - (\alpha+\beta)\,(A_\alpha(u_\alpha), A_\beta(u_\beta))_H \\
&\quad \geq \alpha\,\|A_\alpha(u_\alpha)\|_H^2 + \beta\,\|A_\beta(u_\beta)\|_H^2 - \alpha\left(\|A_\alpha(u_\alpha)\|_H^2 + \frac{1}{4}\,\|A_\beta(u_\beta)\|_H^2\right) \\
&\qquad - \beta\left(\frac{1}{4}\,\|A_\alpha(u_\alpha)\|_H^2 + \|A_\beta(u_\beta)\|_H^2\right) \\
&\quad \geq -\frac{\alpha+\beta}{4}\,K^2\,,
\end{aligned}
$$

where $K := \sup\big\{\|A_\alpha(u_\alpha(t))\|_H \ :\ 0 \leq t \leq T \quad \text{and} \quad \alpha > 0\big\}$. Notice that this supremum is indeed finite, as the sequence $\{A_\alpha(u_\alpha)\}$ is bounded in $C([0,T],H)$. Hence, upon writing the time dependence explicitly again,

$$-\big(A_\alpha(u_\alpha(t)) - A_\beta(u_\beta(t)), u_\alpha(t) - u_\beta(t)\big)_H \leq \frac{\alpha+\beta}{4}\,K^2\,.$$

Putting this estimate into (2.6), we find that

$$\frac{d}{dt}\,\|u_\alpha(t) - u_\beta(t)\|_H^2 \leq 2\omega\,\|u_\alpha(t) - u_\beta(t)\|_H^2 + \frac{\alpha+\beta}{2}\,K^2\,,$$

and using Gronwall inequality with $\delta = 0$ (cf. Lemma 2.2), we discover

$$\|u_\alpha(t) - u_\beta(t)\|_H^2 \leq \frac{\alpha+\beta}{4\,\omega}\,K^2\,(e^{2\omega\,t} - 1) \qquad \forall\, t \in [0,T]\,.$$

Therefore, $\{u_\alpha\}$ is uniformly Cauchy and, consequently, there exists a function $u \in C([0,T],H)$ satisfying

$$\|u_\alpha(t) - u(t)\|_H^2 \leq \frac{\alpha}{4\,\omega}\,K^2\,(e^{2\omega\,t} - 1) \qquad \forall\, t \in [0,T]\,, \quad \forall\,\alpha > 0\,,$$

which, in particular, gives us that $u_\alpha \to u$ in $\mathrm{L}^2(0,T;H)$. Since $\{A_\alpha(u_\alpha)\}_\alpha$ is bounded in $C([0,T];H)$, we also have that is bounded in $\mathrm{L}^2(0,T;H)$, and, consequently, there is a subsequence (not relabeled) such that $A_\alpha(u_\alpha) \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;H)$, for some $\xi \in \mathrm{L}^2(0,T;H)$. Analogously for $\{u'_\alpha\}_\alpha$, it follows that there exists a subsequence weakly convergent. Moreover, by taking $\alpha \to 0$ in the equation

$$u_\alpha(t) = u_0 + \int_0^t u'_\alpha(s)\,ds\,,$$

we discover that $u'_\alpha \rightharpoonup u'$. On the other hand, we recall from Definition 1.2 that $u_\alpha - J_\alpha(u_\alpha) = \alpha\,A_\alpha(u_\alpha)$, which along with the boundedness of $\{A_\alpha(u_\alpha)\}_\alpha$ yields

$$\|u_\alpha - J_\alpha(u_\alpha)\|_{C([0,T],H)} \leq \alpha\,\|A_\alpha(u_\alpha)\|_{C([0,T],H)} \xrightarrow{\alpha\to 0} 0\,,$$

whence $J_\alpha(u_\alpha) \to u$ in $C([0,T],H)$ and, thus, we also obtain convergence in $\mathrm{L}^2(0,T;H)$. Therefore, we can take limit in (2.4) to find that

$$u' + \xi = \omega\,u + f \quad \text{in} \quad \mathrm{L}^2(0,T;H)\,. \tag{2.9}$$

