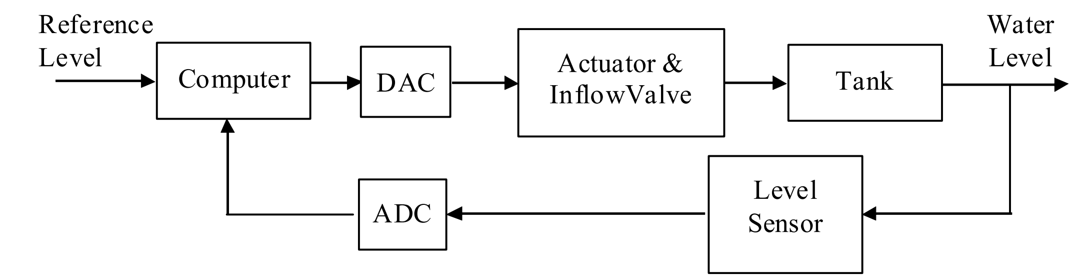
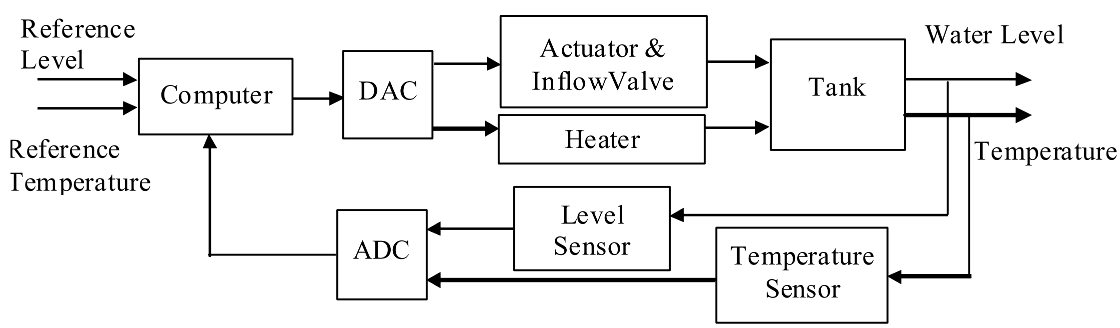
Chapter 1 Solutions

1.1 A fluid level control system includes a tank, a level sensor, a fluid source and an actuator to control fluid inflow. Consult any classical control text[1] to obtain a block diagram of an analog fluid control system. Modify the block diagram to show how the fluid level could be digitally controlled.



Block diagram of water level digital control system.

1.2 If the temperature of the fluid of Problem 1.1 is to be regulated together with its level, modify the analog control system to achieve the additional control (Hint: an additional actuator and sensor are needed). Obtain a block diagram for the two-input-two-output control system with digital control.



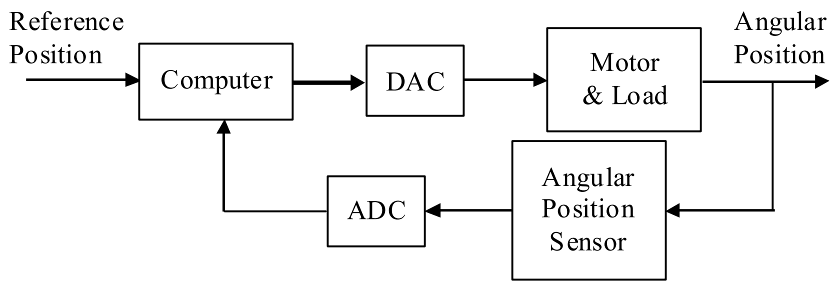
Block diagram of water level and temperature digital control system.

Note that the DAC and ADC can have more than one input and output channel.

1.3 Position control servos are discussed extensively in classical control texts. Draw a block diagram for a DC motor position control system after consulting your classical control text. Modify the block diagram to obtain a digital position control servo.

For the angular position sensor we could use a potentiometer, which is often packaged with an ADC to give a digital output.

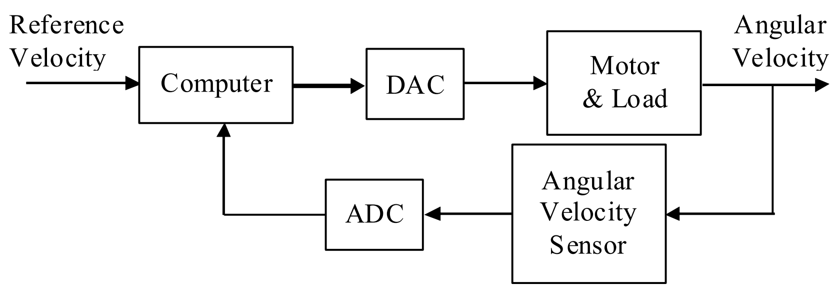
[1]see for example: J. Van deVegte, *Feedback Control Systems*, Prentice Hall, Englewood Cliffs, NJ, 1994.



Block diagram of DC motor digital position control system.

1.4 Repeat Problem 1.3 for a velocity control servo.

For the angular velocity sensor we could use a tachometer, which is often combined with an ADC to give a digital output. Alternatively, we could use an optical encoder , which has a digital output.



Block diagram of DC motor digital velocity control system.

1.5 A ballistic missile is required to follow a predetermined flight path by adjusting its angle of attack α (the angle between its axis and its velocity vector v). The angle of attack is controlled by adjusting the thrust angle δ (angle between the thrust direction and the axis of the missile). Draw a block diagram for a digital control system for the angle of attack including a gyroscope to measure the angle α and a motor to adjust the thrust angle δ.

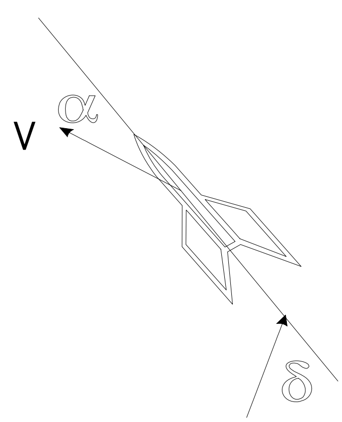
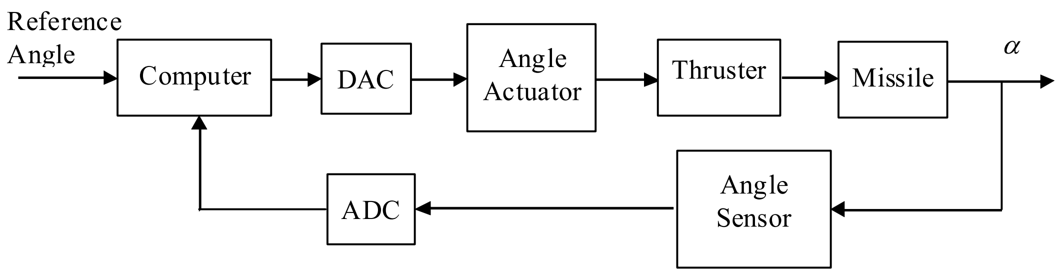


Fig. P1.1 **Missile angle of attack control.**

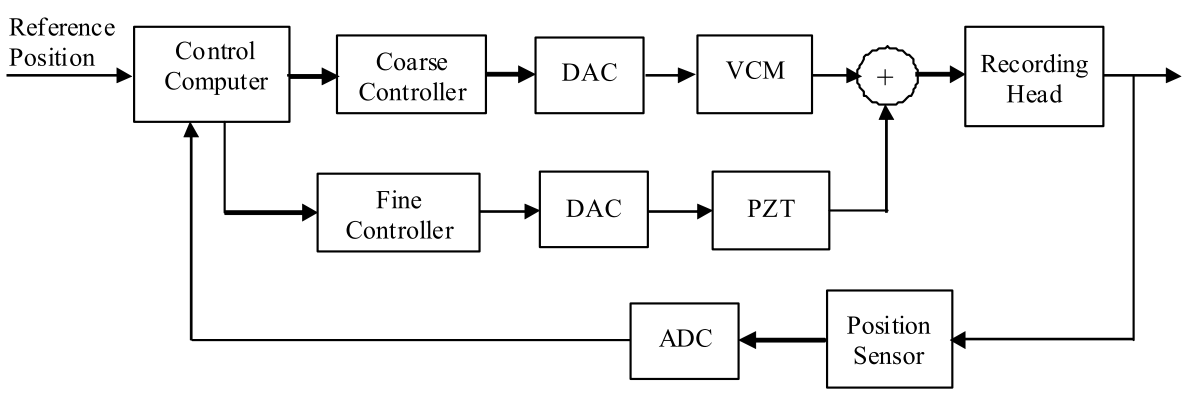


Block diagram of digital missile control system.

1.6 A system is proposed to remotely control a missile from an earth station. Due to cost and technical constraints, the missile coordinates would be measured every 20 seconds for a missile speed of up to 500 m/s. Is such a control scheme feasible? What would the designers need to do to eliminate potential problems?

If the missile is only observed every 20 seconds with speeds of up to 500 m/s, the missile position could change drastically between measurements. This makes the control scheme unrealistic. The missile coordinates need to be measured at a much higher rate.

1.7 The control of the recording head of a dual actuator hard disk drive (HDD) requires two types of actuators to achieve the required high areal density. The first is a coarse voice coil motor (VCM) with large stroke but slow dynamics and the second is a fine piezo-electric transducer (PZT) with a small stroke and fast dynamics. A sensor measures the head position and the position error is fed to a separate controller for each actuator. Draw a block diagram for a dual actuator digital control system for the HDD[2].



[2] J. Ding, F. Marcassa, S.-C. Wu, and M. Tomizuka, "Multirate control for Computational Saving", *IEEE Trans. Control Systems Tech.*, Vol. 14, No. 1, January 2006, pp. 165-169.

Chapter 2 Solutions

2.1 Derive the discrete-time model of Example 2.1 from the solution of the system differential equation with initial time kT and final time$(k+1)T$.

The volumetric fluid balance gives the analog mathematical model

$$\frac{dh}{dt}+\frac{h}{\tau}=\frac{q_i}{C}$$

where $\tau = R\,C$ is the fluid time constant for the tank. The solution of this equation is

$$h(t)=e^{-(t-t_0)/\tau}h(t_0)+\frac{1}{C}\int_{t_0}^{t}e^{-(t-\lambda)/\tau}q_i(\lambda)d\lambda$$

Let q_i be constant over each sampling period T, i.e. $q_i(t) = q_i(k)$ = constant, for t in the interval $[kT, (k+1)T)$. Then

(i) Let $t_0 = kT$, $t = (k + 1)T$

(ii) Simplify the integral as follows with $\nu := (k+1)T-\lambda$

$$\begin{aligned}
&\frac{1}{C}\int_{kT}^{(k+1)T}e^{-[(k+1)T-\lambda]/\tau}q_i(kT)d\lambda \\
&=\frac{1}{C}\left\{\int_{kT}^{(k+1)T}e^{-[(k+1)T-\lambda]/\tau}d\lambda\right\}q_i(kT) && d\nu := -d\lambda \\
&=\frac{1}{C}\left\{\int_{T}^{0}e^{-\nu/\tau}(-d\nu)\right\}q_i(kT) && \nu=\begin{cases}T, & \lambda = kT\\ 0, & \lambda=(k+1)T\end{cases} \\
&=\frac{\tau}{C}\left\{1-e^{-T/\tau}\right\}q_i(kT)
\end{aligned}$$

We thus reduce the differential equation to the difference equation

$$h(k+1)=e^{-T/\tau}h(k)+R\left[1-e^{-T/\tau}\right]q_i(k)$$

2.2 For each of the following equation, determine the order of the equation then test it for
(i) Linearity. (ii) Time-invariance. (iii) Homogeneousness.

(a) $y(\mathrm{k}+2) = y(k+1)\,y(k) + u(k)$
(b) $y(k+3) + 2\,y(k) = 0$
(c) $y(k+4) + y(k-1) = u(k)$
(d) $y(k+5) = y(k+4) + u(k+1) - u(k)$
(e) $y(\mathrm{k}+2) = y(k)\,u(k)$

The results are summarized below

Problem	Order	Linear	Time-invariant	Homogeneous
(a)	2	No	Yes	No
(b)	3	Yes	Yes	Yes
(c)	5	Yes	Yes	No
(d)	5	Yes	Yes	No
(e)	2	No	Yes	No

2.3 Find the transforms of the following sequences using Definition 2.1

(a) $\{0, 1, 2, 4, 0, 0,...\}$ (b) $\{0, 0, 0, 1, 1, 1, 0, 0, 0,...\}$

(c) $\{0, 2^{-0.5}, 1, 2^{-0.5}, 0, 0, 0, ...\}$

From Definition 2.1, $\{u_0, u_1, u_2, ..., u_k, ...\}$ transforms to $U(z) = \sum_{k=0}^{\infty} u_k z^{-k}$. Hence:

(a) $\mathcal{Z}\{0,1,2,4,0,0,...\} = z^{-1} + 2z^{-2} + 4z^{-3}$ (b) $\mathcal{Z}\{0,0,0,1,1,1,0,0,...\} = z^{-3} + z^{-4} + z^{-5}$

(c) $\mathcal{Z}\{0,2^{-0.5},1,2^{-0.5},0,0,...\} = 2^{-0.5}z^{-1} + z^{-2} + 2^{-0.5}z^{-3}$

2.4 Obtain closed forms of the transforms of Problem 2.3 using the table of z-transforms and the time delay property.

Each sequence can be written in terms of transforms of standard functions

(a) $\{0, 1, 2, 4,0,0,...\} = \{0, 1, 2, 4, 8, 16,...\} - \{0, 0, 0, 0, 8, 16,...\} = \{f(k)\} - \{g(k)\}$

$$\frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

where $f(k) = \begin{cases} 2^{k-1}, & k > 0 \\ 0, & k \le 0 \end{cases}$ $\quad g(k) = \begin{cases} 8 \times 2^{k-4}, & k > 4 \\ 0, & k \le 4 \end{cases}$

$$\mathcal{Z}\{0,1,2,4,0,0,...\} = z^{-1}\frac{z}{z-2} - z^{-4}\frac{8z}{z-2} = \frac{z^3-8}{z^3(z-2)}$$

(b) $\{0, 0, 0, 1, 1, 1, 0, 0,...\} = \{0, 0, 0, 1, 1, 1, 1, 1,...\} - \{0, 0, 0, 0, 0, 0, 1, 1, 1, 1,...\}$
$= \{f(k)\} - \{g(k)\}$

where $f(k) = \begin{cases} 1, & k > 3 \\ 0, & k \le 3 \end{cases}$ $\quad g(k) = \begin{cases} 1, & k > 6 \\ 0, & k \le 6 \end{cases}$

$$\mathcal{Z}\{0,0,0,1,1,1,0,0,...\} = z^{-3}\frac{z}{z-1} - z^{-6}\frac{z}{z-1} = \frac{z^3-1}{z^5(z-1)}$$

(c) $\{0,2^{-0.5},1,2^{-0.5},0,0,...\} = \{0,2^{-0.5},1,2^{-0.5},0,-2^{-0.5},-1,-2^{-0.5},0,...\} + \{0,0,0,0,2^{-0.5},1,2^{-0.5},0,-2^{-0.5},-1,-2^{-0.5},0,...\}$

$= \{f(k)\} + \{g(k)\}$

where $f(k) = \begin{cases} \sin(k\pi/4), & k > 0 \\ 0, & k \le 0 \end{cases}$ $\quad g(k) = \begin{cases} \sin(k\pi/4), & k > 4 \\ 0, & k \le 4 \end{cases}$

$$\mathcal{Z}\{0,2^{-0.5},1,2^{-0.5},0,0,0,...\} = \frac{\sin(\pi/4)\,z}{z^2 - 2\cos(\pi/4)\,z + 1} - z^{-4}\frac{\sin(\pi/4)\,z}{z^2 - 2\cos(\pi/4)\,z + 1} = \frac{2^{-0.5}(z^4-1)}{z^3\left[z^2 - 2^{0.5}z + 1\right]}$$

2.5 Prove the linearity and time delay properties of the z-transform from basic principles.

To prove linearity, we must prove homogeneity and additivity using Definition 2.1,

(i) Homogeneity: $\mathcal{Z}\{\alpha f(k)\} = \alpha\,\mathcal{Z}\{f(k)\}$

$$\mathcal{Z}\{f(0), f(1), f(2), \ldots, f(i), \ldots\} = f(0) + f(1)z^{-1} + f(2)z^{-2} + \ldots + f(i)z^{-i} + \ldots = \sum_{i=0}^{\infty} f(i)z^{-i}$$

$$\mathcal{Z}\{\alpha f(0), \alpha f(1), \alpha f(2), \ldots, \alpha f(i), \ldots\} = \alpha f(0) + \alpha f(1)z^{-1} + \alpha f(2)z^{-2} + \ldots + \alpha f(i)z^{-i} + \ldots = \alpha\sum_{i=0}^{\infty} f(i)z^{-i}$$

(ii) Additivity $\qquad \mathcal{Z}\{f(k) + g(k)\} = \mathcal{Z}\{f(k)\} + \mathcal{Z}\{g(k)\}$

$$\begin{aligned}
\mathcal{Z}\{f(k) + g(k)\} &= \mathcal{Z}\{f(0) + g(0), f(1) + g(1), f(2) + g(2), \ldots, f(i) + g(i), \ldots\} \\
&= \left(f(0) + g(0)\right) + \left(f(1) + g(1)\right)z^{-1} + \left(f(2) + g(2)\right)z^{-2} + \ldots + \left(f(i) + g(i)\right)z^{-i} + \ldots \\
&= \sum_{i=0}^{\infty} f(i)z^{-i} + \sum_{i=0}^{\infty} g(i)z^{-i} = \mathcal{Z}\{f(k)\} + \mathcal{Z}\{g(k)\}
\end{aligned}$$

To prove the time delay property, we write the transform of the delayed sequence

$$\begin{aligned}
\mathcal{Z}\{0, f(0), f(1), f(2), \ldots, f(i), \ldots\} &= f(0)z^{-1} + f(1)z^{-2} + f(2)z^{-3} + \ldots + f(i)z^{-i+1} + \ldots \\
&= z^{-1}\sum_{i=0}^{\infty} f(i)z^{-i} = z^{-1}\mathcal{Z}\{f(k)\}
\end{aligned}$$

2.6 Use the linearity of the z-transform and the transform of the exponential function to obtain the transforms of the discrete-time functions.

(a) $\sin(k\omega T)$ $\qquad$ (b) $\cos(k\omega T)$

(a) $\sin(k\omega T) = \dfrac{e^{jk\omega T} - e^{-jk\omega T}}{2j}$

$$\begin{aligned}
\mathcal{Z}\{\sin(k\omega T)\} &= \frac{1}{2j}\left[\mathcal{Z}\{e^{jk\omega T}\} - \mathcal{Z}\{e^{-jk\omega T}\}\right] \\
&= \frac{1}{2j}\left[\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right] \\
&= \frac{1}{2j}\left[\frac{\left(e^{j\omega T} - e^{-j\omega T}\right)z}{z^2 - \left(e^{j\omega T} + e^{-j\omega T}\right)z + 1}\right] = \frac{\sin(\omega T)z}{z^2 - 2\cos(\omega T)z + 1}
\end{aligned}$$

(b) $\cos(k\omega T) = \dfrac{e^{jk\omega T} + e^{-jk\omega T}}{2}$

$$\begin{aligned}
\mathcal{Z}\{\cos(k\omega T)\} &= \frac{1}{2}\left[\mathcal{Z}\{e^{jk\omega T}\} + \mathcal{Z}\{e^{-jk\omega T}\}\right] \\
&= \frac{1}{2}\left[\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right] \\
&= \frac{1}{2}\left[\frac{2z^2 - \left(e^{j\omega T} + e^{-j\omega T}\right)z}{z^2 - \left(e^{j\omega T} + e^{-j\omega T}\right)z + 1}\right] = \frac{z^2 - \cos(\omega T)z}{z^2 - 2\cos(\omega T)z + 1}
\end{aligned}$$

2.7 Use the multiplication by exponential property to obtain the transforms of the discrete-time functions.

(a) $e^{-\alpha kT}\sin(k\omega T)$ $\qquad$ (b) $e^{-\alpha kT}\cos(k\omega T)$

The multiplication by exponential property with $a^{-k} = \left(e^{\alpha T}\right)^{-k} = e^{-\alpha kT}$ gives

$$\mathcal{Z}\left\{e^{-\alpha kT}\mathrm{f}(k)\right\} = F(e^{\alpha T}z)$$

(a) $$\mathcal{Z}\left\{e^{-\alpha kT}\sin(k\omega T)\right\} = \frac{\sin(\omega T)\left(e^{\alpha T}z\right)}{\left(e^{\alpha T}z\right)^2 - 2\cos(\omega T)\left(e^{\alpha T}z\right)+1} = \frac{\sin(\omega T)e^{-\alpha T}z}{z^2 - 2\cos(\omega T)e^{-\alpha T}z + e^{-2\alpha T}}$$

(b) $$\mathcal{Z}\left\{e^{-\alpha kT}\cos(k\omega T)\right\} = \frac{\left(e^{\alpha T}z\right)^2 - \cos(\omega T)\left(e^{\alpha T}z\right)}{\left(e^{\alpha T}z\right)^2 - 2\cos(\omega T)\left(e^{\alpha T}z\right)+1} = \frac{z^2 - \cos(\omega T)e^{-\alpha T}z}{z^2 - 2\cos(\omega T)e^{-\alpha T}z + e^{-2\alpha T}}$$

2.8 Find the inverse transforms of the following functions using Definition 2.1 and, if necessary, long division

(a) $F(z) = 1 + 3z^{-1} + 4z^{-2}$ (b) $F(z) = 5z^{-1} + 4z^{-5}$

(c) $F(z) = \dfrac{z}{z^2 + 0.3z + 0.02}$ (d) $F(z) = \dfrac{z - 0.1}{z^2 + 0.04z + 0.25}$

Use Definition 2.1 to obtain

(a) $\mathcal{Z}\left\{1 + 3z^{-1} + 4z^{-2}\right\} = \{1,3,4,0,0,0,..\}$ (b) $\mathcal{Z}\left\{5z^{-1} + 4z^{-5}\right\} = \{0,5,0,0,0,4,0,0,..\}$

(c)

$$\begin{array}{r|l} & z^{-1} - 0.3z^{-2} + 0.07z^{-3} + \ldots\ldots \\ \hline z^2 + 0.3z + 0.02 & z \\ & \underline{z + 0.3 + 0.02z^{-1}} \\ & -0.3 - 0.02z^{-1} \\ & \underline{-0.3 - 0.09z^{-1} - 0.006z^{-2}} \\ & 0.07z^{-1} + 0.006z^{-2} \end{array}$$

$$F(z) = \frac{z}{z^2 + 0.3z + 0.02} = z^{-1} - 0.3z^{-2} + 0.07z^{-3} + \ldots\ldots \qquad \{\mathrm{f}(k)\} = \{0,1,-0.3,0.07,\ldots.\}$$

(d)

$$\begin{array}{r|l} & z^{-1} - 0.14z^{-2} - 0.244z^{-3} + \ldots\ldots \\ \hline z^2 + 0.04z + 0.25 & z - 0.1 \\ & \underline{z + 0.04 + 0.25z^{-1}} \\ & -0.14 - 0.25z^{-1} \\ & \underline{-0.14 - 0.0056z^{-1} - 0.035z^{-2}} \\ & -0.244z^{-1} + 0.035z^{-2} \end{array}$$

$$F(z) = \frac{z}{z^2 + 0.04z + 0.25} = z^{-1} - 0.14z^{-2} - 0.244z^{-3} + \ldots.$$

$$\{\mathrm{f}(k)\} = \{0,1,-0.14,-0.244,\ldots.\}$$

2.9 For Problems 2.8.(c), (d), find the inverse transforms of the functions using partial fraction expansion and table look-up.

(c) $$\frac{F(z)}{z}=\frac{1}{z^2+0.3z+0.02}=\frac{1}{(z+0.1)(z+0.2)}=10\left\{\frac{1}{z+0.1}-\frac{1}{z+0.2}\right\}$$

$$F(z)=10\left\{\frac{z}{z+0.1}-\frac{z}{z+0.2}\right\}\quad \{\mathrm{f}(k)\}=10\left[(-0.1)^k-(-0.2)^k\right]$$

(d) $$\frac{F(z)}{z}=\frac{z-0.1}{z\left(z^2+0.04z+0.25\right)}=-\frac{0.4}{z}+\frac{0.4z+1.016}{z^2+0.04z+0.25}$$

We obtain $F(z)=-0.4+\dfrac{0.4z^2+1.016z}{z^2+0.04z+0.25}$ and use the identities

$$Z\left\{e^{-\alpha k}\sin(k\omega_d)\right\}=\frac{e^{-\alpha}\sin(\omega_d)z}{z^2-2e^{-\alpha}\cos(\omega_d)z+e^{-2\alpha}}$$

$$Z\left\{e^{-\alpha k}\cos(k\omega_d)\right\}=\frac{z[z-e^{-\alpha}\cos(\omega_d)]}{z^2-2e^{-\alpha}\cos(\omega_d)z+e^{-2\alpha}}$$

$$e^{-\alpha}=\sqrt{0.25}=0.5 \qquad \cos(\omega_d)=-0.04\Rightarrow\omega_d=1.611\,\mathrm{rad}$$

$$\frac{0.4z^2+1.016z}{z^2+0.04z+0.25}=\frac{0.4\left(z^2+0.02z\right)+1.008z}{z^2+0.04z+0.25}=\frac{0.4\left(z^2+0.02z\right)+2.018(0.4996)z}{z^2+0.04z+0.25}$$

$$\begin{aligned}\{\mathrm{f}(k)\}&=-0.4\delta(k)+(0.5)^k\left[0.4\cos(1.611k)+2.018\sin(1.611k)\right]\\&=-0.4\delta(k)+2.057(0.5)^k\sin(1.611k+0.196)\end{aligned}$$

$$2.057=\sqrt{(0.4)^2+(2.018)^2} \qquad 0.196=\sin^{-1}\left(\frac{0.4}{2.057}\right)$$

sin(A+B) = sin(A) cos(B) + cos(A) sin(B)

2.10 Solve the following difference equations

(a)	$y(k+1) - 0.8\,y(k) = 0$,	y(0) = 1
(b)	$y(k+1) - 0.8\,y(k) = 1(k)$,	y(0) = 0
(c)	$y(k+1) - 0.8\,y(k) = 1(k)$,	y(0) = 1
(d)	$y(k+2) + 0.7\,y(k+1) + 0.06\,y(k) = \delta(k)$,	y(0)=0, y(1)=2

(a) $y(k+1) - 0.8\,y(k) = 0$, y(0) = 1

z-transform

$$zY(z)-z-0.8Y(z)=0\Rightarrow Y(z)=\frac{z}{z-0.8} \qquad f(k)=(0.8)^k, k=0,1,2,\ldots$$

(b) $y(k+1) - 0.8\,y(k) = 1(k)$, y(0) = 0

z-transform

$$(z-0.8)Y(z)=\frac{z}{z-1}\Rightarrow Y(z)=\frac{z}{(z-0.8)(z-1)}$$

$$\frac{Y(z)}{z}=\frac{1}{(z-0.8)(z-1)}=5\left[\frac{1}{z-1}-\frac{1}{z-0.8}\right]$$

$$f(k) = 5\left[1-(0.8)^k\right], k = 0,1,2,...$$

(c) $y(k+1) - 0.8\, y(k) = 1(k)$, $\quad$ y(0) = 1

The solution is the sum of the solutions from (a) and (b)

$$f(k) = 5\left[1-(0.8)^k\right] + (0.8)^k, k = 0,1,2,...$$

(d) $y(k+2) + 0.7\, y(k+1) + 0.06\, y(k) = \delta(k)$, $\quad$ y(0)=0, y(1)=2

z-transform

$$(z^2 + 0.7z + 0.06)Y(z) = 1 + 2z \Rightarrow Y(z) = \frac{2z+1}{(z+0.1)(z+0.6)}$$

$$\frac{Y(z)}{z} = \frac{2z+1}{z(z+0.1)(z+0.6)} = \frac{16.667}{z} - \frac{16}{z+0.1} - \frac{0.667}{z+0.6}$$

$$Y(z) = 16.667 - \frac{16z}{z+0.1} - \frac{0.667z}{z+0.6}$$

$$y(k) = 16.667\delta(k) - 16(-0.1)^k - 0667(-0.6)^k$$

2.11 Find the transfer functions corresponding to the difference equations of Problem 2.2 with input $u(k)$ and output $y(k)$. If no transfer function is defined, explain why.

(a) and (e) are nonlinear and (b) is homogeneous. They have no transfer functions.

(c) $y(k+4) + y(k-1) = u(k)$

Z-transform $(z^4 - z^{-1})Y(z) = U(z)$ $\qquad$ $G(z) = \dfrac{z}{z^5 + 1}$

(d) $y(k+5) = y(k+4) + u(k+1) - u(k)$

z-transform $(z^5 - z^4)Y(z) = (z-1)U(z)$ $\qquad$ $G(z) = \dfrac{z-1}{z^5 - z^4} = \dfrac{1}{z^4}$

2.12 Test the linearity with respect to the input of the systems for which you found transfer functions in 2.11.

(c) $y(k+4) + y(k-1) = u(k)$

The transfer function of the system is

$$G(z) = \frac{z}{z^5 + 1}$$

For inputs $u_1(k)$ and $u_2(k)$, we have outputs

$$Y_i(z) = G(z)U_i(z) = \frac{z}{z^5+1}U_i(z), i = 1,2$$

We now as input try the linear combination

$$u(k) = \alpha u_1(k) + \beta u_2(k)$$

$$Y(z) = G(z)U(z) = \alpha \frac{z}{z^5+1} U_1(z) + \beta \frac{z}{z^5+1} U_2(z)$$
$$= \alpha Y_1(z) + \beta Y_2(z)$$

(d) $y(k+5) = y(k+4) + u(k+1) - u(k)$

Repeat above steps using the transfer function of (d).

2.13 If the rational functions of Problems 2.8.(c), (d), are transfer functions of LTI systems, find the difference equation governing each system.

(c) $$F(z) = \frac{z}{z^2 + 0.3z + 0.02}$$

$y(k+2 + 0.3$ y(k+1) + 0.02 y(k) $= u(k+1)$

(d) $$F(z) = \frac{z - 0.1}{z^2 + 0.04z + 0.25}$$

$y(k+2 + 0.04$ y(k+1) + 0.25 y(k) $= u(k+1) - 0.1\ u(k)$

2.14 We can use z-transforms to find the sum of integers raised to various powers. This is accomplished by first recognizing that the sum is the solution of the difference equation

$$f(k) = f(k-1) + a(k)$$

where $a(k)$ is the k^{th} term in the summation. Evaluate the following summations using z-transforms

(a) $\sum_{k=1}^{n} k$ (b) $\sum_{k=1}^{n} k^2$

(a) We consider the difference equation

$$f(k) = f(k-1) + k$$

Z-transform

$$F(z) = z^{-1}F(z) + \frac{z}{(z-1)^2} = \frac{z^2}{(z-1)^3} = \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2}\right]$$

Inverse z-transform

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(b) We consider the difference equation

$$f(k) = f(k-1) + k^2$$

$$F(z) = z^{-1}F(z) + \frac{z(z+1)}{(z-1)^3}$$

$$F(z) = \frac{z^2(z+1)}{(z-1)^4} = \frac{z(z^2+4z+1)}{3(z-1)^4} + \frac{z(z+1)}{2(z-1)^3} + \frac{z}{6(z-1)^2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{1}{6}n(n+1)(2n+1)$$

2.15 Find the impulse response functions for the systems governed by the following difference equations

(a) $y(k+1) - 0.5\,y(k) = u(k)$
(b) $y(k+2) - 0.1\,y(k+1) + 0.8\,y(k) = u(k)$

(a) $y(k+1) - 0.5\,y(k) = u(k)$

$$G(z) = \frac{1}{z-0.5} = z^{-1}\frac{z}{z-0.5} \qquad g(k) = \begin{cases} (0.5)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

(b) $y(k+2) - 0.1\,y(k+1) + 0.8\,y(k) = u(k)$

$$G(z) = \frac{1}{z^2 - 0.1z + 0.8} = z^{-1}\frac{z}{z^2 - 0.1z + 0.8} = \frac{Ae^{-\alpha}\sin(\omega_d)z}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$

Equating coefficients, we solve for $e^{-\alpha}$ and ω_d then use the tables and the delay theorem

$$g(k) = \begin{cases} 1.12(0.893)^{k-1}\sin(1.515(k-1)), & k \geq 1 \\ 0, & k < 1 \end{cases}$$

2.16 Find the final value for the functions if it exists

(a) $F(z) = \dfrac{z}{z^2 - 1.2z + 0.2}$ (b) $F(z) = \dfrac{z}{z^2 + 0.3z + 2}$

(a) $$f(\infty) = \frac{z}{z^2 - 1.2z + 0.2} \times \frac{z-1}{z}\bigg|_{z\to 1} = \frac{z-1}{(z-1)(z-.2)}\bigg|_{z\to 1} = \frac{1}{0.8} = 1.25$$

(b) $$F(z) = \frac{z}{z^2 + 0.3z + 2} = \frac{z}{z^2 - 2e^{-\alpha}\cos(\omega_d)z + e^{-2\alpha}}$$
$$= \frac{z}{\left(z - e^{-\alpha + j\omega_d}\right)\left(z - e^{-\alpha - j\omega_d}\right)}$$

The denominator has complex conjugate poles with magnitude $\sqrt{2}$ greater than unity. Therefore the corresponding time sequence is unbounded and the final value theorem does not apply.

2.17 Find the steady-state response of the systems due to the sinusoidal input $u(k) = 0.5\sin(0.4\,k)$

(a) $H(z) = \dfrac{z}{z-0.4}$ (b) $H(z) = \dfrac{z}{z^2 + 0.4z + 0.03}$

Sinusoidal input $\quad u(k) = 0.5\sin(0.4k)$

(a) $$H(z) = \frac{z}{z-0.4} = \frac{1}{1-0.4z^{-1}}$$
$$H\left(e^{j0.4}\right) = \frac{1}{1-0.4e^{-j0.4}} = 1.537\angle -0.242$$

$u(k)$ =0.5 × 1.537 sin(0.4 k – 0.242) = 0.769 sin(0.4k – 0.242)

(b) $$H(z) = \frac{z}{z^2+0.4z+0.03}$$
$$H\left(e^{j0.4}\right) = \frac{1}{e^{j0.4}+0.4+0.03e^{-j0.4}} = 0.714\angle -0.273$$

$u(k)$ =0.5 × 0.714 sin(0.4 k – 0.273) = 0.357 sin(0.4 k – 0.273)

2.18 Find the frequency response of a noncausal system whose impulse response sequence is given by

$$\{h(k), h(k) = h(k+K), k = -\infty, \ldots, \infty\}$$

Hint: Express the periodic impulse response sequence with period K as

$$h^*(t) = \sum_{l=0}^{K-1} \sum_{m=-\infty}^{\infty} h(l+mK)\delta(t-l-mK)$$

Then Laplace transform it.

Laplace transform the sequence then let $s = j\omega$

$$H^*(s) = \sum_{l=0}^{K-1} \sum_{m=-\infty}^{\infty} h(l+mK)e^{-(l+mK)s}$$
$$H^*(j\omega) = \sum_{l=0}^{K-1} \sum_{m=-\infty}^{\infty} h(l+mK)e^{-j(l+mK)\omega}$$

2.19 The well known Shannon reconstruction theorem states that:
Any bandlimited signal $u(t)$ with bandwidth $\omega_s/2$ can be exactly reconstructed from its samples at a rate $\omega_s = 2\pi/T$. The reconstruction is given by

$$u(t) = \sum_{k=-\infty}^{\infty} u(k) \frac{\sin\left[\frac{\omega_s}{2}\left(t-kT\right)\right]}{\frac{\omega_s}{2}\left(t-kT\right)}$$

Use the convolution theorem to justify the above expression.

By the sampling theorem, the signal can be recovered from its samples using a LPF of bandwidth ω_s. Multiplication in the frequency domain is equivalent to convolution with the inverse transform, the sinc function in the time domain. Convolution of the samples and the sinc function yields the expression.

2.20 Obtain the convolution of the two sequences {1,1,1} and {1,2,3}
(a) Directly (b) Using z-transformation.

Convolution of the two sequences $\{f(k)\}=\{1, 1, 1\}$ and $\{g(k)\}=\{1, 2, 3\}$

(a) Directly

$$y(0) = f(0).g(0)= 1\times1=1$$
$$y(1) = f(1).g(0) + f(0).g(1) = 1\times1+1\times2=3$$
$$y(2) = f(2).g(0) + f(1).g(1) + f(0).g(2) = 1\times1+1\times2 + 1\times3 = 6$$
$$y(3) = f(2).g(1) + f(1).g(2) = 1\times2 + 1\times3 = 5$$
$$y(4) = f(2).g(2) = 1\times3 = 3$$
$$y(k) = 0,\ k > 4$$

(b) Using z-transformation

$$F(z) = 1 + z^{-1} + z^{-2} \quad G(z) = 1 + 2z^{-1} + 3z^{-2}$$

$$Y(z) = F(z).G(z) = 1+ 3z^{-1} + 6 z^{-2} + 5z^{-3} + 3z^{-4}$$

$$\{y(k)\} = \{1, 3, 6, 5, 3, 0, 0, ...\}$$

2.21 Obtain the modified z-transforms for the functions of Problems (2.6) and (2.7).

For 2.6-(a), $\sin(k\omega T) = \dfrac{e^{jk\omega T} - e^{-jk\omega T}}{2j}$

$$Z_m\{\sin(k\omega T)\} = Z_m\left\{\frac{e^{jk\omega T} - e^{-jk\omega T}}{2j}\right\}$$
$$= \frac{1}{2j}\left\{\frac{e^{jm\omega T}}{z-e^{j\omega T}} - \frac{e^{-jm\omega T}}{z-e^{-j\omega T}}\right\} = \frac{\sin(m\omega T)z + \sin[(1-m)\omega T]}{z^2 - 2\cos(\omega T)z + 1}$$

For 2.6-(b), $\cos(k\omega T) = \dfrac{e^{jk\omega T} + e^{-jk\omega T}}{2}$

$$Z_m\{\cos(k\omega T)\} = Z_m\left\{\frac{e^{jk\omega T} + e^{-jk\omega T}}{2}\right\}$$
$$= \frac{1}{2}\left\{\frac{e^{jm\omega T}}{z-e^{j\omega T}} + \frac{e^{-jm\omega T}}{z-e^{-j\omega T}}\right\} = \frac{\cos(m\omega T)z - \cos[(1-m)\omega T]}{z^2 - 2\cos(\omega T)z + 1}$$

For 2.7-(a) ,

$$\mathcal{Z}_m\left\{e^{-akT}\sin(k\omega T)\right\} = \mathcal{Z}_m\left\{\frac{e^{jk\omega T-akT} - e^{-jk\omega T-akT}}{2j}\right\}$$

$$= \frac{1}{2j}\left\{\frac{e^{jm\omega T-maT}}{z-e^{j\omega T-aT}} - \frac{e^{-jm\omega T-maT}}{z-e^{-j\omega T-aT}}\right\}$$

$$= e^{-maT}\frac{\sin(m\omega T)z + e^{-aT}\sin[(1-m)\omega T]}{z^2-2e^{-aT}\cos(\omega T)z+e^{-2aT}}$$

For 2.7-(b),

$$\mathcal{Z}_m\left\{e^{-akT}\cos(k\omega T)\right\} = \mathcal{Z}_m\left\{\frac{e^{jk\omega T-akT} + e^{-jk\omega T-akT}}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{e^{jm\omega T-maT}}{z-e^{j\omega T-aT}} + \frac{e^{-jm\omega T-maT}}{z-e^{-j\omega T-aT}}\right\}$$

$$= e^{-maT}\frac{\cos(m\omega T)z + e^{-aT}\sin[(1-m)\omega T]}{z^2-2e^{-aT}\cos(\omega T)z+e^{-2aT}}$$

2.22 Using the modified z-transform, examine the intersample behavior of the functions $h(k)$ of: (a) Problem 2.15, and (b) 2.16. Use delays of (i) 0.3*T*, (ii) 0.5*T*, and (iii) 0.8*T*.

