Introduction to Logistic Regression

Overview for Today

Today we will be learning about:

- All about simple and multiple logistic regression
- How to interpret the results of a logistic regression model

Generalized Linear Models: A Family of Statistical Models

Generalized linear models (GLMs) are a family of statistical models that **generalize** the methods of linear regression to outcome variables that are neither continuous, nor normally distributed.

The Components of a GLM

GLMs are built from three separate components:

1. **Random component** that specifies the probability distribution of the outcome variable.
2. **Linear predictor** that describes how the predictor variables relate to the outcome variable.
3. **Link function** that **links** the linear predictor to the **mean** of the outcome variable's probability distribution.

Linear Regression as a Generalized Linear Model

When considered as a GLM, we can specify a simple linear regression model as:

1. **Random Component**: Normal distribution
2. **Linear Predictor**: $\beta_0 + \beta_1 X_1$
3. **Link Function**: $g(\mu) = \beta_0 + \beta_1 X_1$

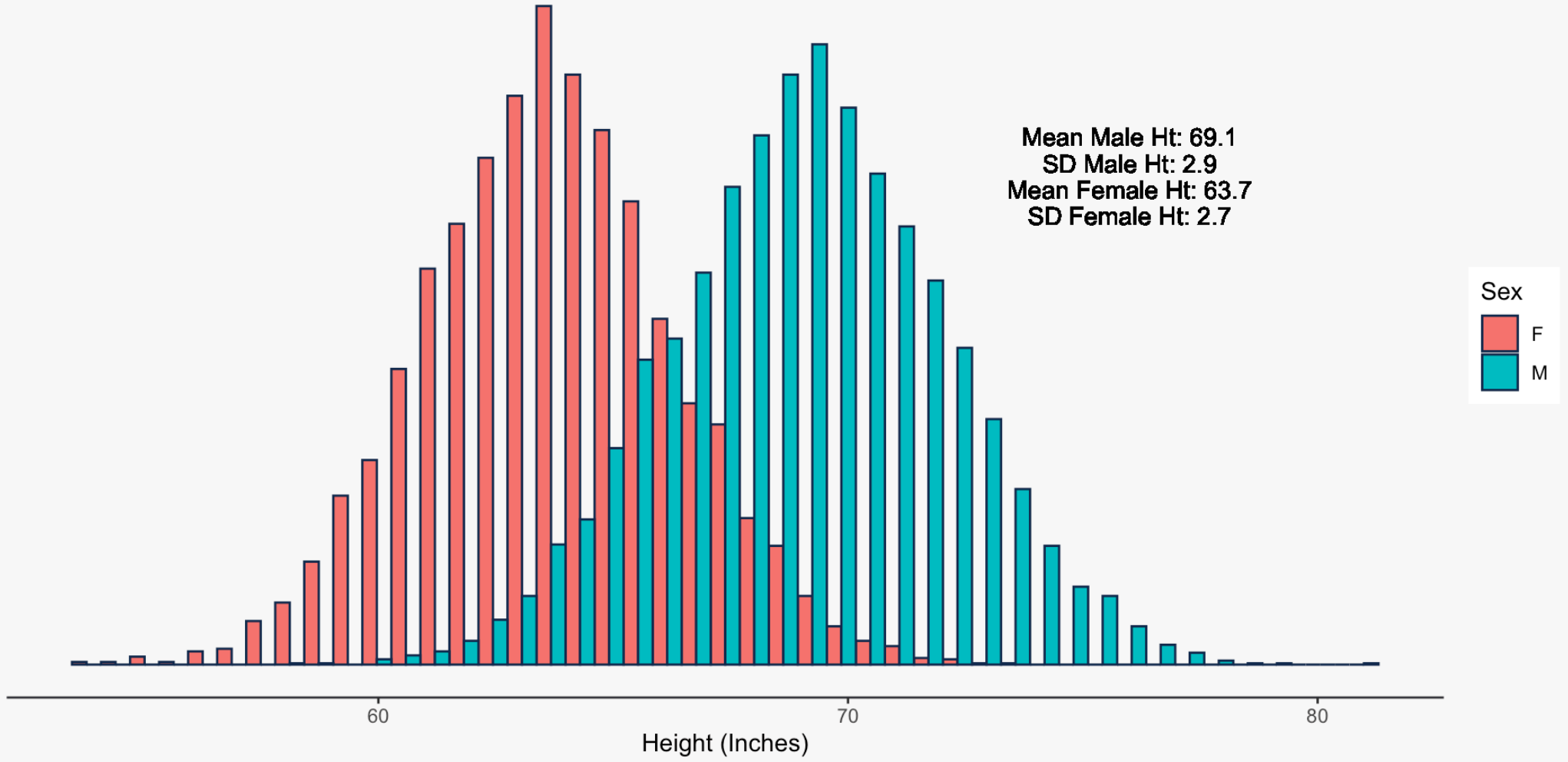
Linear Regression as a Generalized Linear Model

We can write the linear regression model as a generalized linear model where the mean of the normal distribution is just set equal to the linear predictor, $\beta_0 + \beta_1 X_1$.

$$Y|X \sim N(\text{mn.} = \mu = g^{-1}(x),\ \text{s.d.} = \sigma)$$

$$Y|X \sim N(\text{mn.} = \beta_0 + \beta_1 X_1,\ \text{s.d.} = \sigma)$$

Linear Regression: An Example with US Heights by Gender



Linear Regression: An Example with US Heights by Gender

```
1 mod_height <- lm(ht ~ sex, data = data_ht)
```

```
Call:
lm(formula = ht ~ sex, data = data_ht)

Residuals:
     Min       1Q   Median       3Q      Max
-11.1388  -1.8635   0.0065   1.8557  11.6430

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 63.68069    0.04038 1576.85   <2e-16 ***
sexM         5.39268    0.05610   96.13   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 2.803 on 9998 degrees of freedom
Multiple R-squared:  0.4803,    Adjusted R-squared:  0.4803
F-statistic:  9242 on 1 and 9998 DF,  p-value: < 2.2e-16
```