It remains to prove that $\xi \in A(u)$ to obtain (2.3). To that end, we introduce the realization $\mathcal{A} \subset \mathrm{L}^2(0,T;H) \times \mathrm{L}^2(0,T;H)$ as a relation defined by

$$(z,w) \in \mathcal{A} \quad \text{if and only if} \quad w(t) \in A(z(t)) \quad \text{for a.e.} \quad t \in (0,T)\,. \tag{2.10}$$

Since A is m-accretive in H, it is not difficult to see that $\mathcal{A}$ is m-accretive in $\mathrm{L}^2(0,T;H)$. Then, putting $x_\alpha := J_\alpha(u_\alpha)$ and $y_\alpha := A_\alpha(u_\alpha)$, we have that $(x_\alpha, y_\alpha) \in \mathcal{A}$. We already know that $x_\alpha \to u$ in $\mathrm{L}^2(0,T;H)$ and $y_\alpha \rightharpoonup \xi$ in $\mathrm{L}^2(0,T;H)$. Consequently,

$$\int_0^T (x_\alpha(t), y_\alpha(t))_H \, dt \xrightarrow{\alpha\to 0} \int_0^T (u(t), \xi(t))_H \, dt\,. \tag{2.11}$$

By applying Lemma 1.3 to the realization $\mathcal{A}$ (cf. (2.10)) on the Hilbert space $\mathrm{L}^2(0,T;H)$, together with the sequences $\{x_\alpha\}$ and $\{y_\alpha\}$ and the convergence in (2.11), we obtain that $(u,\xi) \in \mathcal{A}$; that is, $\xi(t) \in A(u(t))$ for a.e. $t \in (0,T)$. Therefore, (2.9) becomes

$$u'(t) + A(u(t)) \ni \omega\, u(t) + f(t) \qquad \text{for a.e.} \quad t \in (0,T)\,,$$

which proves (2.3). Furthermore, since $u_\alpha \to u$ in $C([0,T],H)$, we have that $u_\alpha(0) \to u(0)$, i.e. $u(0) = u_0$. We now prove that u is Lipschitz continuous. In fact, each u_α is Lipschitz, since for all $t \geq s$,

$$\|u_\alpha(t) - u_\alpha(s)\|_H \leq \int_s^t \|u'_\alpha(t) - u'_\alpha(s)\|_H \leq C\,(t-s)\,,$$

where C is independent of t and s, owing to the uniformly boundedness of u'_α (cf. (2.5)). Hence,

$$\|u(t) - u(s)\|_H = \lim_{\alpha\to 0} \|u_\alpha(t) - u_\alpha(s)\|_H \leq C\,(t-s)\,,$$

so u is Lipschitz continuous. This completes the proof. □

3. Interlude: Semi-inner product spaces and duality

Let E be a vector space endowed with the inner product $(\cdot,\cdot)$. We assume that E is a pre-Hilbert space, i.e. E with the induced norm is not necessarily complete. Recall that the dual space E', endowed with the usual dual norm $\|\cdot\|_{E'}$ is a Banach space. Let $T : E \to E'$ be the map defined by

$$[T(u), v] := (u,v) \qquad \forall\, u,v \in E\,,$$

where $[\cdot,\cdot]$ denotes the dual pairing between E' and E. It is easy to show that T is a linear isometry, so it is injective. Notice that if T were surjective, then it would coincide with the Riesz map and E would be complete, which, in general, is not the case here. Our purpose is to show that $R(T)$ is dense in E' and that E' is actually a Hilbert space. To that end, we first define the inner product $(\!(\cdot,\cdot)\!) : R(T) \times R(T) \to \mathbb{R}$ by

$$(\!(f,g)\!) := (T^{-1}(f), T^{-1}(g)) \qquad \forall\, f,g \in R(T)\,,$$

which is well defined due to the injectivity of T. We now extend this definition to $\overline{R(T)}$ by density. Let $f,g \in \overline{R(T)}$ and $\{f_n\}_{n\in\mathbb{N}}, \{g_n\}_{n\in\mathbb{N}} \subset R(T)$ such that $f_n \to f$ and $g_n \to g$ in E'. We define