Solution for 2.15

2.15(a)

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z-.2)} = \frac{1}{0.8}\left[\frac{1}{z-1} - \frac{1}{z-0.2}\right]$$

$$F(z,m) = 1.25\left[\frac{1}{z-1} - \frac{(0.2)^m}{z-0.2}\right]$$

For any value m

$$F(z,m) = 1.25\left[\frac{1}{z-1} - \frac{(0.2)^m}{z-0.2}\right]$$

$$\Rightarrow F(k,m) = 1.25\left[1-(0.2)^m(0.2)^{k-1}\right], k = 1,2,3,\ldots \text{and zero elsewhere}$$

(i) 0.3*T*, m = 0.7 (ii) 0.5*T*, m = 0.5, and (iii) 0.8*T*, m = 0.2.

2.15(b)

$$F(z) = \frac{z}{z^2+0.3z+2} = \frac{z}{(z+0.15-j1.406)(z+0.15+j1.406)}$$

$$= \frac{z}{\left(z-\sqrt{2}e^{j1.677}\right)\left(z-\sqrt{2}e^{-j1.677}\right)}$$

$$\frac{F(z)}{z} = \frac{3.5556e^{-j\pi}}{z-\sqrt{2}e^{j1.677}} + \frac{3.5556e^{j\pi}}{z-\sqrt{2}e^{-j1.677}}$$

$$f(k) = 0.7111\left(\sqrt{2}\right)^k\sin(1.6771k)$$

Use the results of problem 2.18 to obtain the answer.

$$\mathcal{Z}_m\left\{e^{-akT}\sin(k\omega T)\right\} = e^{-maT}\,\frac{\sin(m\omega T)z + e^{-aT}\sin[(1-m)\omega T]}{z^2 - 2e^{-aT}\cos(\omega T)z + e^{-2aT}}$$

$$\mathcal{Z}_m\left\{0.7111\left(\sqrt{2}\right)^k \sin(1.6771k)\right\} = 0.7111\left(\sqrt{2}\right)^m \frac{\sin(1.6771m)z + \sqrt{2}\sin[1.6771(1-m)]}{z^2 + 0.3z + 2}$$

$$2\sqrt{2}\cos(1.6771) = -0.3$$

(i) $m = 0.7$

$$F(z,0.7) = 0.7111\left(\sqrt{2}\right)^{0.7}\frac{\sin(1.1740)z + \sqrt{2}\sin[0.5031]}{z^2 + 0.3z + 2}$$

$$= 0.7111\left(\sqrt{2}\right)^{0.7}\frac{0.9223z + 0.6819}{z^2 + 0.3z + 2}$$

$$f(k,0.7) = 0.7111\left(\sqrt{2}\right)^{0.7}\left(\sqrt{2}\right)^{k-1}\sin[1.6771(k-1) + 1.1740],\ k = 1,2,3,\ldots.\ and\ zero\ elsewhere$$

(ii) $m = 0.5$ Similarly

$$F(z,0.5) = 0.7111\left(\sqrt{2}\right)^{0.5}\frac{\sin(0.8386)\left(z + \sqrt{2}\right)}{z^2 + 0.3z + 2}$$

$$= 0.7111\left(\sqrt{2}\right)^{0.5}\frac{0.8844\left(z + \sqrt{2}\right)}{z^2 + 0.3z + 2}$$

$$f(k,0.7) = 0.7111\left(\sqrt{2}\right)^{0.5}\left(\sqrt{2}\right)^{k-1}\sin[1.6771(k-1) + 0.8386],\ k = 1,2,3,\ldots.\ and\ zero\ elsewhere$$

(iii) $m = 0.2$

$$F(z,0.2) = 0.7111\left(\sqrt{2}\right)^{0.2}\frac{\sin(0.3354)z + \sqrt{2}\sin(0.0839)}{z^2 + 0.3z + 2}$$

$$= 0.7111\left(\sqrt{2}\right)^{0.2}\frac{0.3528z + 1.4762}{z^2 + 0.3z + 2}$$

$$f(k,0.7) = 0.7111\left(\sqrt{2}\right)^{0.2}\left(\sqrt{2}\right)^{k-1}\sin[1.6771(k-1) + 0.3354],\ k = 1,2,3,\ldots.\ and\ zero\ elsewhere$$

Solution for 2.16

2.16(a) $H(z,m) = \dfrac{(0.4)^m}{z - 0.4}$

(iv) $m = 0.7$ $H(z,0.7) = \dfrac{(0.4)^{0.7}}{z - 0.4} \Rightarrow h(k,0.7) = (0.4)^{0.7}(0.4)^{k-1},\ k = 1,2,3,\ldots.\ and\ zero\ elsewhere$

(v) $m = 0.5$ $H(z,0.5) = \dfrac{(0.4)^{0.5}}{z - 0.4} \Rightarrow h(k,0.5) = (0.4)^{0.5}(0.4)^{k-1},\ k = 1,2,3,\ldots.\ and\ zero\ elsewhere$

(iii) $m = 0.2$ $H(z,0.2) = \dfrac{(0.4)^{0.2}}{z - 0.4} \Rightarrow h(k,0.2) = (0.4)^{0.2}(0.4)^{k-1},\ k = 1,2,3,\ldots.\ and\ zero\ elsewhere$

2.16(b) $H(z) = \dfrac{z}{z^2 + 0.4z + 0.03} = \dfrac{5z}{z + 0.1} - \dfrac{5z}{z + 0.3}$

$$H(z,m) = 5\left\{\frac{(-0.1)^m}{z+0.1} - \frac{(-0.3)^m}{z+0.3}\right\}$$

$$\Rightarrow h(k,m) = 5\left\{(-0.1)^m(-0.1)^{k-1} - (-0.3)^m(-0.3)^{k-1}\right\},\ k = 1,2,3,....$$

and zero elsewhere

$(-0.1)^m$ and $(-0.3)^m$ are complex numbers. Thus, the sequence is not defined between sampling points.

Obtain $H(z, m)$ for m = 0.7, 0.5, 0.2, as in (a).

2.23 The following open-loop systems are to be digitally feedback controlled. Select a suitable sampling period for each if the closed-loop system is to be designed for the given specifications

(a) $G_{ol}(s) = \dfrac{1}{s+3}$ Time Constant = 0.1 s

(b) $G_{ol}(s) = \dfrac{1}{s^2+4s+3}$ Undamped natural frequency = 5 rad/s, Damping ratio = 0.7

(a) For a time constant = 0.1 s, let T = 0.1/40 = 0.0025 s

(b) For ω_n = 5 rad/s, ζ = 0.7, we have ω_d = 3.57 rad/s,

$$T = \frac{2\pi}{\omega_s} = \frac{2\pi}{70\omega_d} = 0.025s$$ Let T = 25 ms.

2.24 Repeat problem 2.23 if the systems have sensor delays of : (a) 0.025 s (b) 0.03 s

(a) T = 0.025 s (b) T = 0.03 s. (cannot sample faster than the sensor delay)

Computer Exercises

2.25 Consider the closed-loop system of Problem 2.23(a)

a. Find the impulse response of the **closed-loop** transfer function and obtain the impulse response sequence for a sampled system output.
b. Obtain the z-transfer function by z-transforming the impulse response sequence.
c. Using MATLAB, obtain the frequency response plots for the analog system and for sampling frequencies $\omega_s = k\,\omega_b$, k = 5, 35, 70.
d. Comment on the choices of sampling periods of part (b).

The closed-loop transfer function is $G(s) = \dfrac{1}{0.1s+1} = \dfrac{10}{s+10}$

(a) The impulse response is

$$g(t) = 10e^{-10t}$$

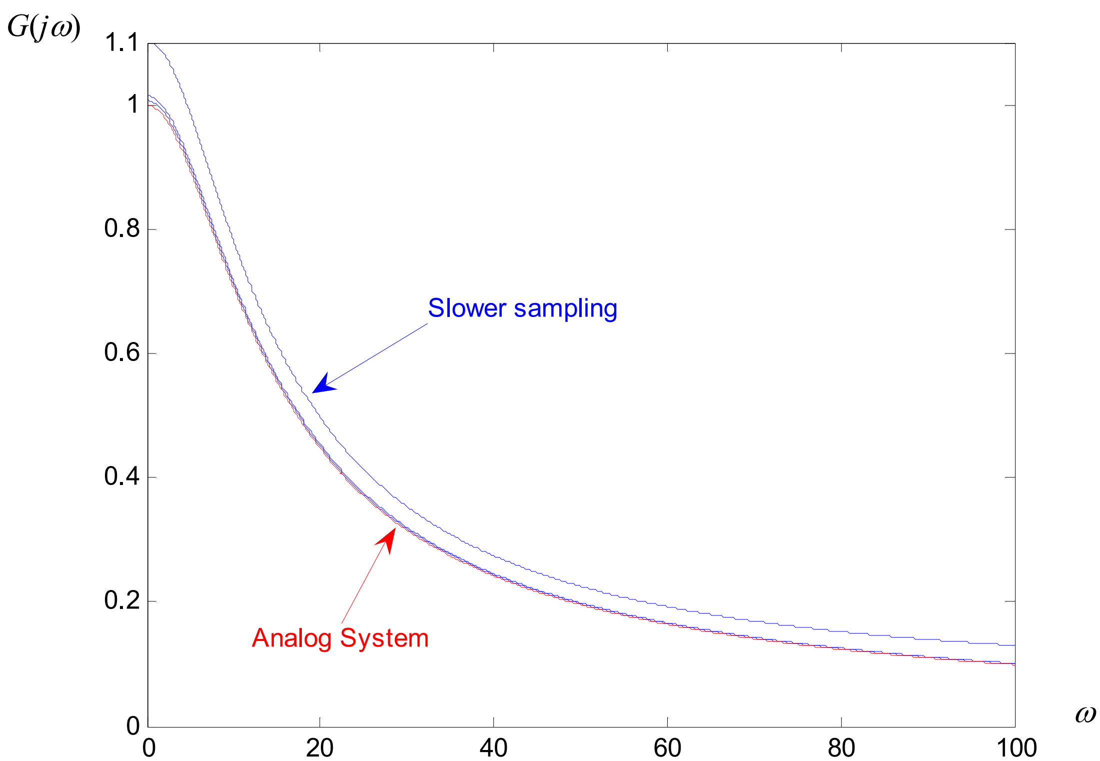
and the impulse response sequence for a sampled system output is $g(kT) = 10e^{-10kT}$

(b) The z-transform of the impulse response is

$$G(z) = \frac{10z}{z-e^{-10T}}$$

(c) The corresponding frequency response plots for sampling periods $T = 0.1$, 0.05, 0.02, 0.1 s, as well as for the analog system can be obtained using the MATLAB commands

```
% Exercise 2.22 Digital control text
clf
tau=0.1;% 1/wb=time constant
T=tau*[1/5, 1/35, 1/70];
num=[10,0];
w=[.1:.05:100];
for i=1:3
den=[1,-exp(-10*T(i))];
g=tf(num,den, T(i));
[mag,ang]= bode(g,w); % Frequency response
mm=mag(:); % Change mag to vector
plot(w,T(i)*mm)
hold on
end
nc=1; dc=[.1, 1];
w=.1:.05:100;
[mc,ac, w]=bode(nc,dc,w); plot(w,mc,'r')
```



Frequency response plots for sampling frequencies $\omega_s = k\ \omega_b$, k = 5, 35, 70 and for the analog system for Problem 2.25.

The frequency response plots are normalized (multiplied by T) to simplify their comparison. The plots for the discrete time system are closer to the analog frequency response for faster sampling. The discrete time plots are significantly different from the analog plot for $T = 0.1$ s and almost indistinguishable for $T = 0.1/35$ and 0.1/70s. This verifies the rule of thumb for the selection of the sampling rate.

2.26 Repeat Problem 2.25 for the second order closed-loop system of Problem 2.23(b) with plots for sampling frequencies $\omega_s = k\,\omega_d$, $k = 5, 35, 70$.

The closed-loop transfer function is $G(s) = \dfrac{25}{s^2 + 7s + 25} = \dfrac{7.0014 \times 3.5707}{(s+3.5)^2 + (3.5707)^2}$

(a) The impulse response is

$$g(t) = 7.0014\sin(3.5707t)e^{-3.5t}$$

and the impulse response sequence for a sampled system output is

$$g(kT) = 7.0014\sin(3.5707kT)e^{-3.5kT}$$

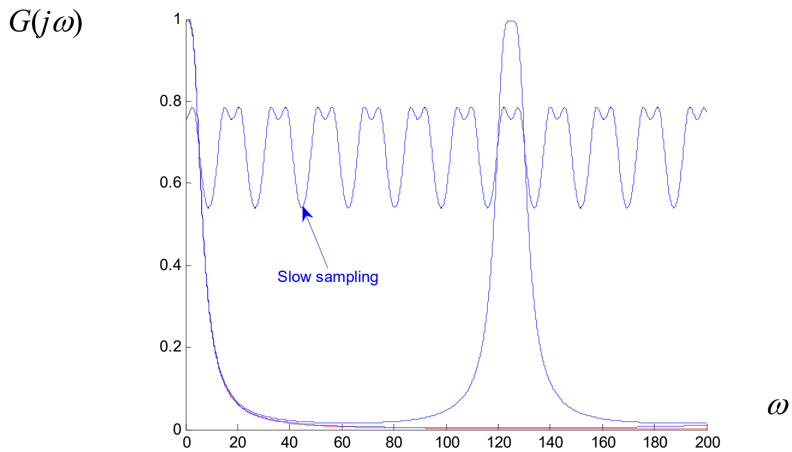
(b) The z-transform of the impulse response is

$$G(z) = \frac{7.0014e^{-3.5T}\sin(3.5707T)z}{z^2 - 2e^{-3.5T}\cos(3.5707T)z + e^{-7T}}$$

(c) The corresponding frequency response plots for sampling periods $T = 2\pi/(k\omega_d)$s, k=5, 35, 70, as well as for the analog system can be obtained using the MATLAB commands

```
% Exercise 2_24
clf
hold on
wn=5;zeta=0.7; % Closed-loop data
wd=wn*sqrt(1-zeta^2); % Damped natural frequency
ttt=2*pi/wd; T=[ttt/5,ttt/35, ttt/70]; % Sampling periods
w=[.1:.1:200];
gc=tf(wn^2,[1,2*zeta*wn,wn^2]); % Analog transfer function
% Plot the frequency response for the analog system
w=.1:.1:200;
[mc,ac, w]=bode(gc,w); plot(w,mc(:),'r')
% Calculate and plot discrete frequency responses
for i=1:length(T)
ti=T(i);
% numerator and denominator od z-transfer function
num=[7.0014*exp(-3.5*ti)*sin(3.5707*ti),0];
den=[1,-2*exp(-3.5*ti)*cos(3.5707*ti), exp(-7*ti)];
g=tf(num,den,ti);
[mm,aa,w]= bode(g,w);
plot(w,ti*mm(:))
end
```

(d) The frequency response show little aliasing in the frequency range of interest for $T = 2\pi/(70\omega_d)$ s, some aliasing for $T = 2\pi/(35\omega_d)$ s, and unacceptable aliasing $T = 2\pi/(5\omega_d)$ s,. The analog plot (red) is similar to that of the two faster rates at low frequencies and differs from the $T = 2\pi/(35\omega_d)$ s plot close to the folding frequency. The results confirm that the rule of thumb gives a reasonable estimate of the required sampling rate.

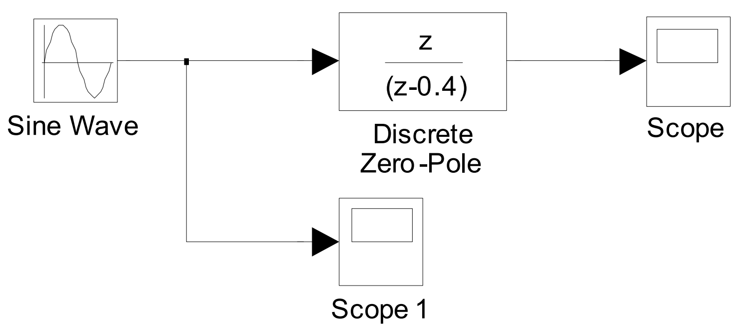


Frequency response plots for sampling frequencies $\omega_s = k\ \omega_b$, k = 5, 35, 70 and for the analog system for Problem 2.26.

2.27 Use SIMULINK with a sampling period of 1s. to verify the results of Problem 2.17. Simulate the system for 300 s then change the axes to display the last 50 s only.

Use SIMULINK with a sampling period of 1s to verify the result of Problem 2.17. Simulate the system for 300 s then change the axes to display the last 50 s only.

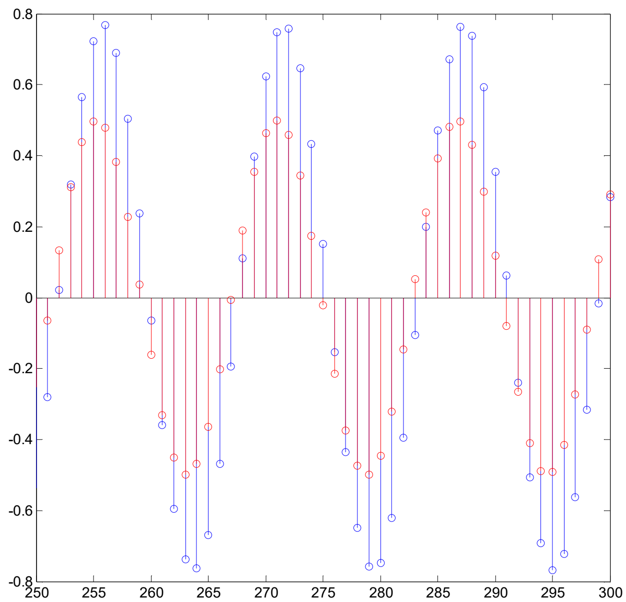
(a) $H(z) = \dfrac{z}{z-0.4}$



Simulation diagram for Problem 2.17(a) using SIMULINK.

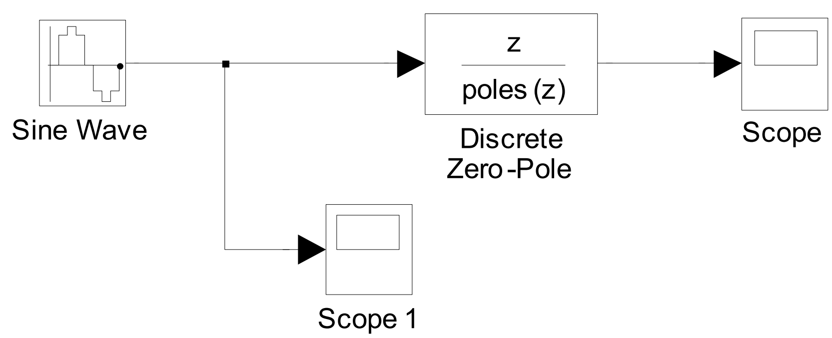
Problem 2.17(a) gives the steady-state response

$u(k) = 0.769 \sin(0.4k - 0.242)$



Sampled sinusoidal input (red) and steady-state sinusoidal (blue) for Problem 2.17(a).

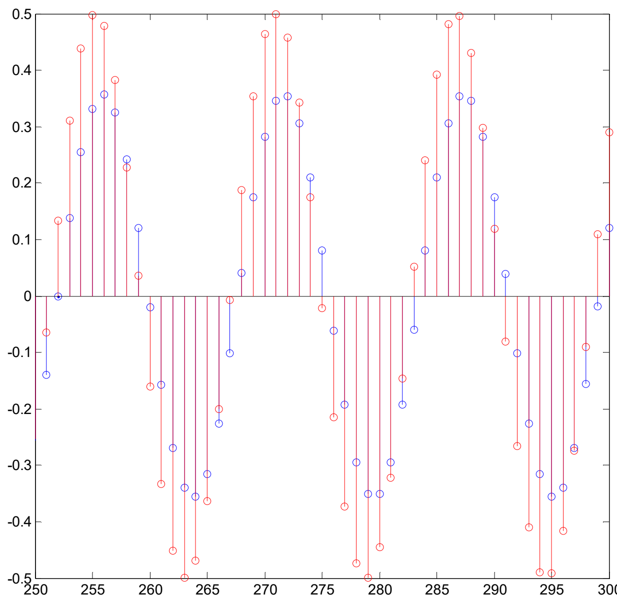
(b) $$H(z) = \frac{z}{z^2 + 0.4z + 0.03}$$



Simulation diagram for Problem 2.17(b) using SIMULINK.

Problem 2.17(b) gives the steady-state response

$u(k) = 0.357 \sin(0.4\,k - 0.273)$



Sampled sinusoidal input (red) and steady-state sinusoidal (blue) for Problem 2.17(a).

2.28 The following difference equation describes the evolution of the expected price of a commodity[1]

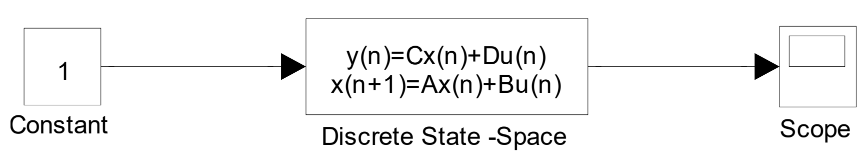
$$p_e(k+1) = (1-\gamma)\, p_e(k) + \gamma\, p(k)$$

where $p_e(k)$ is the expected price after k quarters, $p(k)$ is the actual price after k quarters, and γ is a constant.

a) Simulate the system with $\gamma = 0.5$ and a fixed actual price of one unit and plot the actual and expected prices. Discuss the accuracy of the model prediction
b) Repeat part (a) for an exponentially decaying price $p(k) = (0.4)^k$.
c) Repeat part (a) for an exponentially decaying price $p(k) = (0.95)^k$.
d) Discuss the predictions of the model referring to your simulation results.

The recursion describing the solution can be easily simulated using a discrete state-space block. Although discrete state-space equations are introduced in Chapter 7, they reduce to the simple recursion of our model for the case of scalar vector x(k), where x(k) is the price $p_e(k)$. We could also avoid the use of state-space blocks by z-transforming to obtain the corresponding transfer function.

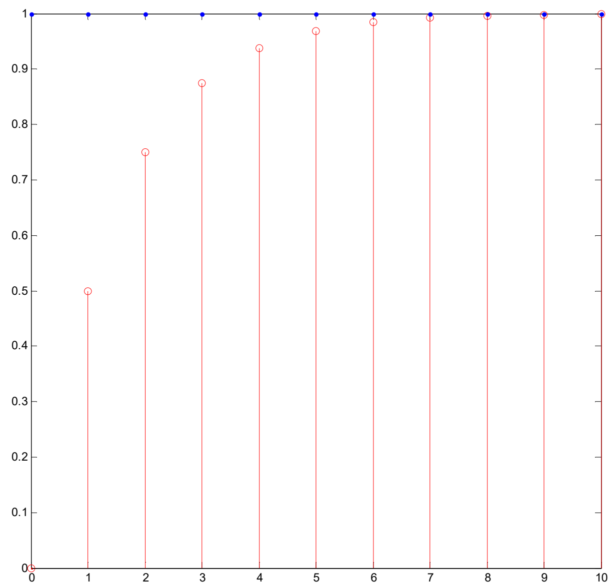
a) Simulate the system with $\gamma = 0.5$ and a fixed actual price of one unit and plot the actual and expected prices. Discuss the accuracy of the model prediction.



Simulation diagram for constant price using SIMULINK.

[1] D. N. Gujarate, *Basic Econometrics*, McGraw Hill, NY, 1988, pp. 547.

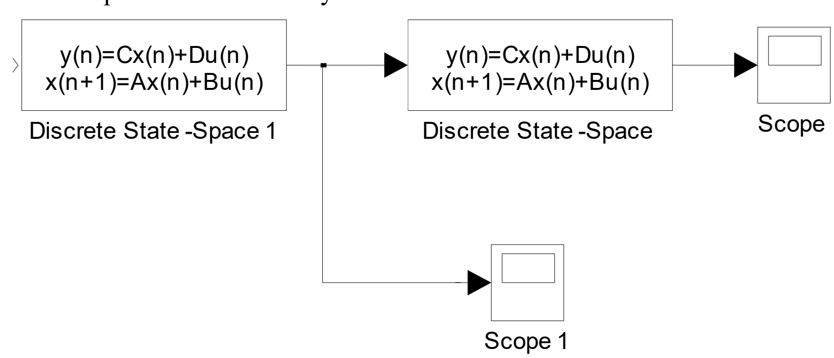
The model converges to the correct estimate after a few sample points. At $k = 5$, the error is less than 5%.. This is a reasonable estimate assuming that the sampling period is small relative to the time after which the price estimate is used.



Time response of price estimator for a constant price.

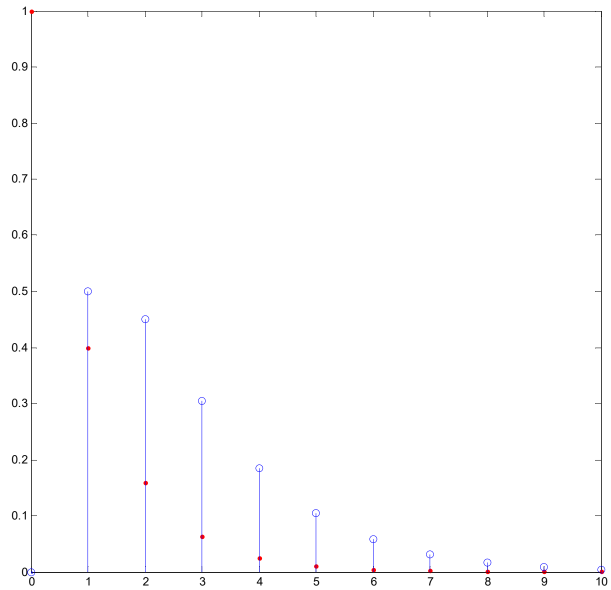
b) Repeat part (a) for an exponentially decaying price $p(k) = (0.4)^k$.

We use a state space block with unity initial condition and A=0.4.



Simulation diagram for exponentially decaying price using SIMULINK.

The dynamics of the model are too slow to track the exponentially decaying price. The actual price decays much faster than the model predictions.

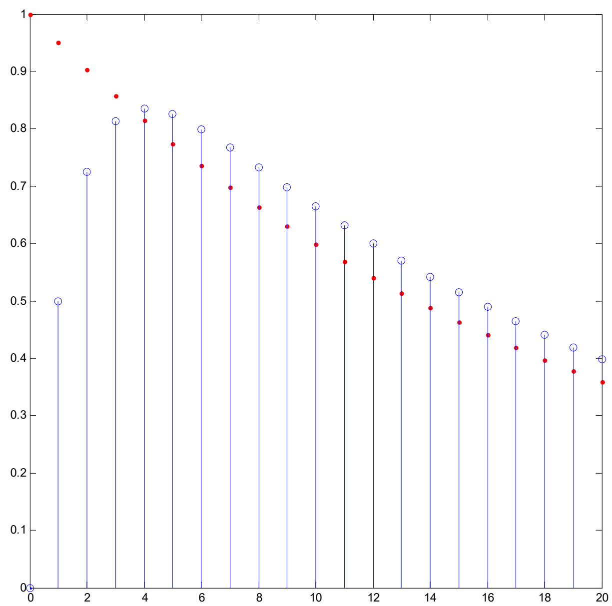


Time response of price estimator for a fast exponentially decaying price.

c) Repeat part (a) for an exponentially decaying price $p(k) = (0.95)^k$.

We use a state space block with unity initial condition and A=0.95 and with the same simulation diagram as part (b).

The dynamics of the model are able to track the exponentially decaying price since the decay is very slow.



Time response of price estimator for a slow exponentially decaying price.

d) Discuss the predictions of the model referring to your simulation results.

The price estimator dynamics are able to estimate a constant price but are unable to estimate a decaying exponential if the rate of decay is fast relative to the filter dynamics. If the price decay is very slow, then the estimator is able to track the price with some error.

Chapter 3 Solutions

3.1. Find magnitude and phase at frequency ω=1 rad/s of a zero-order hold with sampling period T=0.1 s.

The magnitude results from the expression

$$|G_{ZOH}(j\omega)| = T\left|\sin c\left(\frac{\omega T}{2}\right)\right| = T\frac{\sin\left(\omega\frac{T}{2}\right)}{\omega\frac{T}{2}} = 0.1\frac{\sin\left(1\frac{0.1}{2}\right)}{1\frac{0.1}{2}} = 0.1$$

while the phase is given by

$$\angle G_{ZOH}(j\omega) = -\omega\frac{T}{2} = -1\frac{0.1}{2} = -0.05 \text{ rad}$$

3.2. The first-order hold uses the last two numbers in a sequence to generate its output. The output of both the zero-order hold and the first order hold is given by

$$u(t) = u(kT) + a\frac{u(kT) - u[(k-1)T]}{T}(t - kT), kT \le t \le (k+1)T$$

$$k = 0,1,2,...$$

with a = 0,1, respectively.

(a) For a discrete impulse input, obtain and sketch the impulse response for the above equation with a = 0 and a = 1.

(b) Write an impulse sampled version of the above impulse response and show that the transfer function is given by

$$G_H(s) = \frac{1}{s}\left(1 - e^{-sT}\right)\left[1 - ae^{-sT} + \frac{a}{sT}\left(1 - e^{-sT}\right)\right]$$

(c) Obtain the transfer functions for the zero-order hold and for the first-order hold from the above transfer function. Verify that the transfer function of the zero-order hold is the same as that obtained in Section 3.3.

The output of the zero-order hold is of the form

$$u(t) = u(kT), \quad kT \le t < (k+1)T$$

The first order hold uses the last two numbers of a sequence to generate the output

$$u(t) = u(kT) + \frac{u(kT) - u([k-1]T)}{T}[t - kT], \quad kT \le t < (k+1)T$$

The two outputs are represented by the equation

$$u(t) = u(kT) + a\frac{u(kT) - u([k-1]T)}{T}[t - kT], \quad kT \le t < (k+1)T$$

where $a = 0, 1$, is the order of the hold circuit.

(a) For a zero-order hold (ZOH), the output for each sampling period is a constant equal to the input at the beginning of the period. Thus it is equal to one for the first period and zero thereafter. For a first-order hold (FOH), the output increases steadily from an initial level = u(0) =1, to reach a level of 2 at $t = T$. For the second sampling period, $u(T)$ is zero and $u(0)$ is unity. Thus the output decrease steadily to a level of -1 at $t = 2T$. At $t = 3T$, the output goes to zero since both $u(T)$ and $u(2T)$ are zero. The results of this analysis are shown in Figure P3.0.

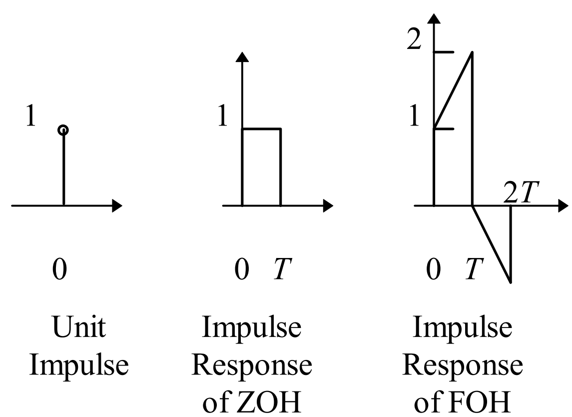


Figure P3.0 Unit impulse and impulse responses for ZOH and FOH.

(b) The output of Problem 3.1 for $u(0) = 1$ and $u(k) = 0$, $k > 0$, gives the impulse sampled form

$$u^*(t) = u(0)\{1(t) - 1(t-T)\} + a\frac{u(0)}{T}\{t.1(t) - (t-T)1(t-T)\} - au(0)1(t-T)$$
$$- a\frac{u(0)}{T}\{(t-T).1(t-T) - (t-2T)1(t-2T)\} + au(0)1(t-2T)$$

with Laplace transform the impulse response to obtain

$$U^*(s) = \frac{1}{s}\left\{1 - e^{-sT}\right\} + a\frac{1}{T}\frac{1}{s^2}\left\{1 - 2e^{-sT} + e^{-2sT}\right\} - ae^{-sT}\left\{1 - e^{-sT}\right\}$$
$$= \frac{1}{s}\left\{1 - e^{-sT}\right\}\left[1 - ae^{-sT} + \frac{a}{sT}\left\{1 - e^{-sT}\right\}\right]$$

i.e. $$G_H(s) = \frac{1}{s}\left(1 - e^{-sT}\right)\left[1 - a\,e^{-sT} + \frac{a}{sT}\left(1 - e^{-sT}\right)\right]$$

(c) Thus for $a = 0$, we have the impulse response of the zero-order hold (as given in the notes)

$$G^*_{ZOH}(s) = \frac{1}{s}\left\{1 - e^{-sT}\right\}$$

and for $a = 1$, we have the impulse response of the first-order hold

$$G^*_{FOH}(s) = \frac{1}{s}\left\{1 - e^{-sT}\right\}\left[1 - e^{-sT} + \frac{1}{sT}\left\{1 - e^{-sT}\right\}\right]$$

3.3. Many chemical processes can be modelled by the following transfer function:

$$G(s) = \frac{K}{\tau s + 1} e^{-T_d s}$$

where K is the gain, τ is the time constant and T_d is the time delay. Obtain the transfer function $G_{ZAS}(z)$ for the system in terms of the system parameters. Assume that the time delay T_d is a multiple of the sampling time T.

Obtain the partial fraction expansion $\frac{G(s)}{s} = \frac{K}{s(\tau s + 1)} e^{-T_d s} = \left(\frac{K}{s} - \frac{K}{s + 1/\tau}\right) e^{-T_d s}$

The z-transfer function $G_{ZAS}(z)$ for the system is given by

$$\begin{aligned}
G_{ZAS}(z) &= \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\left(\frac{K}{s} - \frac{K}{s + 1/\tau}\right)e^{-T_d s}\right\} \\
&= \left(1 - z^{-1}\right)\left\{\frac{K}{1 - z^{-1}} - \frac{K}{1 - e^{-T/\tau} z^{-1}}\right\} z^{-T_d / T} \\
&= \left(K - \frac{K(z - 1)}{z - e^{-T/\tau}}\right) z^{-T_d / T} \\
&= \frac{K\left(1 - e^{-T/\tau}\right)}{z - e^{-T/\tau}} z^{-T_d / T}
\end{aligned}$$

3.4. Obtain the transfer function of a point mass (m) with force as input and displacement as output neglecting actuator dynamics then find $G_{ZAS}(z)$ for the system.

The equation of motion for a point mass is $m\ddot{x}(t) = f(t)$

The transfer function for the system is $G(s) = \frac{X}{F} = \frac{1}{ms^2}$

The z-transfer function $G_{ZAS}(z)$ for the system is given by

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{1}{ms^3}\right\}$$
$$= \frac{1}{m}\left(1 - z^{-1}\right)\frac{T^2 z^{-1}\left(1 + z^{-1}\right)}{\left(1 - z^{-1}\right)^3}$$
$$= \frac{T^2}{m}\frac{(z+1)}{(z-1)^2}$$

3.5. For an internal combustion engine, the transfer function with injected fuel flow rate as input and fuel flow rate into the cylinder as output is given by[1]

$$G(s) = \frac{\varepsilon \tau s + 1}{\tau s + 1}$$

where τ is a time constant and ε is known as the fuel split parameter. Obtain the transfer function $G_{ZAS}(z)$ for the system in terms of the system parameters.

Obtain the partial fraction expansion $\dfrac{G(s)}{s} = \dfrac{\varepsilon \tau s + 1}{s(\tau s + 1)} = \dfrac{1}{s} + \dfrac{\varepsilon - 1}{s + 1/\tau}$

The z-transfer function $G_{ZAS}(z)$ for the system is given by

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{1}{s} + \frac{\varepsilon - 1}{s + 1/\tau}\right\}$$
$$= \left(1 - z^{-1}\right)\left\{\frac{1}{1 - z^{-1}} + \frac{\varepsilon - 1}{1 - e^{-T/\tau} z^{-1}}\right\}$$
$$= 1 + \frac{(\varepsilon - 1)(z - 1)}{z - e^{-T/\tau}}$$

3.6. Repeat Problem 3.5 including a delay of 25 ms in the transfer function with a sampling period of 10 ms.

For a delay of 25 ms with T = 10 ms we have

$$25 = 3 \times 10 - 0.5 \times 10$$

Hence, m = 3 and f = 0.5. The partial fraction coefficient corresponding to the s-domain pole at $(-1/\tau)$ is modified by the factor $e^{-5/\tau}$ where the time constant is assumed to be given in ms. The transfer function for the system with ZOH and sampler is given by

[1] J. Moskwa, *Automotive Engine Modeling and Real Time Control*, MIT Doctoral Thesis, 1988.