Linear Regression: An Example with US Heights by Gender

The probability model estimated by our regression is:

$$\text{US Ht.}|\text{Sex} \sim N(\text{mn.} = 63.68 + 5.39 \times \text{Sex},\ \text{s.d.} = 2.80$$

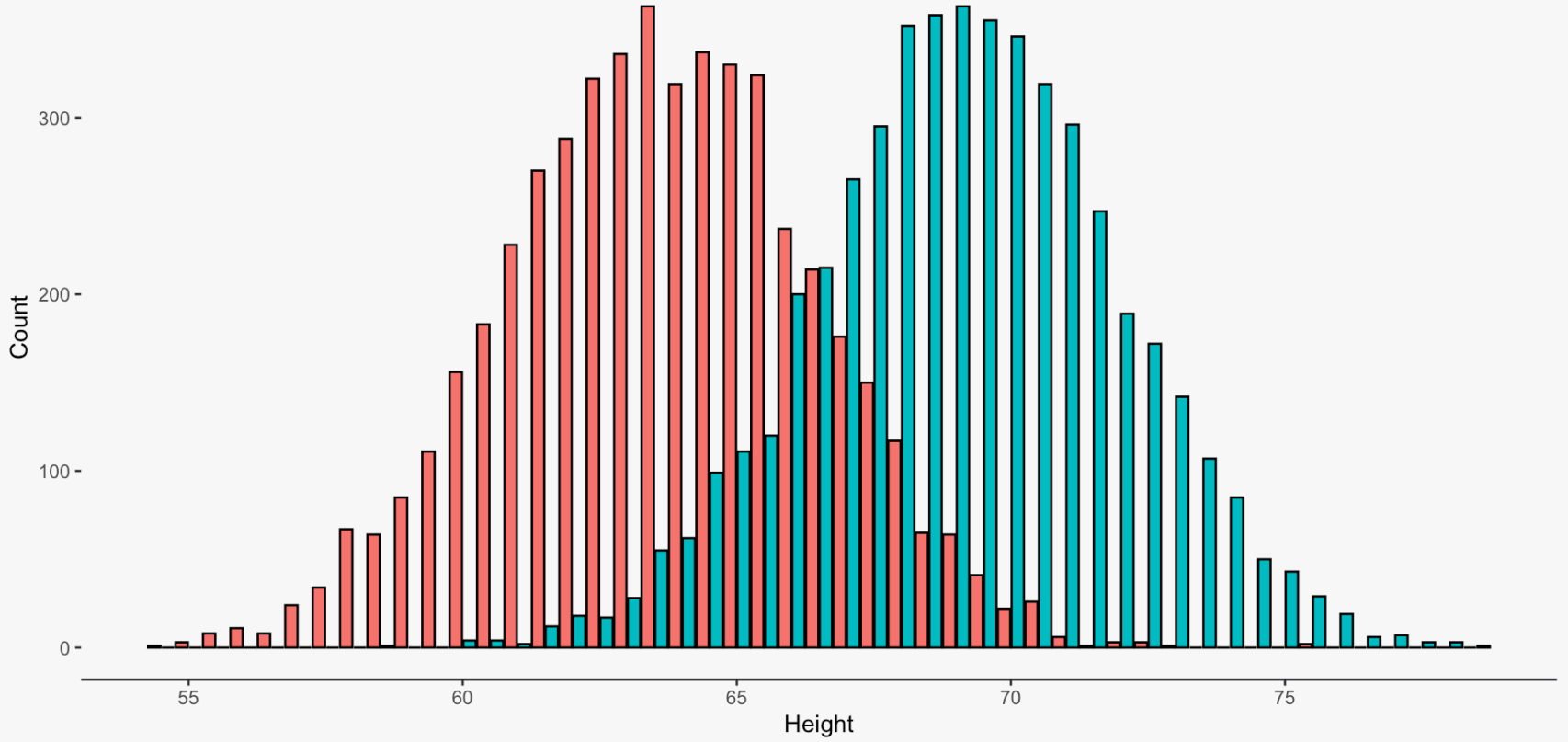
$$\text{US Ht.}|\text{Sex} = \text{Male} \sim N(\text{mn.} = 63.68 + 5.39,\ \text{s.d.} = 2.$$

$$\text{US Ht.}|\text{Sex} = \text{Female} \sim N(\text{mn.} = 63.68,\ \text{s.d.} = 2.80)$$

Linear Regression: An Example with US Heights by Gender

Simulating the data according to our model:

```r
ht_female <- rnorm(5000, mean = 63.68 + 5.39 * 0, sd = 2.80)
ht_male <- rnorm(5000, mean = 63.68 + 5.39 * 1, sd = 2.80)
```



What Happens When Our Outcome Isn't Normal?

The power of GLMs is that they open up a whole new world of probability distributions for us to specify when our outcome doesn't follow a normal distribution like:

- Bernouli Distribution
- Gamma Distribution
- Poisson Distribution & more!

But how?

Non-Profit Donation: An Example of a Bernouli Distribution

Our outcome is whether or not a shopper decided to donate to a non-profit the store at which they were shopping supported.