$$(\!(f,g)\!) := \lim_{n\to\infty} (T^{-1}(f_n), T^{-1}(g_n))\,. \tag{3.1}$$

This definition clearly extends the previous one. We proceed to prove that $(\!(\cdot,\cdot)\!)$ is properly defined, which means that the limit exists and the definition does not depend on the particular sequences. To prove the former, we use the triangle and Cauchy–Schwarz inequalities along with the fact that T is an isometry to obtain

$$\begin{aligned}
&\big|(T^{-1}(f_n),T^{-1}(g_n)) - (T^{-1}(f_m),T^{-1}(g_m))\big| \\
&\quad = \big|(T^{-1}(f_n)-T^{-1}(f_m),T^{-1}(g_n)) + (T^{-1}(f_m),T^{-1}(g_n)-T^{-1}(g_m))\big| \\
&\quad \leq \|T^{-1}(f_n)-T^{-1}(f_m)\|\,\|T^{-1}(g_n)\| + \|T^{-1}(f_m)\|\,\|T^{-1}(g_n)-T^{-1}(g_m)\| \\
&\quad = \|f_n-f_m\|_{E'}\,\|g_n\|_{E'} + \|f_n\|_{E'}\,\|g_n-g_m\|_{E'} \xrightarrow{n,m\to\infty} 0\,.
\end{aligned}$$

Thus, $\{(T^{-1}(f_n),T^{-1}(g_n))\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, and, consequently, it is convergent.

On the other hand, (3.1) does not depend on the particular choice of sequences. In fact, let $\{\widetilde{f}_n\}_{n\in\mathbb{N}},\{\widetilde{g}_n\}_{n\in\mathbb{N}}\subset R(T)$ be another pair of sequences satisfying that $\widetilde{f}_n\to f$ and $\widetilde{g}_n\to g$ in E'. Then,

$$\begin{aligned}
&\big|(T^{-1}(f_n),T^{-1}(g_n)) - (T^{-1}(\widetilde{f}_n),T^{-1}(\widetilde{g}_n))\big| \\
&\quad = \big|(T^{-1}(f_n)-T^{-1}(\widetilde{f}_n),T^{-1}(g_n)) + (T^{-1}(\widetilde{f}_n),T^{-1}(g_n)-T^{-1}(\widetilde{g}_n))\big| \\
&\quad \leq \|T^{-1}(f_n)-T^{-1}(\widetilde{f}_n)\|\,\|T^{-1}(g_n)\| + \|T^{-1}(\widetilde{f}_n)\|\,\|T^{-1}(g_n)-T^{-1}(\widetilde{g}_n)\| \\
&\quad = \|f_n-\widetilde{f}_n\|_{E'}\,\|g_n\|_{E'} + \|\widetilde{f}_n\|_{E'}\,\|g_n-\widetilde{g}_n\|_{E'} \xrightarrow{n\to\infty} 0\,,
\end{aligned}$$

so the limits coincide, which proves that (3.1) is well defined.

Now, notice that $(\!(f,f)\!)^{1/2}=\|f\|_{E'}$ for all $f\in\overline{R(T)}$. Therefore, we have proven that $(\overline{R(T)},(\!(\cdot,\cdot)\!))$ is a Hilbert space. In what follows, we prove that $\overline{R(T)}=E'$.

Given $f\in E'$, we may define $\phi_f:\overline{R(T)}\to\mathbb{R}$ by

$$[\phi_f,g] := \lim_{n\to\infty}[f,T^{-1}(g_n)] \qquad \forall\, g\in R(T)\,,$$

where $\{g_n\}_{n\in\mathbb{N}}$ denotes any sequence in $R(T)$ such that $g_n\to g$ in E'. Analogously as before, it can be proven that this definition does not depend on the choice of the sequence. Furthermore, it is straightforward to prove that $\phi_f\in\overline{R(T)}'$. Since $\overline{R(T)}$ is a Hilbert space, we may use the Riesz representation theorem to deduce that there exists a unique $\widehat{f}\in\overline{R(T)}$ such that

$$[\phi_f,g] = (\!(\widehat{f},g)\!) \qquad \forall\, g\in\overline{R(T)}\,.$$

Moreover, given a sequence $\{\widehat{f}_n\}_{n\in\mathbb{N}}\subset R(T)$ converging to $\widehat{f}$, we have that, for all $g\in R(T)$,

$$\begin{aligned}
(\!(\widehat{f},g)\!) &= \lim_{n\to\infty}(T^{-1}(\widehat{f}_n),T^{-1}(g)) = \lim_{n\to\infty}[T(T^{-1}(\widehat{f}_n)),T^{-1}(g)] \\
&= \lim_{n\to\infty}[\widehat{f}_n,T^{-1}(g)] = [\widehat{f},T^{-1}(g)]\,.
\end{aligned}$$