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$
$$= \left(1 - z^{-1}\right)\left\{\frac{1}{1 - z^{-1}} + \frac{(\varepsilon - 1)e^{-5/\tau}}{1 - e^{-T/\tau}z^{-1}}\right\}z^{-3}$$
$$= \left[1 + \frac{(\varepsilon - 1)(z - 1)e^{-5/\tau}}{z - e^{-T/\tau}}\right]z^{-3}$$

Hence, $m = 3$ and $f = 0.5$. The partial fraction coefficient corresponding to the s-domain pole at $(-1/\tau)$ is modified by the factor $e^{-5/\tau}$ where the time constant is assumed to be given in ms. The transfer function for the system with ZOH and sampler is given by

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$
$$= \left(1 - z^{-1}\right)\left\{\frac{1}{1 - z^{-1}} + \frac{(\varepsilon - 1)e^{-5/\tau}}{1 - e^{-T/\tau}z^{-1}}\right\}z^{-3}$$
$$= \left[1 + \frac{(\varepsilon - 1)(z - 1)e^{-5/\tau}}{z - e^{-T/\tau}}\right]z^{-3}$$

3.7. Find the equivalent sampled impulse response sequence and the equivalent z-transfer function for the cascade of the two analog systems with sampled input

$$H_1(s) = \frac{1}{s + 6} \qquad H_2(s) = \frac{10}{s + 1}$$

(a) If the systems are directly connected.
(b) If the systems are separated by a sampler.

(a) In the absence of samplers between the systems, the overall transfer function is

$$H(s) = \frac{10}{(s + 6)(s + 1)}$$
$$= -\frac{2}{s + 6} + \frac{2}{s + 1}$$

The impulse response of the cascade is

$$h(t) = -2e^{-6t} + 2e^{-t}$$

and the sampled impulse response is

$$h(kT) = -2e^{-6kT} + 2e^{-kT}, k = 0,1,2,...$$

Hence the z-domain transfer function is

$$H(z) = -\frac{2z}{z-e^{-6T}} + \frac{2z}{z-e^{-T}} = \frac{2\left(e^{-T}-e^{-6T}\right)z}{\left(z-e^{-6T}\right)\left(z-e^{-T}\right)}$$

(b) If the analog systems are separated by a sampler then each has a z-domain transfer function and the transfer functions are given by

$$H_1(z) = \frac{z}{z-e^{-6T}} \qquad H_2(z) = \frac{10z}{z-e^{-T}}$$

The overall transfer function for the cascade is

$$H(z) = \frac{10z^2}{\left(z-e^{-6T}\right)\left(z-e^{-T}\right)}$$

The partial fraction expansion of the transfer function is

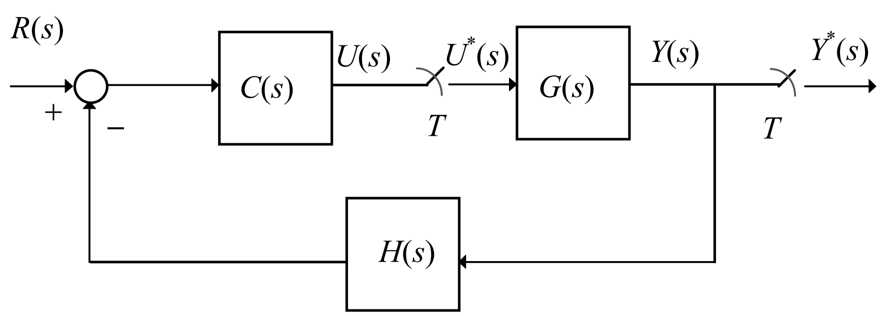
$$H(z) = \frac{10}{e^{-6T}-e^{-T}}\left[\frac{e^{-6T}z}{z-e^{-6T}} - \frac{e^{-T}z}{z-e^{-T}}\right]$$

Inverse z-transforming gives the impulse response sequence

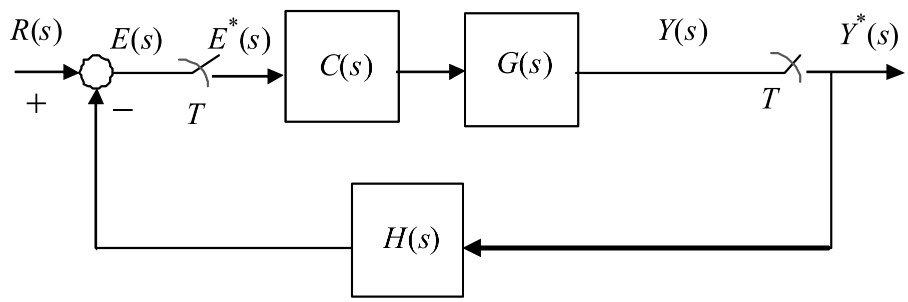
$$\begin{aligned} h(kT) &= \frac{10}{e^{-6T}-e^{-T}}\left[e^{-6T}e^{-6kT} - e^{-T}e^{-kT}\right] \\ &= \frac{10}{e^{-6T}-e^{-T}}\left[e^{-6(k+1)T} - e^{-(k+1)T}\right], k = 0,1,2,\ldots \end{aligned}$$

3.8. Obtain expressions for the analog and sampled outputs from the block diagrams of Figure P3.1

(a)



(b)



(c)

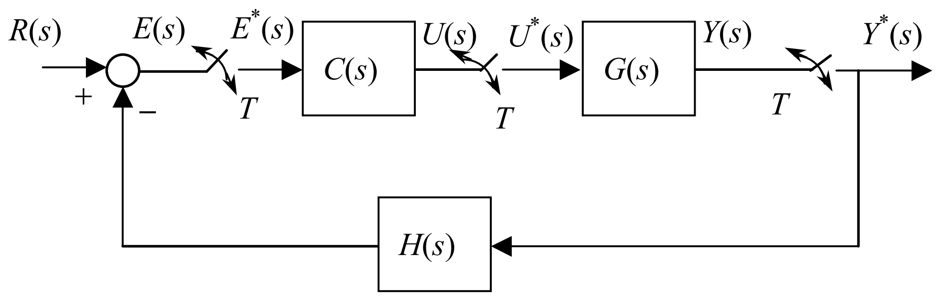


Figure P3.1 Block diagrams of control loops.

(a) From the block diagram $U(s) = C(s)R(s) - C(s)H(s)G(s)U^*(s)$

Then sampling gives $U^*(s) = (CR)^*(s) - (CHG)^*(s)U^*(s)$

Solving for $U^*(s)$, we obtain $U^*(s) = \dfrac{(CR)^*(s)}{1+(CHG)^*(s)}$

The analog output is $Y(s) = G(s)U^*(s) = \dfrac{G(s)(CR)^*(s)}{1+(CHG)^*(s)}$

The sampled output is $Y(s) = \dfrac{G^*(s)(CR)^*(s)}{1+(CHG)^*(s)}$

(b) The analog output is $Y(s) = C(s)G(s)E^*(s)$

The sampled output is $Y^*(s) = (CG)^*(s)E^*(s)$

The error is given by

$$E(s) = R(s) - H(s)Y^*(s) = R(s) - H(s)(CG)^*(s)E^*(s)$$

Sampling gives $E^*(s) = R^*(s) - H^*(s)(CG)^*(s)E^*(s)$

Solving for $E^*(s)$, we obtain $E^*(s) = \dfrac{R^*(s)}{1+H^*(s)(CG)^*(s)}$

Hence, we have the output $Y(s) = \dfrac{C(s)G(s)R^*(s)}{1+H^*(s)(CG)^*(s)}$

and the sampled output $$Y^*(s) = \frac{(CG)^*(s)R^*(s)}{1+H^*(s)(CG)^*(s)}$$

(c) Since there are samplers between all blocks, we use the feedback rule for block diagrams to obtain

$$U^*(s) = \frac{C^*(s)R^*(s)}{1+H^*(s)G^*(s)C^*(s)}$$
$$Y^*(s) = \frac{G^*(s)C^*(s)R^*(s)}{1+H^*(s)G^*(s)C^*(s)}$$

The analog output is $Y(s) = G(s)U^*(s) = \dfrac{G(s)C^*(s)R^*(s)}{1+H^*(s)G^*(s)C^*(s)}$

3.9. For the shown unity feedback system of Figure P3.2, we are given the analog subsystem

$$G(s) = \frac{s+8}{s+5}$$

The system is digitally controlled with a sampling period of 0.02 s. The controller transfer function was selected as

$$C(z) = \frac{0.35z}{z-1}$$

(a) Find the z-transfer function for the analog subsystem with DAC and ADC.
(b) Find the closed-loop transfer function and characteristic equation.
(c) Find the steady-state error due to a sampled unit step and a sampled unit ramp. Comment on the effect of the controller on steady-state error.

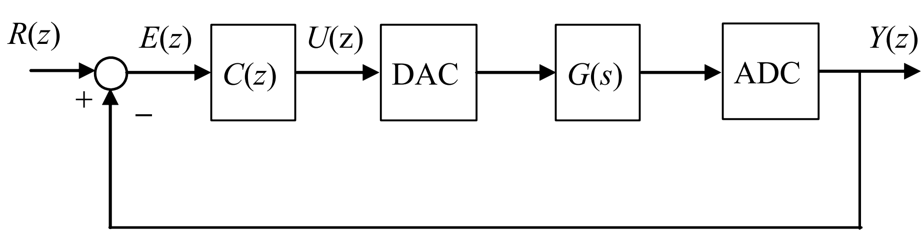
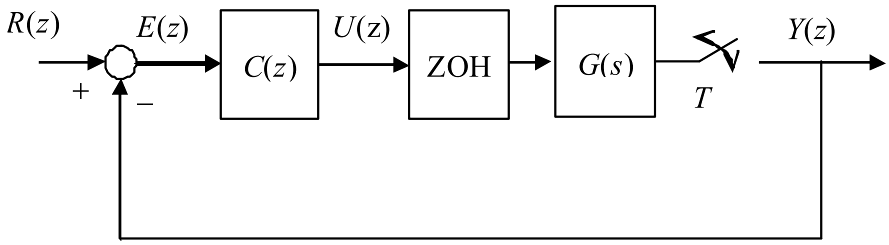


Figure P3.2 Block diagram for a closed-loop system with digital control.

(a)



Obtain the partial fraction expansion

$$\frac{G(s)}{s} = \frac{s+8}{s(s+5)} = \frac{1.6}{s} - \frac{0.6}{s+5}$$

The transfer function with DAC (ZOH) and ADC (sampler) is

$$\begin{aligned}
G_{ZAS}(z) &= \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\
&= \frac{z-1}{z}\left\{\frac{1.6z}{z-1} - \frac{0.6z}{z - e^{-5\times 0.02}}\right\} \\
&= 1.6 - \frac{0.6(z-1)}{z - e^{-0.1}} \\
&= \frac{z - 0.845}{z - 0.905}
\end{aligned}$$

(b) The closed-loop transfer function is

$$\begin{aligned}
G_{cl}(z) &= \frac{C(z)G_{ZAS}(z)}{1 + C(z)G_{ZAS}(z)} \\
&= \frac{0.35z(z-0.848)}{(z-1)(z-0.905) + 0.35z(z-0.848)} \\
&= \frac{0.35z(z-0.848)}{1.35z^2 - 2.202z + 0.905} \\
&= \frac{0.259z(z-0.848)}{z^2 - 1.631z + 0.670}
\end{aligned}$$

The closed-loop characteristic equation is

$$0 = z^2 - 1.631z + 0.670 = (z - 0.815)^2 + 0.07^2$$

(c) The system is type 1 and therefore has zero steady-state error due to a step and finite steady-state error due to a ramp. The error due to a unit ramp is

$$e(\infty)\% = \left. \frac{100T}{(z-1)C(z)G_{ZAS}(z)} \right|_{z=1} \%$$

$$= \left. \frac{2}{0.35\left(1.6 - \dfrac{z-1}{z-e^{-0.1}}\right)} \right|_{z=1} \% = 3.57\%$$

The steady-state error is improved because the controller increases the type of the system by 1.

3.10. Find the steady-state error due to a unit step disturbance input for the systems of Figure P3.3 with a sampling period of 0.03 s and the transfer functions

$$G_d(s) = \frac{2}{s+1} \qquad G(s) = \frac{4(s+2)}{s(s+3)} \qquad C^*(s) = \frac{e^{sT} - 0.95}{e^{sT} - 1}$$

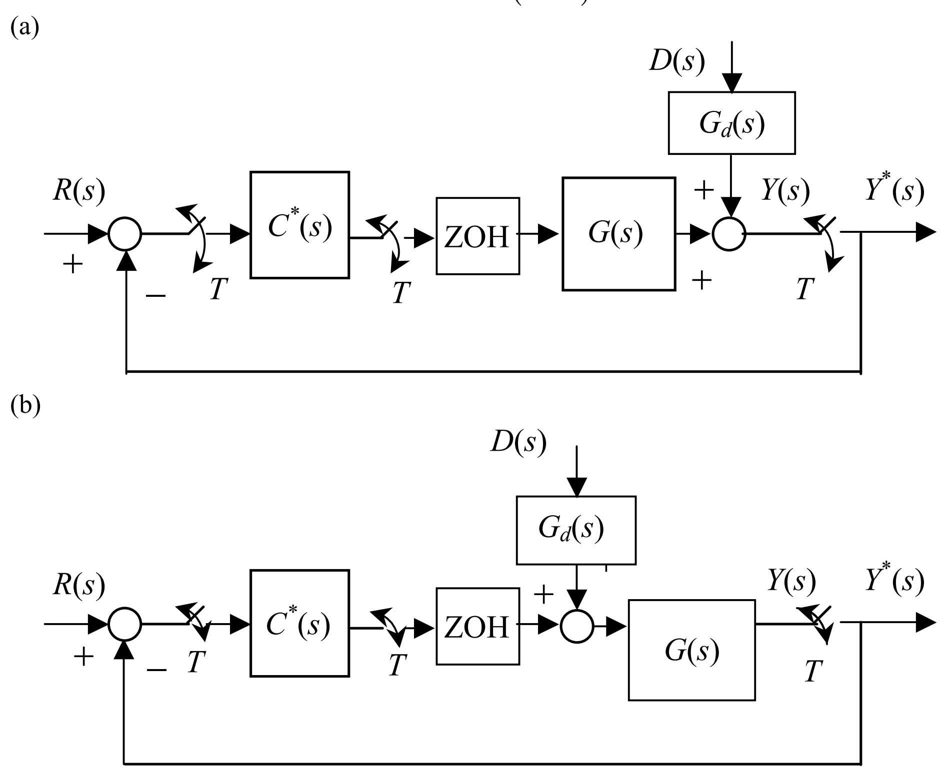


Figure P3.3 Block diagrams for systems with disturbance inputs.

The disturbance $D(s)$ $1/s$ gives

$$G_d(s)D(s) = \frac{2}{s(s+1)} = 2\left[\frac{1}{s} - \frac{1}{s+1}\right]$$

Sampling gives

$$(G_d D)(z) = 2\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] = 2\left[\frac{z}{z-1} - \frac{z}{z-0.970}\right] = \frac{0.911\times 10^{-2} z}{(z-1)(z-0.970)}$$

The transfer function for the plant with ZOH and sampler is

$$\begin{aligned}
G_{ZAS}(z) &= \left(1-z^{-1}\right)\mathcal{Z}\left\{\frac{4(s+2)}{s^2(s+3)}\right\} \\
&= \left(1-z^{-1}\right)\mathcal{Z}\left\{\frac{8/3}{s^2} + \frac{4}{9}\left(\frac{1}{s} - \frac{1}{s+3}\right)\right\} \\
&= \left(1-z^{-1}\right)\left\{\frac{8/3}{\left(1-z^{-1}\right)^2} + \frac{4}{9}\left(\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-3\times 0.03}z^{-1}}\right)\right\} \\
&= \frac{8/3}{1-z^{-1}} + \frac{4}{9}\left(1 - \frac{1-z^{-1}}{1-0.914z^{-1}}\right) \\
&= \frac{0.118(z-0.942)}{(z-1)(z-0.914)}
\end{aligned}$$

$$C(z) = \frac{z-0.95}{z-1}$$

$$Y(z) = \frac{(G_d D)(z)}{1+C(z)G_{ZAS}(z)} = \left\{\frac{0.911\times 10^{-2} z}{(z-1)(z-0.97)}\right\} \Bigg/ \left\{1 + 0.118\frac{(z-0.942)(z-0.95)}{(z-1)^2(z-0.914)}\right\}$$

$$y(\infty) = (z-1)Y(z)\big|_{z=1} = 0$$

(b) The disturbance now has a different route and we need

$$G(s)G_d(s)D(s) = \frac{8(s+2)}{s^2(s+1)(s+3)} = 4\left[\frac{4/3}{s^2} - \frac{10/9}{s} + \frac{1}{s+1} + \frac{1/9}{s+3}\right]$$

Sampling gives

$$\begin{aligned}
(GG_d D)(z) &= 4\left[\frac{(4/3)z}{(z-1)^2} - \frac{(10/9)z}{z-1} + \frac{z}{z-e^{-T}} + \frac{(1/9)z}{z-e^{-3T}}\right] \\
&= \frac{3.529\times 10^{-3} z(z+0.980)(z-0.942)}{(z-1)^2(z-0.970)(z-0.914)}
\end{aligned}$$

The transfer function for the plant with ZOH and sampler is (as in part a)

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{4(s+2)}{s^2(s+3)}\right\}$$
$$= \frac{0.118(z-0.942)}{(z-1)(z-0.914)}$$

$$C(z) = \frac{z-0.95}{z-1}$$

$$Y(z) = \frac{(G_d D)(z)}{1 + C(z) G_{ZAS}(z)}$$
$$= \left\{\frac{3.529\times 10^{-3} z(z+0.980)(z-0.942)}{(z-1)^2(z-0.970)(z-0.914)}\right\} \bigg/ \left\{1 + 0.1182\frac{(z-0.942)(z-0.95)}{(z-1)^2(z-0.914)}\right\}$$

$$y(\infty) = (z-1)Y(z)\Big|_{z=1} = 0$$

3.11. For the following systems with unity feedback, find

a) The position error constant.
b) The velocity error constants.
c) The steady state error due to a unit step input.
d) The steady-state error due to a unit ramp input.

(i) $G(z) = \frac{0.4(z+0.2)}{(z-1)(z-0.1)}$ (ii) $G(z) = \frac{0.5(z+0.2)}{(z-0.1)(z-0.8)}$

a) The position error constant:

(i) The system is Type 1 and has an infnite position error constant.

(ii) $K_p = G(1) = \dfrac{0.5(1+0.2)}{(1-0.1)(1-0.8)} = 3.333\dot{3}$

b) The velocity error constants:

(i) $K_v = \dfrac{1}{T}(z-1)G(z)\Big|_{z=1} = \dfrac{0.4(1+0.2)}{T(1-0.1)} = \dfrac{0.533\dot{3}}{T}$

(ii) The system is Type 0 and has zero velocity error constant.

c) The steady state error due to a unit step input:

(i) The system is Type 1 and has zero steady-state error due to step.

(ii) $e(\infty)\% = \dfrac{100}{1+K_p}\% = \dfrac{100}{1+3.333\dot{3}}\% = 23.08\%$

d) The steady-state error due to a unit ramp input:

(i) $e(\infty)\% = \dfrac{100}{.K_v}\% = \dfrac{100T}{5.33\dot{3}}\% = 187.5T\%$

(ii) The system is Type 0 and has infinite steady-state error due to a ramp.

Computer Exercises

3.12. For the analog system with a sampling period of 0.05 s

$$G(s) = \frac{10(s+2)}{s(s+5)}$$

(a) Obtain the transfer function for the system with sampled input and output.
(b) Obtain the transfer function for the system with DAC and ADC.
(c) Obtain the unit step response of the system with sampled output and analog input.
(d) Obtain the poles of the systems in (a), (b) and the output of (c) and comment on the differences between them.

(a) Using the MATLAB command

```
>> g=zpk(-2,[0,-5],10)
>> gd=c2d(g,.05,'imp')
```

we obtain the transfer function

$$G(z) = \frac{10z(z-0.9115)}{(z-1)(z-0.7788)}$$

(b) Using the MATLAB command

```
>> g=zpk(-2,[0,-5],10)
>> gd = c2d(g,.05,'zoh')
```

we obtain the transfer function $G(z) = \dfrac{0.4654(z-0.905)}{(z-1)(z-0.7788)}$

(c) Obtain the output with the MATLAB command

```
>> [y,t]=step( tf([10,20],[1,5,0]),[0:0.05:2]) % Sampled analog step response
>> plot(t, y, '*') % Plot samples with period 0.05
```

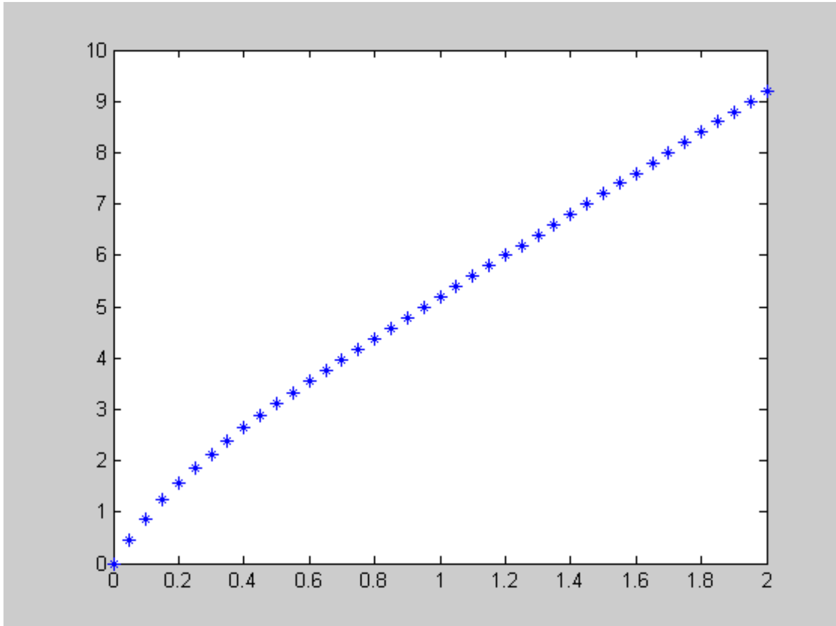


Figure P3.4 Step response of system with analog input and sampled output.

(d) The two systems of (a) and (b) have the same poles at (1, 0.779) given by $e^{0.05\,p_s}$ with p_s = 0, -5 (the poles of the analog system). The output of (c) has an extra pole at unity resulting from the step input with its pole at the origin.

3.13. For the system of Problem 3.9

(a) Obtain the transfer function for the analog subsystem with DAC and ADC.

(b) Obtain the step response of the open-loop analog system and the closed-loop digital control system and comment on the effect of the controller on the time response.

(c) Obtain the frequency response of the digital control system and verify that 0.02 s is an acceptable choice of sampling period. Explain briefly why the sampling period is chosen based on the closed-loop rather than the open-loop dynamics.

(a) The transfer function for the analog subsystem with DAC and ADC of Figure P3.3 is obtained using the MATLAB command

```
gd = c2d( tf([1, 8], [1, 5]), 0.02, 'zoh')
```

(b) The responses show that the steady-state error due to a unit step has been drastically reduced using the controller. The transient response has been somewhat adversely affected. The deterioration in the time response is the price paid for improving the steady-state error. A different controller must be used if the transient response is unacceptable.

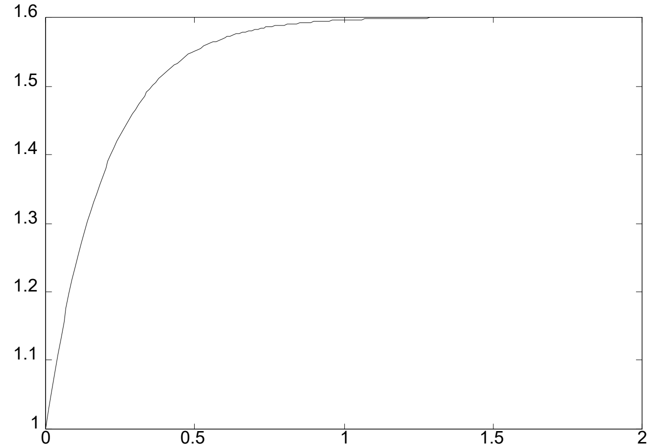


Figure P3.5 Step response of open-loop analog system.

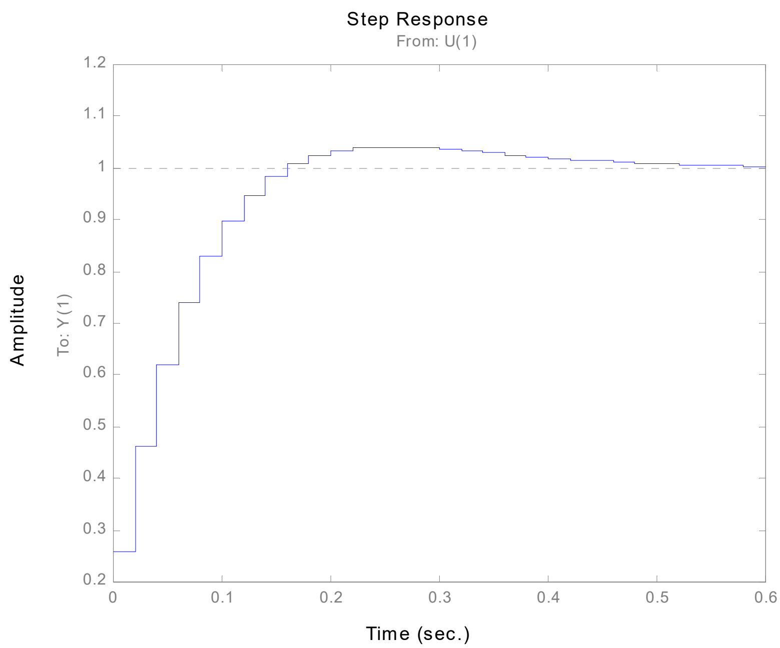


Figure P3.6 Sampled step response of digital system.

(c) The frequency response plot shows that the magnitude at the folding frequency is relatively low. Thus the sampling frequency is acceptable.
The sampling frequency is chosen based on the actual dynamics of the system after the loop is closed since these are the conditions under which the system actually operates. The open-loop dynamics are irrelevant and may be much slower than the closed-loop dynamics.

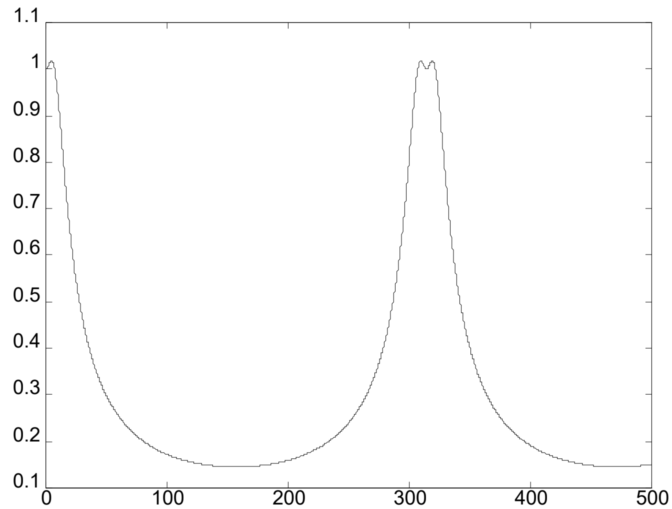


Figure P3.7 Frequency response plot of digital system

3.14. Consider the internal combustion engine model of Problem 3.4. Assume that, for the operational conditions of interest, the time constant τ is approximately 1.2 s while the parameter ε can vary in the range 0.4 to 0.6. The digital cascade controller

$$C(z) = \frac{0.02z}{z-1}$$

was selected to improve the time response of the system with unity feedback and a sampling period of 4 ms. Simulate the digital control system with $\varepsilon = 0.4$, 0.5, 0.6, and discuss the behavior of the system in each case.

We use SIMULINK to simulate the system with the numerical values of the parameters entered in MATLAB. The simulation diagram is shown in Figure P3.8. Alternatively, the following MATLAB commands yield the same results.

```
% Prob3_14 Problem 3.14 solution: step response of IC Engine
T=0.01; % Sampling period
tau=1.2; % Time constant
hold on
for eps=0.4:0.1:0.6
g=tf([eps*tau,1],[tau,1]); % Engine transfer fucntion
gd=c2d(g,T); % Transfer function with ZOH and sampler
c=zpk(0,1,0.02,T); % Controller transfer function
gc=feedback(gd*c,1); % Closed-loop transfer function
step(gc) % Step response
end
```

The step responses plotted using MATLAB are shown in Figure P3.9. As the value of the parameter ε increases, the zero of the system gets closer to the origin and its effect on the time response increases. This results in a higher percentage overshoot and a faster response. The controller includes an integral term which results in zero steady-state error due to step for all three parameter values.

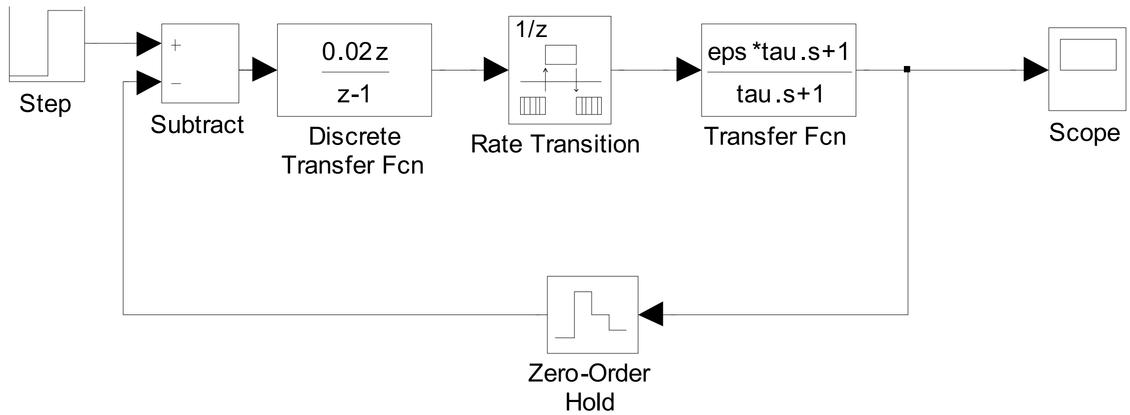


Figure P3.8 Simulation diagram for the closed-loop control of the internal combustion engine.

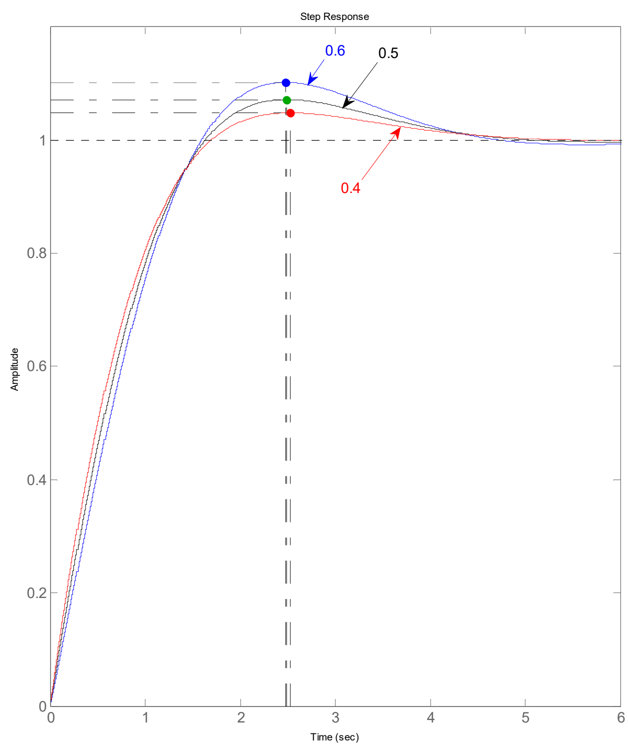


Figure P3.9 Step response for the internal combustion engine with ε= 0.4, 0.5, and 0.6.

3.15. Simulate the continuous-discrete system of Problem 3.7 and examine the behavior of both the continuous output and the sampled output. Repeat the simulation with a %10 error in the plant gain. Discuss the simulation results and comment on the effect of the parameter error on disturbance rejection.

The SIMULINK scheme employed for the simulation of the control system is shown in Figure Figure P3.10.

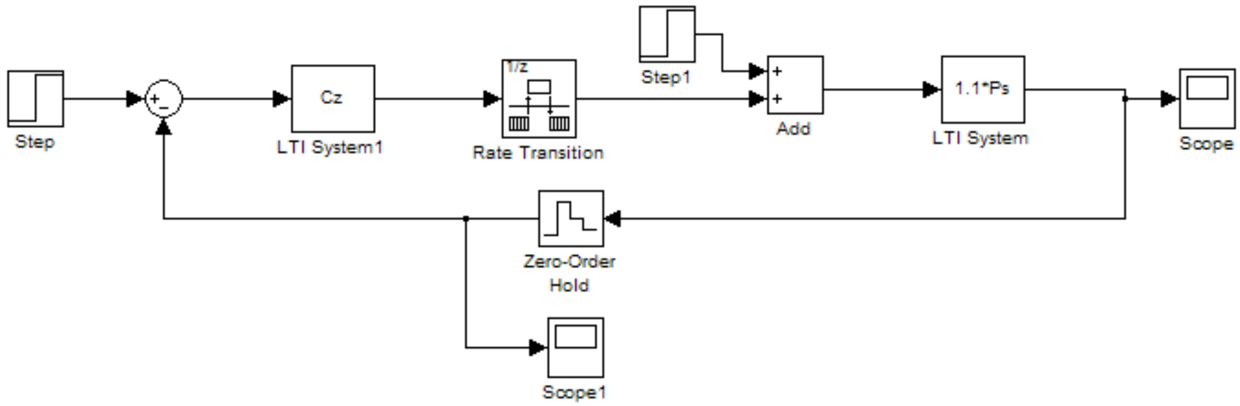


Figure P3.10 SIMULINK block diagram for Problem 3.13

A step reference input is applied at time t=0 s and a step disturbance is applied at time t=1 s. The resulting continuous and sampled process output are shown in Figure P3.11.

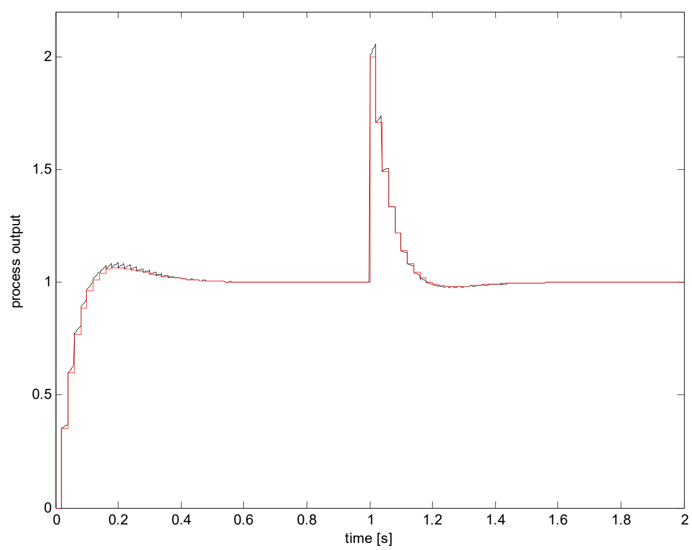


Figure P3.11 Time response for the nominal system of Problem 3.15. Black: analog output. Red: sampled output.

The results obtained when the gain of the process is increased by 10% are shown in Figure P3.12. The steady-state error is zero despite the gain modification because of the presence of an integrator in the controller.

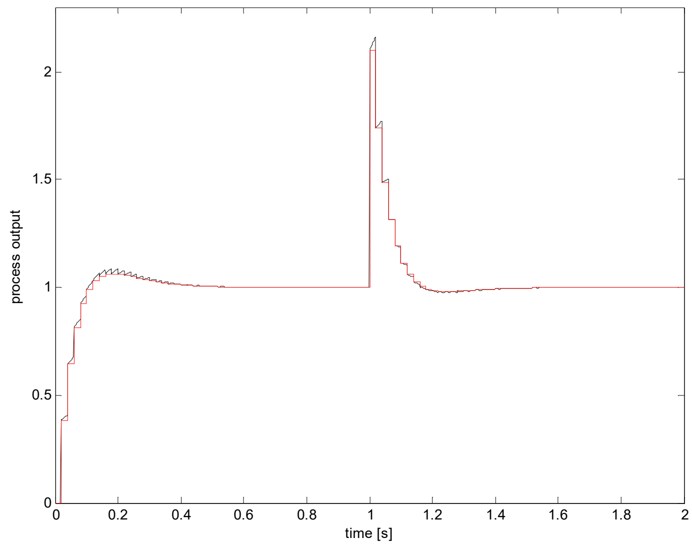


Figure P3.12 Time response for Problem 3.15 when the gain is increased by 10%. Black: analog output. Red: sampled output.

Chapter 4. Solutions

4.1 Determine the asymptotic stability and the BIBO stability of the following systems

a) $$y(k+2)+0.8y(k+1)+0.07y(k)=2u(k+1)+0.2u(k) \qquad k=0,1,2,\ldots$$

b) $$y(k+2)-0.8y(k+1)+0.07y(k)=2u(k+1)+0.2u(k) \qquad k=0,1,2,\ldots$$

c) $$y(k+2)+0.1y(k+1)+0.9y(k)=3.0u(k) \qquad k=0,1,2,\ldots$$

To determine the asymptotic stability or BIBO stability, we z-transform to obtain the transfer function. The system is both BIBO stable and asymptotically stable if its poles are all inside the unit circle (without transfer function reduction).

a) $$y(k+2)+0.8y(k+1)+0.07y(k)=2u(k+1)+0.2u(k) \qquad k=0,1,2,\ldots$$

$$G(z)=\frac{2z+0.2}{z^2+0.8z+0.07}=\frac{2(z+0.1)}{(z+0.7)(z+0.1)}$$

Stable: $|\,0.7\,|<1$ and $|\,0.1\,|<1$

b) $$y(k+2)-0.8y(k+1)+0.07y(k)=2u(k+1)+0.2u(k) \qquad k=0,1,2,\ldots$$

$$G(z)=\frac{2z+0.2}{z^2-0.8z+0.07}=\frac{2(z+0.1)}{(z-0.7)(z-0.1)}$$

Stable: $|-0.7\,|<1$ and $|-0.1\,|<1$

c) $$y(k+2)+0.1y(k+1)+0.9y(k)=3.0u(k) \qquad k=0,1,2,\ldots$$

$$G(z)=\frac{3}{z^2+0.1z+0.9}=\frac{3}{(z+0.05)^2+0.9-0.05^2}$$

Stable: Complex conjugate poles with square magnitude $|\,0.9\,|<1$

4.2 Biochemical reactors are used in different processes such as waste treatment and alcohol fermentation. By considering the dilution rate as the manipulated variable and the biomass concentration as the measured output, the biochemical reactor can be modeled by the following transfer function in the vicinity of an unstable steady-state operating point[1]

$$G(s) = \frac{5.8644}{-5.888s + 1}$$

Determine $G_{ZAS}(z)$ with a sampling rate T=0.1 and then consider the feedback controller

$$C(z) = -\frac{z - 1.017}{z - 1}$$

Verify that the resulting feedback system is not internally stable.