- Donate: Yes/No
- Perceived Corporate Social Responsibility of corporation: 1-7
- Does customer identify with the corporation: Yes/No

Non-Profit Donation: An Example of Simple Logistic Regression

We can write our statistical model as:

$$\text{Donate} \sim \mathit{Bern.}\left(\text{mn.} = \pi,\ \text{s.d.} = \sqrt{\pi(1-\pi)}\right)$$

Non-Profit Donation: Linking CSR to Donations

We are interested in understanding if a shopper's perceptions of the corporation's corporate social responsibility is related to their decision to donate or not. How can we model this?

$$\text{Donate}|\text{CSR} \sim Bern.\left(\text{mn.} = g^{-1}(x),\ \text{s.d.} = \sqrt{\pi(1-\pi}\right.$$

How should a good link function for π behave? (Hint: the linear predictor can take on any negative or positive value.)

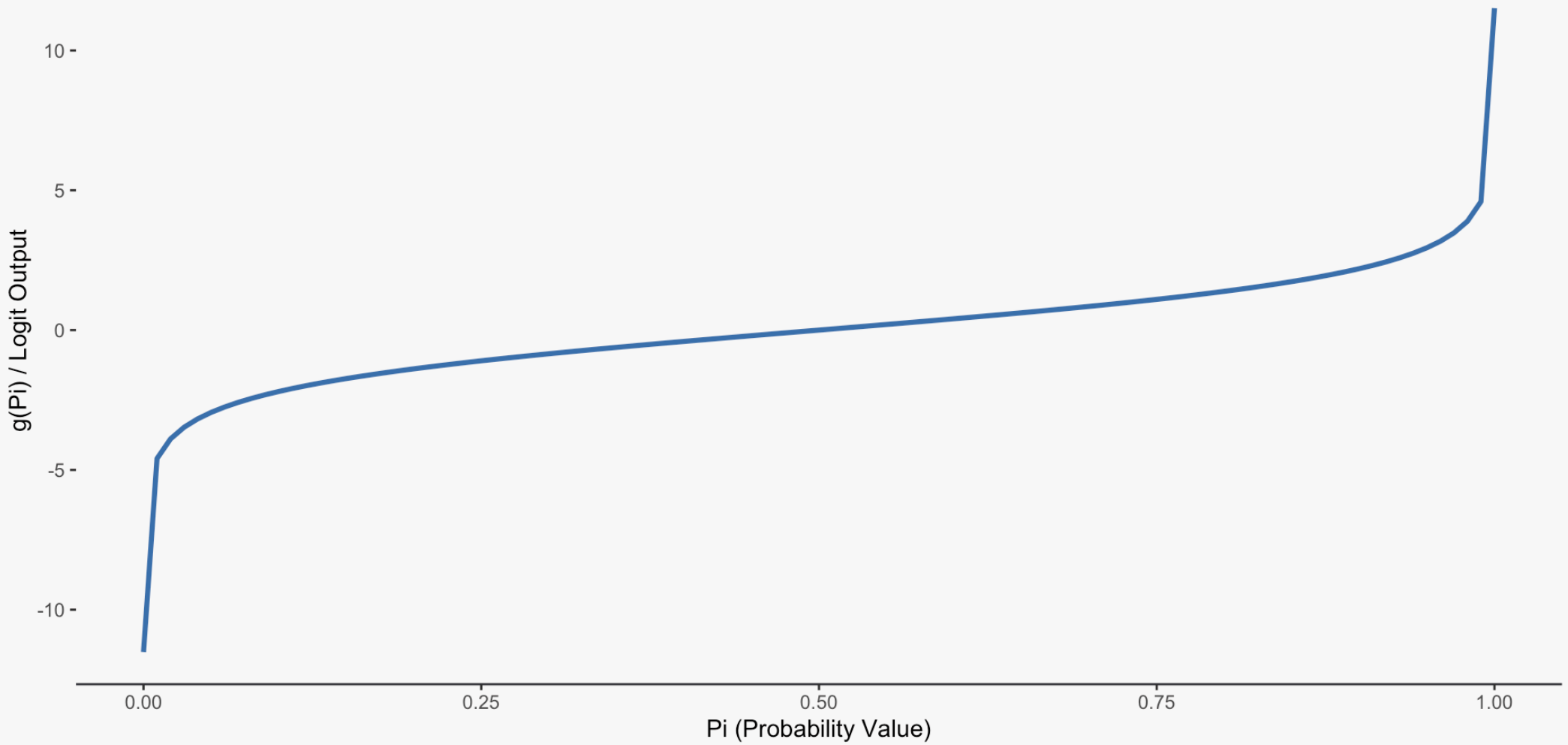
The Logistic Regression Link Function: The Logit

It turns out there is a link function that works very well: the logit or log-odds.