Then,

$$\|f-\widehat{f}\|_{E'} = \sup_{0\neq u\in E}\frac{[f-\widehat{f},u]}{\|u\|} = \sup_{0\neq g\in R(T)}\frac{[f-\widehat{f},T^{-1}(g)]}{\|T^{-1}(g)\|} = \sup_{0\neq g\in R(T)}\frac{[\phi_f,g]-(\!(\widehat{f},g)\!)}{\|g\|} = 0\,,$$

which means that $f=\widehat{f}$. Hence, $f\in\overline{R(T)}$ and, consequently, $\overline{R(T)}=E'$. In particular, E' is a Hilbert space.

Certainly, we have shown that E' is a Hilbert space containing a dense subspace that is isometrically isomorphic to E. The unique space satisfying these properties, up to isometric isomorphism, is the completion of E. Hence, E' can be identified with the completion of E. We summarize these results in the following theorem.

Theorem 3.1. *Let E be a vector space endowed with the inner product $(\cdot,\cdot)$. Then, E' is a Hilbert space and it is identified with the completion of E.*

For the purposes of this manuscript, the situation in which the vector space E is endowed with a semi-inner product is even more relevant. In this case, we can also obtain a similar result. We shall consider then the following setting. Denote by $E^\star$ the algebraic dual of E and let $\mathcal{N} : E \to E^\star$ be a linear, symmetric and monotone operator, and define the semi-inner product

$$\begin{aligned} (\cdot,\cdot)_* : E \times E &\to \mathbb{R} \\ (x,y) &\mapsto (x,y)_* := [\mathcal{N}(x), y]\,. \end{aligned}$$

Denote by $|\cdot|_*$ the induced seminorm and by E_* the corresponding seminormed space. Let $\mathcal{K}$ be the kernel of $\mathcal{N}$. The topological dual of E_* is defined as

$$E'_* := \Big\{ f \in E^\star \;\Big|\; \exists\, C > 0 \;:\; \forall\, x \in E \;:\; |f(x)| \le C\,|x|_* \Big\}\,,$$

and it is endowed with the norm

$$\|f\|_{E'_*} := \sup_{u \in E\setminus\mathcal{K}} \frac{|f(u)|}{|u|_*} \qquad \forall\, f \in E'_*\,.$$

Now, define the quotient space

$$\widehat{E}_* := E/\mathcal{K} = \Big\{ [u] \;:\; u \in E \Big\}, \quad \text{where} \quad [u] := \Big\{ v \in E \;:\; \; v - u \in \mathcal{K} \Big\}\,,$$

and the quotient map $\pi : E \to \widehat{E}_*$ by $\pi(u) := [u]$ for all $u \in E$. We endow the space $\widehat{E}_*$ with the inner product

$$(\widehat{u}, \widehat{v})_{\widehat{E}_*} := (u,v)_*\,, \tag{3.2}$$

for all $\widehat{u}, \widehat{v} \in \widehat{E}_*$, where $u \in \pi^{-1}(\widehat{u})$ and $v \in \pi^{-1}(\widehat{v})$. It is easy to verify that this definition does not depend on the particular choice of representative of each class. Denote by $\|\cdot\|_{\widehat{E}_*}$ the induced norm and by W the completion of $\widehat{E}_*$. Hence, by Theorem 3.1, it follows that $\widehat{E}'_*$ is a Hilbert space and it is isometrically isomorphic to W.