The discretized process transfer function is

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{-0.1004}{z - 1.017}$$

Although the transfer function from the reference input to the system output

$$\frac{Y(z)}{R(z)} = \frac{0.1004}{z - 0.8997}$$

appears to be asymptotically stable (note that the pole is inside the unit circle), the controller results in pole-zero cancellation outside the unit circle and therefore the system is not internally stable according to Theorem 4-5. This can also be verified by considering the transfer function between the disturbance input D and the system output Y

$$\frac{Y(z)}{D(z)} = \frac{-0.10045(z-1)}{(z - 1.017)(z - 0.0997)}$$

The pole at 1.017 is outside the unit circle and therefore the system output is unbounded when a disturbance occurs in the system.

4.3 Use the Routh-Hurwitz criterion to investigate the stability of the systems

a) $$G(z) = \frac{5(z-2)}{(z-0.1)(z-0.8)}$$

b) $$G(z) = \frac{10(z+0.1)}{(z-0.7)(z-0.9)}$$

The Routh-Hurwitz criterion requires the substitution

---

[1] B. W. Bequette, *Process Control – Modeling, Design, and Simulation*, Prentice Hall, Upper Saddle River (NJ), 2003.

$$z = \frac{1+w}{1-w}$$

The systems investigated in (a) and (b) are both second order and the following conditions (see Example 4.4) can be used. For the second order polynomial

$$F(z) = a_2 z^2 + a_1 z + a_0 = 0$$

The stability conditions are:

i) $a_2 - a_1 + a_0 > 0$
ii) $a_2 - a_0 > 0$
iii) $a_2 + a_1 + a_0 > 0$

a) $G(z) = \dfrac{5(z-2)}{(z-0.1)(z-0.8)} = \dfrac{5(z-2)}{z^2 - 0.9z + 0.08}$
i) $1 - (-0.9) + 0.08 > 0$
ii) $1 - 0.08 > 0$
iii) $1 + (-0.9) + 0.08 > 0$

Hence, the system is stable. This is obvious since its poles are inside the unit circle.

b) $G(z) = \dfrac{10(z+.1)}{(z-0.7)(z-0.9)} = \dfrac{10(z+1)}{z^2 - 1.6z + 0.63}$
i) $1 - (-1.6) + 0.63 > 0$
ii) $1 - 0.63 > 0$
iii) $1 + (-1.6) + 0.63 > 0$

Hence, the system is stable. This is obvious since its poles are inside the unit circle.

4.4 Repeat Problem 4.3 using the Jury criterion.

The conditions for the Jury criterion are equivalent to those of Problem 4.3. The conditions are:

$F(1) > 0$
$F(-1) > 0$
$a_2 > | a_0 |$

a) $G(z) = \dfrac{5(z-2)}{(z-0.1)(z-0.8)} = \dfrac{5(z-2)}{z^2 - 0.9z + 0.08}$
i) $1 + (-0.9) + 0.08 > 0$
ii) $1 - (-0.9) + 0.08 > 0$
iii) $1 > 0.08$

Hence, the system is stable.

b) $G(z) = \dfrac{10(z+0.1)}{(z-0.7)(z-0.9)} = \dfrac{10(z+0.1)}{z^2 - 1.6z + 0.63}$

i) $1 + (-1.6) + 0.63 > 0$
ii) $1 - (-1.6) + 0.63 > 0$
iii) $1 > 0.63$

Hence, the system is stable.

4.5 Obtain the impulse response for the systems of Problem 4.3 and verify the results obtained using the Routh-Hurwitz criterion. Also determine the exponential rate of decay for each impulse response sequence.

The impulse response sequence is the inverse z-transform of the z-transfer function.

a)
$$\frac{G(z)}{z} = \frac{5(z-2)}{z(z-0.1)(z-0.8)} = -\frac{125}{z} + \frac{135.714}{z-0.1} - \frac{10.714}{z-0.8}$$
$$G(z) = \frac{5(z-2)}{(z-0.1)(z-0.8)} = \frac{13.5714}{z-0.1} - \frac{8.5714}{z-0.8}$$

$$g(t) = \begin{cases} -125\delta(k) + 135.714(0.1)^k - 10.714(0.8)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$
$$= \begin{cases} 13.5714(0.1)^{k-1} - 8.5714(0.8)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

Exponential rate of decay is $(0.8)^k$, which clearly goes to zero asymptotically. We have either of the bounds

$$\sum_{i=0}^{\infty} |h(i)| < 2|135.714| \sum_{i=0}^{\infty} 0.8^i = \frac{271.428}{1-0.8} = 1357.14 < \infty$$
$$\sum_{i=0}^{\infty} |h(i)| < 2|13.5714| \sum_{i=0}^{\infty} 0.8^i = \frac{27.1428}{1-0.8} = 135.714 < \infty$$

b)
$$\frac{G(z)}{z} = \frac{10(z+0.1)}{z(z-0.7)(z-0.9)} = -\frac{1.587}{z} - \frac{57.143}{z-0.7} + \frac{55.556}{z-0.9}$$
$$G(z) = \frac{10(z+0.1)}{(z-0.7)(z-0.9)} = -\frac{40}{z-0.7} + \frac{50}{z-0.9}$$

$$g(t) = \begin{cases} -1.587\delta(k) - 57.143(0.7)^k + 55.556(0.9)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$$
$$= \begin{cases} -40(0.7)^{k-1} + 50(0.9)^{k-1}, & k \geq 1 \\ 0, & k < 1 \end{cases}$$

Exponential rate of decay is $(0.9)^k$, which clearly goes to zero asymptotically. We have either of the bounds

$$\sum_{i=0}^{\infty} |h(i)| < 2|57.143| \sum_{i=0}^{\infty} 0.9^i = \frac{114.286}{1-0.9} = 1142.86 < \infty$$

$$\sum_{i=0}^{\infty} |h(i)| < 2|50| \sum_{i=0}^{\infty} 0.9^i = \frac{100}{1-0.9} = 1000 < \infty$$

4.6 Use the Routh-Hurwitz criterion to find the stable range of K for the closed-loop unity feedback systems with loop gain

a) $$G(z) = \frac{K(z-1)}{(z-0.1)(z-0.8)}$$

b) $$G(z) = \frac{K(z+0.1)}{(z-0.7)(z-0.9)}$$

The characteristic equation for a system with forward gain $G(z)$ and unity feedback is

$$1 + G(z) = 0 \Rightarrow N(z) + D(z) = 0$$

where $N(z)$ and $D(z)$ are the numerator and denominator of the transfer function respectively.

a) $$G(z) = \frac{K(z-1)}{(z-0.1)(z-0.8)}$$

The closed loop characteristic equation is

$$(z-0.1)(z-0.8) + K(z-1) = z^2 + (K-0.9)z + 0.08 - K = 0$$

This is a second order polynomial and the Routh-Hurwitz test yields the stability conditions are

i) $a_2 - a_1 + a_0 > 0$
ii) $a_2 - a_0 > 0$
iii) $a_2 + a_1 + a_0 > 0$

Hence

(i) $1 - (K - 0.9) + (0.08 - K) > 0 \Rightarrow 1.98 - 2K > 0 \Rightarrow K < 0.99$
(ii) $1 - (0.08 - K) > 0 \Rightarrow K > -0.92$
(iii) $1 + (K - 0.9) + (0.08 - K) > 0 \Rightarrow 0.18 > 0$

The stable range of K is: $-0.92 < K < 0.99$

b) $$G(z) = \frac{K(z+0.1)}{(z-0.7)(z-0.9)}$$

The closed loop characteristic equation is
$$(z-0.7)(z-0.9)+K(z+0.1)=z^2+(K-1.6)z+0.63+0.1K=0$$

This is a second order polynomial and the Routh-Hurwitz test yields the conditions of Problem 4.3. The stability conditions are

(i) $1-(K-1.6)+(0.63+0.1K)>0 \Rightarrow 3.23-0.9K>0 \Rightarrow K<3.589$
(ii) $1-(0.63+0.1K)>0 \quad \Rightarrow K<3.7$
(iii) $1+(K-1.6)+(0.63+0.1K)>0 \Rightarrow 1.1K-0.03>0 \Rightarrow K>-0.0273$

The stable range of K is: $-0.0273<K<3.589$

4.7 Repeat Problem 4.6 using the Jury criterion.

The systems are all second order and the Jury criterion gives the same three conditions obtained in Problem 4.6 for each systems. The remainder of the analysis is identical to Problem 4.6.

4.8 Use the Jury criterion to dermine the stability of the following polynomials

a) $z^5+0.2z^4+z^2+0.3z-0.1=0$

b) $z^5-0.25z^4+0.1z^3+0.4z^2+0.3z-0.1=0$

a) $z^5+0.2z^4+z^2+0.3z-0.1=0$

The first two conditions require the evaluation of $F(z)$ at $z=\pm 1$

(1) $F(1)=1+0.2+1+0.3-0.1=2.4>0$

(2) $(-1)^5F(-1)=(-1)(-1+0.2+0.3-0.1)=0.2>0$

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	–0.1	0.3	1	0	0.2	1
2	1	0.2	0	1	0.3	–0.1
3	–0.99	–0.23	–0.1	–1	–0.32	
4	–0.32	–1	–0.1	–0.23	–0.99	
5	0.8777	0.0923	0.0067	0.9164		
6	0.9164	0.0067	0.0923	0.8777		
7	–0.0199	–0.1277	–0.131			

(3) $|a_0|=0.1<a_2$

(4) $|b_0|=0.99>|b_4|=0.32$

(5) $|c_0|=0.8777<|c_3|=0.9164$ Condition violated

We conclude that the polynomial is unstable. The polynomial has roots outside the unit circle.

b) $z^5-0.25z^4+0.1z^3+0.4z^2+0.3z-0.1=0$

The first two conditions require the evaluation of $F(z)$ at $z=\pm 1$

(1) $F(1)=1-0.25+0.1+0.4+0.3-0.1=1.45>0$

(2) $(-1)^5F(-1)=(-1)(-1-0.25-0.1+0.4-0.3-0.1)=1.35>0$

Row	z^0	z^1	z^2	z^3	z^4	z^5
1	–0.1	0.3	0.4	0.1	– 0.25	1
2	1	– 0.25	0.1	0.4	0.3	–0.1
3	–0.99	0.22	–0.14	–0.41	–0.275	
4	–0. 275	–0.41	–0.14	0.22	–0.99	
5	0.9045	0.3306	0.1001	0.4664		
6	0. 4664	0. 1001	0. 3306	0. 9045		
7	0.812	–0.2752	0.2288			

(3) $|a_0| = 0.1 < a_2$

(4) $|b_0| = 0.99 > |b_4| = 0.275$

(5) $|c_0| = 0.9045 > |c_3| = 0.9164$

(6) $|d_0| = 0.812 > |d_2| = 0.2288$

We conclude that the polynomial is stable. The polynomial has no roots outside the unit circle.

4.9 Determine the stable range of the parameter a for the closed-loop unity feedback systems with loop gain

a) $$G(z) = \frac{1.1(z-1)}{(z-a)(z-0.8)}$$

b) $$G(z) = \frac{1.2(z+0.1)}{(z-a)(z-0.9)}$$

a) $$G(z) = \frac{1.1(z-1)}{(z-a)(z-0.8)}$$

The closed loop characteristic equation is

$$(z-a)(z-0.8)+1.1(z-1) = z^2+(0.3-a)z+0.8a-1.1=0$$

This is a second order polynomial and the Routh-Hurwitz test yields the stability conditions are

i) $a_2 - a_1 + a_0 > 0$

ii) $a_2 - a_0 > 0$

iii) $a_2 + a_1 + a_0 > 0$

Hence

(i) $1 - (0.3 - a) + (0.8a - 1.1) > 0 \Rightarrow 1.8a - 0.4 > 0 \Rightarrow a > 2/9$

(ii) $1 - (0.8a - 1.1) > 0 \quad \Rightarrow a < 21/8$

(iii) $1 + (0.3 - a) + (0.8a - 1.1) > 0 \Rightarrow 0.2 - 0.2a > 0 \Rightarrow a < 1$

The stable range of a is: $2/9 < a < 1$

b) $$G(z) = \frac{1.2(z+0.1)}{(z-a)(z-0.9)}$$

The closed loop characteristic equation is

$$(z-a)(z-0.9)+1.2(z+0.1) = z^2 + (0.3-a)z + 0.9a + 0.12 = 0$$

This is a second order polynomial and the Routh-Hurwitz test yields the stability conditions are

(i) $1 - (0.3 - a) + (0.9a + 0.12) > 0 \Rightarrow 1.9a + 0.82 > 0 \Rightarrow a > -41/95$
(ii) $1 - (0.9a - 0.12) > 0 \quad \Rightarrow 0.88 - 0.9a > 0 \Rightarrow a < 44/45$
(iii) $1 + (0.3 - a) + (0.9a + 0.12) > 0 \Rightarrow 1.42 - 0.1a > 0 \Rightarrow a < 14.2$

The stable range of a is: $-41/95 < a < 44/45$

4.10 For a gain of 0.5, derive the gain margin and phase margin of the systems of Problem 4.6 analytically. Let $T = 1$ with no loss of generality since the value of ωT in radians is all that is needed for the solution. Explain why the phase margin is not defined for the system of Problem 4.6(a).

Hint: The gain margin is obtained by finding the point where the imaginary part of the frequency response is zero. The phase margin is obtained by finding the point where the magnitude of the frequency respone is unity.

a) $$G(z) = \frac{K(z-1)}{(z-0.1)(z-0.8)}$$

The frequency transfer function is

$$\begin{aligned} G(e^{j\omega}) &= \frac{0.5(e^{j\omega}-1)}{e^{j2\omega} - 0.9e^{j\omega} + 0.08} \\ &= \frac{0.5[(\cos(\omega)-1) + j\sin(\omega)]}{\cos(2\omega) - 0.9\cos(\omega) + 0.08 + j[\sin(2\omega) - 0.9\sin(\omega)]} \end{aligned}$$

The phase crossover occurs when the imaginary part is zero

$$\begin{aligned} &\sin(\omega)\{2\cos^2(\omega) - 0.9\cos(\omega) - 0.92\} \\ &-(\cos(\omega)-1)\{2\sin(\omega)\cos(\omega) - 0.9\sin(\omega)\} \\ &= 2\sin(\omega)(\cos(\omega) - 0.91) = 0 \end{aligned}$$

which yields

$$\sin(\omega_{pc}) = 0 \quad or \quad \cos(\omega_{pc}) = 0.91$$

The second value corresponds to the positive real-axis intersection

$$G(e^{j\omega})\Big|_{\omega=\cos^{-1}(0.91)} = 0.5435$$

The first value corresponds to the negative-real axis intersection we seek and to the value $z = -1$. The intersection with the negative real axis is given by

$$G(-1) = -0.5051$$

The gain margin is given by

$$GM = \frac{1}{|0.5051|} = 1.98$$

$$GM\ dB = 20\log(1.98) = 5.93\,dB$$

At a gain of 0.5, the magnitude of the frequency response remains below unity for all frequencies. Hence, the phase margin is not defined.

b) $G(z) = \dfrac{0.5(z+0.1)}{(z-0.7)(z-0.9)}$

The frequency transfer function is

$$\begin{aligned} G(e^{j\omega}) &= \frac{0.5(e^{j\omega}+0.1)}{e^{j2\omega}-1.6e^{j\omega}+0.63} \\ &= \frac{0.5[(\cos(\omega)+0.1)+j\sin(\omega)]}{\cos(2\omega)-1.6\cos(\omega)+0.63+j[\sin(2\omega)-1.6\sin(\omega)]} \end{aligned}$$

The phase crossover occurs when the imaginary part is zero

$$\begin{aligned} &\sin(\omega)\{2\cos^2(\omega)-1.6\cos(\omega)-0.37\} \\ &-(\cos(\omega)+0.1)\{2\sin(\omega)\cos(\omega)-1.6\sin(\omega)\} = \sin(\omega)(0.2\cos(\omega)-0.21) = 0 \end{aligned}$$

which yields

$$\sin(\omega_{pc}) = 0$$

Thus, the negative-real axis intersection corresponds to the value $z = -1$. The intersection with the negative real axis is given by

$$G(-1) = -0.1393$$

The gain margin is given by

$$GM = \frac{1}{|0.1393|} = 7.1778$$

$$GM\ dB = 20\log(7.1778) = 17.112\,dB$$

To find the phase margin, we evaluate the magnitude squared of the frequency response

$$\left|G(e^{j\omega})\right|^2 = \frac{0.25\left[(\cos(\omega)+0.1)^2+\sin^2(\omega)\right]}{(\cos(2\omega)-1.6\cos(\omega)+0.63)^2+(\sin(2\omega)-1.6\sin(\omega))^2}$$

$$= \frac{0.25[0.2\cos(\omega)+1.01]}{2.52\cos^2(\omega)-5.216\cos(\omega)+2.6969} = 1$$

This yields the equation

$$\cos^2(\omega)-2.08968\cos(\omega)+0.97=0$$

We solve for the frequency and obtain the positive solution $\omega = 0.801$ rad/s

$$PM = 180^\circ + \angle G(e^{j\omega}) = 25.872^\circ$$

Computer Exercises

4.11 Write a computer program to perform the Routh-Hurwitz test using a suitable CAD tool.

A MATLAB function to bilinearly transform a given polynomial and test its stability is given below.

```
%  Function to perform the bilinear transformation
%  then test stability using the Routh-Hurwitz criterion
%  The function returns the stability status (con) and the
%  transformed polynomial sump.

function [con,sump]=drouth(pol)
%  First perform the bilinear transformation to obtain the
%  polynomial sump  i.e z=(w+1)/(w-1)
  np=[1,1]; nn=[1,-1]; sn=nn; l=length(pol);
  sump=pol(1)*np+pol(2)*nn;
for i=3:l
sn=conv(sn,nn); sump=conv(sump,np)+pol(i)*sn; end
  dl=length(sump);
%  Test the polynomial coefficient using the necessary
%  stability condition that all must be positive
   for i=1:dl
    if sump(i)<0; con = 'Unstable'; return; end
   end
%  If all coefficients are positive, use the Routh-Hurwitz
%  criterion
dlp = dl; a = sump;
   for j=1:dl-2
   ni=floor(dlp/2);
     for i = 1:ni-1
     b(i) = a(2*i+1)-a(1)*a(2*i+2)/a(2);
     if b(1) <= 0; con = 'Unstable'; return; end
     end
```

```
 if dlp/2 > ni; b(ni)=a(dlp); end
    for k = 2:2:dlp
    a(k-1) = a(k); a(k) = b(k/2); end
dlp = dlp-1; end
con = 'stable';
```

4.12 Write a computer program to perform the Jury test using a suitable CAD tool.

A MATLAB function to test the stability of polynomial using the Jury test is given below.

```
%  Function to perform the Jury stability test
%  The function returns the stability status (con)
% Use 1=stable, 0 = unstable
function con=jurytest(pol)
l = length(pol); n = l-1;
if polyval(pol,1)<=0 |  (-1)^n*polyval(pol,-1)<=0|abs(pol(l))>pol(1)
con=0;return;end
if l <= 3; con = 1; return; end
%  criterion
   dlp = l;  a = pol;
   for i = 1:l-3
    for j=1:dlp-1
    b(j) = a(dlp)*a(dlp+1-j)-a(1)*a(j); end
    dlp = dlp-1;
    if abs(b(1)) <= abs(b(dlp)); con = 0; return; end
    for k = 1:dlp; a(k)=b(k); end
   end
con=1;
end
```

Note: We could assign character values to con, `'unstable'` or `'stable'` but we prefer to use number to simplify the questions that follow)

4.13 Write a computer program that uses the Jury test program of 4.10 to determine the stability of a system with an uncertain gain K in a given range [K_{min} , $Kmax$]. Verify the answers obtained for Problem 4.5 using your program.

The following program shows that the stability ranges obtained for Problem 4.5 are correct. It calls the function jurytest. If Kmax=0.99 or Kmin=−0.92, instability results.

```
% Program JuryRange
% Set K = Kmin, call jury, increment K until K>= Kmax or an
unstable
% polynomial is encountered
Kmin =-0.921; Kmax=0.989; Kstep=0.001; % Define Kmin, Kmax,
Kstep
K=Kmin;
while K< Kmax
pol=[1, K-0.9, 0.08-K];% Statement to evaluate the coefficient
of pol
con=jurytest(pol);
if con==0; 'unstable', K, return; end
K=K+Kstep;
```

```
end
'stable'
```

4.14 Show how the program of Problem 4.11 can be used to test the stability of a system with uncertain zero location. Use the program to test the effect of a ± 20% variation in the location of the zero for the systems of Problem 4.5, with a fixed gain equal to half the critical value.

The open-loop transfer function is $G(z) = \dfrac{0.495(z-a)}{(z-0.1)(z-0.8)}, \qquad 0.8 \le a \le 1.2$

The closed loop characteristic equation is

$0.495(z-a)+(z-0.1)(z-0.8) = z^2 - 0.405z + 0.08 - 0.495a,\ 0.8 \le a \le 1.2$

The program JuryRange with Kmin=0.8, Kmax=1.2, and the above polynomial shows that the system is stable for the given range of zero locations.

4.15 Show how the program of Problem 4.11 can be used to test the stability of a system with uncertain pole location. Use the program to test the effect of a ± 20% variation in the location of the first pole for the systems of Problem 4.5, with a fixed gain equal to half the critical value.

The open-loop transfer function is $G(z) = \dfrac{0.495(z-1)}{(z-0.1)(z-p)}, \qquad 0.64 \le p \le 0.96$

The closed loop characteristic equation is

$0.495(z-1)+(z-0.1)(z-p) = z^2 + (0.395-p)z + 0.1p - 0.495,\ 0.64 \le p \le 0.96$

The program JuryRange with Kmin=0.64, Kmax=0.96, and the above polynomial shows that the system is stable for the given range of pole locations.

4.16 Simulate the closed-loop systems of Problem 4.5 with a unit step input and (i) gain K equal to half the critical gain, (ii) gain K equal to the critical gain. Discuss their stability using your simulation results.

a)For $K = K_{cr} = 0.99$, we obtain the unstable step response.

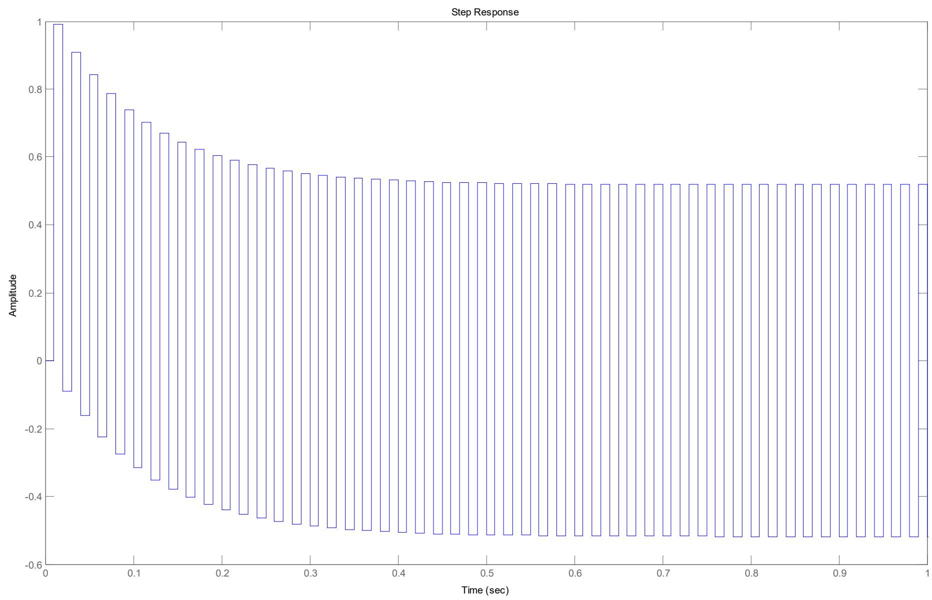


Figure 4-1 Step response of the unstable system of Problem 4.5(a) at the critical gain.

For $K = K_{cr}/2 = 0.495$, we obtain a stable step response but because the system has a zero at 1, the response decays to zero. This is similar to the response to the response of an analog system with a differentiator or numerator *s*.

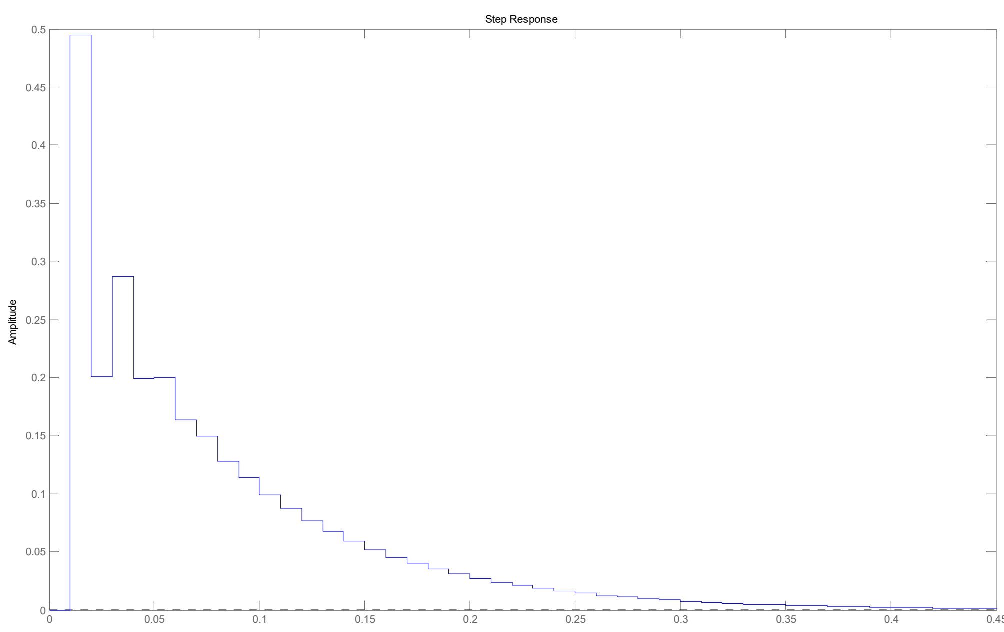


Figure 4-2 Step response of the system of Problem 4.5(a) at half the critical gain.

b) For $K = K_{cr} = 3.589$, we obtain the unstable step response.

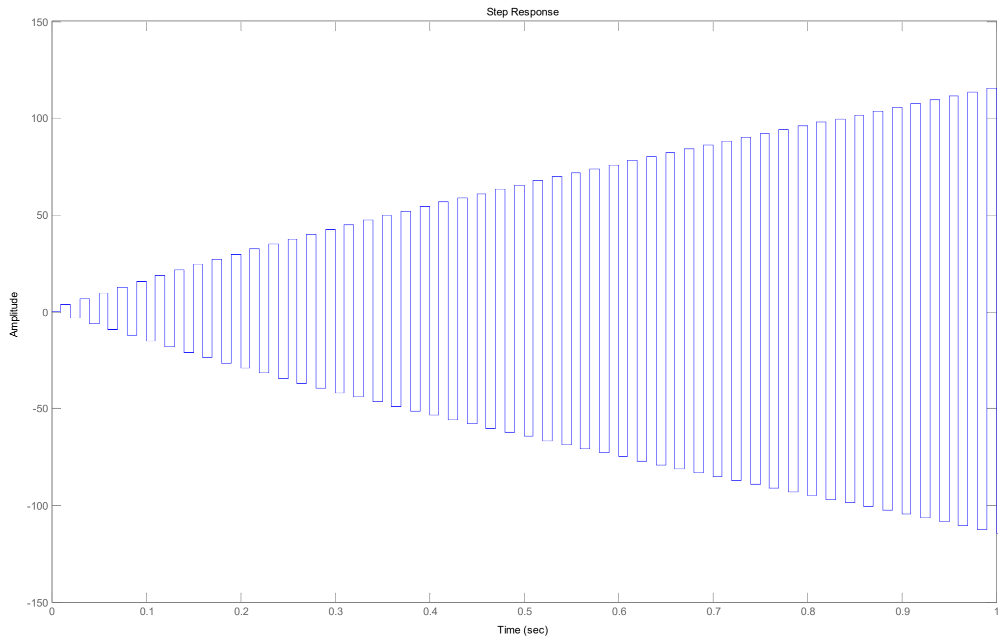


Figure 4-3 Step response of unstable system of Problem 4.5(b) at the critical gain.

For $K = K_{cr}/2 = 1.8945$, we obtain a stable step response. The system is type zero and the steady-stae error is small but not zero.

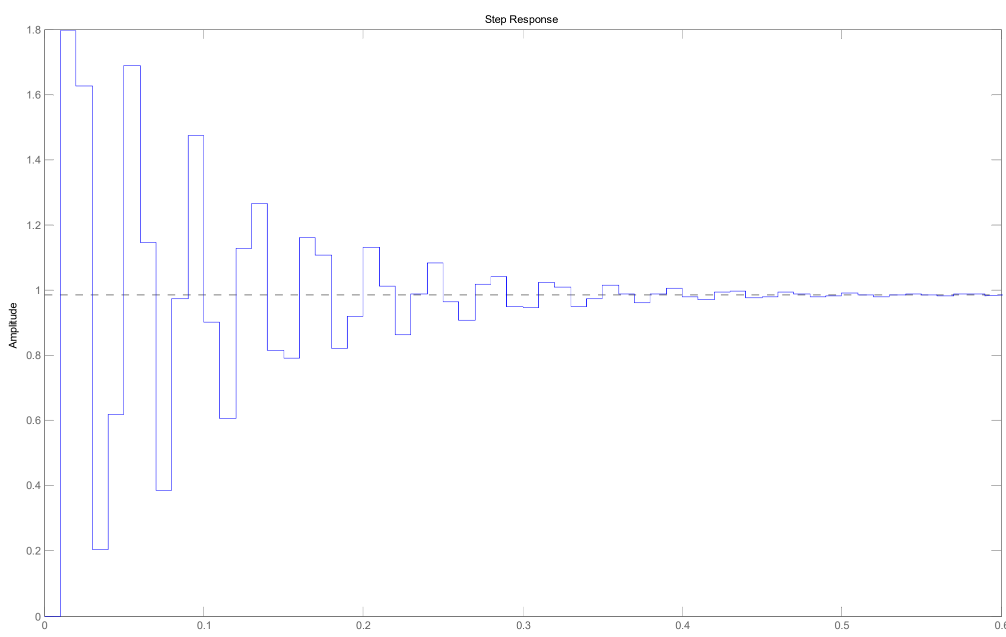


Figure 4-4 Step response of unstable system of Problem 4.5(b) at half the critical gain.

4.17 For unity gain, obtain the Nyquist and Bode plots of the systems of Problem 4.5 using MATLAB and determine

i) The intersection with the real axis using the Nyquist plot then using the Bode plot.
ii) The stable range of positive gains K for the closed-loop unity feedback systems.
iii) The gain margin and phase margin for a gain $K = 0.5$

a) $$G(z) = \frac{(z-1)}{(z-0.1)(z-0.8)}$$

The MATLAB commands are

```
>> z=tf('z')
 Transfer function:
z
 Sampling time: unspecified

>> g=(z-1)/(z-.1)/(z-.8)

Transfer function:
     z - 1
------------------
z^2 - 0.9 z + 0.08

Sampling time: unspecified

>> nyquist(g)
>> bode(g)
```

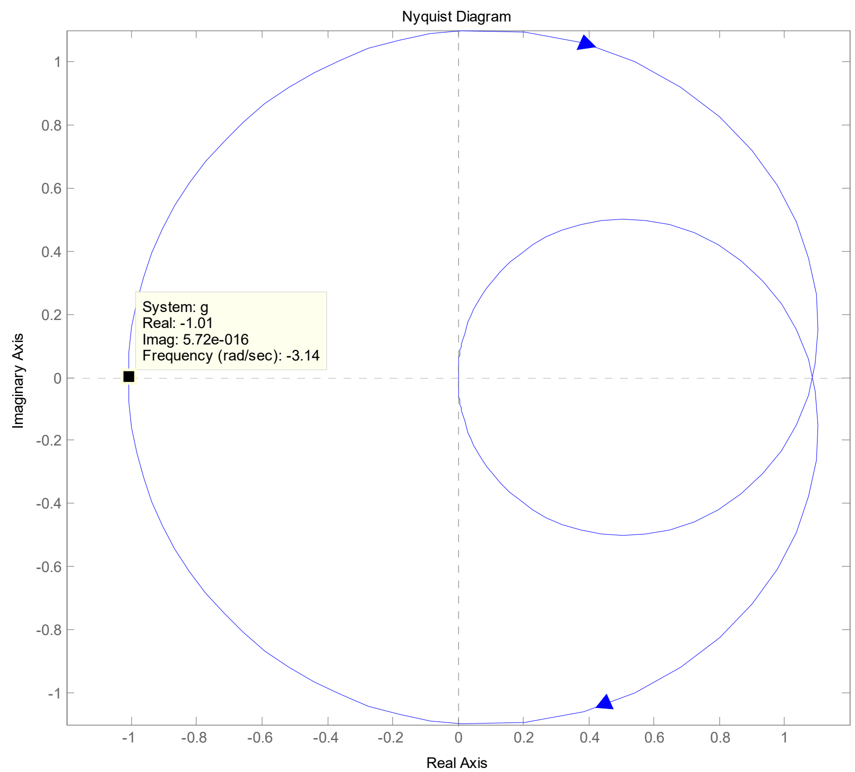


Figure 4-5 Nyquist plot for the system of Problem 4.5(a) with unity gain.

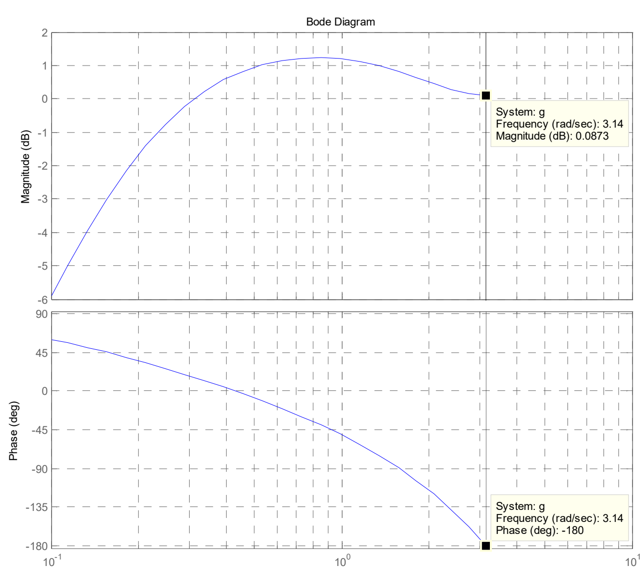


Figure 4-6 Bode plot for the system of Problem 4.5(a) with unity gain.

iv) The intersection with the real axis using the Nyquist plot then using the Bode plot.

From the Nyquist plot, the intersection with the real axis is at -1.01 and the system is closed-loop unstable. From the Bode plot, at an angle of $-180°$ the corresponding magnitude is 0.0873 dBs, i.e. magnitude greater than unity. The corresponding magnitude is

$$|G(j3.24)| = 10^{0.0873/20} \approx 1.01$$

v) The stable range of positive gains K for the closed-loop unity feedback systems.

To put the system on the verge of instability, we need to reduce the gain to the critical gain $K_{cr} = 0.99$ (as shown in Problem 4.5).

vi) The gain margin and phase margin for a gain $K = 0.5$

We use the MATLAB commands

```
>> margin(g*.5)
```

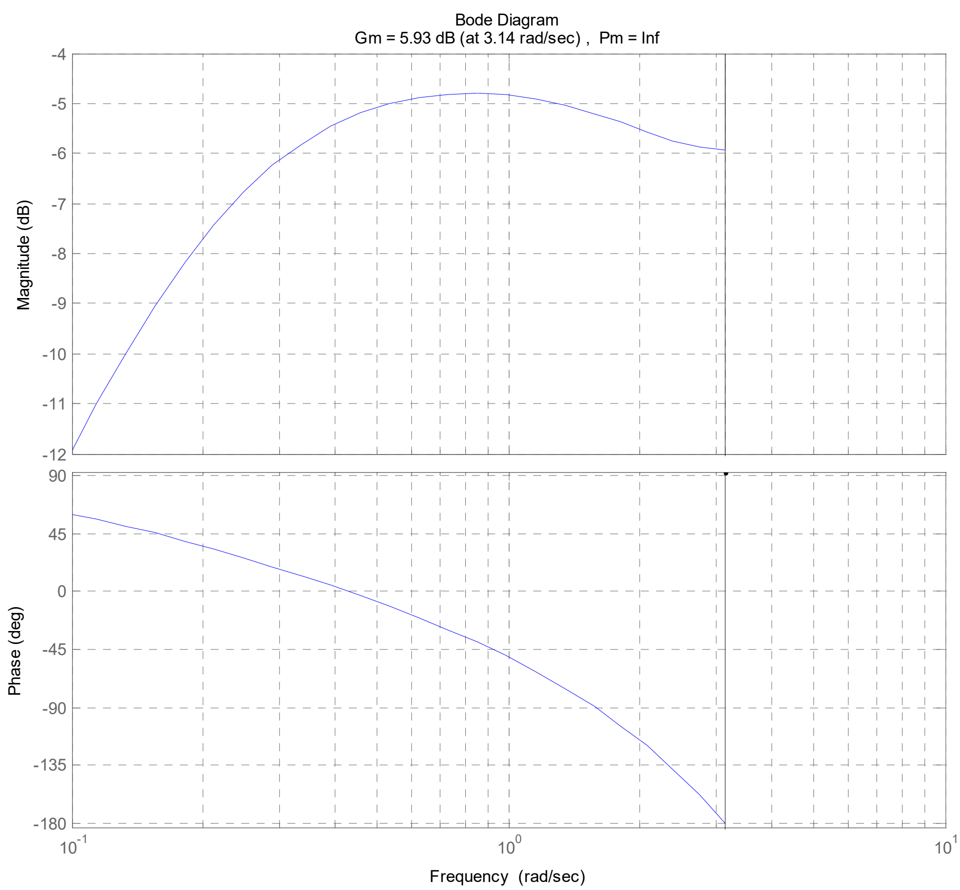


Figure 4-7 Phase margin and gain margin for the system of Problem 4.5(a) with a gain of 0.5.