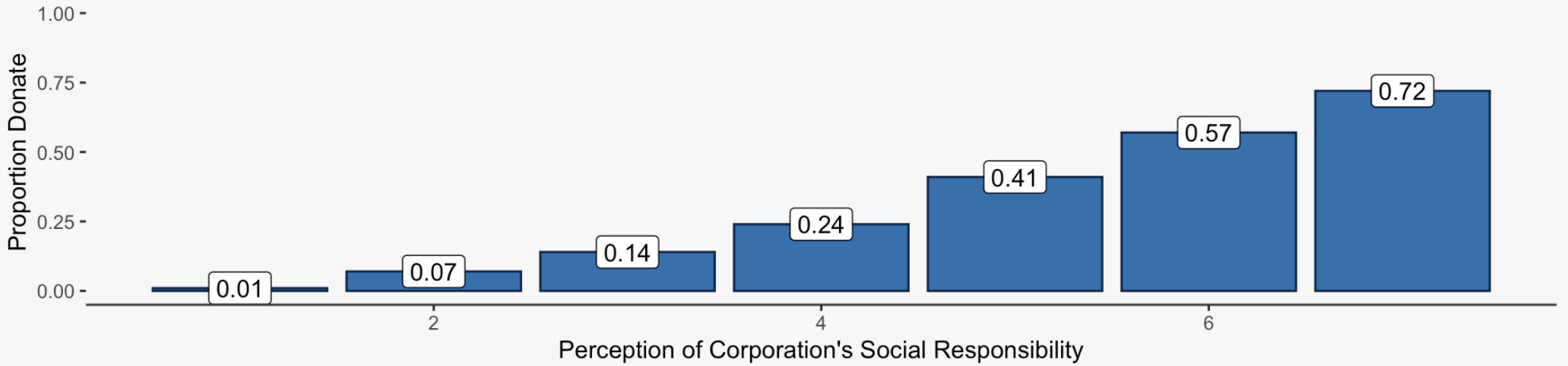
$$g(\pi) = \ln(\frac{\pi}{1-\pi})$$

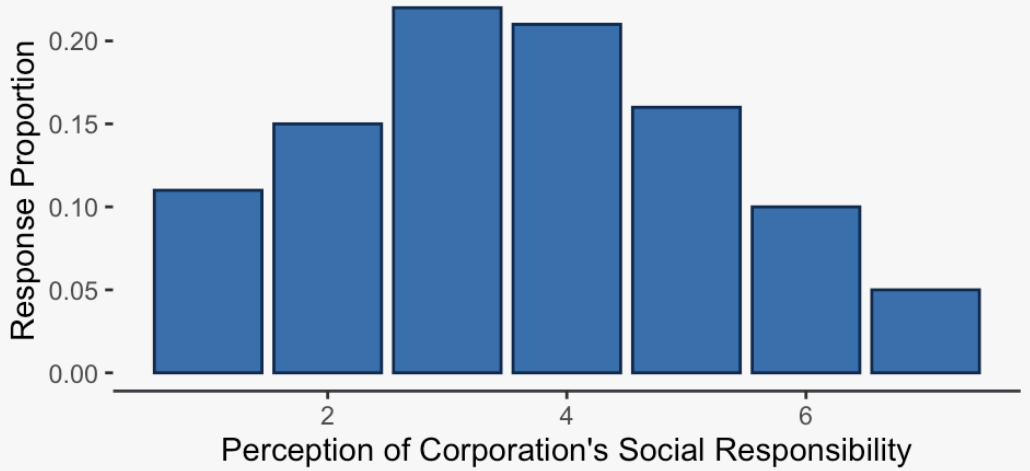
The Logistic Regression Link Function: The Logit

Let's see what value the Logit function outputs at different values of π:



The Relationship Between CSR and Donations





Using a Chi-Square Test

We could explore the relationship between Perceptions of CSR and donations by using a chi-squared test:

```
1 donate_csr_table <- xtabs(~x_csr + donate, data_donate)
2 chisq.test(donate_csr_table)
```

```
    Pearson's Chi-squared test

data:  donate_csr_table
X-squared = 510.06, df = 6, p-value < 2.2e-16
```

Modeling the Relationship Between CSR and Donations

A better and more informative way to model the relationship between CSR and donations is by building and estimating a logistic regression equation:

$$\ln \frac{\pi_{Don.}}{1 - \pi_{Don.}} = \beta_0 + \beta_1 \mathrm{CSR}$$

Note that the outcome we are modeling is now the log-odds (logit) of the probability of donating.

Estimating a Simple Logistic Regression Model with `glm`

```
mod_csr <- glm(donate ~ x_csr, family = binomial(link = "logit"),
               data = data_donate)
```

We can use the function `glm` to estimate a logistic regression model in R. We need to tell `glm`:

- the linear predictor: `donate ~ x_csr`
- the random component: `family = binomial`
- the link function: `link = "logit"`.

The Results of a Logistic Regression Model

```
1 summary(mod_csr)
```

```
Call:
glm(formula = donate ~ x_csr, family = binomial(link = "logit"),
    data = data_donate)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept) -4.03572    0.16903  -23.88   <2e-16 ***
x_csr        0.72472    0.03652   19.85   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2837.7  on 2499  degrees of freedom
Residual deviance: 2315.0  on 2498  degrees of freedom
AIC: 2319
```

What Do We Look for in the Results?

Here is a checklist of things to focus on in the model summary:

1. The sign and magnitude of the slope estimates
2. The size of the standard error compared to the estimate
3. The p-value

The Difficulty with Interpreting the Logistic Regression Parameters

Because of the nonlinearity of the link function, it is difficult to interpret the estimated parameters of a logistic regression model. There are two things we can know immediately though:

1. A positive slope estimate means that increases in the predictor variable lead to increases in the probability of observing the event.
2. A Z-value greater than ~|2| signals that the slope is significantly different from 0

Thankfully, there are ways we can transform the slopes to make more sense of them!

The Slope as a Change in the Odds Ratio

Because thinking in logits is weird (and hard), let us transform the coefficients into something more interpretable: an odds ratio.

$$\beta_1 = \log \frac{\text{Odds}_{X+1}}{\text{Odds}_X}$$

$$\exp(\beta_1) = \frac{\text{Odds}_{X+1}}{\text{Odds}_X}$$

The Slope as a Change in the Odds Ratio

A one unit increase in CSR results in a 2.06 (106 %) increase in the odds of donating to the corporation's charity of choice.

```
1 exp(mod_csr$coefficients)
```

Coef. Name	Estimate	Exp. Estimate	SE	Z	p
(Intercept)	-4.04	0.02	0.17	-23.88	0
x_csr	0.72	2.06	0.04	19.85	0