Since W is the completion of $\widehat{E}_*$, there exists a linear and bounded operator $j : \widehat{E}_* \to W$ that is injective, isometric and has dense range. Let us define the linear and continuous operator $q := j \circ \pi : E_* \to W$. Notice that

$$\|q(u)\|_W = \|j(\pi(u))\|_W = \|\pi(u)\|_{\widehat{E}_*} = |u|_* \qquad \forall\, u \in E_*\,,$$

so $\ker(q) = \mathcal{K}$, which means that q is not injective. In addition, since π is surjective and j has dense range, it follows that q has dense range. Let us introduce the adjoint of q as the operator $q' : W' \to E'_*$ defined by

$$q'(f) := f \circ q \qquad \forall\, f \in W'\,, \tag{3.3}$$

which is a linear and bounded operator. It is straightforward to prove that q' is an isometry. Since W' and E'_* are normed vector spaces, this fact implies that q' is injective.

We proceed to prove that it is also surjective. Let $g \in E'_*$ and define the map $\widehat{g} : \widehat{E}_* \to \mathbb{R}$ by

$$\widehat{g}(\widehat{u}) := g(u) \qquad \forall\, \widehat{u} \in \widehat{E}_* \quad \text{with} \quad u \in \pi^{-1}(\widehat{u})\,.$$

Using that g is continuous in E_*, it can be proven that $\widehat{g}$ is well defined, meaning that it does not depend on the representative of the class. Moreover, $\widehat{g} \in \widehat{E}'_*$. Then, we may define $f : R(j) \to \mathbb{R}$ by

$$f(w) := \widehat{g}(j^{-1}(w))\,,$$

which is properly defined, since j is injective. One can verify that f is a bounded and linear functional on $R(j)$. By Hahn–Banach theorem, it follows that there is a continuous linear extension $F \in W'$. We now notice that, for each $u \in E_*$,

$$q'(F)(u) = F(q(u)) = F(j(\pi(u))) = f(j(\pi(u))) = \widehat{g}(\pi(u)) = g(u)\,,$$

which means that $q'(F) = g$. Therefore, q' is surjective.

We have shown that q' is an isometric isomorphism between W' and E'_*. Since W is a Hilbert space, the Riesz representation theorem implies that W and W' are isometrically isomorphic. We thus obtain the following result.

Theorem 3.2. *Let E be a real vector space and let $\mathcal{N} : E \to E^\star$ be a linear, symmetric and monotone operator, whose kernel is denoted by $\mathcal{K}$. Let $(\cdot,\cdot)_*$ be the semi-inner product induced by $\mathcal{N}$, and denote by $|\cdot|_*$ the corresponding seminorm. Set $E_* := (E, |\cdot|_*)$ and define the quotient space $\widehat{E}_* := E/\mathcal{K}$ endowed with the inner product (3.2). Furthermore, let W denote the completion of $\widehat{E}_*$ and let T be the isometric isomorphism from $\widehat{E}'_*$ onto W. Additionally, let $\mathcal{R} : W \to W'$ be the Riesz mapping and $q' : W' \to E'_*$ be defined as in (3.3). Then, we have the following isometric isomorphisms:*

$$\widehat{E}'_* \xrightarrow{\;T\;} W \xrightarrow{\;\mathcal{R}\;} W' \xrightarrow{\;q'\;} E'_*\,.$$

4. Finale: Solvability of an evolution problem

In what follows, we adopt the same notations and definitions as in Theorem 3.2 and, additionally, we consider a monotone operator $\mathcal{M} : \mathcal{D} \to E'_*$, where $\mathcal{D} \subset E$ is the domain of $\mathcal{M}$. In this section, our aim is to study the solvability of the problem

$$\frac{d}{dt}\big(\mathcal{N}(u(t))\big) + \mathcal{M}\big(u(t)\big) \;=\; f(t) \quad \text{for a.e.} \quad 0 < t < T\,,$$

where $f : [0,T] \to E'_*$ and $u(0) = u_0$ are given. To this end, we employ the tools developed in the previous section to reformulate this problem within the framework of Kato's theorem (cf. Theorem 2.3).