The gain margin is 5.93 dB at a frequency of 3.14 rad/s, which is the phase crossover frequency obtained earlier. The phase magin is infinite since the frequency response magnitude remains below unity or zero dBs.

b) $G(z) = \dfrac{(z+0.1)}{(z-0.7)(z-0.9)}$

The MATLAB commands are

```
>> z=tf('z')
 Transfer function:
z
 Sampling time: unspecified

>> g=(z+.1)/(z-.7)/(z-.9)

Transfer function:
    z + 0.1
------------------
z^2 - 1.6 z + 0.63

Sampling time: unspecified

>> nyquist(g)
>> bode(g)
```

i) The intersection with the real axis using the Nyquist plot then using the Bode plot.

From the Nyquist plot, the intersection with the real axis is at −0.279 and the system is closed-loop stable. From the Bode plot, at an angle of −180° the corresponding magnitude is −11.1 dBs, i.e. magnitude greater than unity. The corresponding magnitude is

$$\left|G(j3.24)\right| = 10^{-11.1/20} \approx 0.279$$

ii) The stable range of positive gains K for the closed-loop unity feedback systems.

To put the system on the verge of instability, we need to reduce the gain to the critical gain K_{cr} = 3.584 (which is approximately the same as the value of 3.589 obtained in Problem 4.5).

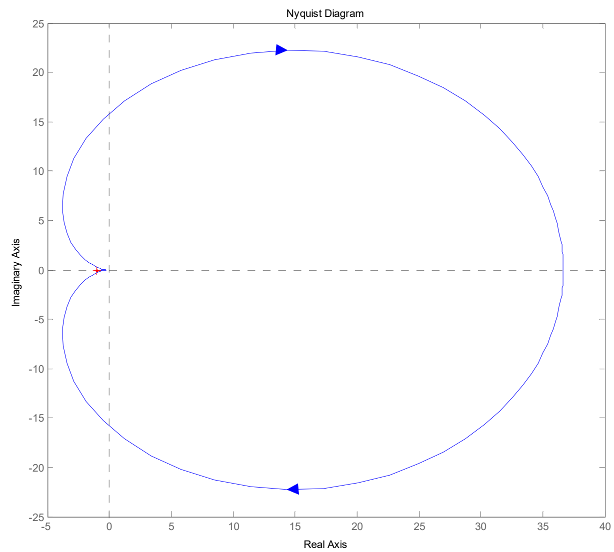


Figure 4-8 Nyquist plot for the sytem of Problem 4.5(b).

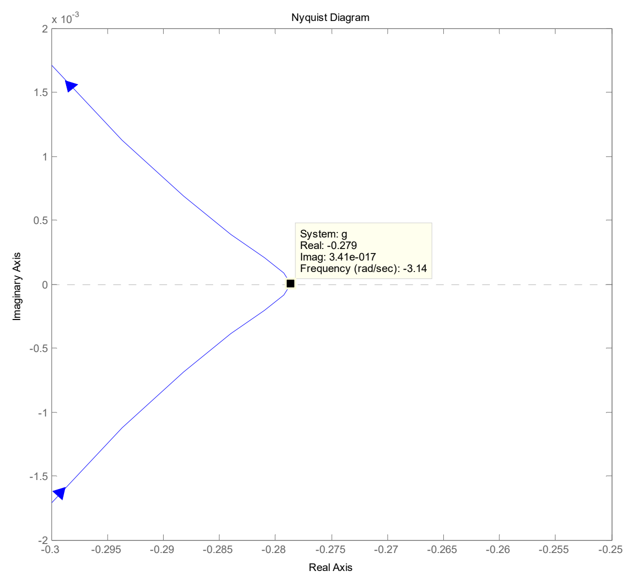


Figure 4-9 Nyquist plot for the sytem of Problem 4.5(b) showing intersection with the real axis.

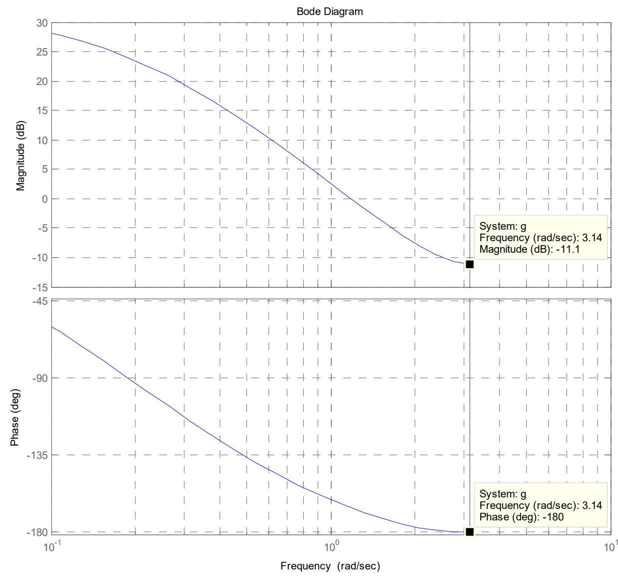


Figure 4-10 Bode plot for the system of Problem 4.5(b) with unity gain.

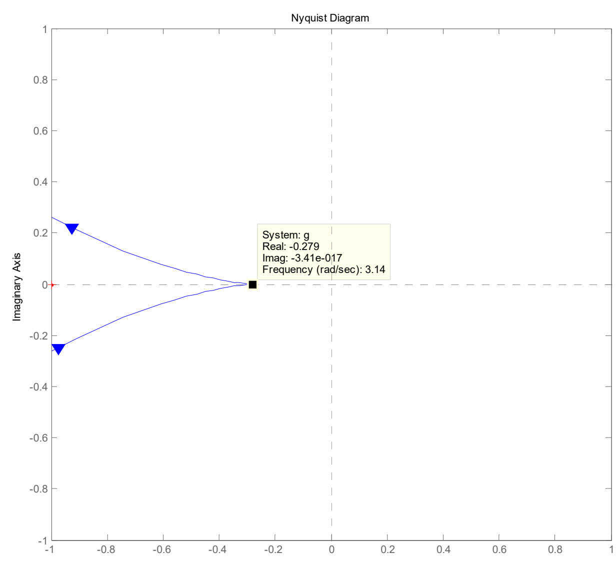


Figure 4-11 Nyquist plot for the sytem of Problem 4.5(b) with unity gain showing intersection with the real axis.

vii) The gain margin and phase margin for a gain $K = 0.5$

We use the MATLAB commands

```
>> margin(g*.5)
```

The gain margin is 17.1 dB at a frequency of 3.14 rad/s, which is the phase crossover frequency obtained earlier. The phase magin is 25.9° at a frequency of 0.8 rad/s. We can also obtain the phase margin and gain margin using the Nyquist plot as shown in Figure Figure 4-13 by clicking on the plot and selecting

Characteristics

All stability margins

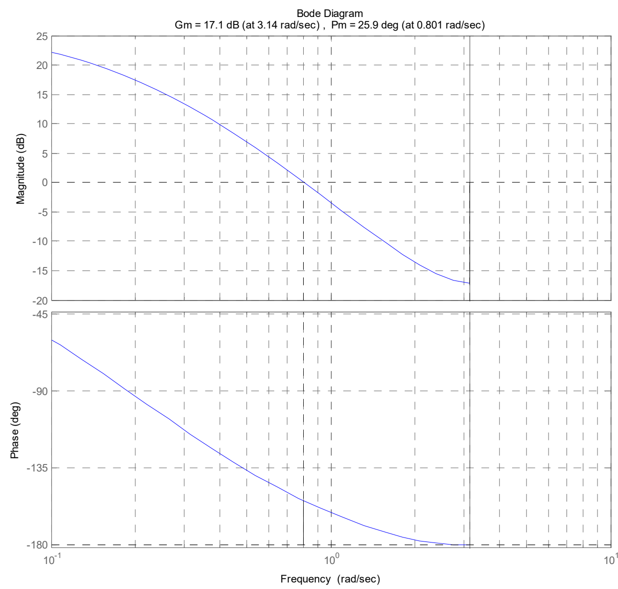


Figure 4-12 Phase margin and gain margin for the system of Problem 4.5(b) at a gain of 0.5 using the Bode plot.

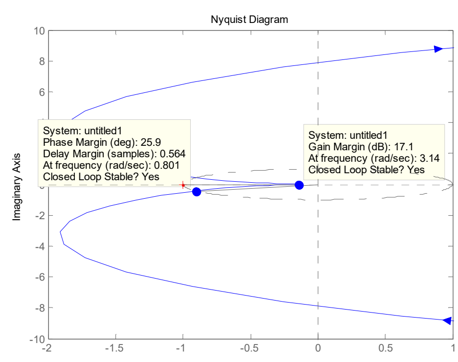


Figure 4-13 Phase margin and gain margin for the system of Problem 4.5(b) at a gain of 0.5 using the Nyquist plot.

4.18 For twice the nominal gain, use MATLAB to obtain the Nyquist and Bode plots of the oven control system of Example 4.10 with a sampling period of 0.01 and determine

i) The intersection with the real axis using the Nyquist plot then using the Bode plot.
ii) The stable range of additional positive gains K for the closed-loop unity feedback systems.
iii) The gain margin and phase margin for twice the nominal gain.

i) The intersection with the real axis using the Nyquist plot then using the Bode plot.

We obtain the Nyquist plot (Figure 4.14), zoom in on the point (–1, 0) of the Nyquist plot (Figure 4.15), and the Bode plot (Figure 4.16). To obtain more accurate values for the intersection in the Nyquist plot, we use the command

>> **nyquist(gdt*2,[5:.01:6]) % Obtain Nyquist plot for 5 to 6 rad/s.**

The intersection with the negative real axis is at –0.576 which corresponds to the value on the Bode plot since

$20 \log_{10}(0.562) = -4.8$

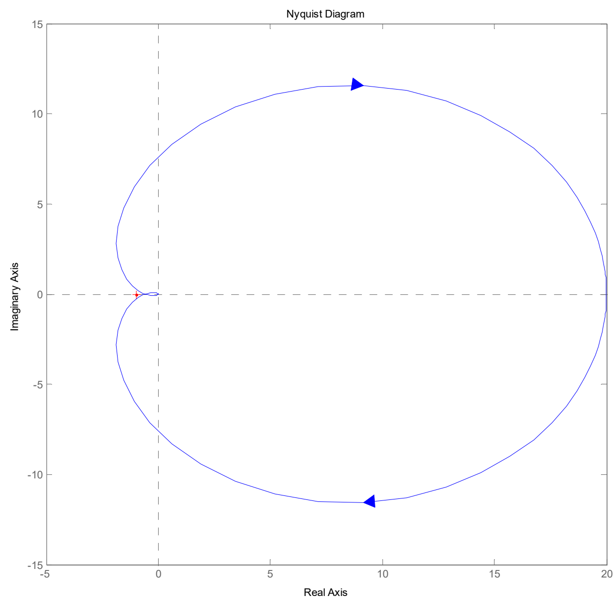


Figure 4-14 Nyquist plot for the oven control system.

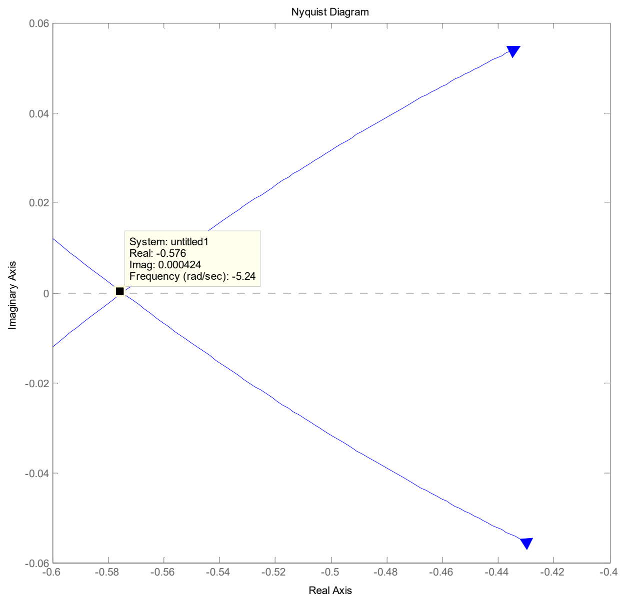


Figure 4-15 Nyquist plot for the oven control system in the vicinity of the point (–1, 0).

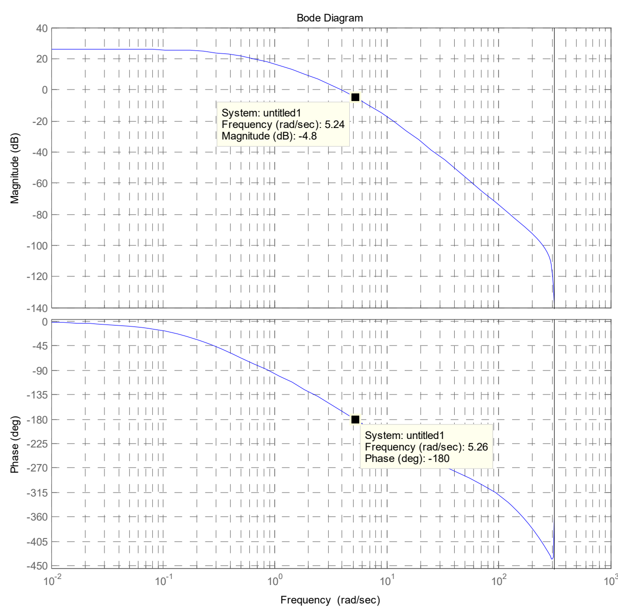


Figure 4-16 Bode plot for the oven control system.

ii)The stable range of additional positive gains K for the closed-loop unity feedback systems.

The gain margin is given by

$$GM = -\frac{1}{0.576} \approx 1.74$$

To put the system on the verge of instability, we can increase the gain to the critical value K_{cr} = 1.74 and the allowable range of positive gains is [0, 1.74]

iii) The gain margin and phase margin for twice the nominal gain.

We use the MATLAB command

```
>> [gm,pm]=margin(gdt*2)
gm =
    1.7409
pm =
   15.5247
```

The Bode plot including the margins of Figure 4.17 is obtained using

```
>> margin(gdt*2)
```

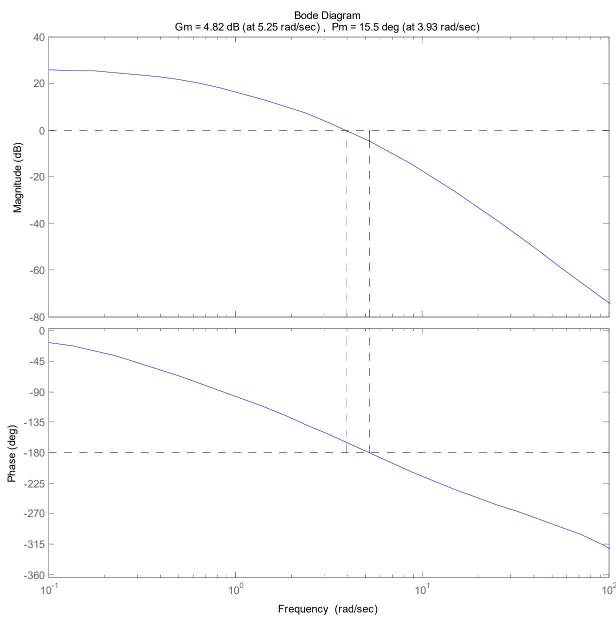


Figure 4-17 Phase and gain margins show on the Bode plot for the oven control system.

Th gain and phase margins can also be obtained using the Nyquist command as shown in Figure 4.18 by clicking on the plot and selecting

Characteristics

All stability margins

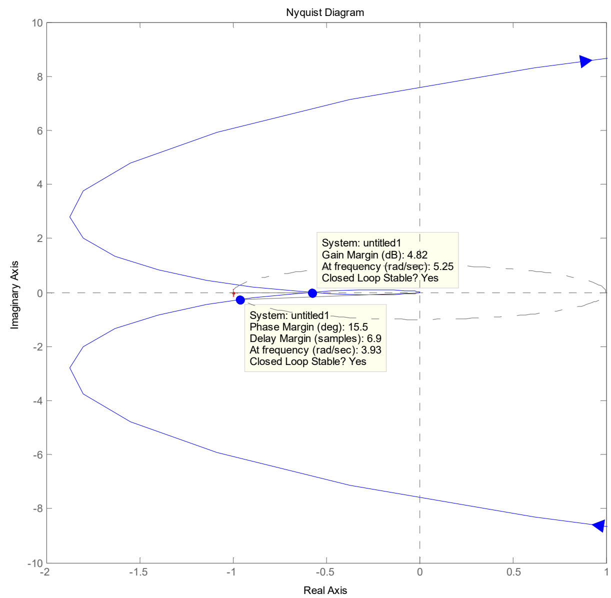


Figure 4-18 Phase and gain margins show on the Nyquist plot for the oven control system.

4.19 In many applications, there is a need for accurate position control at the nanometer scale. This is known as **nano-positioning** and is now feasible due to advances in nanotechnology. The following transfer function represents a single-axis nanopositioning system[2]

$$G(s) = 4.29E10 \frac{\left(s^2 + 631.2s + 9.4E6\right)}{\left(s^2 + 178.2s + 6E6\right)\left(s^2 + 412.3s + 16E6\right)} \frac{\left(s^2 + 638.8s + 45E6\right)}{\left(s^2 + 209.7s + 56E6\right)\left(s + 5818\right)}$$

a) Obtain the DAC-analog system-ADC transfer function for a sampling period of 100 ms and determine its stability using the Nyquist criterion.
b) Obtain the DAC-analog system-ADC transfer function for a sampling period of 1 ms and determine its stability using the Nyquist criterion.
c) Plot the closed-loop step response of the system of (b) and explain the stability results of (a) and (b) based on your plot.

d) Obtain the DAC-analog system-ADC transfer function for a sampling period of 100 ms and determine its stability using the Nyquist criterion.

```
>> gd=c2d(g,.1)

Transfer function:
0.5801 z^6 - 0.0001393 z^5 + 1.051e-008 z^4 - 3.084e-013 z^3 + 3.704e-018 z^2 + 4.488e-027 z + 2.376e-036
-------------------------------------------------------------------------------------------
z^7 - 0.0003086 z^6 + 3.139e-008 z^5 - 1.068e-012 z^4 + 1.424e-017 z^3 + 2.807e-026 z^2 + 1.769e-035 z - 4.254e-055
```

[2] A. Sebastian and S. M. Salapaka, "Design Methodologies for Robust Nano-Positioning," *IEEE TRANS. CONTROL SYSTEMS TECH.*, VOL.13, NO.6, November, 2005.

```
>> nyquist(gd)
```

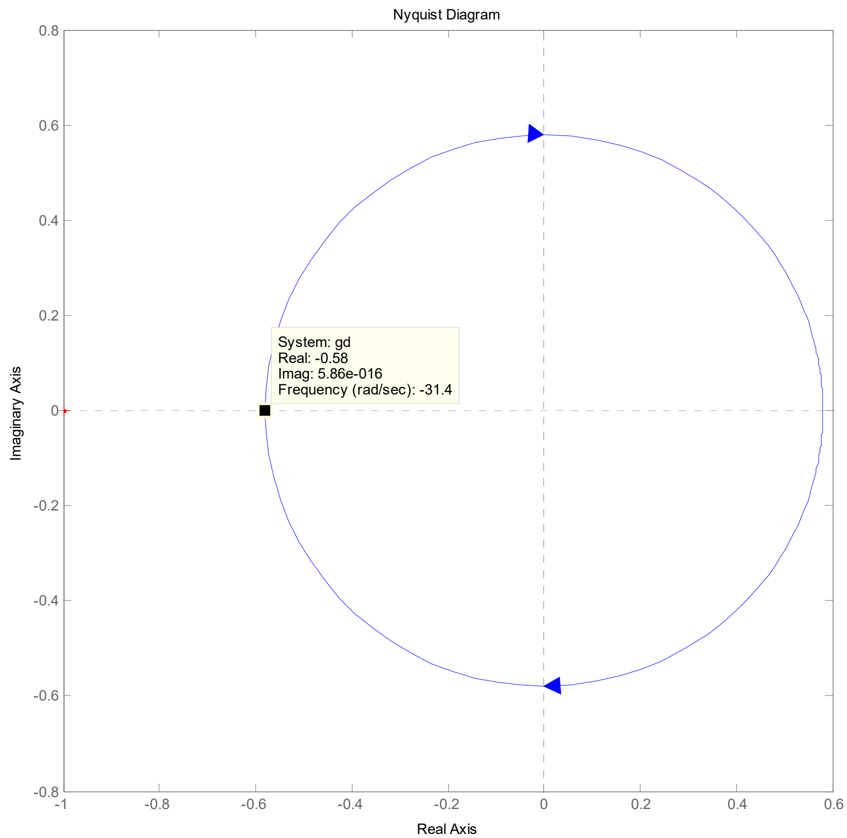


Figure 4-19 Nyquist plot for the nano-positioning system with a sampling period *T*=100ms.

e) Obtain the DAC-analog system-ADC transfer function for a sampling period of 1 ms and determine its stability using the Nyquist criterion.

```
>> gd1=c2d(g,.001)

Transfer function:
  0.9063 z^6 + 1.227 z^5 + 1.16 z^4 + 1.114 z^3 + 0.9899 z^2 + 0.4766 z + 0.05919
---------------------------------------------------------------------------------
z^7 + 1.82 z^6 + 2.191 z^5 + 1.865 z^4 + 1.79 z^3 + 1.114 z^2 + 0.4459 z - 0.001336

Sampling time: 0.001
>> nyquist(gd1)
```

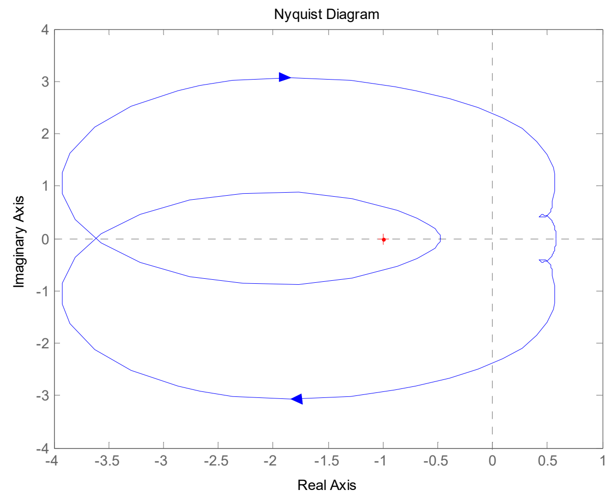


Figure 4-20 Nyquist plot for the nano-positioning system with a sampling period *T*=1ms.

f) Plot the closed-loop step response of the of the system of (b) and explain the stability results of (a) and (b) based on your plot.

The step response of the system with sampling period 1 ms is shown in Figure 4-21 and the system is clearly unstable as predicted by the Nyquist criterion. The analog subsystem is unstable at a gain of unity but at a slow sampling rate, the digital system appears stable. The system will have intersample oscillations and this is revealed by the instability of the digital system with the faster sampling rate of 1 ms.

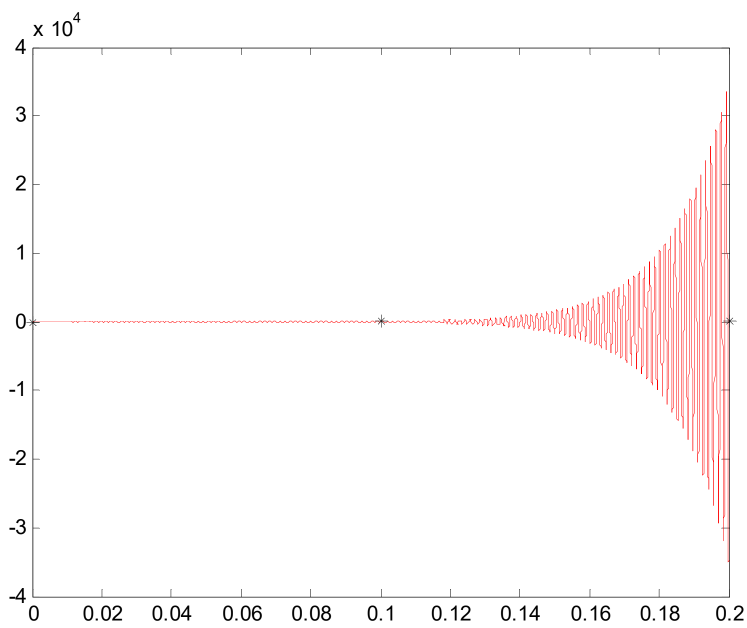


Figure 4-21 Step response for the closed-loop digital system with a samping rate *T*=1 ms.

Chapter 5 Solutions

5.1 For the transfer function

$$G(s) = \frac{s+a}{(s+p_1)(s+p_2)}, \qquad a < p_1 < p_2 \quad or \quad p_1 < p_2 < a$$

The breakaway and break-in points are obtained by minimizing the gain

$$K = -\frac{(\sigma + p_1)(\sigma + p_2)}{\sigma + a}$$

A necessary condition for a minimum or maximum is that the first derivative with respect to σ be zero. The first derivative is

$$\frac{dK}{d\sigma} = -\frac{\left[(\sigma + p_1) + (\sigma + p_2)\right](\sigma + a) - (\sigma + p_1)(\sigma + p_2)}{\left(\sigma + a\right)^2}$$

$$= -\frac{2\sigma^2 + (p_1 + p_2 + 2a)\sigma + a(p_1 + p_2) - \sigma^2 - (p_1 + p_2)\sigma - p_1 p_2}{\left(\sigma + a\right)^2}$$

$$= -\frac{\sigma^2 + 2a\sigma + a(p_1 + p_2) - p_1 p_2}{\left(\sigma + a\right)^2}$$

Equating to zero and solving for σ gives

$$\sigma_b = -a \pm \sqrt{a^2 - a(p_1 + p_2) + p_1 p_2}$$

$$= -a \pm \sqrt{(a - p_1)(a - p_2)}$$

For $a < p_1 < p_2$ or $p_1 < p_2 < a$, there are two real solutions for σ_b . One solution corresponds to a breakaway point since it lies between the two poles and will yield a positive gain *K*. The second is a break-in point that corresponds to a negative gain. For *a* outside the specified range, the root locus is entirely on the real axis with no breakaway or break-in points. The breakaway point is where *K* is at a maximum and the break-in point is where *K* is at a minimum.

5.2 The transfer function

$$KG(s) = \frac{K(s+4)}{s(s+2)}$$

has a breakaway point at

$$\sigma_b = -4 + \sqrt{(4-0)(4-2)}$$
$$= -1.1716$$

The second solution obtained using the results of Problem 5.1 is not between the two open-loop poles (σ_b =-6.8284) and corresponds to the break-in point associated with a negative gain.

The root locus off the real line is a circle centered at the zero (-4) with radius $(8)^{1/2}$
The root locus is shown in Figure 5-1.

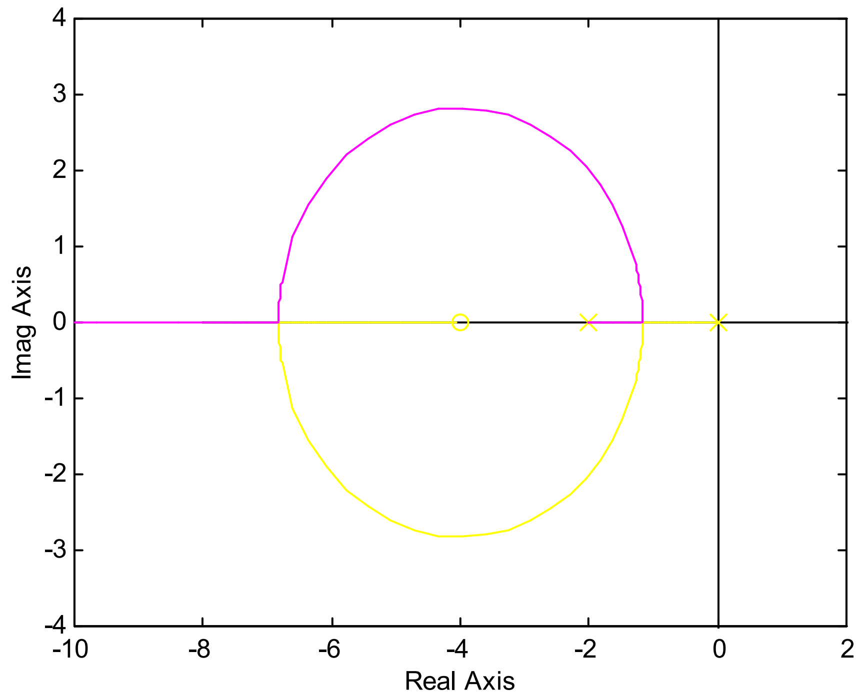


Figure 5-1 Root locus for Problem 5.2.

5.3 (i) $KG(s) = \dfrac{K}{s(s+2)(s+5)}$ (ii) $KG(s) = \dfrac{K(s+2)}{s(s+3)(s+5)}$

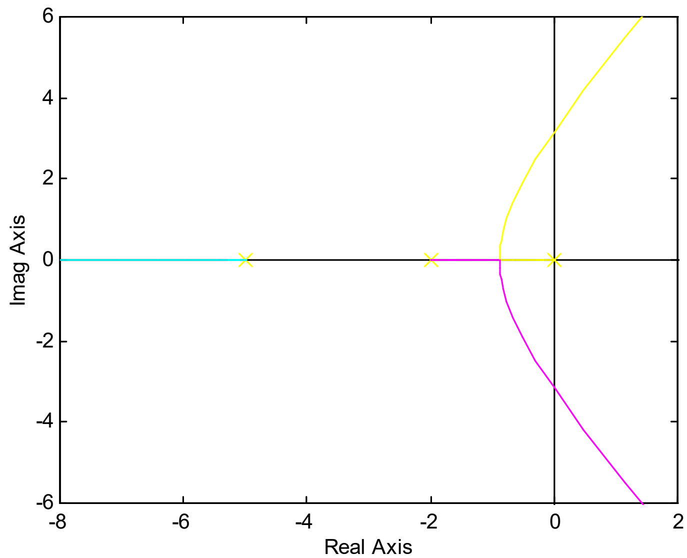


Figure 5-2 Root locus for Problem 5.3 (i)

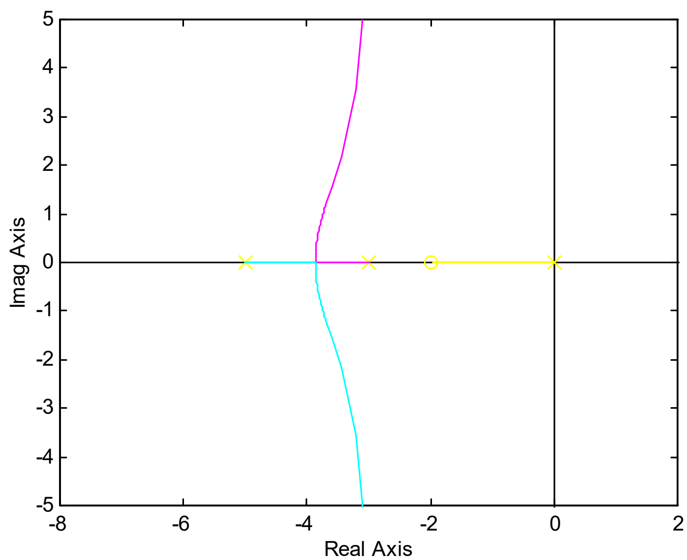


Figure 5-3 Root locus for Problem 5.3 (ii)

5.4 Consider the system in 5.3(ii) with a required steady-state error of 20%, and an adjustable PI controller zero location. Show that the corresponding closed-loop characteristic equation is given by

$$1 + K\left(\frac{s+a}{s}\right)\frac{1}{(s+3)(s+5)} = 0$$

Next, rewrite the equation as

$$1 + K_f G_f(s) = 0$$

where $K_f = K$, $K\,z$ is constant and $G_f(s)$ is a function of s, and examine the effect of shifting the zero on the closed-loop poles.

a) Design the system for a dominant second order pair with a damping ratio of 0.5. What is ω_n for this design?

For the transfer function $$KG(s) = \frac{K(s+\text{a})}{s(s+3)(s+5)}$$

For the required steady-state error of 20%, K_v = 100/20 = 5

Hence $K_v = \underset{s\to 0}{Lim}\, sG(s) = \dfrac{Ka}{15} = 5$

i.e. $K\,a = 5 \times 15 = 75$

The closed-loop characteristic equation with unity feedback and cascade PD control is

$$1 + KG(s)C(s) = 1 + \frac{Ks + 75}{s(s+3)(s+5)} = 0$$

Multiply by the denominator of the loop gain then divide by all terms that do not include K to obtain

$$1 + \frac{Ks}{s^3 + 8s^2 + 15s + 75} = 0$$

where $K\,a$ =75 is fixed. To examine the effect of shifting the zero on the closed-loop poles, we obtain the root locus for the new transfer function using the MATLAB command **rlocus** and obtain the desired design suing the cursor.

For a dominant second order pair with a damping ratio of 0.5, ω_n =5.04 rad/s, K = 25.3,
a =75/25.3 = 2.96.

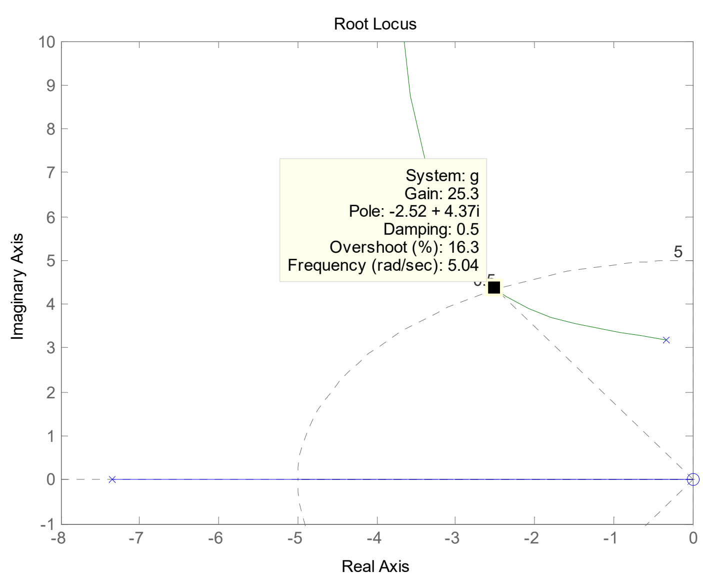


Figure 5-4 Part (a) Root locus for *K a* = 75.

b) Obtain the time response using a CAD program. How does the time response compare with that of a second order systems with the same ω_n and ζ as the dominant pair? Give reasons for the differences.

The time response of the system is shown below together with the response of a second order systems with the same ω_n and ζ as the dominant pair. For the second order system, the percentage overshoot is given by

$$PO = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}\,\% = 16.3\%$$

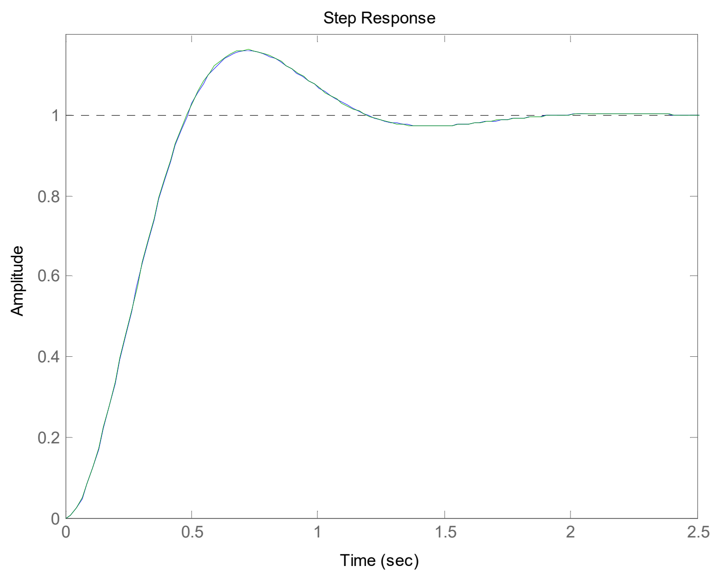


Figure 5-5 Time response for *K z* = 75 and a zero at –2.96.

Our system behaves almost exactly like the second order underdamped pair because it has a zero at –2.96 very close to a third pole at –3. So it has PO = 16.3 %. The system has time to first peak is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{4.37} \approx 0.72s$$

c) Discuss briefly the tradeoff between error, speed of response and relative stability in this problem.

The root locus of the system of Figure 5-6 for a zero fixed at –2.96 shows that increasing the gain of the system to improve its steady-state error would result in a decrease in the damping ratio. Increasing the gain would speed up the system but would decrease its relative stability and the PO will be unacceptably high. For example, if the gain is increased to 40 we obtain the time response of Figure 5-7 with a PO of 25.4%. Thus the choice of 20% steady-state error due to ramp is a compromise that yield an acceptable transient response.

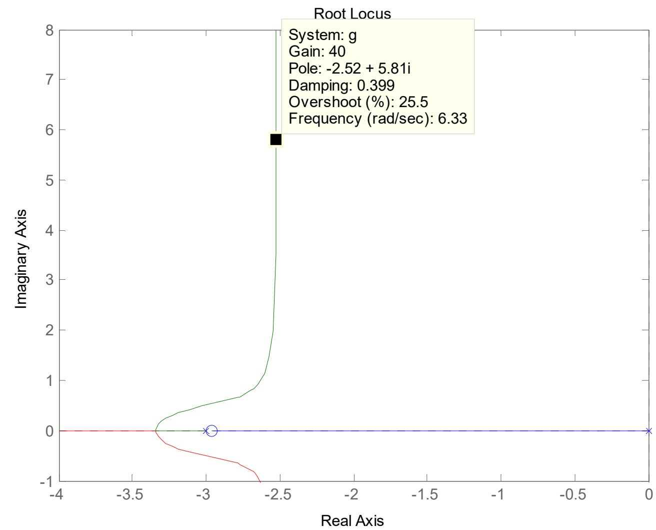


Figure 5-6 Root locus for $K = 40$ with a zero at –2. 7.