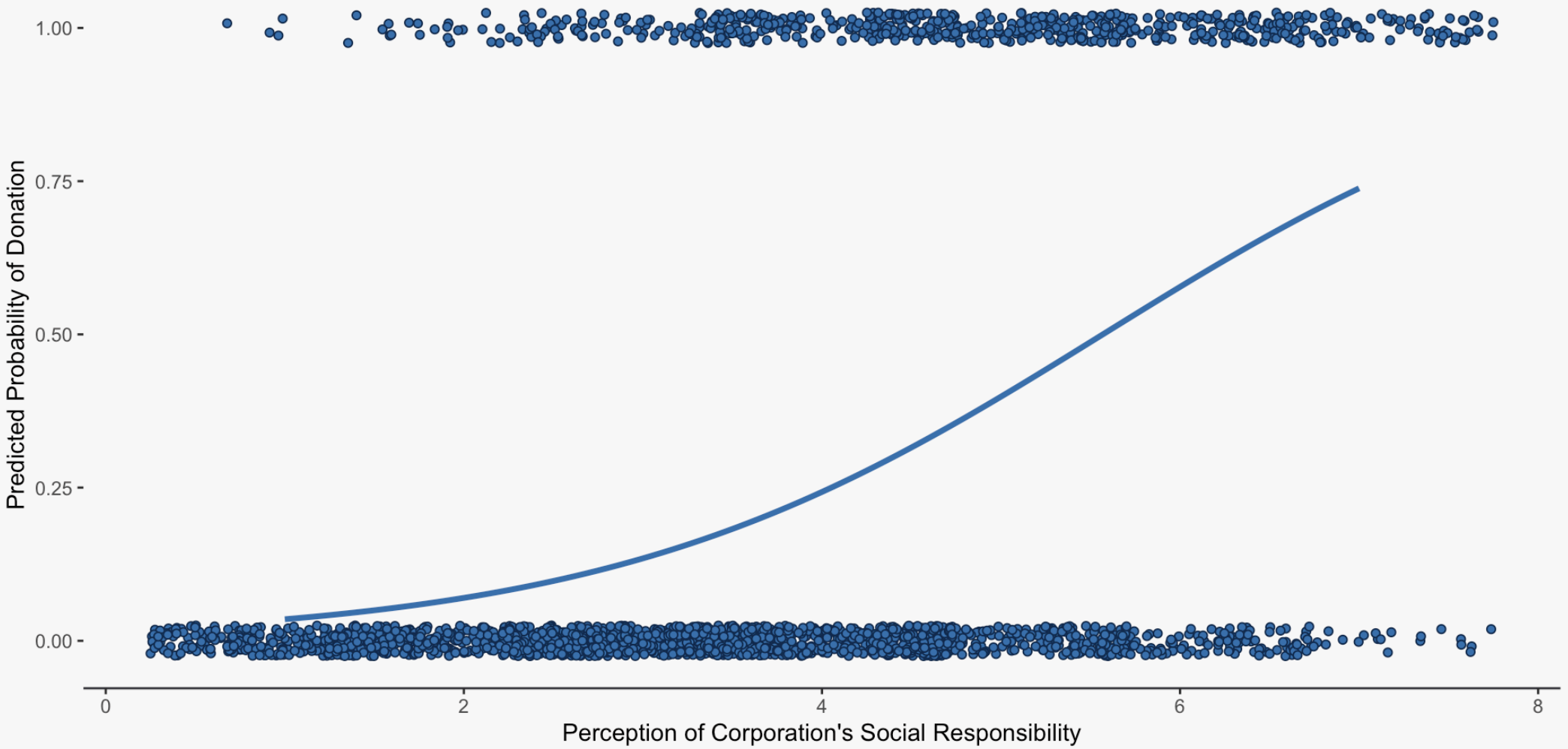
Interpreting the Logistic Regression Slope: Predicted Probability

Odds ratios are also kind of hard to interpret, so I prefer to interpret the logistic regression slope as a change in the predicted probability of the outcome occurring (donating, in our example).

```
predicted_probability <- predict(mod_csr, type = "response")
```

CSR	Pred. Prob.
1	0.04
2	0.07
3	0.13
4	0.24
5	0.40
6	0.58
7	0.74

Predicted Probability Curve



How Do We Interpret the Predicted Probability Plot?