Using the definitions of $(\cdot,\cdot)_*$ and $(\cdot,\cdot)_{\widehat{E}_*}$, together with the fact that j is an isometry and the definition of the mapping q (cf. (3.3)), we find that, for all $x, y \in E$,

$$[\mathcal{N}(x), y] = (x,y)_* = (\pi(x), \pi(y))_{\widehat{E}_*} = (j(\pi(x)), j(\pi(y)))_W = (q(x), q(y))_W\,. \tag{4.1}$$

Moreover, employing now the Riesz map $\mathcal{R} : W \to W'$, which satisfies

$$[\mathcal{R}(v), w] = (v,w)_W \qquad \forall\, v, w \in W\,,$$

and the definition of q' (cf. (3.3)), the identity (4.1) is rewritten as

$$[\mathcal{N}(x), y] = [\mathcal{R}(q(x)), q(y)] = \big[(q' \circ \mathcal{R} \circ q)(x), y\big] \qquad \forall\, x, y \in E\,,$$

which proves the following factorization of $\mathcal{N}$,

$$\mathcal{N} = q' \circ \mathcal{R} \circ q\,. \tag{4.2}$$

On the other hand, to obtain a similar factorization for $\mathcal{M}$, let us introduce the relation $\mathcal{M}_0 \subset W \times W'$ defined by

$$g \in \mathcal{M}_0(\widehat{x}) \quad \text{if and only if} \quad \exists\, x \in \mathcal{D}\ :\ q(x) = \widehat{x} \quad \text{and} \quad q'(g) = \mathcal{M}(x)\,. \tag{4.3}$$

In general, $\mathcal{M}_0$ is not a single-valued operator, since q is not injective. Observe also that $\mathcal{D}(\mathcal{M}_0) = \big\{q(x) \in W\ :\ x \in \mathcal{D}\big\}$. Moreover,

$$\mathcal{M} = q' \circ \mathcal{M}_0 \circ q\,. \tag{4.4}$$

The following lemma presents two equivalences that relate all these operators. The proof follows directly from (4.2), (4.3), and (4.4), and is therefore omitted.

Lemma 4.1. *There hold:*

(1) For each $\widehat{x} = q(x) \in W$, $q'(g) = \mathcal{M}(x)$ if and only if $g \in \mathcal{M}_0(\widehat{x})$, and, in this case, we have $[q'(g), x] = g(\widehat{x})$.

(2) The equality $q'(g) = (\mathcal{N} + \mathcal{M})(x)$ holds if and only if $g \in (\mathcal{R} + \mathcal{M}_0)(q(x))$.

Using this lemma, it is straightforwardly obtained the following result.

Corollary 4.2. *There hold:*

(1) $\mathcal{M} : \mathcal{D} \to E'_*$ is monotone if and only if $\mathcal{M}_0 : \mathcal{D}(\mathcal{M}_0) \to W'$ is monotone.

(2) q' is a bijection of $R(\mathcal{R} + \mathcal{M}_0)$ onto $R(\mathcal{N} + \mathcal{M})$.

We are finally ready to establish the main result of this note.

Theorem 4.3. *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given from the real vector space E to its algebraic dual $E^\star$, and let E'_* be the Hilbert space which is the dual of the seminormed space $(E, |\cdot|_*)$, where*

$$|x|_* = [\mathcal{N}(x), x]^{1/2} \qquad \forall\, x \in E\,.$$

Let $\mathcal{M} : \mathcal{D} \to E'_*$ be a monotone operator, with $\mathcal{D} \subset E$, and suppose that $R(\mathcal{N} + \mathcal{M}) = E'_*$. Then, for each $f \in \mathrm{W}^{1,1}(0, T; E'_*)$ and for each $u_0 \in \mathcal{D}$, there is a solution $u : [0, T] \to E$ of

$$\frac{d}{dt}\big(\mathcal{N}(u(t))\big) + \mathcal{M}\big(u(t)\big) \;=\; f(t) \quad \textit{for a.e.} \quad 0 < t < T\,, \tag{4.5}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0, T; E'_*)\,, \quad u(t) \in \mathcal{D} \quad \textit{for all } 0 \le t \le T\,, \quad \textit{and} \quad \mathcal{N}(u(0)) = \mathcal{N}(u_0)\,.$$