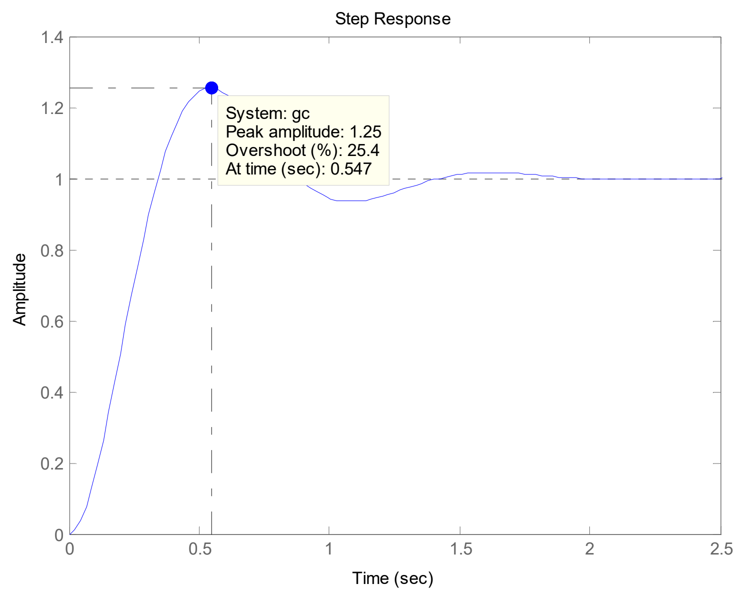


Figure 5-7 Time response for $K = 40$ with a zero at –2.7.

5.5 Prove equations (5.13) and (5.14) and justify the design Procedures 5.1, 5.2.

By the angle condition, the sum of the angles of all the blocks in the loop must be ±180° at any closed-loop pole. If the compensator angle is θ_c and the loop gain (before compensation) is $L(s)$, the angle conditon gives

$$\theta_C = \pm 180^\circ - \angle L(s_{cl}) \qquad (5.13)$$

Equation (5.14) follows from the geometry of Figure 5-8

$$a = \frac{\omega_d}{\tan(\theta_C)} + \zeta\omega_n \tag{5.14}$$

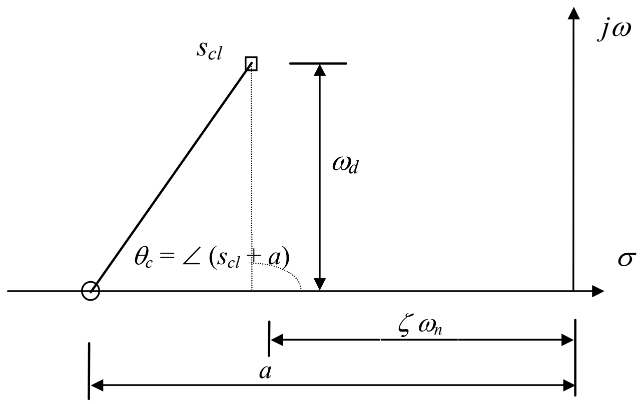


Figure 5-8 Zero location for a PD controller.

5.6 Show that a PI feedback controller is undesirable since it results in a differentiator in the forward path. Discuss the step response of the closed-loop system.

Let the feedforward transfer function be $G(s)$ and the feedback controller transfer function be

$$C(s) = K\frac{s+a}{s}$$

The closed-loop transfer function of the system is

$$\begin{aligned}
T(s) &= \frac{G(s)}{1 + C(s)G(s)} \\
&= \frac{G(s)}{1 + K\dfrac{s+a}{s}G(s)} \\
&= \frac{sG(s)}{s + K(s+a)G(s)}
\end{aligned}$$

The corresponding steady-state error is

$$\begin{aligned}
e(\infty) &= \operatorname*{Lim}_{s\to 0}\left[1 - T(s)\right]sR(s) \\
&= \operatorname*{Lim}_{s\to 0}\left[1 - \frac{sG(s)}{s + K(s+a)G(s)}\right]sR(s) \\
&= \operatorname*{Lim}_{s\to 0} sR(s)
\end{aligned}$$

Thus the system has a large error due to step and cannot follow a ramp or a parabolic.

5.7 Design a controller for the transfer function

$$G(s) = \frac{1}{(s+1)(s+5)}$$

to obtain (i) zero steady-state error due to step, (ii) a settling time of less than 2 s, and (iii) an undamped natural frequency of 5 rad/s. Obtain the response due to a unit step and find the percentage overshoot, the time to the first peak and steady-state error percent due to a ramp input.

The plant transfer function is $G(s) = \dfrac{1}{(s+1)(s+5)}$

For zero steady-state error due to step, a PI controller is needed. The simplest design is by cancellation of the pole at −1 with the controller zero. This leaves the loop gain

$$L(s) = C(s)G(s) = \frac{1}{s(s+5)}$$

The corresponding root locus includes a vertical line at −2.5 i.e. for sufficiently high gain we have an underdamped system with $\zeta\,\omega_n = 2.5$ and $T_s = 4/2.5 < 2$. For $\zeta = 0.5$, $\omega_n = 2.5/0.5 = 5$ rad/s. From the symmetry of the root locus, the gain K at the desired closed-loop pole is equal to the square of the distance from the location to the origin i.e. $K = 5^2 = 25$. The same results can be obtained using a CAD tool or analytically, since the compensated system is simple.

The design yields a second order system and the remaining design criteria can be computed analytically as follows

$$PO = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}\,\% = 16.3\%$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{5\sqrt{1-(.5)^2}} = 0.726s$$

$$K_v = sC(s)G(s)\big|_{s\to 0} = \left.\frac{25}{s+5}\right|_{s\to 0} = 5 \Rightarrow e(\infty)\% = \frac{100}{K_v} = 20\%$$

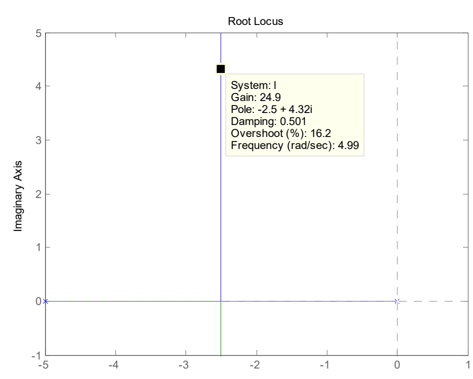


Figure 5-9 Root locus for problem 5.7

5.8 Repeat Problem 5.7 with a required settling time less than 0.5 s and an undamped natural frequency of 10 rad/s.

The plant transfer function is $G(s) = \dfrac{1}{(s+1)(s+5)}$

For zero steady-state error due to step, a PI controller is needed but the transient response specifications can not be met with pole-zero cancellation. For proportional control, the root locus includes a vertical line at -3 which yields a settling time of $4/3 > 0.5$ s. Hence, PI control cannot meet the design specifications and PID control is needed.

We begin the design by selecting a PD controller using a CAD tool. We choose ζ=0.8 and ω_n = 10 rad/s which give a settling time of 0.5 s. We obtain the values K= 10 and z = 9.5. Next, we add a PI controller with a zero at $\zeta\,\omega_n/10 = 0.8$. The PID controller is now of the form

$$C(s) = K\frac{(s+0.8)(s+9.5)}{s}$$

Using a CAD tool, we obtain a gain K = 11.8 for ζ = 0.8. The corresponding time response is good with PO of 6%, a time to first peak of 0.27 s, and a settling time of about 0.44 s.

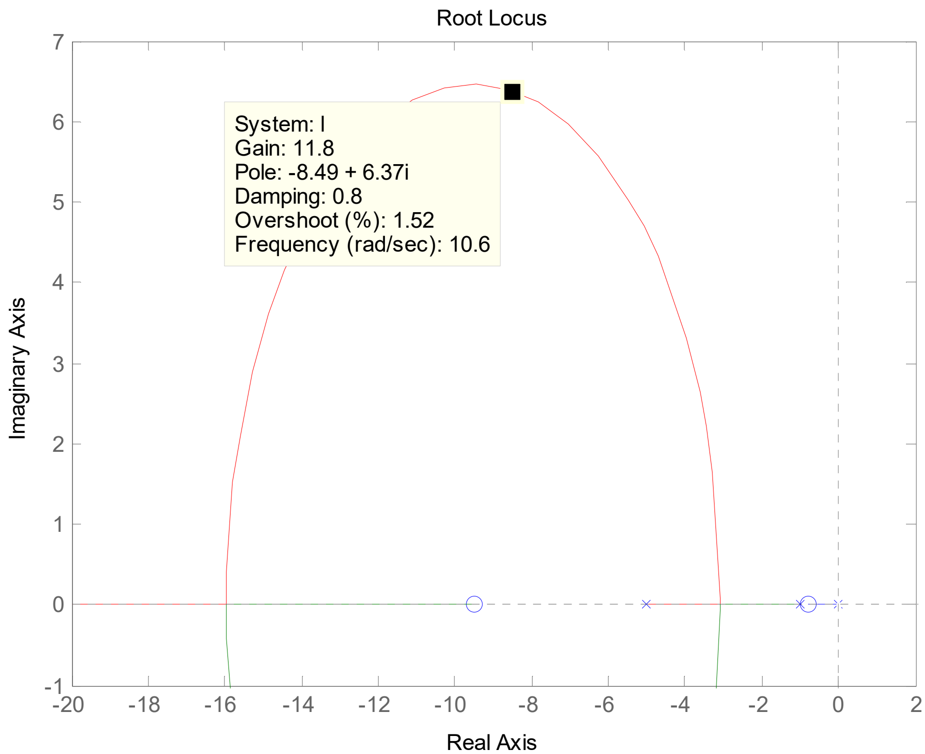


Figure 5-10 Root locus for the PID controlled system of Problem 5.8

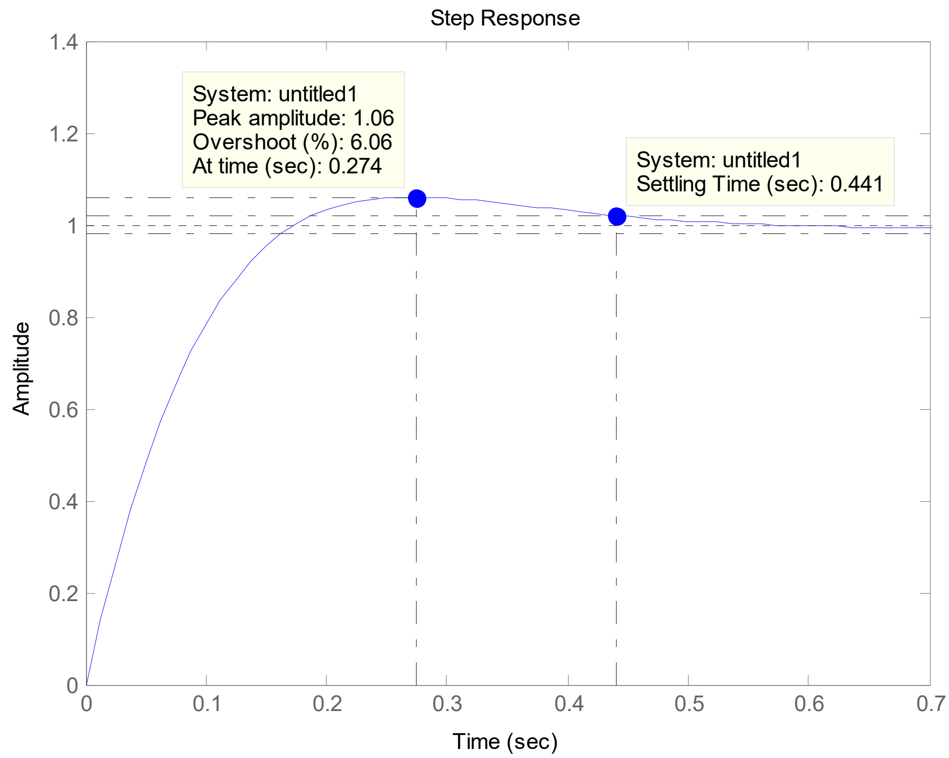


Figure 5-11 Time response for the PID controlled system of Problem 5.8

5.9 Consider the oven temperature control system of Example 3.5 with transfer function

$$G(s) = \frac{K}{s^2 + 3s + 1}$$

a) Design a proportional controller for the system to obtain a percentage overshoot less than 5 %

The percentage overshoot specification yields

$$\zeta \geq \frac{|\ln(.05)|}{\sqrt{|\ln(.05)|^2 + \pi^2}} = .69$$

We select a damping ratio of 0.7. We can solve the problem analytically since the system is second order. It is more convenient to use MATLAB to obtain the gain value. We obtain the root locus plot of Figure 5-13 for the system and observe that for a gain of about 3.5 we have the desired damping ratio.

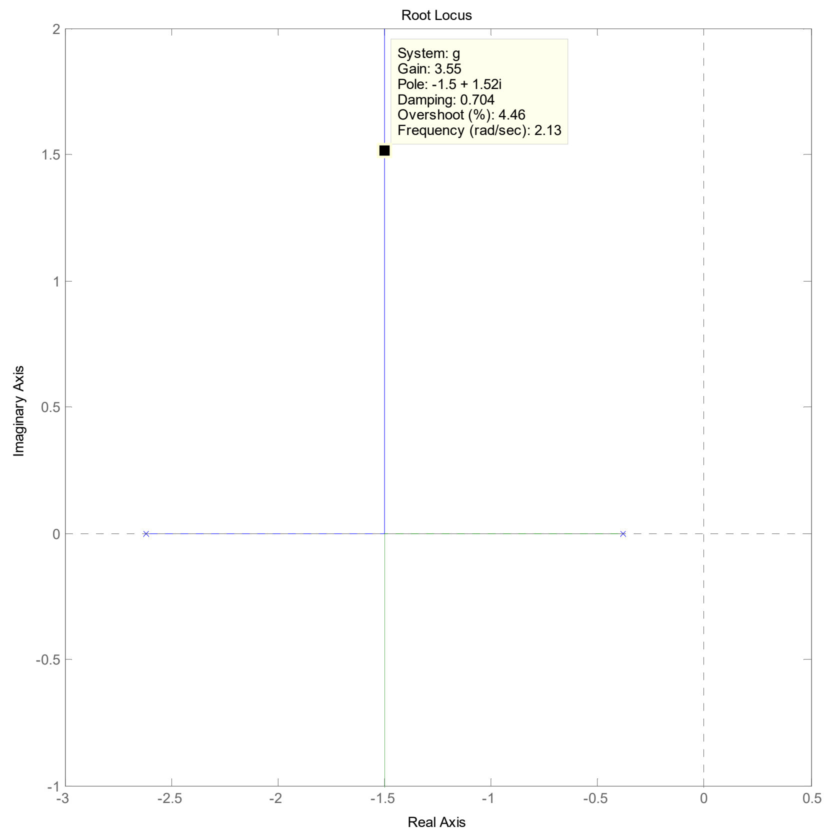


Figure 5-12 Root locus for the oven temperature control system.

b) Design a controller for the system to reduce the steady-state error due to step to zero without significant deterioration in the transient response.

We need a PI controller to reduce the steady-state error to zero. The proportional control design yields a pole with real part equal to 1.5. The PI controller zero is at one tenth this value, which is approximately 0.1 and the controller transfer function is

$$C(s) = K\frac{s+0.1}{s}$$

For a gain of 3.5, we have a dominant pair of poles with about the same damping ratio as for proportional control. However, the step response for the system is sluggish and the settling time is very large. We increase the gain to $K = 7$ and obtain the step response of Figure 5-14. Although the 2% settling time is large (over 17s), the 5 % settling time is about half this value and the response is acceptable with an overshoot less than 4 %.

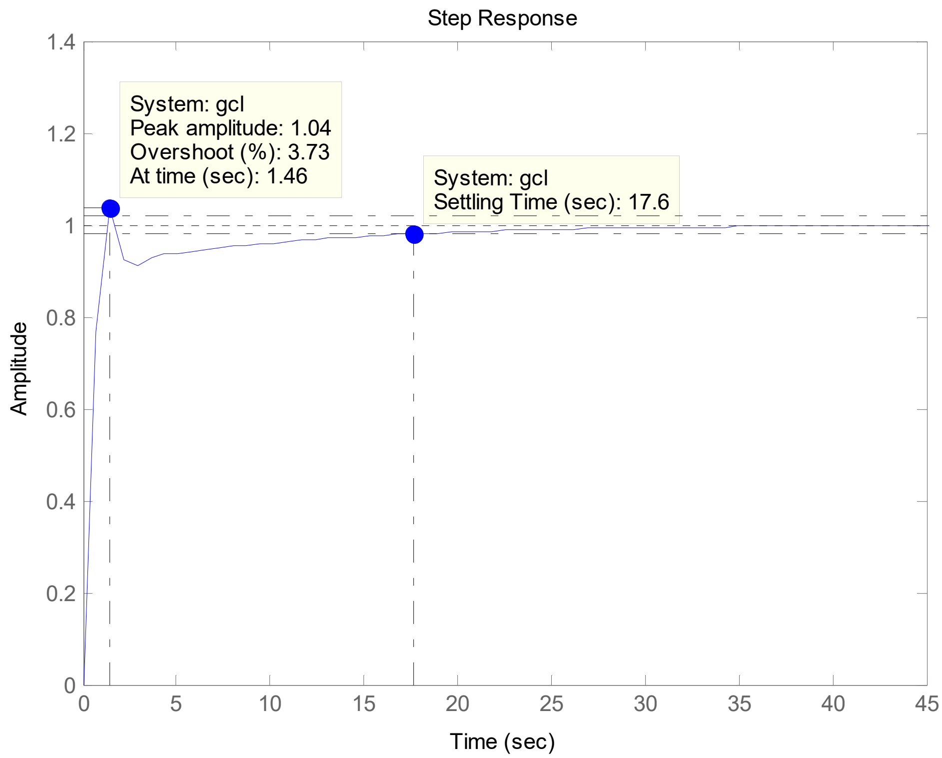


Figure 5-13 Step response for the system with PI control and a gain of 7.

5.10 For the inertial system governed by the differential equation

$$\ddot{\theta} = \tau$$

Design a feedback controller to stabilize the system and reduce the percentage overshoot below 10% with a settling time of less that 4 s.

The transfer function of the system is given by

$$G(s) = \frac{1}{s^2}$$

We use PD control to stabilize the system. The feedback control does not result in a closed-loop zero. The closed-loop system is second-order with the damping ratio constraint

$$\zeta \geq \frac{|\ln(.1)|}{\sqrt{|\ln(.1)|^2 + \pi^2}} = .591$$

We choose $\zeta = 0.6$ and calculate the settling time constraint

$$\omega_n \geq \frac{4}{T_s \zeta} = \frac{4}{4 \times 0.6} = 1.667$$

We choose $\omega_n = 2$ rad/s. The corresponding desired closed-loop pole is at

$$s_{cl} = -1.2 + j1.6$$

The angle of the compensator is

$$\theta_C = -\pi - \angle L(s_{cl}) = 1.287\, rad/s$$

The compensator zero is calculate from

$$a = \frac{\omega_d}{\tan(\theta_C)} + \zeta\omega_n = 1.6667$$

The closed-loop transfer function with additional precompensator gain is

$$G(s) = \frac{4}{s^2 + 2.4s + 4}$$

For this system, analytical design is also simple since the closed-loop chacacteristic equation is

$$s^2 + K(s+a) = s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2.4s + 4$$

The step response of the system shown in Figure 5-14 meets the design specifications.

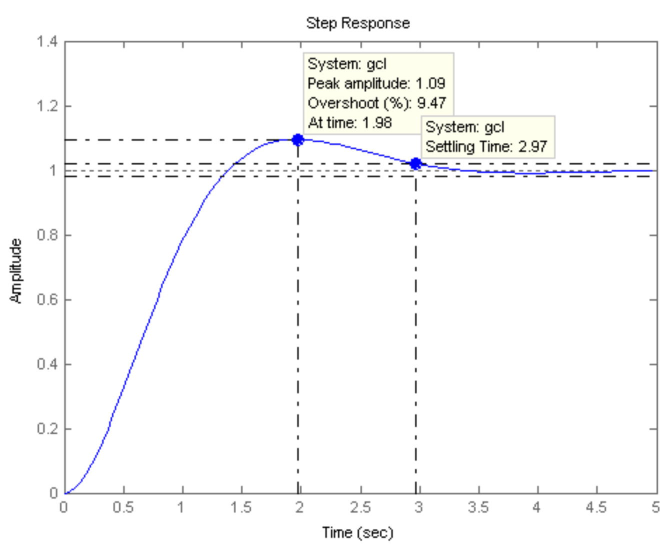


Figure 5-14 Step response for the compensated system of Problem 5.9

Computer Exercises

5.11 Consider the oven temperature control system of Example 3.5 with transfer function

$$G(s) = \frac{K}{s^2 + 3s + 10}$$

c) Obtain the step response of the system with the a PD cascade controller with gain 80 and a zero at –5.

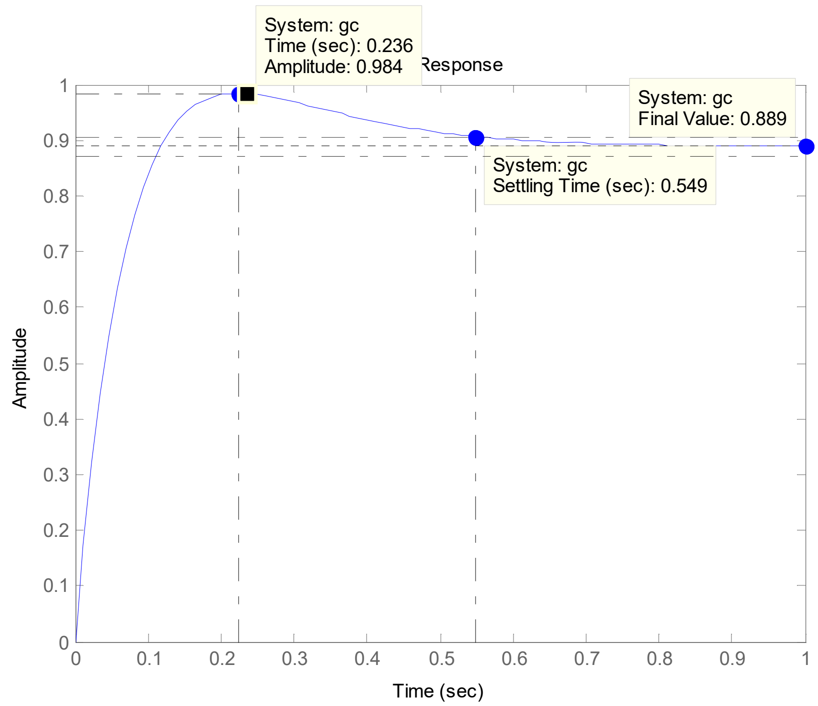


Figure 5-15 Step response of cascade compensated system.

d) Obtain the step response of the system with the a PD feedback controller with a zero at –5 and unity gain and a forward gain of 80.

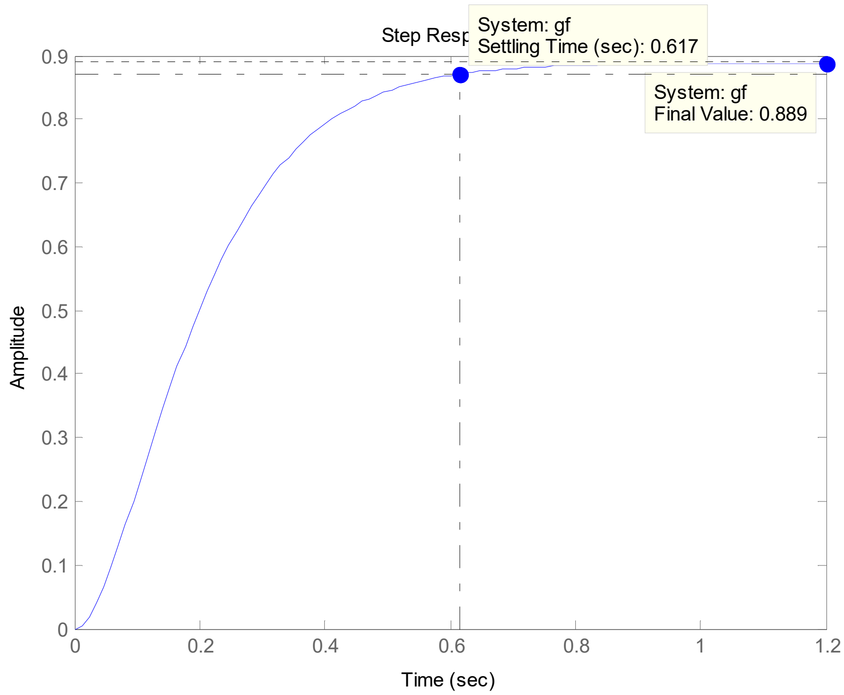


Figure 5-16 Step response of feedback compensated system.

e) Why is the root locus identical for both systems?

For both configurations, the loop gain is given by

$$L(s) = C(s)G(s) = 16\frac{s+5}{s^2+3s+10}$$

Hence, the root locus is identical for the two configurations.

f) Why are the time responses different although the systems have the same loop gains?

The time responses are different because the cascade configuration has a closed loop zero at −5 while the feedback configuration has no closed-loop zero.

g) Complete a comparison table using the responses of (a) and (b) including the percentage overshoot, the time to first peak, the settling time, and the steady-state error. Comment on the results and explain the reason for the differences in the response.

The step responses give the values shown in Table P5.1. The percentage overshoot for cascade compensation is calculated using the equation

$$\%OS = \frac{y_{peak} - y_{final}}{y_{final}} \times 100\% = \frac{0.984 - 0.889}{0.889} \times 100\% = 10.7\%$$

The system with cascade compensation has a faster response due to the closed-lop zero at −5 but the response is more oscillatory. The two systems have the same steady-state error of 11%.

Table P5.1 Summary of simulation results for cascade and feedback compensation.

Configuration	%OS	T_p	T_s	$e(\infty)$%
Cascade	10.7%	0.236	0.549	11%
Feedback	No overshoot	Not defined	0.617	11%

5.12 Use SIMULINK to examine a practical implementation of the cascade controller of Problem 5.11. The compensator transfer function includes a pole since PD control is only approximately realizable. The controller transfer is of the form

$$C(s) = 80\frac{0.2s+1}{0.02s+1}$$

(a) Simulate the system with a step reference input both with and without a saturation block with saturation limits ±5 between the controller and plant. Export the output to MATLAB for plotting (you can use a Scope block and select "Save data to workspace").

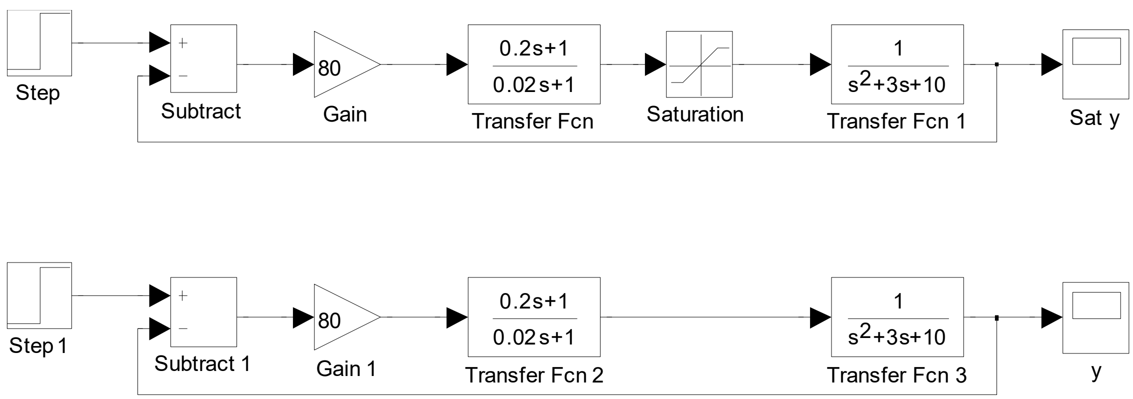


Figure 5.17 Simulation diagram for the cascade control system with and without saturation.

(b) Plot the output of the system with and without saturation together and comment on the difference between the two step responses.

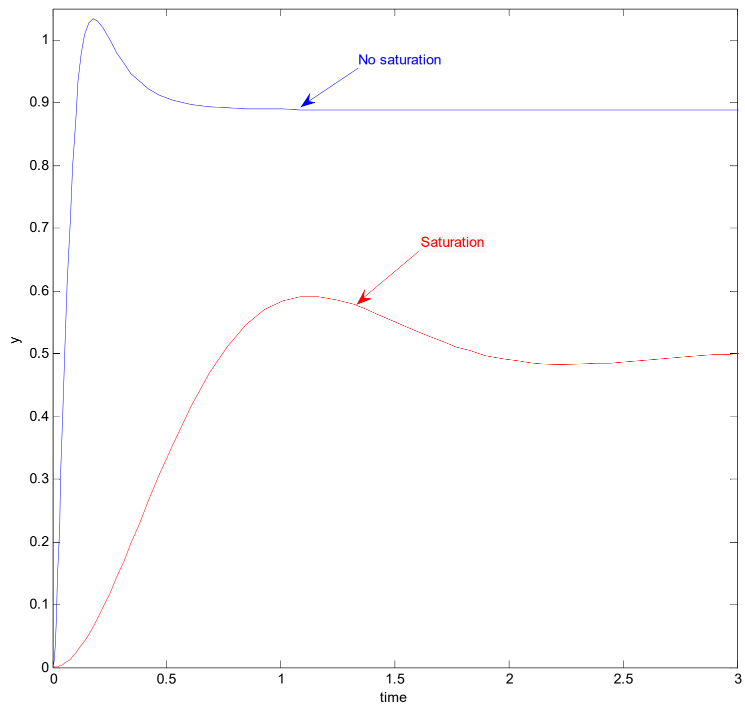


Figure 5.18 Step response of the cascade control system with and without saturation.

The saturation block reduces the output of the controller resulting in a slower response with a smaller peak and a much smaller steady-state value.

5.13 Consider the system

$$G(s) = \frac{1}{(s+1)^4}$$

and apply the Ziegler-Nichols procedure to design a PID controller. Obtain the response due to a unit step input as well as a unit step disturbance signal.

The application of an open-loop unit step input gives the response shown in Figure 5-19. It can be seen that K=1 and L=1.3. Since the value of $\tau+L$ is the time interval between the application of the step input and the time when the process output reaches 63.2% of its final value, we have τ=3. The Ziegler-Nichols rules given in Table 5.1 provide the following PID parameters: K_p=2.77, T_i=2.6, and T_d=0.65. The resulting process output when a step is applied to the set-point signal at time t=0 and a load disturbance signal is applied at time t=35 is plotted in Figure 5-20.

Conversely, if we compute the sum $\tau+L$ as the time interval between the application of the step input and the intersection of the tangent line with the straight representing the final steady-state value of the process output, we obtain $\tau+L$=6, and therefore τ=3. Thus, we can determine the following PID parameters: K_p=4.34, T_i=2.6, and T_d=0.65. The resulting process output when a step is applied to the set-point signal at time t=0 and to the load disturbance signal at time t=60 is shown in Figure 5-21. It appears that in this case the output is more oscillatory than in the previous case. This actually occurs in general because the estimate of the dominant time constant is usually higher when estimated using the intersection of the tangent line with the steady-state value of the process output. This implies that the proportional gain estimate is higher than the estimate obtained by considering the time when the process output reaches 63.2% of its final value.

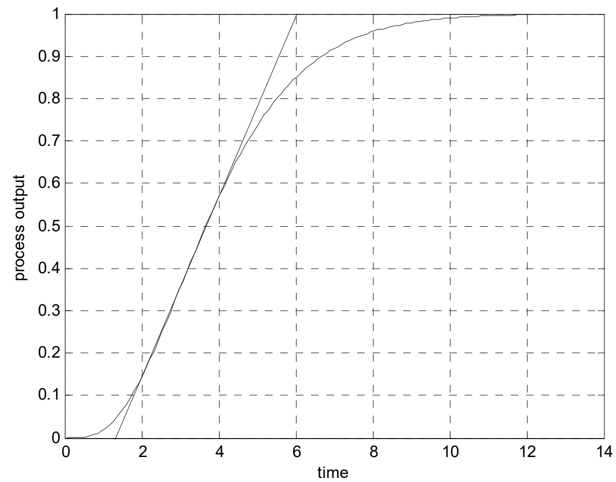


Figure 5-19 Open-loop step response

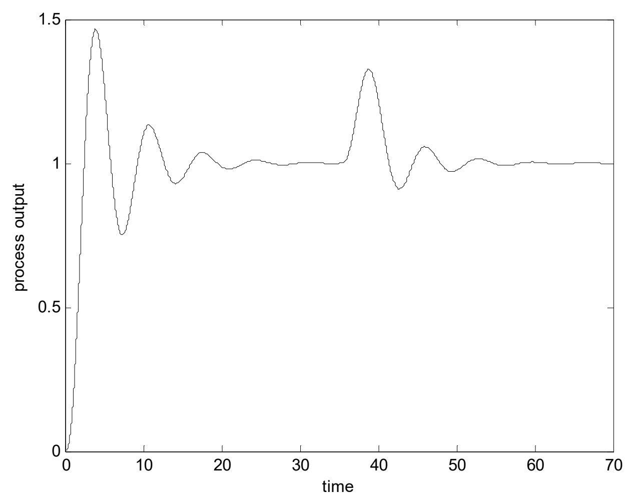


Figure 5-20 Set-point and load disturbance step response for K_p=2.77, T_i=2.6, and T_d=0.65

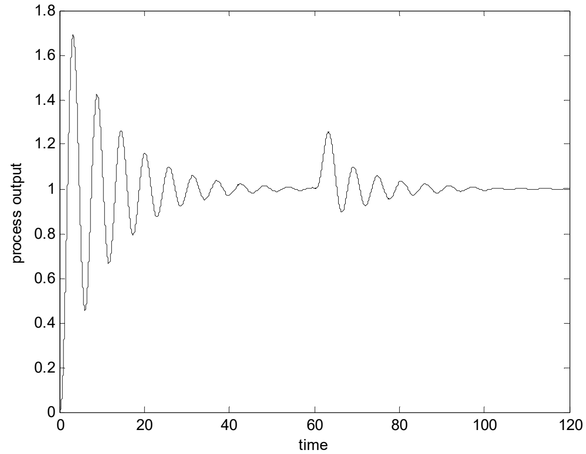


Figure 5-21 Set-point and load disturbance step response for K_p=4.34, T_i=2.6, and T_d=0.65

5.14 Write a computer program that implements the estimation of a first-order-plus-dead-time transfer function with the tangent method and then determines the PID parameters using the Ziegler-Nichols formula. Apply the program to the system

$$G(s) = \frac{1}{(s+1)^8}$$

and simulate the response of the control system when a set-point step change and a load disturbance step are applied. Discuss the choice of the time constant value based on the results.

The following Matlab function that implements the Ziegler-Nichols procedure:

```
function [Kp,Ti,Td]=zn(Gs)

[y,t]=step(Gs); % step response
tfinal=1.5*t(end); % refinement to reduce numerical
problems
t=0:tfinal/1000:tfinal;
[y,t]=step(Gs);

% tangent method
sampling_time=t(2)-t(1);
dy=diff(y);
[diff_max,max_pos]=max(dy);
t_max=t(max_pos);
dy_max=diff_max/sampling_time;
L=t_max-y(max_pos)/dy_max;
Ttop=t_max+(y(end)-y(max_pos))/dy_max;
T=Ttop-L; % the time constant is determined by considering
          % the intersection of the tangent line with the
          % straight line representing the final
          % steady-state value of the process output
```

```
% T=t(find(y>0.632*y(end),1))-L;
% use the second form of T to determine the time constant
% as the time interval between the application of the
% step input and the time when the process output attains
% the 63.2% of its final value

K=y(end);

% Ziegler-Nichols tuning
Kp=1.2*T/K/L;
Ti=2*L;
Td=0.5*L;
```

The open-loop step response with the tangent is plotted in Figure 5-22.

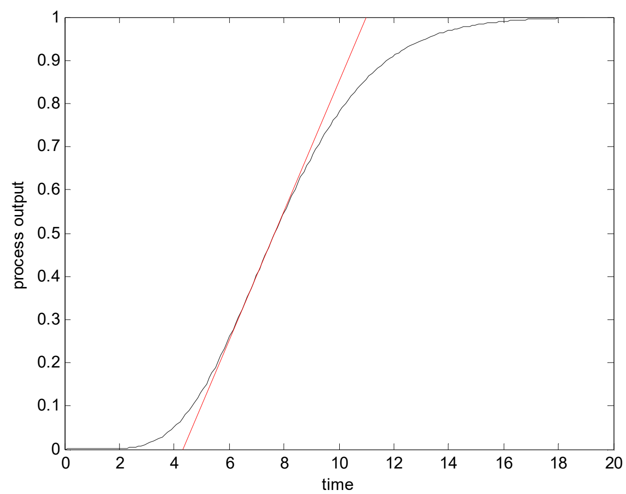


Figure 5-22Figure Open-loop step response for Problem 5.11

If we run the MATLAB function with the given process transfer function, two options are available for calculating the time constant. The first is by considering τ+L as the time interval between the application of the step input and the intersection of the tangent line with the steady-state output level. We obtain K=1, τ=6.7 and L=4.3. Thus, we have K_p=1.87, T_i=8.6, and T_d=2.15. The set-point step response is shown in Figure 5-23, while the load disturbance response is shown in Figure 5-24Figure 5-23. Both responses are highly oscillatory. This is not surprising as the system is of high-order and the approximation with a first-order-plus-dead-time model is not accurate.

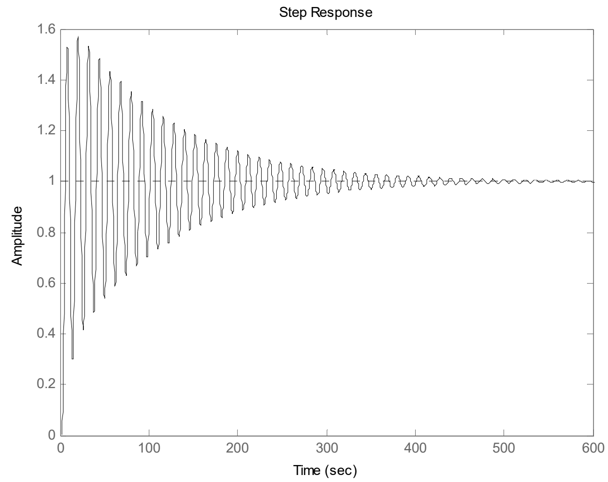


Figure 5-23 Set-point step response for Problem 5.11 with τ=6.7

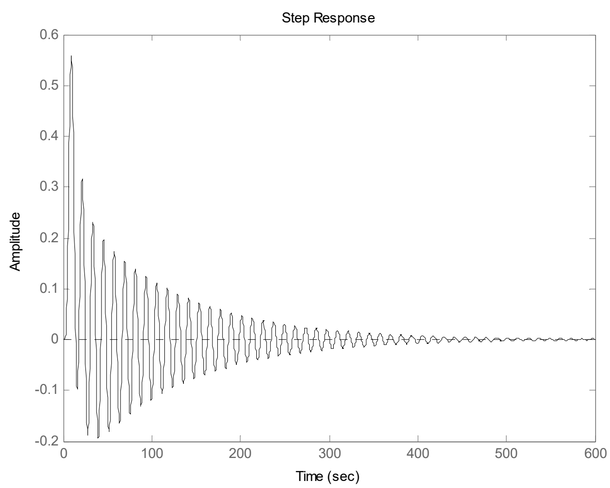


Figure 5-24 Load disturbance step response for Problem 5.11 with τ=6.7

The second approach is to take the value of $\tau+L$ as the time interval between the application of the step input and the time when the output attains 63.2% of its final value. We obtain K=1, τ=4.2 and L=4.3. This yields K_p=1.23, T_i=8.6, and T_d = 2.15. The resulting set-point step response is shown in Figure 5-25, while the load disturbance response is shown in Figure 5-26. Note that in the second case the time constant is smaller than the first and the process output is far less oscillatory.