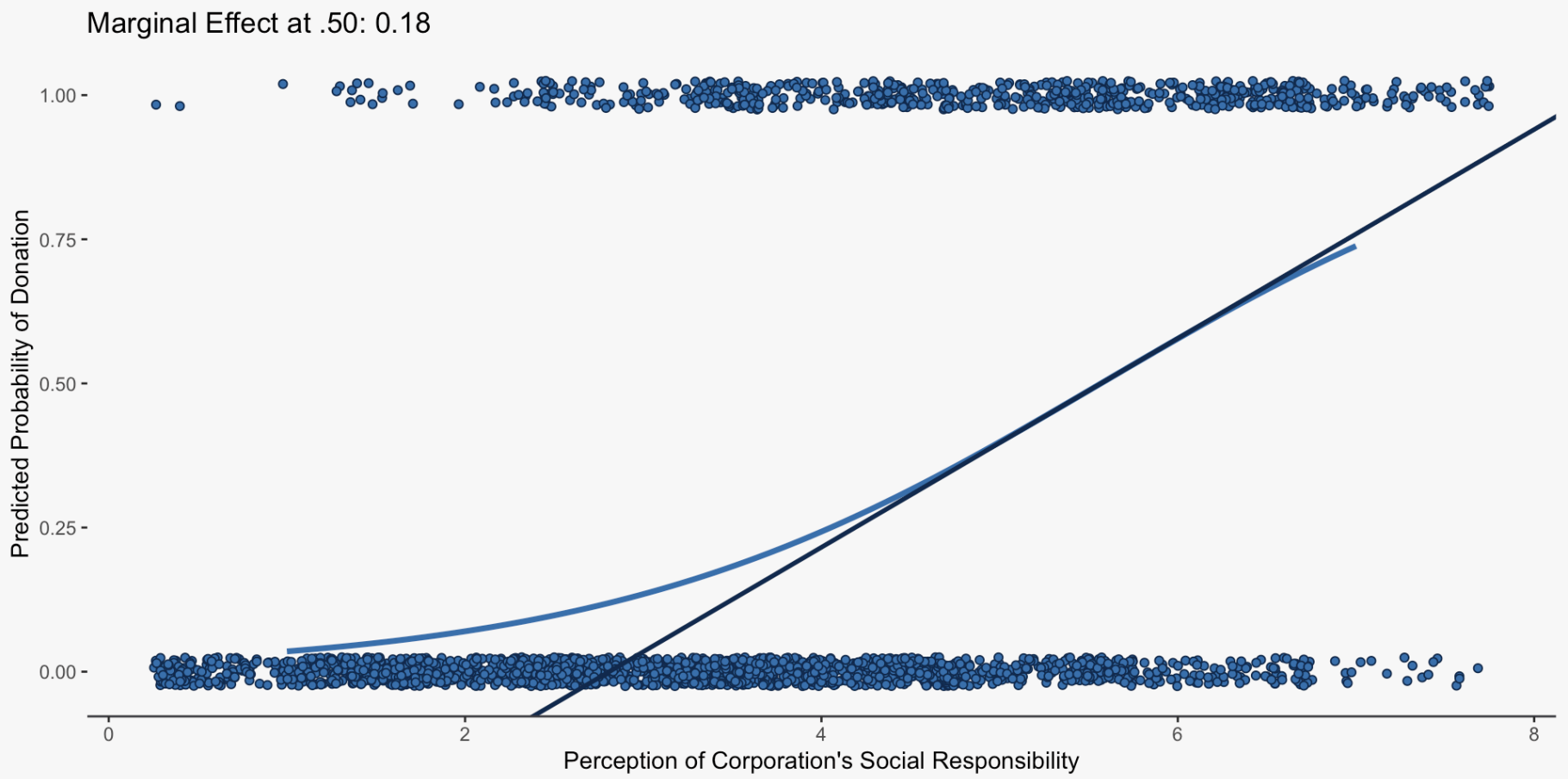
The probability curve is nonlinear, so the effect that our predictor variable (corporate social responsibility) has on our outcome (donating or predicted probability of donating) differs depending on what the predicted probability is:

- The effect of of X on Y is low when the predicted probability is around ~.05.
- The effect of X on Y is moderate when the predicted probability is around ~.50.
- The effect of X on Y begins to level off after predicted probability of ~.75.

So how do we provide a summary of these effects?

Interpreting the Logistic Regression Slope: Marginal Effect

One way to solve the interpretability solution is to calculate the effect of the predictor at a specific value of the predicted probability:



Interpreting the Logistic Regression Slope: The Average & Max Marginal Effect

Instead of calculating the marginal effect a single value of the predicted probability, it would be even better to calculate the **average marginal effect** and the **maximum marginal effect**:

```
1  mfx::logitmfx(mod_csr, atmean = FALSE, data = data_donate) # Average ME
2  mod_csr$coefficients[2] / 4 # Maximum ME
```

Avg. ME	Max ME
0.11	0.18

So...What is the Effect of CSR on Donating?

Here is a summary of our different interpretations:

1. A one unit increase in CSR (predictor) leads to a 2.06 increase in the odds of donating.
2. A one unit increase in CSR will lead to at most a 18 point increase in the probability of donating and roughly a 11 point increase, on average.

Simple Logistic Regression: Categorical Predictor

Now we would like to know if customers' identification with the company (a categorical predictor—yes or no) is related to whether they donate to the company's preferred charity.

A Quick Reminder on Indicator Coding

Indicator coding takes a categorical variable with K categories (2 in our case) and transforms them into K - 1 indicator variables (0 or 1).

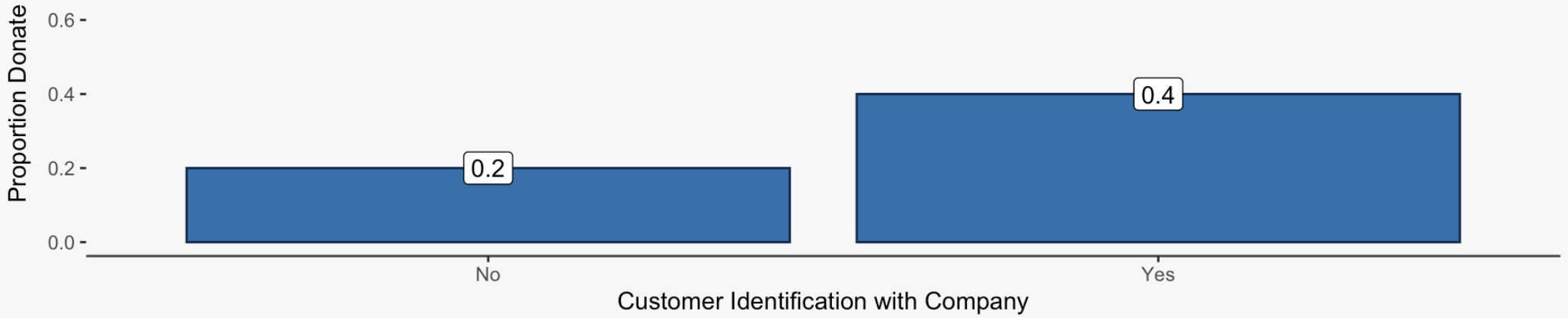
- An indicator variable for each category except the reference category.
- An indicator variable takes on a value of 1 if the observation is a member of the category else it takes on 0.
- The reference category is identified by taking on 0s across all of the indicator variables.