Proof. We begin by showing that the existence of a solution to (4.5) is equivalent to the existence of a solution to a related problem formulated within the framework of (2.3). Suppose that $u : [0, T] \to E$ is a solution to (4.5). Then, since $q' : W' \to E'_*$ is an isomorphism, by combining (4.2) and (4.4), we find that $\widehat{u} := \mathcal{R} \circ q \circ u : [0,T] \to W'$ satisfies

$$\frac{d\widehat{u}}{dt}(t) + (\mathcal{M}_0 \circ \mathcal{R}^{-1})(\widehat{u}(t)) \ni \widehat{f}(t) \quad \text{for a.e.} \quad 0 < t < T\,, \tag{4.6}$$

where $\widehat{f} := (q')^{-1} \circ f$. Observe that we have lost the equality, which has been replaced by an inclusion, since $\mathcal{M}_0$ is not a single-valued operator. We now suppose that $\widehat{u} : [0,T] \to W'$ solves (4.6). Then, $\mathcal{R}^{-1} \circ \widehat{u}(t) \in \mathcal{D}(\mathcal{M}_0)$ for a.e. $t \in (0,T)$, which implies that, for almost every t, there exists $u(t) \in \mathcal{D}$ such that $\mathcal{R}^{-1} \circ \widehat{u}(t) = q(u(t))$. Hence, using once again the isomorphism q' together with (4.2) and (4.4), we deduce that u satisfies the original equation (4.5).

Now, in order to use Theorem 2.3, we must prove that $\mathcal{M}_0 \circ \mathcal{R}^{-1} \subset W' \times W'$ is an m-accretive operator. Since $\mathcal{M}$ is monotone, by Corollary 4.2 it follows that $\mathcal{M}_0$ is monotone. Then, for all $(f_i, g_i) \in W' \times W'$ such that $(\mathcal{M}_0 \circ \mathcal{R}^{-1})(f_i) = g_i$, $i \in \{1,2\}$, there holds

$$\begin{aligned}(g_1 - g_2, f_1 - f_2)_{W'} &= (\mathcal{R}^{-1}(g_1 - g_2), \mathcal{R}^{-1}(f_1 - f_2))_W = [g_1 - g_2, \mathcal{R}^{-1}(f_1 - f_2)] \\ &= [\mathcal{M}_0(\mathcal{R}^{-1}(f_1)) - \mathcal{M}_0(\mathcal{R}^{-1}(f_2)), \mathcal{R}^{-1}(f_1) - \mathcal{R}^{-1}(f_2)] \geq 0\,,\end{aligned}$$

where we used the monotonicity of $\mathcal{M}_0$. This proves that $\mathcal{M}_0 \circ \mathcal{R}^{-1}$ is accretive. It remains to prove the range condition $R(I + \mathcal{M}_0 \circ \mathcal{R}^{-1}) = W'$. From the hypothesis on the range and Corollary 4.2, we have that q' is a bijection of $R(\mathcal{R} + \mathcal{M}_0)$ onto E'_*. This means that $R(\mathcal{R} + \mathcal{M}_0) = W'$. Thus, for all $g \in W'$, there exists $w \in W$ such that

$$(\mathcal{R} + \mathcal{M}_0)(w) = g\,.$$

Since $\mathcal{R}$ is an isomorphism, the above equation is equivalent to say that for all $g \in W'$, there exists $f \in W'$ such that

$$(I + \mathcal{M}_0 \circ \mathcal{R}^{-1})(f) = g\,,$$

where $f = \mathcal{R}(w)$. Thus, $R(I + \mathcal{M}_0 \circ \mathcal{R}^{-1}) = W'$. Therefore, $\mathcal{M}_0 \circ \mathcal{R}^{-1}$ is m-accretive.

Finally, by applying Theorem 2.3 in the context of (4.6), we conclude the desired result. □

References

[Bus25] Alonso J. Bustos. "A Banach Space Mixed Formulation for the Unsteady Brinkman Problem with Spatially Varying Porosity". Undergraduate thesis. Universidad de Concepción, 2025. URL: https://repositorio.udec.cl/items/bc57f140-1aec-435e-b914-1752e345c409.

[Kat53] Tosio Kato. "Integration of the equation of evolution in a Banach space". In: *J. Math. Soc. Japan* 5 (1953), pp. 208–234.

[Sho97] R.E. Showalter. *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Mathematical surveys and monographs. American Mathematical Society, 1997.