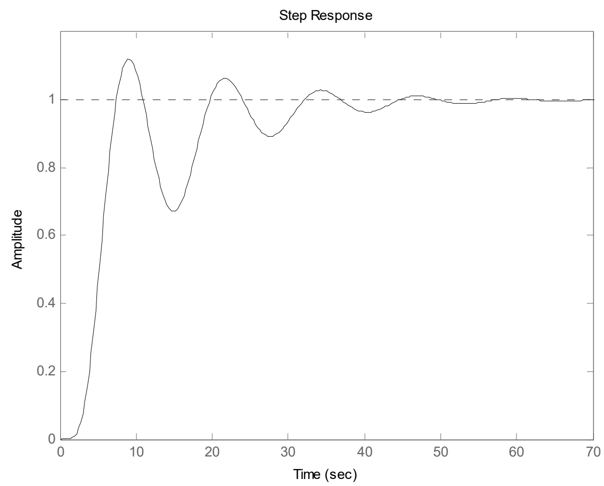


Figure 5-25 Set-point step response for Problem 5.11 with τ=4.2

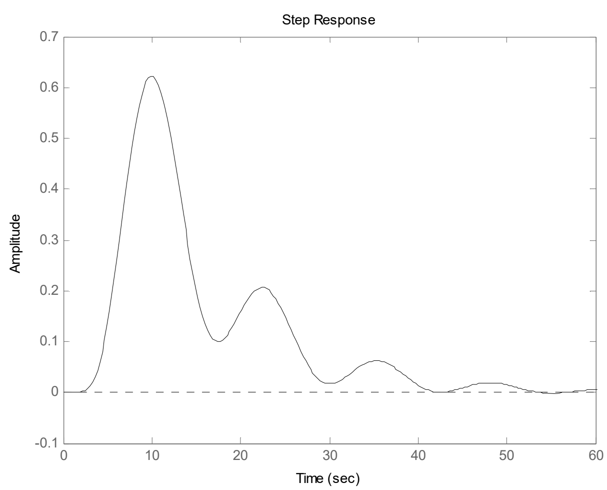


Figure 5-26 Load disturbance step response for Problem 5.11 with τ=4.2

5.15 Apply the script of Problem 5.14 to the system

$$G(s) = \frac{1}{(s+1)^2}$$

and simulate the response of the control system when a set-point step change and a load disturbance step are applied. Compare the result obtained with those of Problem 5.14.

The application of the script of Problem 5.14, where we take the value of $\tau+L$ as the time interval between the application of the step input and the time when the output attains 63.2% of its final value, yields K=1, τ=1.94 and L=0.28. This yields K_p=8.30, T_i=0.56, and T_d= 0.14. The resulting set-point step response is shown in Figure 5-27, while the load disturbance response is shown in Figure 5-28. It can be seen that, while the response is still oscillatory, as it is typical of the Zeigler-Nichols tuning rules, the settling time is much less than that obtained in Problem 5.14. In fact, being the process of second order instead of eighth order, the approximating first order model has a smaller time constant and a smaller ratio L/T. Similar considerations holds for the case where

$\tau+L$ is considered as the time interval between the application of the step input and the intersection of the tangent line with the steady-state output level.

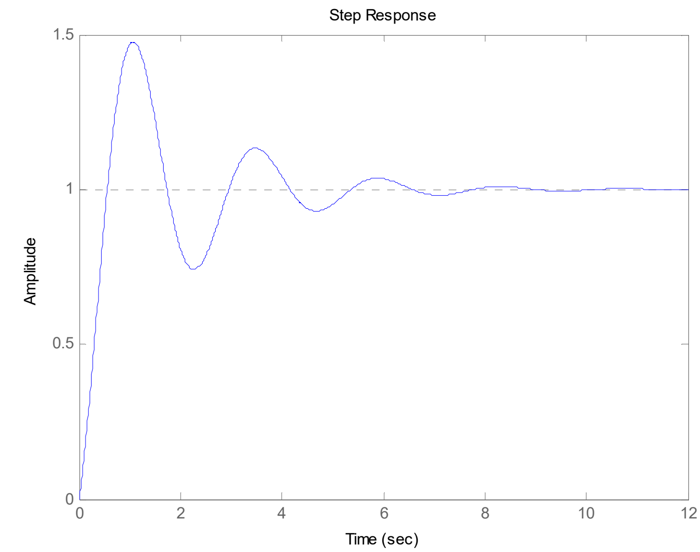


Figure 5-27 Set-point step response for Problem 5.13

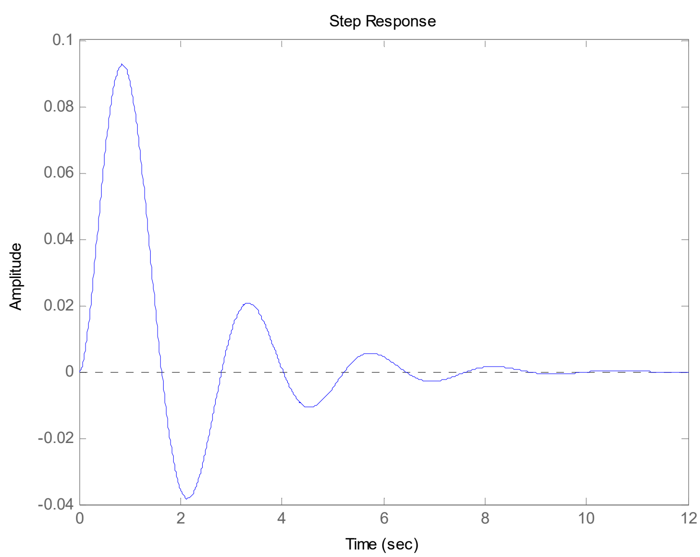


Figure 5-28 Load disturbance step response for Problem 5.13.

Chapter 6 Solutions

6.1 Sketch the z-domain root locus and find the critical gain for the following systems

(i) $G(z) = \dfrac{K}{z-0.4}$ (ii) $G(z) = \dfrac{K}{(z+0.9)(z-0.9)}$

(iii) $G(z) = \dfrac{Kz}{(z-0.2)(z-1)}$ (iv) $G(z) = \dfrac{K(z+0.9)}{(z-0.2)(z-0.8)}$

To sketch the z-domain root locus by hand, we apply the root locus rules. The plots can more easily be obtained using the MATLAB command **rlocus**.

(i) For $G(z) = \dfrac{K}{z-0.4}$, the closed-loop characteristic equation is $K + z - 0.4 = 0$

At $z = -1$, we obtain the critical gain, $K_{cr} = 1.4$ (see also the root locus plot)

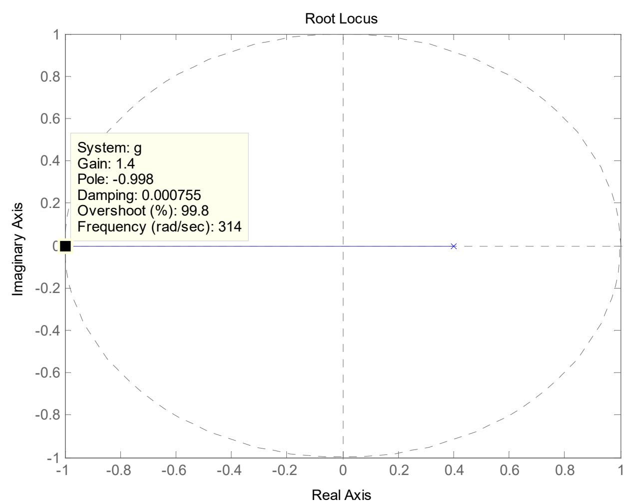


Figure 6-1 Root locus for Problem 6.1(i)

(ii) For $G(z) = \dfrac{K}{(z+0.9)(z-0.9)}$, the closed-loop characteristic equation is $K + z^2 - 0.81 = 0$

On the unit circle, $z^2 = -1$ and we obtain $K_{cr} = 1.81$

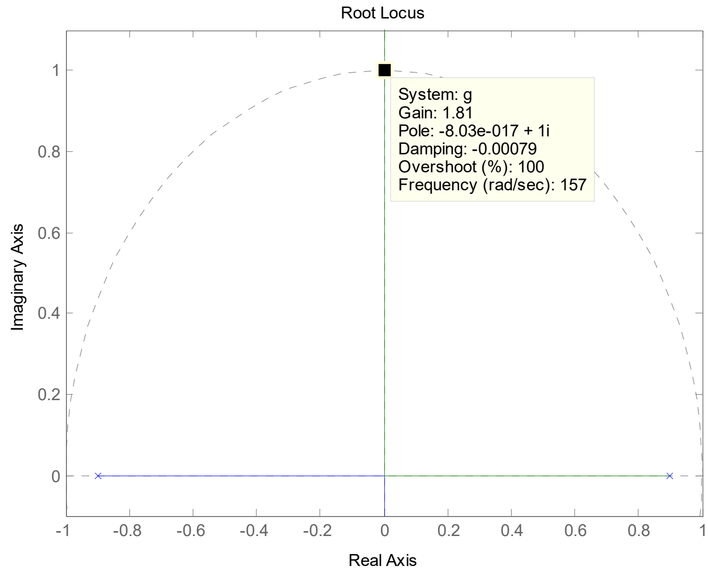


Figure 6-2 Root locus for Problem 6.1(ii)

(iii) For $G(z) = \dfrac{Kz}{(z-0.2)(z-1)}$, the closed-loop characteristic equation is

$K\,z + z^2 - 1.2\,z + 0.2 = 0$. On the unit circle, $z = -1, -K + 1 + 1.2 + 0.2 = 0$ and the critical gain is

$K_{cr} = 2.4$

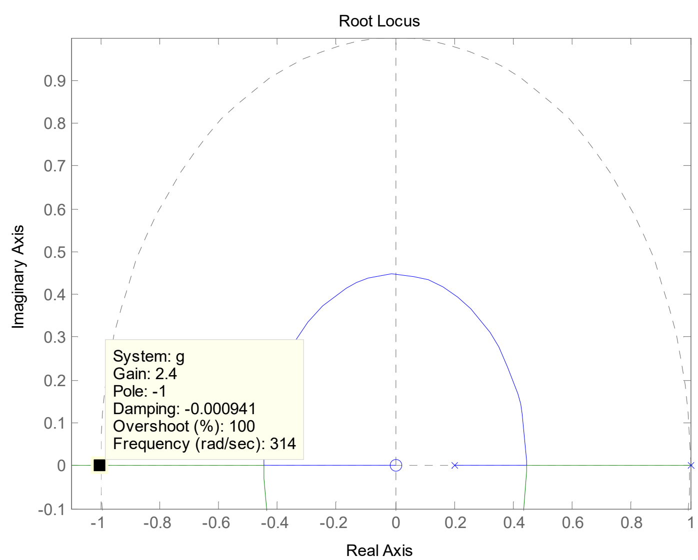


Figure 6-3 Root locus for Problem 6.1(iii)

(iv) For $G(z) = \dfrac{K(z+0.9)}{(z-0.2)(z-0.8)}$, the closed-loop characteristic equation is

$z^2 - z + 0.16 + K\,z + 0.9\,K = 0$. On the unit circle, $|z|^2 = 1 = 0.9\,K + 0.16$ and we obtain $K_{cr} = 0.933$

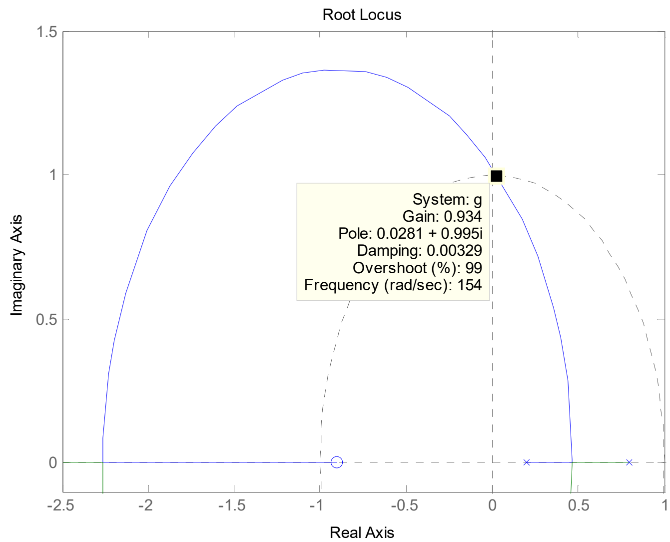


Figure 6-4 Root locus for Problem 6.1(iv)

6.2 Prove that expression (6.6) describes a constant ω_n contour in the z-plane.

It is required to prove that constant ω_n contours are defined by the equation $|z| = e^{-\sqrt{(\omega_n T)^2 - \theta^2}}$

We first write $z_{1,2} = e^{-\zeta\omega_n T}\angle \pm \omega_d T = |z|\angle \pm \theta$, i.e. we have an angle equation

$\theta = \omega_d T = \omega_n T\sqrt{1-\zeta^2}$ and a magnitude equation $|z| = e^{-\zeta\omega_n T}$. We solve the angle equation for ζ and obtain

$$\zeta = \sqrt{1 - \left(\frac{\theta}{\omega_n T}\right)^2} \quad \text{and} \quad \zeta\omega_n T = \sqrt{(\omega_n T)^2 - \theta^2}$$

Substituting in the magnitude equation we obtain the result.

6.3 Hold equivalence is a digital filter design approach which approximates an analog filter using

$$C(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left\{ \mathcal{L}^{-1}\left[\frac{C_a(s)}{s}\right]^* \right\}$$

(a) Obtain the hold equivalent digital filter for the PD, PI, and PID controllers. Modify the results as necessary to obtain a realizable filter with finite frequency response at the folding frequency.

PD: $$C(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left\{ \mathcal{L}^{-1}\left[K\frac{s+a}{s}\right]^* \right\}$$

PI: $$C(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left\{ \mathcal{L}^{-1}\left[K\frac{s+a}{s^2} \right]^* \right\}$$

PID: $$C(z) = \left(\frac{z-1}{z}\right) \mathcal{Z}\left\{ \mathcal{L}^{-1}\left[K\frac{(s+a)(s+b)}{s^2} \right]^* \right\}$$

Use symbolic manipulation to simplify the above expressions.

(b) Why are the filters obtained using hold equivalence always stable?

The s-domain poles p_s are mapped to z-domain poles p_z using the rule

$$p_z = e^{p_s T}$$

where T is the sampling period. This maps LHP poles p_s to poles p_z inside the unit circle.

6.4 Show that the bilinear transformation of the PID controller expression (5.20) yields expression (6.32).

Substituting for s in the filter transfer function gives

The transfer function considered for the PID controller is

$$C(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

If we substitute $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$ we trivially have $C(z) = K_p\left(1 + \frac{1}{T_i}\frac{T}{2}\frac{z+1}{z-1} + T_d\frac{2}{T}\frac{z-1}{z+1}\right)$

6.5 Design proportional controllers for the systems of Problem 6.1 to meet the following specifications where possible. If the design specification cannot be met, explain why and suggest a more appropriate controller.

(a) A damping ratio of 0.7.
(b) A steady-state error of 10 % due to a unit step.
(c) A steady-state error of 10 % due to a unit ramp.

(i) $G(z) = \dfrac{K}{z - 0.4}$

(a) A damping ratio of 0.7
This is a first order system and does not have an oscillatory response. Hence, the design specification cannot be met with proportional control.

A PI controller would give a system with the desired damping ratio but it is probably preferable to use proportional control to obtain a good time response with the appropriate time constant unless the steady-state error is required to be zero.

(b) A steady-state error of 10 % due to a unit step
The position error constant is

$$K_p = G(1) = \frac{K}{1-0.4} = \frac{100}{e(\infty)\%} - 1 = 9$$

Hence, the gain for 10% error due to step is $K = 5.4$. Unfortunately, the critical gain for the system is 1.4 and a gain of 5.4 would make the system unstable and the design specifications cannot be met with proportional control.

Using a PD controller would allow us to increase the gain without causing instability. Alternatively, we could use a PI controller to reduce the error due to a step to zero, which would meet the desired specifications.

(c) A steady-state error of 10 % due to a unit ramp.
The system is type 0 and cannot track a ramp input. Hence, the design specification cannot be met with proportional control.

A PI controller is needed for the system to become type 1 and have a finite steady-state error due to ramp.

(ii) $G(z) = \dfrac{K}{(z+0.9)(z-0.9)}$

(a) A damping ratio of 0.7

The closed-loop characteristic polynomial is
$z^2 - 0.81 + K = z^2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T}z + e^{-2\zeta\omega_n T}$

For a gain K>0.81, we have imaginary poles so that the angle of the pole is
$\omega_d T = \omega_n T\sqrt{1-\zeta^2} = \omega_n T\sqrt{0.51} = \pi/2$

The square magnitude of the poles is
$K - 0.81 = e^{-2\zeta\omega_n T} = e^{-1.4\times 2.1996} = 0.04599$
The required gain is
$K = 0.85599$

Note that we only need the value of the product $\omega_n T$ to solve the problem rather than each of the two values. Hence, the result can be verified with MATLAB using a unity sampling period and the command rlocus.

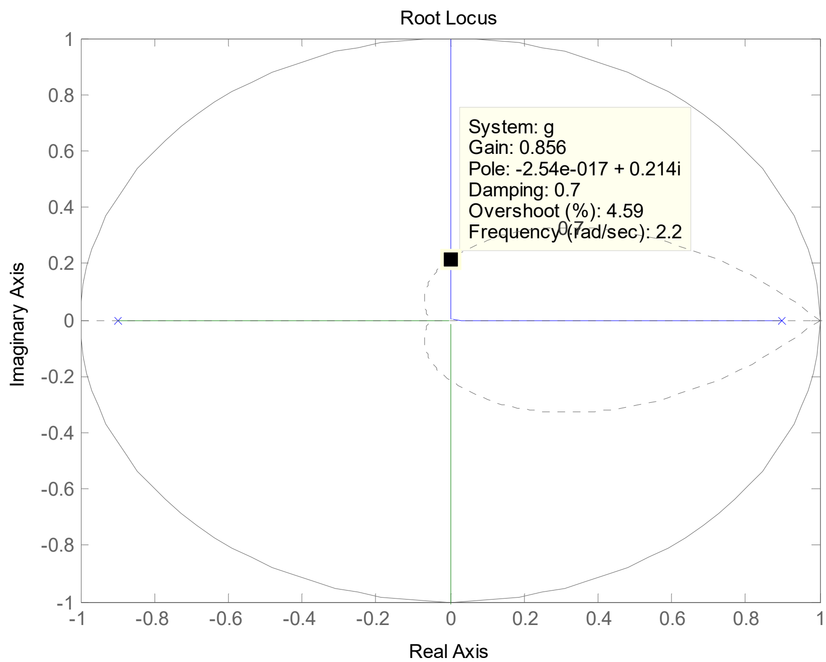


Figure 6.5 Root locus for Problem 6.5 (ii) and the gain for a damping ratio of 0.7.

(b) A steady-state error of 10 % due to a unit step.

The steady-state error is given by

$$e(\infty)\% = \frac{100}{1+K_p} = 10$$

The error constant for the system is

$$K_p = 10 - 1 = 9$$

$$= G(1) = \frac{K}{(1+0.9)(1-0.9)}$$

We solve for the gain

$$K = 9 \times 1.9 \times 0.1 = 1.71$$

The specification can be met because the gain is less than the critical gain.

(c) A steady-state error of 10 % due to a unit ramp.
The system is type 0 and cannot track a ramp input. Hence, the design specification cannot be met with proportional control.

A PI controller is needed for the system to become type 1 and have a finite steady-state error due to ramp.

(iii) $$G(z) = \frac{Kz}{(z-0.2)(z-1)}$$

(a) A damping ratio of 0.7

The closed-loop characteristic polynomial is

$$z^2 + (K - 1.2)z + 0.2 = z^2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T} z + e^{-2\zeta\omega_n T}$$

Equating coefficients, we have

$$e^{-\zeta\omega_n T} = \sqrt{0.2} = 0.4472$$

$$\omega_n T = -\ln\left(\sqrt{0.2}\right)/0.7 = 1.1496$$

$$\cos(\omega_d T) = \cos\left(\omega_n T\sqrt{1-\zeta^2}\right) = \cos\left(1.1496\sqrt{0.51}\right) = 0.6815$$

$$K - 1.2 = -2\cos(\omega_d T)e^{-\zeta\omega_n T} = -2 \times 0.6815 \times 0.4472 = -0.6096$$

$$K = 1.2 - 0.6096 = 0.5904$$

We obtain approximately the same answer using the MALAB command rlocus.

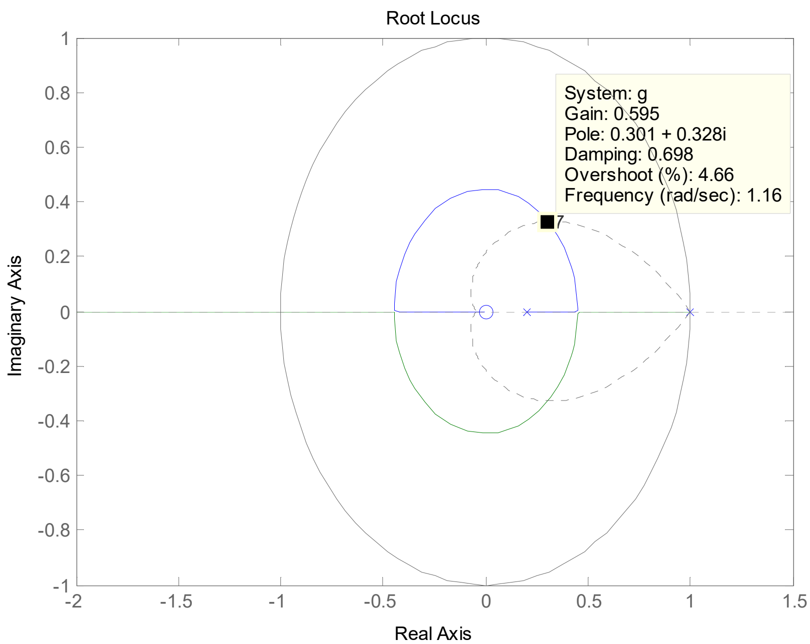


Figure 6.6 Root locus for Problem 6.5 (iii) and the gain for a damping ratio of 0.7.

(b) A steady-state error of 10 % due to a unit step.

The steady-state error is zero since the system is type I and the specification can be met for any stable gain. The gain can be selected as in Part (a) to obtain a satisfactory transient response.

(c) A steady-state error of 10 % due to a unit ramp.
The system is type 1 and can track a ramp input. The velocity error constant is

$$K_v = \frac{1}{T}(z-1)G(z)\Big|_{z=1} = \frac{1}{T}\frac{K}{(1-0.2)} = 10$$

$$K = 8T$$

The sampling period is typically sufficiently small to result in a small gain that is less than the critical value and the specifications can be met.

(iv) $$G(z) = \frac{K(z+0.9)}{(z-0.2)(z-0.8)}$$

(a) A damping ratio of 0.7

The closed-loop characteristic polynomial is

$$z^2 + (K-1)z + 0.16 + 0.9K = z^2 - 2\cos(\omega_d T)e^{-\zeta\omega_n T} z + e^{-2\zeta\omega_n T}$$

Equating coefficients, we have

$$e^{-\zeta\omega_n T} = \sqrt{0.16 + 0.9K}$$

$$\omega_n T = -\ln(0.16 + 0.9K)/1.4$$

$$\cos(\omega_d T) = \cos\left(\omega_n T\sqrt{1-\zeta^2}\right) = \cos\left(-\ln(0.16+0.9K)\sqrt{0.51}/1.4\right)$$

$$K - 1 = -2\cos(\omega_d T)e^{-\zeta\omega_n T} = -2\cos\left(-\ln(0.16+0.9K)\sqrt{0.51}/1.4\right)\times\left(\sqrt{0.16+0.9K}\right)$$

This is a nonlinear equation in K and must be solved numerically. By trial and error, we obtain gain value K=0.14034. Approximately, the same value can be obtained using the MATLAB command rlocus.

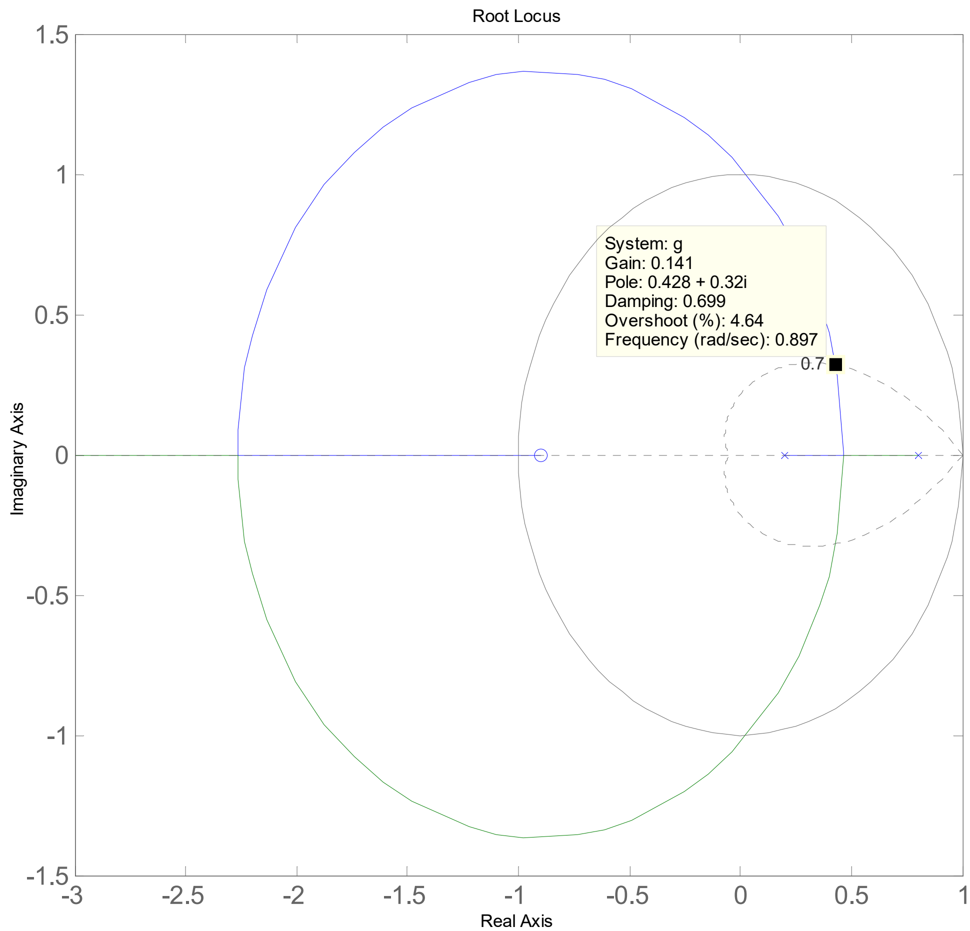


Figure 6.7 Root locus for Problem 6.5 (iv) and the gain for a damping ratio of 0.7.

(b) A steady-state error of 10 % due to a unit step.

The steady-state error is given by

$$e(\infty)\% = \frac{100}{1+K_p} = 10$$

The error constant for the system is

$$K_p = 10 - 1 = 9$$

$$= G(1) = \frac{K(1+0.9)}{(1-0.2)(1-0.8)}$$

We solve for the gain

$$K = 9 \times 0.8 \times 0.2/1.9 = 0.758$$

The specification can be met because the gain is less than the critical gain.

(c) A steady-state error of 10 % due to a unit ramp.
The system is type 0 and cannot track a ramp input. Hence, the design specification cannot be met with proportional control.

A PI controller is needed for the system to become type 1 and have a finite steady-state error due to ramp.

6.6 Design digital controllers to meet the desired specifications for the systems of Problems 5.4, 5.7 and 5.8 by bilinearly transforming the analog designs.

Problem 5.4 We first select a sampling period for the digital control system. For our analog design, the damped natural frequency is 4.37 rad/s. Using our rule of thumb for the sampling period

$$T = \frac{2\pi}{70 \times 4.37} \approx 0.02\,s$$

```
>> gd1=c2d(g,.02)
```

The corresponding z-transfer function for the plant with ADC and DAC is

```
Zero/pole/gain:
0.00018965 (z+0.9481)
---------------------
(z-0.9418) (z-0.9048)

Sampling time: 0.02
```

The analog controller from Problem 5.4 is

$$C_a(s) = 25.3\frac{s+2.96}{s}$$

Bilinearly transforming the analog controller gives

```
>> c=c2d(ca,.02,'tustin')
          26.0489 (z-0.9425)
C(z) = -----------------
            (z-1)
```

The root locus for the system is

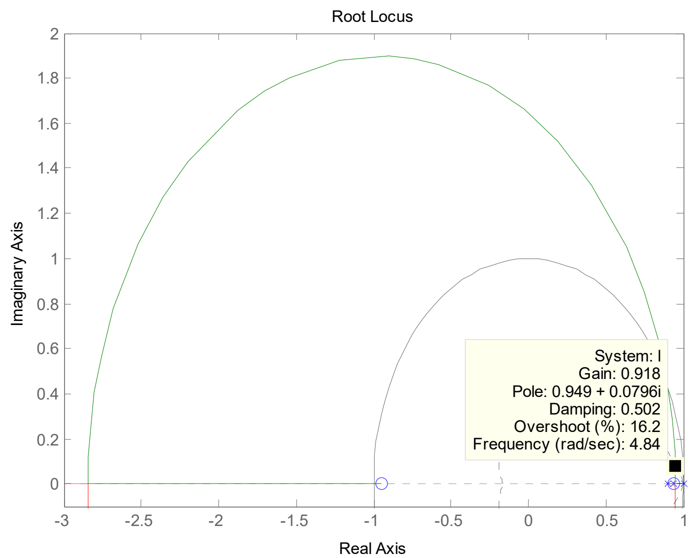


Figure 6-8 Root locus for the indirect digital design of Problem 5.4

The gain has to be reduced to 0.92 of its analog value for a damping ratio of 0.5 and the corresponding error constant is

$$K_v = \frac{1}{T}(z-1)G(z)\bigg|_{z=1} = \frac{0.0045(1-.9425)(1+.9481)}{0.02(1-.9418)(1-.9048)} = 4.594$$

This corresponds to a steady-state error of 21.8%. We have to accept a compromise between the desired steady-state error and damping ratio or repeat the analog design. The error is close to the required value and the time response meets the design requirements. The design would be acceptable in many applications.

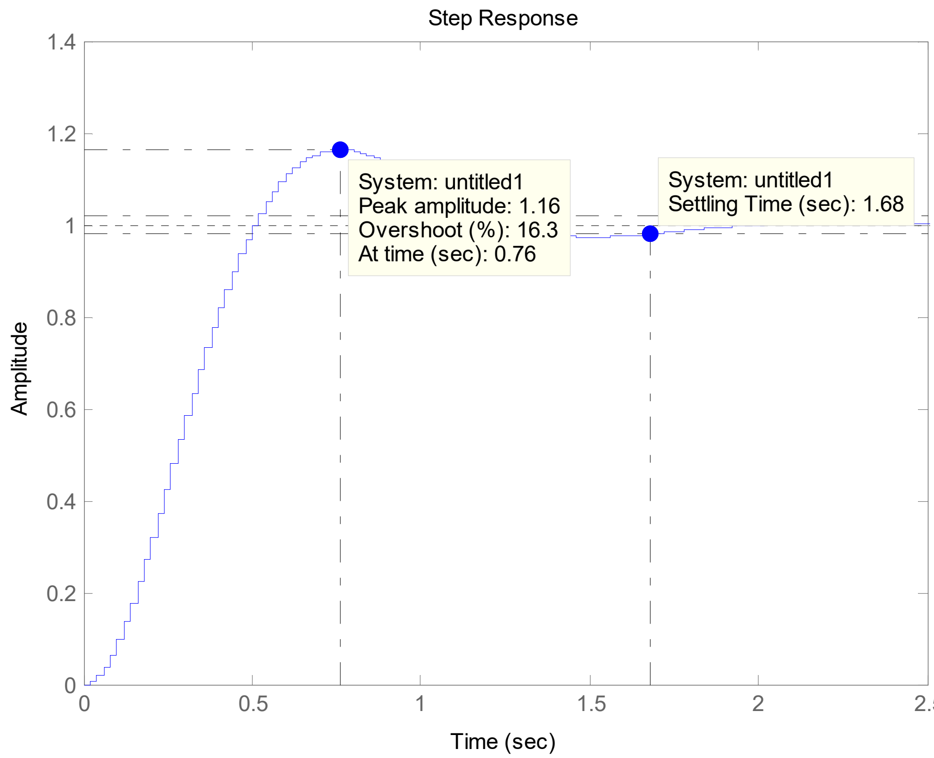


Figure 6-9 Time response for indirect digital design of Problem 5.4

Problem 5.7 The required specifications are (i) zero steady-state error due to step, (ii) a settling time of less than 2 s, and (iii) an undamped natural frequency of 5 rad/s.

From the transient response specifications, we obtain the damping ratio

$$\zeta = \frac{4}{T_s \omega_n} \geq \frac{4}{10}$$

We select a damping ratio $\zeta = 0.5$ which corresponds to an undamped natural frequency $\omega_d = 4.33$ rad/s. We calculate the sampling period as

$$T = \frac{2\pi}{70 \times \omega_d} = \frac{2\pi}{70 \times 4.33} \approx 0.02\, s$$

The corresponding z-transfer function for the plant with ADC and DAC is

```
>> gd=c2d(g,.02)

Zero/pole/gain:
0.0001922 (z+0.9608)
---------------------
(z-0.9802) (z-0.9048)

Sampling time: 0.02
```

The analog design of Problem 5.7 is

$$C_a(s) = \frac{s+1}{s}$$

Bilinearly transforming gives the controller

$$C(z) = \frac{z-0.9802}{z-1}$$

The root locus of the of Figure 6-10, shows that for the desired undamped natural frequency we need a gain of about 25. The corresponding step response of Figure 6-11shows a settling time of 1.65 and 18% overshoot, which is acceptable for the given design specifications.

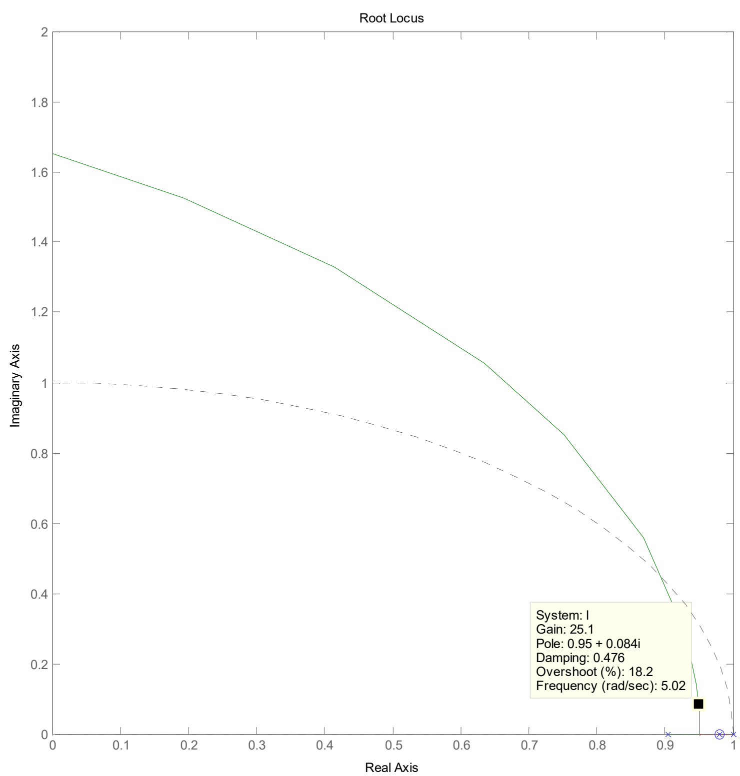


Figure 6-10 Root locus for the indirect digital design of Problem 5.7

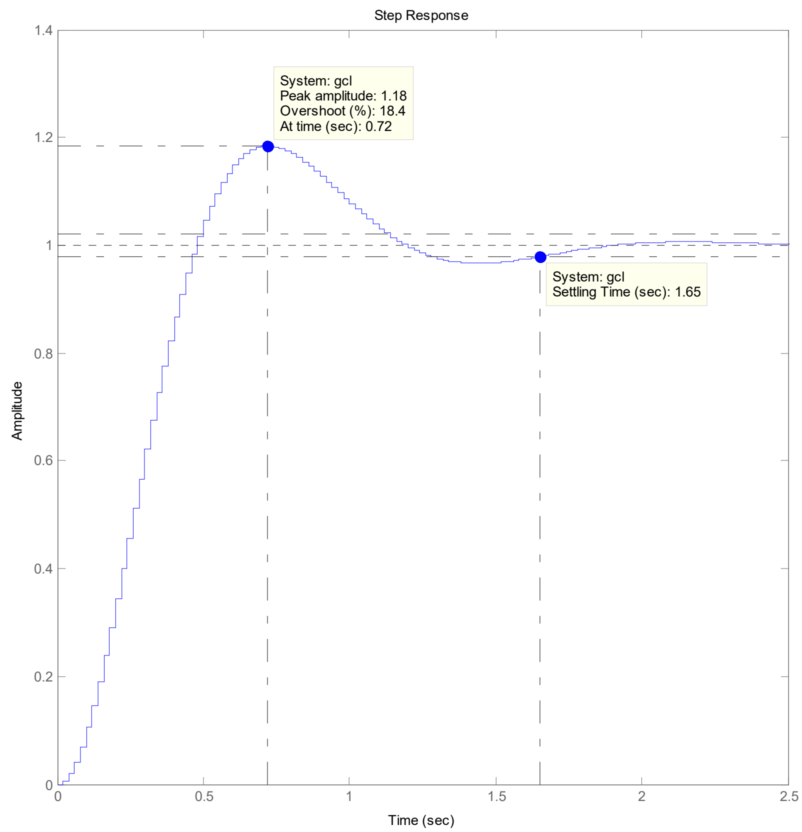


Figure 6-11 Time response for indirect digital design of Problem 5.7

Problem 5.8 The design specifications require a settling time less than 0.5 s and an undamped natural frequency of 10 rad/s. We obtain the damping ratio

$$\zeta = \frac{4}{T_s \omega_n} \geq \frac{4}{5} = 0.8$$

The damping ratio $\zeta = 0.5$ which corresponds to an undamped natural frequency $\omega_d = 6$ rad/s. We calculate the sampling period as

$$T = \frac{2\pi}{70 \times \omega_d} = \frac{2\pi}{70 \times 6} \approx 0.015\, s$$

```
>> gd=c2d(g,.015)

Zero/pole/gain:
0.00010919 (z+0.9704)
---------------------
(z-0.9851) (z-0.9277)

Sampling time: 0.015
```

The analog design of Problem 5.8 is

$$C_a(s) = K\frac{(s+0.8)(s+9.5)}{s}$$

The root locus of Figure 6-12shows that a damping ration of 0.8 corresponds to the high gain of 2200. This gain value gives the step response of Figure 6-13 with 13% overshoot, a peak time of 1.13s, and a settling time less than 0.25 s. The system has a fast response that meets the given specifications.