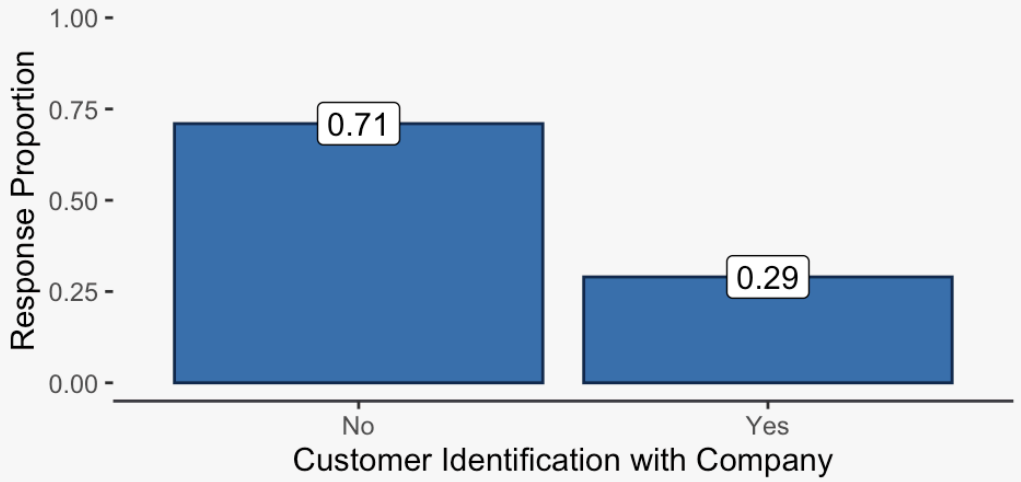
Indicator Coding for Customer Identification

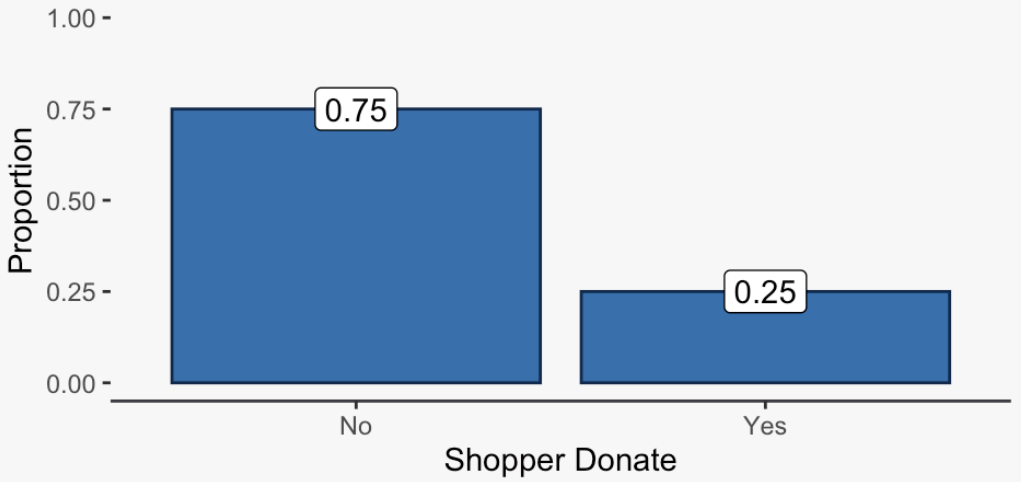
In our data, why do we only need one indicator variable and which category is the reference group?

```
# A tibble: 2,500 × 2
   x_cust_id_indicator x_cust_id
                 <dbl> <chr>
 1                   0 No
 2                   0 No
 3                   1 Yes
 4                   1 Yes
 5                   1 Yes
 6                   1 Yes
 7                   1 Yes
 8                   0 No
 9                   1 Yes
10                   0 No
# ℹ 2,490 more rows
```

The Relationship Between Customer Identification and Donations







Estimation & Interpretation with a Categorical Predictor

Estimating a logistic regression model with a categorical predictor is no different than estimating one with a quantitative predictor and interpretation is a little easier.

```
mod_cust_id <- glm(donate ~ x_cust_id, family = binomial(link = "logit"),
                   data = data_donate)
summary(mod_cid)
```

```
Call:
glm(formula = donate ~ x_cust_id, family = binomial(link = "logit"),
    data = data_donate)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)  -1.40110    0.05950  -23.55   <2e-16 ***
x_cust_idYes  0.98054    0.09671   10.14   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2837.7  on 2499  degrees of freedom
```

Interpreting the Effect of a Categorical Predictor as an Odds Ratio

When we exponentiate the slope for the categorical predictor, we can interpret it as an odds ratio where the numerator is the odds of an event for category K and the denominator is **always** the odds of an event for the reference category.

So, values greater than 1 indicate the odds of an event occurring for category K are greater than they are for the reference category.

```
1 exp(mod_cust_id$coefficients[2])
```

```
x_cust_idYes
        2.67
```

Interpreting the Effect of a Categorical Predictor as a Predicted Probability

Similarly, we can also calculate the predicted probability of the event (donating) for each category:

```
predict(mod_cust_id, type = "response")
```

```
# A tibble: 2 × 2
  x_cust_id pred_prob
  <chr>          <dbl>
1 No               0.2
2 Yes              0.4
```

The Marginal Effect of a Categorical Variable

For a two category categorical variable (like ours), the marginal effect is the difference in the two predicted probabilities. For more than two categories, it is the average difference across the K-1 comparisons.

```
mfx::logitmfx(mod_cust_id, data = data_donate, atmean = FALSE)
prop.test(c(donate_yes, donate_no), c(total_yes, total_no))
```

```
# A tibble: 1 × 4
  `dF/dx` `Std. Err.`     z `P>|z|`
    <dbl>       <dbl> <dbl>   <dbl>
1   0.199       0.021  9.68       0
```

```
    2-sample test for equality of
proportions with continuity correction

data:  c(donate_yes, donate_no) out of
c(total_yes, total_no)
X-squared = 105.51, df = 1, p-value < 2.2e-
16
alternative hypothesis: two.sided
95 percent confidence interval:
 0.1575121 0.2399721
sample estimates:
   prop 1    prop 2
0.3963839 0.1976418
```

Multiple Logistic Regression: A Quantitative & Categorical Predictor

Simple logistic regression is great, but multiple logistic regression is better!

Like multiple linear regression, multiple logistic regression allows us to estimate the effect of one predictor variable while adjusting (controlling) for the effects of the other predictor variables in the model.

What it Means to Adjust for Another Variable

Why are ice cream sales related to shark attacks?

```
# A tibble: 2 × 2
  sales           avg_shark_att
  <chr>                   <dbl>
1 Low Sales Rev.           20.9
2 High Sales Rev.          51.4
```

```
# A tibble: 4 × 3
  season     sales           avg_shark_att
  <chr>      <chr>                   <dbl>
1 Not Summer Low Sales Rev.           9.12
2 Not Summer High Sales Rev.          7.9
3 Summer     High Sales Rev.         62.5
4 Summer     Low Sales Rev.          63.6
```

What it Means to Adjust for Another Variable

Here is what adjusting looks like in a model:

rowname	Estimate	Std. Error	t value	Pr(>|t|)
Int.	51.35	3.28	15.66	0
Low Sales	-30.46	4.59	-6.64	0

rowname	Estimate	Std. Error	t value	Pr(>|t|)
Int.	7.93	1.17	6.79	0.00
Low Sales	1.19	1.18	1.00	0.32
Summer	54.55	1.18	46.07	0.00

Our Updated Example

We now want to know what the impact of both a customer's perception of the corporation's social responsibility efforts and their identification with that corporation have on their willingness to donate to the corporation's preferred charity.

To answer these questions, we will need to use a **multiple logistic regression** model.

Estimating A Multiple Logistic Regression

Estimating a multiple logistic regression is nearly identical to estimating the simple logistic regression equation:

```
mod_donate <- glm(donate ~ x_csr + x_cust_id,
                  family = binomial(link = "logit"),
                  data = data_donate)
```

Interpreting the Multiple Logistic Regression Estimates

We can interpret the estimates in a multiple logistic regression model just like we would the estimates in a simple logistic regression model with the added phrase of:

"while adjusting (or controlling) for the effects of the other predictor variables."

Interpreting the Multiple Logistic Regression Estimates

The easiest way to interpret the effects is to plot the predicted probability curves. Below you will find two curves, one for the relationship between CSR and donation when a customer does not identify with the corporation and another curve for when the customer does identify.

Calculating the Average Marginal Effects

We can also still use `mfx::logitmfx()` to calculate the average marginal effects for each predictor:

```
1 mfx::logitmfx(mod_donate, data = data_donate, atmean = FALSE)
```

```
Call:
mfx::logitmfx(formula = mod_donate, data = data_donate, atmean = FALSE)

Marginal Effects:
                dF/dx Std. Err.      z     P>|z|
x_csr        0.1076262 0.0078272 13.750 < 2.2e-16 ***
x_cust_idYes 0.1842693 0.0178109 10.346 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

dF/dx is for discrete change for the following variables:

[1] "x_cust_idYes"
```

Writing Up Your Interpretation

For every unit increase in a customer's perception of the corporation's social responsibility, we expect the probability of donating to the corporation's preferred charity to increase by 11 points, on average, while adjusting for the customer's identification with the corporation.

What About Statistical Inference?

We can make statistical inferences from the logistic regression model (and all GLMs) just like we did with the ordinary linear regression model:

- Null Hypothesis Significance Testing
- Confidence Intervals

Remember NHST

We setup two hypotheses: Null & Alternative. The Null Hypothesis is a statement that our regression coefficient (slope) takes on a specific value, usually 0. Then we see how far away our actual estimate is from the null value. If it is *far enough away*, then we reject our null and claim that there is a **statistically significant** difference between our null value and estimate.

Steps to NHST

1. Setup your null & alternative hypotheses and your alpha level
2. Calculate your test statistics (Z Value in our output)
3. Calculate the p-value (the probability of seeing a test statistics as extreme or more extreme than ours)
4. If our p-value is less than our alpha level, then we rejoice!

Building Confidence Intervals for Logistic Regression Estimates

To build an approximate confidence interval around the logistic regression estimate, we can use the following formula:

```
csr_se <- summary(mod_donate)$coefficients[2, 2]
ci_95 <- mod_donate$coefficients[2] + c(-qnorm(.975) * csr_se, qnorm(.975) * c
exp(ci_95)
```

Why do we exponentiate the confidence interval?

Interpreting Confidence Intervals for Logistic Regression Estimates

We are 95% confident that the true effect of CSR is between 1.98 and 2.3.

So, at its smallest, a unit increase in CSR will increase the odds of donating by 98% and at its largest, a unit increase in CSR will increase the odds of donating by 130%, while adjusting for customer's identification with the corporation.

```
1 exp(ci_95) |> round(2)
```

```
[1] 1.98 2.30
```