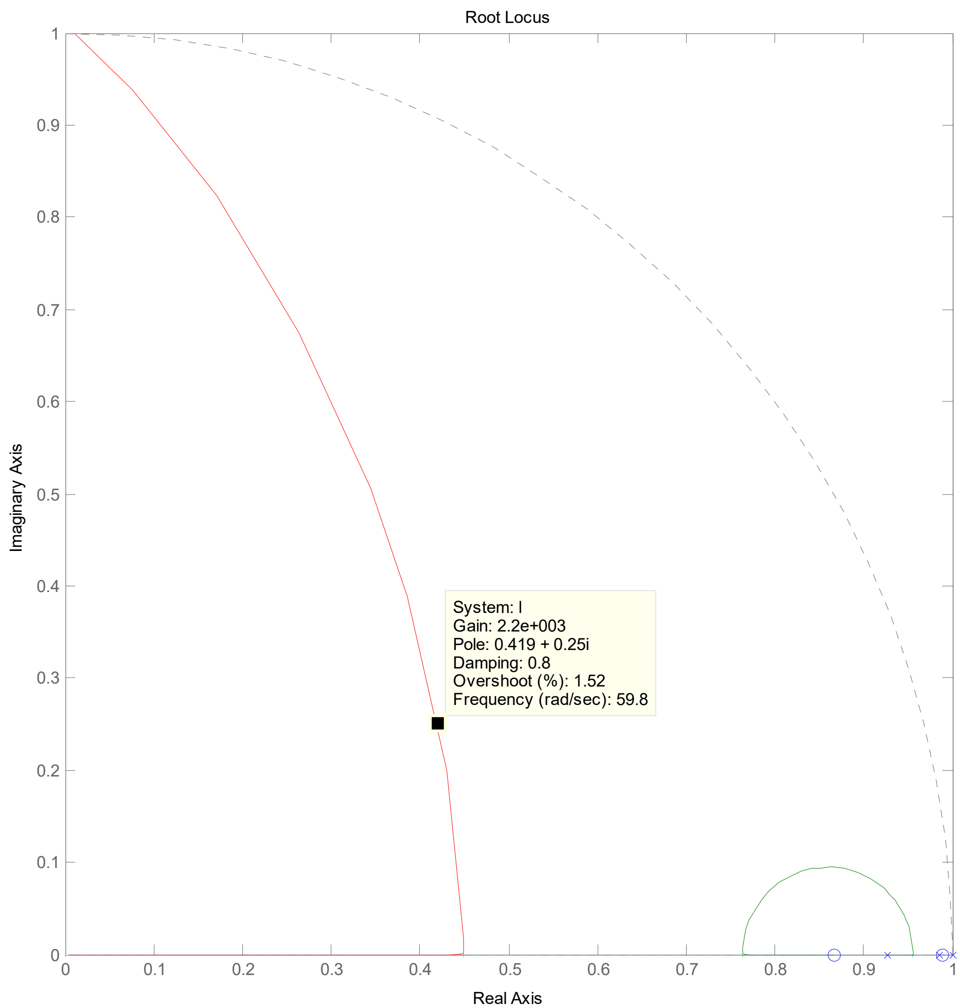


Figure 6-12 Root locus for the indirect digital design of Problem 5.8

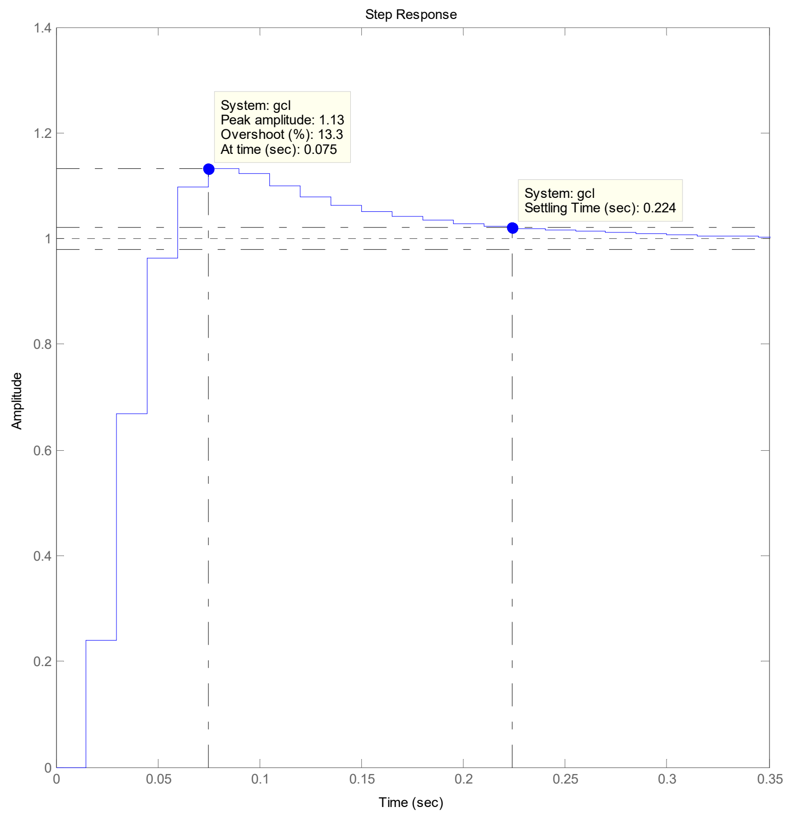


Figure 6-13 Time response for indirect digital design of Problem 5.7

6.7 Design a digital filter by applying the bilinear transformation to the analog (Butterworth) filter

$$C_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

With T=0.1 s. Apply then prewarping at the 3dB frequency.

By substituting $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$ with T=0.1 we have

$$C(z) = 10^{-3}\,\frac{2.329z^2 + 4.659z + 2.329}{z^2 - 1.859z + 0.8682}$$

The 3dB frequency is determined as ω_0=1 and therefore prewarping is applied by substituting

$$s = \frac{\omega_0}{\tan\left(\dfrac{\omega_0 T}{2}\right)}\left[\frac{z-1}{z+1}\right] = \frac{1}{\tan(0.05)}\left[\frac{z-1}{z+1}\right]$$

We obtain the z-transfer function

$$C(z) = 10^{-3}\,\frac{2.333z^2 + 4.666z + 2.333}{z^2 - 1.859 + 0.8681}$$

We observe that he difference between the transfer function with and without prewarping is very small. This is not surprising since tan(0.05) is approximately equal to 005 which gives

$$s = \frac{\omega_0}{\tan\left(\dfrac{\omega_0 T}{2}\right)}\left[\frac{z-1}{z+1}\right] \approx 20\left[\frac{z-1}{z+1}\right] = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

6.8 Design a digital PID controller (with T=0.1) for the plant

$$G(s) = \frac{1}{10s+1}e^{-5s}$$

by applying the Ziegler-Nichols tuning rules of Table 5.1.

The Ziegler-Nichols tuning rules shown in Table 5.1 can be applied by considering K=1, τ=10, L=5. We obtain K_p=2.4, T_i=10, T_d=2.5. Thus, by applying (6.32) we have

$$C(z) = 2.4\left(1 + \frac{1}{10}\frac{0.1}{2}\frac{z+1}{z-1} + 2.5\frac{2}{0.1}\frac{z-1}{z+1}\right) = \frac{122.4z^2 - 240z + 117.6}{z^2 - 1}$$

6.9 Design digital controllers to meet the desired specifications for the systems of Problems 5.4, 5.7 and 5.8 in the z-domain directly.

Problem 5.4 We first select a sampling period for the digital control system. For a plant with poles at −3 and −5 a realistic choice of the damped natural frequency with acceptable percentage overshoot would be about 7 rad/s. Using our rule of thumb for the sampling period

$$T = \frac{2\pi}{70\times 7} \approx 0.01s$$

The corresponding z-transfer function for the plant with ADC and DAC is

```
>> gd=c2d(g,.01)

Zero/pole/gain:
4.8687e-005 (z+0.9737)
----------------------
(z-0.9704) (z-0.9512)
```

The PI controller is in the form

$$C(z) = K\frac{z-a}{z-1}$$

As in the analog design of Problem 5.4, we evaluate the velocity error constant and equate it to a value of 5 corresponding to 20% overshoot.

$$
\begin{aligned}
K_v &= \frac{1}{T}(z-1)G(z)\Big|_{z=1} \\
&= \frac{4.869\times10^{-5}K(1-a)(1+.9737)}{0.01(1-.9704)(1-.9512)} = 4.17K(1-a) = 5 \\
Ka &= K - 0.75
\end{aligned}
$$

The loop gain is

$$
L(z) = \frac{4.869\times10^{-5}K(z-a)(z+.9737)}{(z-1)(z-.9704)(z-.9512)}
$$

We can substitute the above constraint into the characteristic equation for the system to eliminate the product Ka and rewrite it in the form that allows the use of the command **rlocus**

$$
\begin{aligned}
L(z) &= \frac{4.869\times10^{-5}(Kz-K+0.75)(z+.9737)}{(z-1)(z-.9704)(z-.9512)} \\
Ka &= K-0.75 \\
1+L(z) &= (z-1)(z-.9704)(z-.9512) + 4.869\times10^{-5}(0.75)(z+.9737) + K(z-1)(z+.9737) \\
0 &= 1 + K\frac{(z-1)(z+.9737)}{(z-1)(z-.9704)(z-.9512)+4.869\times10^{-5}(0.75)(z+.9737)} \\
&= 1 + K\frac{(z-1)(z+.9737)}{(z-.9283)(z^2-1.993z+0.9944)}
\end{aligned}
$$

We then select the gain value corresponding to a damping ratio of 0.5.

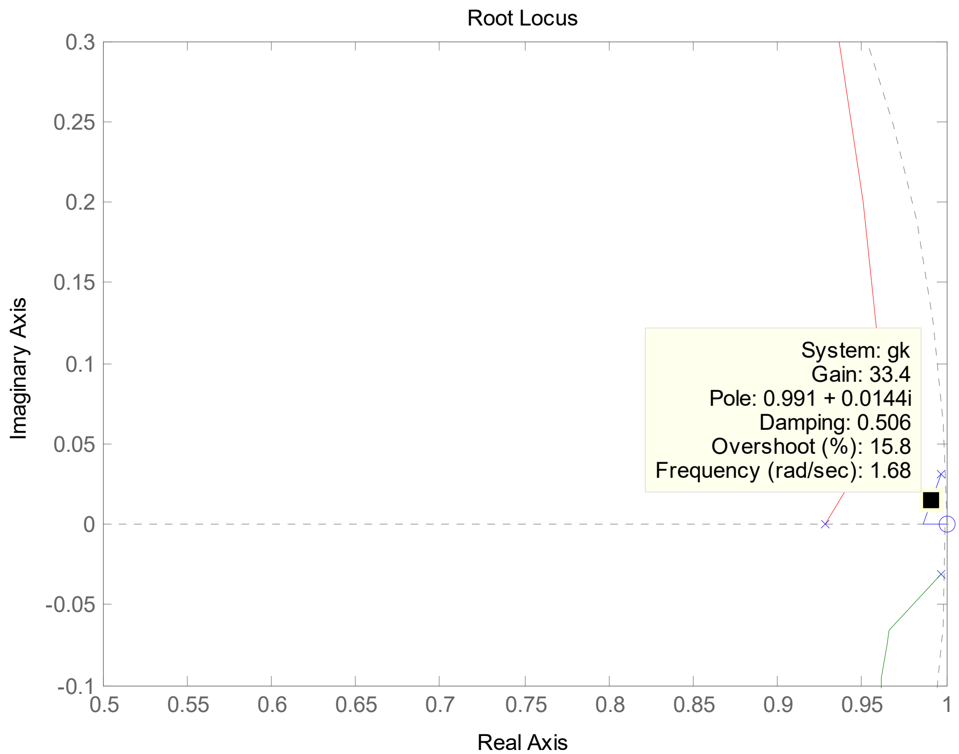


Figure 6-14 For a gain K =33.4, a = 1– 0.75/K = 0.9775

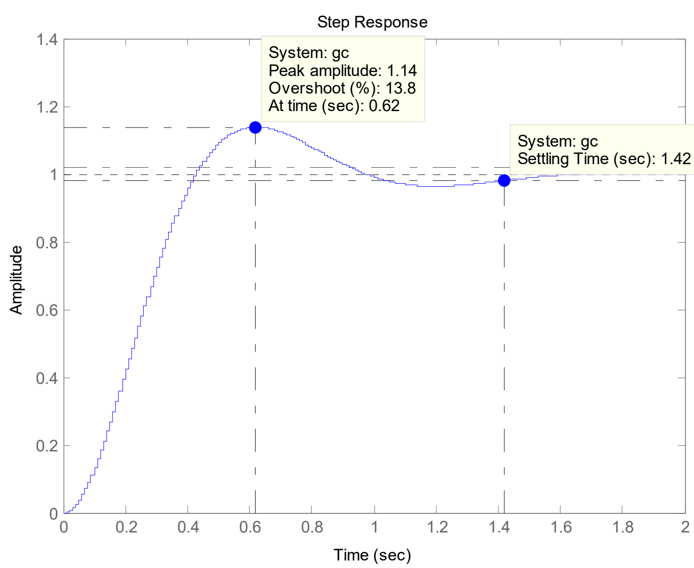


Figure 6-15 Step response for the system with PI control.

The procedure is almost identical to the analog design but no approximation of the controller is involved and the design specifications are met.

Problem 5.7 The required specifications are (i) zero steady-state error due to step, (ii) a settling time of less than 2 s, and (iii) an undamped natural frequency of 5 rad/s.

$$G(s) = \frac{1}{(s+1)(s+5)}$$

From the transient response specifications, we obtain the damping ratio

$$\zeta = \frac{4}{T_s \omega_n} \geq \frac{4}{10}$$

We select a damping ratio $\zeta = 0.5$ which corresponds to an undamped natural frequency $\omega_d = 4.33$ rad/s. We calculate the sampling period as

$$T = \frac{2\pi}{70 \times \omega_d} = \frac{2\pi}{70 \times 4.33} \approx 0.02\, s$$

The corresponding z-transfer function for the plant with ADC and DAC is

```
>> gd=c2d(g,.02)

Zero/pole/gain:
0.0001922 (z+0.9608)
---------------------
(z-0.9802) (z-0.9048)

Sampling time: 0.02
```

We need a PI controller to reduce the error due to step to zero. The controller has a pole at unity and we select its zero to cancel the closest pole to the unit circle. The corresponding root locus is shown in Figure 6-16.

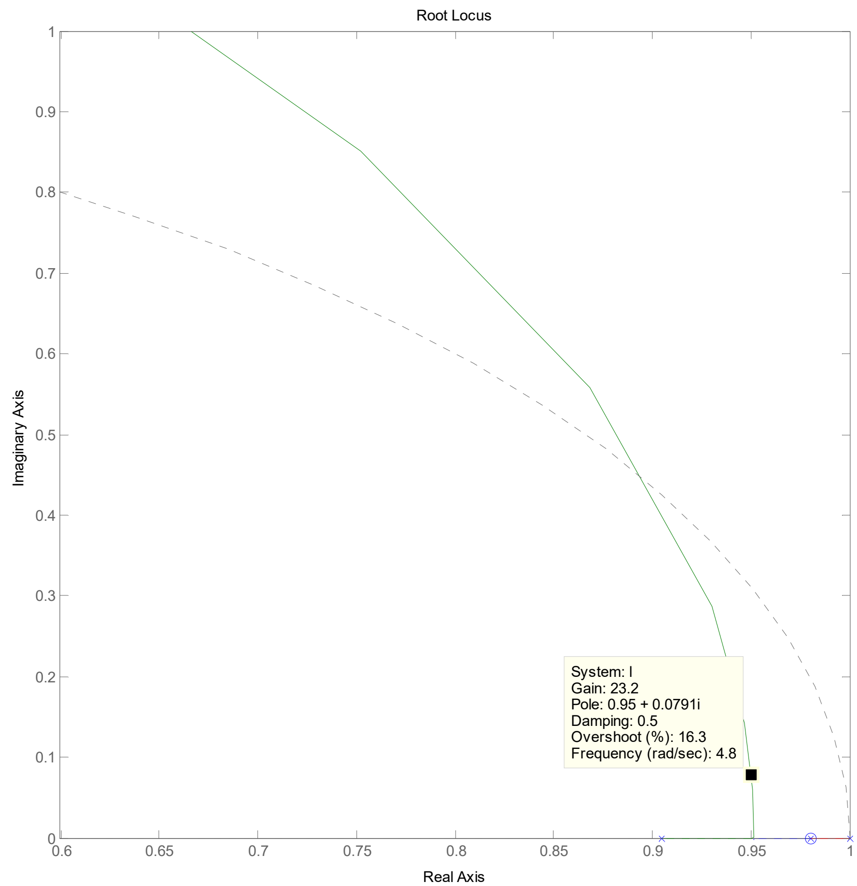


Figure 6-16 Root locus for the digital control of Problem 5.7.

The controller transfer function is

$$C(z) = 23.2\frac{z - 0.9802}{z - 1}$$

The closed-loop step response is shown in Figure 6-17. The system has a settling time of 1.69 s and 17 % overshoot. The system approximately meets the desired specifications since an underdamped second order system with a damping ratio of 0.5 has the percentage overshoot

$$PO = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}\,\% = 16.3\%$$

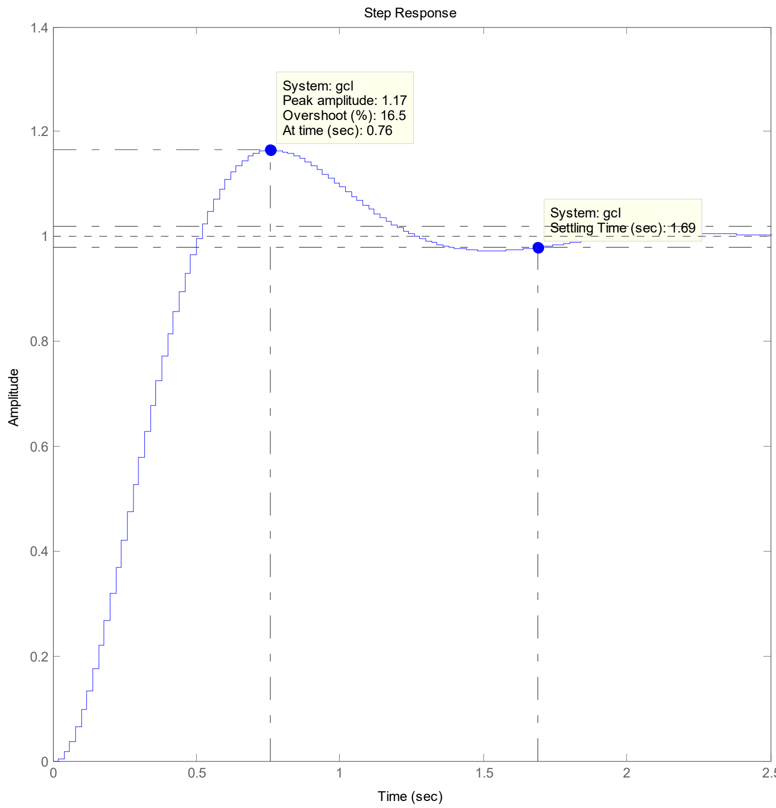


Figure 6-17 Step response of the digital PI-controlled system of Problem 5.7.

Problem 5.8 The design specifications require a settling time less than 0.5 s and an undamped natural frequency of 10 rad/s. We obtain the damping ratio

$$\zeta = \frac{4}{T_s \omega_n} \geq \frac{4}{5} = 0.8$$

The damping ratio ζ = 0.5 which corresponds to an undamped natural frequency ω_d = 6 rad/s. We calculate the sampling period as

$$T = \frac{2\pi}{70 \times \omega_d} = \frac{2\pi}{70 \times 6} \approx 0.015\, s$$

```
>> gd=c2d(g,.015)

Zero/pole/gain:
0.00010919 (z+0.9704)
---------------------
(z-0.9851) (z-0.9277)

Sampling time: 0.015
```

We select a realizable PID control that cancels the two poles closest to the unit circle in the form

$$C(z) = K\frac{(z-0.9851)(z-0.9277)}{z(z-1)}$$

The root locus of the system is shown in Figure 6- 18. For a gain of about 2000, we have the desired damping ratio. The step response of Figure 6-19 shows that the system has 1.25% overshoot and a settling time about 0.06 s. The design easily meets the desired specifications but the high gain may not be acceptable in some applications.

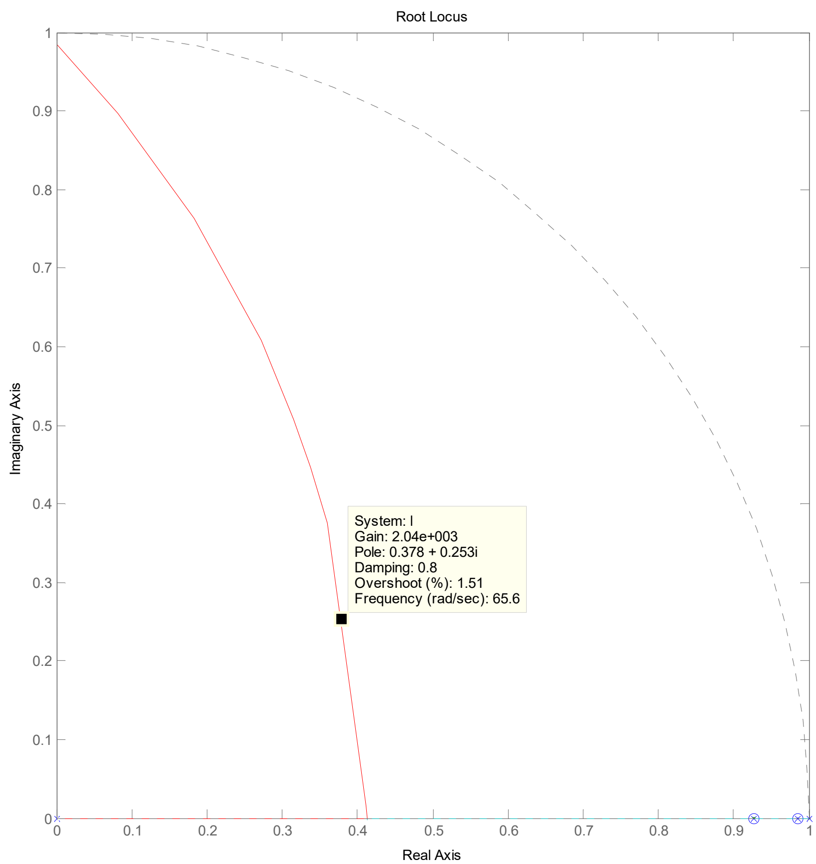


Figure 6- 18 Root locus for the digital control of Problem 5.8.

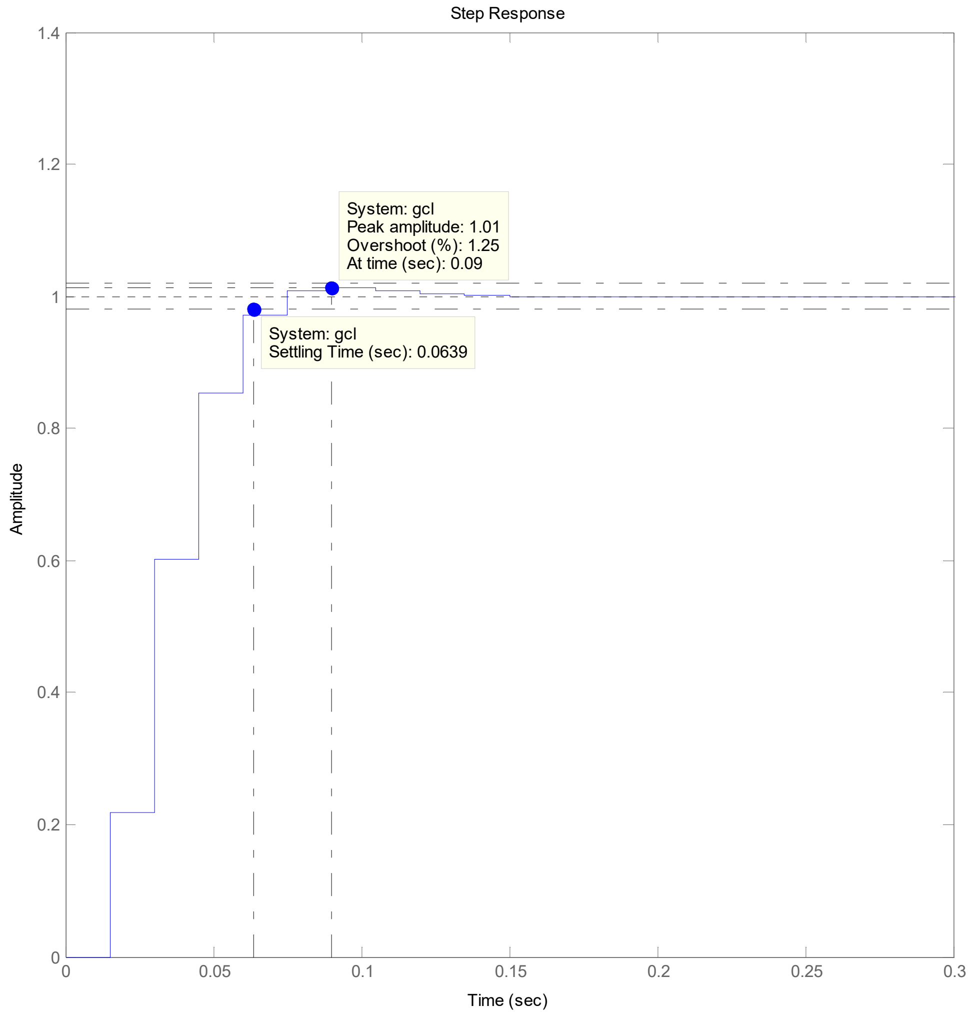


Figure 6-19 Step response of the digital PI-controlled system of Problem 5.8.

6.10 In Example 4.9, we examined the closed-loop stability of the furnace temperature digital control system with proportional control and a sampling period of 0.01 units. We obtained the z-transfer function

$$G_{ZAS}(\mathrm{z}) = 10^{-5}\,\frac{4.95z + 4.901}{z^2 - 1.97z + 0.9704}$$

Design a controller for the system to obtain zero-steady-state error due to a step input without significant deterioration in the transient response.

To reduce the steady-state error to zero we need to design a PI controller for the system. We first examine the root locus of the system (Figure 6-20) and observe that for a gain of about 5. The step response of Figure 6-21 is significantly faster with an overshoot less than 10%, a peak time of 1.6, and the settling time is less than 2.5. However, the steady-state error is large and must be reduced. We now add a pole at unity and a zero to reduce the steady-state error due to step to zero. We select the zero at the location of one of the system poles and obtain the system root-locus of Figure 6-22. The time response of the system with PI control (Figure 6-23) has an overshoot of 11%, a peak time of 1.7, a settling time of about 2.6, and zero steady-state error due to step. Thus, the transient response is only slightly adversely affected but the steady-state error specification is met.

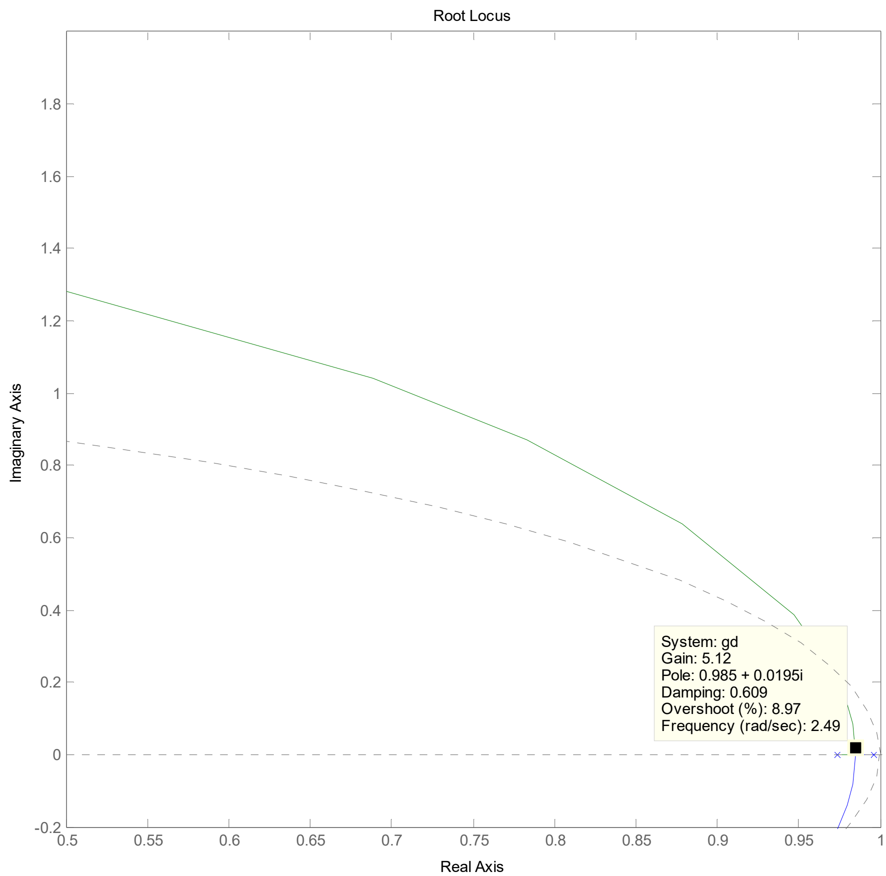


Figure 6-20 Root locus for the furnace control system.

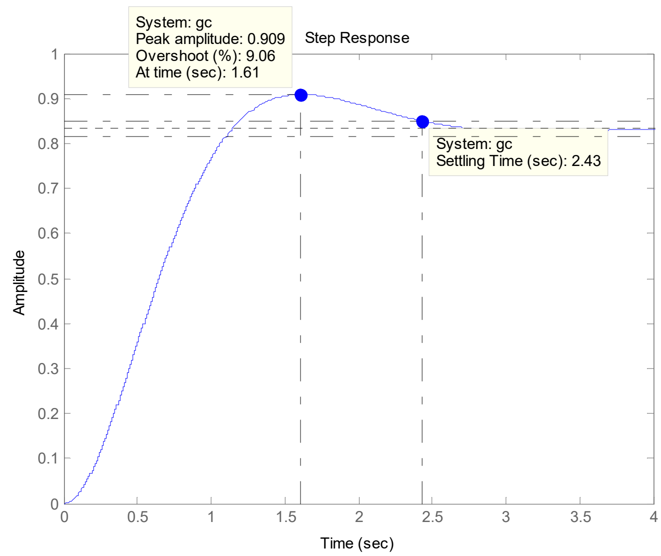


Figure 6-21 Step response for the furnace control system with proportional control and a gain of 5.

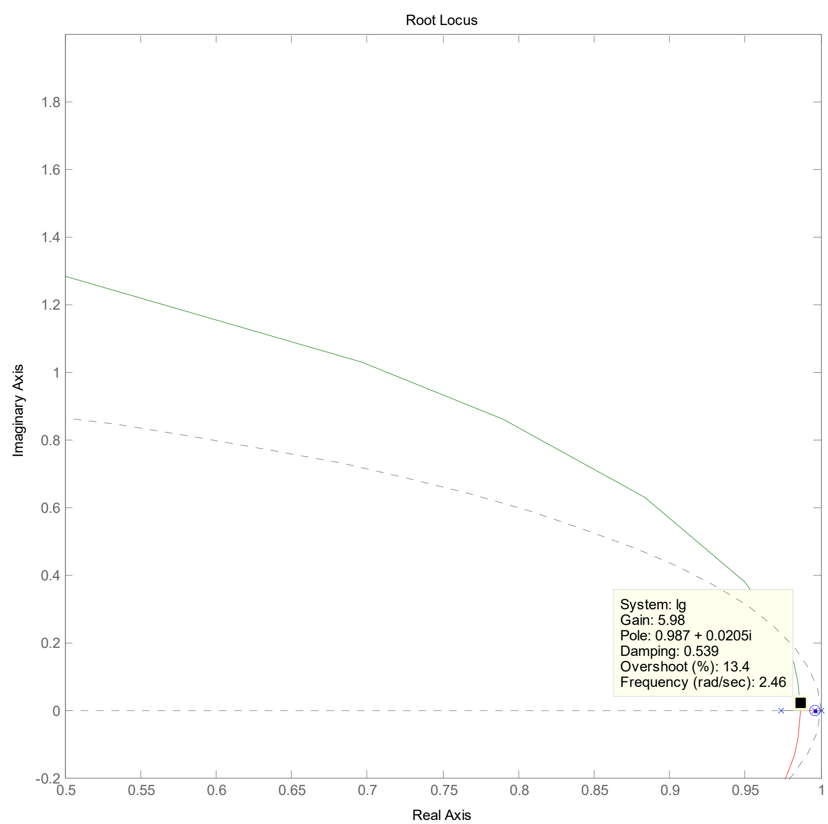


Figure 6-22 Root locus for the system with PI control.

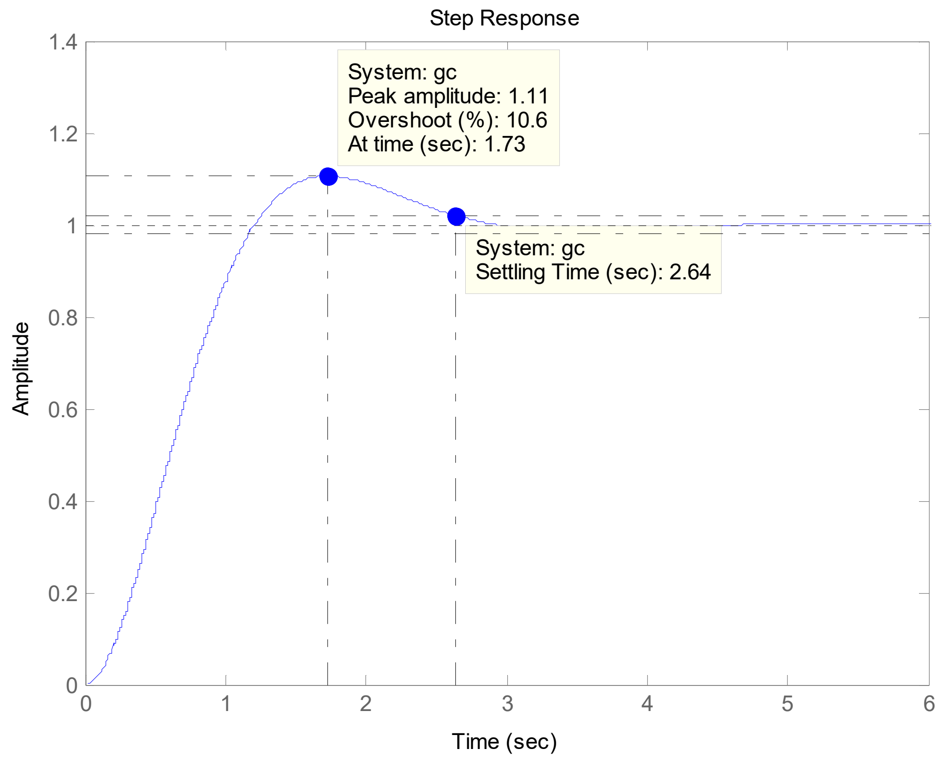


Figure 6-23 Step response for the furnace control system with PI control and a gain of 5.

6.11 Consider the DC motor position control system of Example 3.6, where the (type 1) analog plant has the transfer function

$$G(s) = \frac{1}{s(s+1)(s+10)}$$

and design a digital controller by using frequency response methods to obtain a settling time of about one second and an overshoot of less than 5%.

For 5% overshoot, we calculate the damping ratio as

$$\zeta = \frac{|\ln(0.05)|}{\sqrt{|\ln(0.05)|^2 + \pi^2}} \approx 0.7$$

Then, from (6.42) we have that the system in the w-plane should have a phase margin of about 70 degrees. Further, given the settling time T_s=1, we calculate the undamped natural frequency

$$\omega_n = \frac{4}{\zeta T_s} \approx 5.7\, rad/s$$

and using (6.41), we obtain the desired gain crossover frequency ω_{gc}=3.7 rad/s. A suitable sampling period for the selected dynamics is T=0.01 s. The discretized process is then determined as

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathrm{Z}\left\{\frac{G(s)}{s}\right\} = 1.6217 \times 10^{-7}\,\frac{(z+3.623)(z+0.2606)}{(z-1)(z-0.99)(z-0.9048)}$$

Using (6.36), we obtain the w-plane transfer function

$$G(w) = \frac{4.162 \cdot 10^{-8}\ (w+341)\ (w-352)(s-200)}{w(w+9.992)\ (w+1)}.$$

The simplest design that meets the desired specifications is to cancel the dominant pole in −1 and to increase the gain until the required gain crossover frequency is attained. Thus, the resulting controller transfer function is

$$C(w) = 39.5(w+1)$$

The Bode diagram of the loop transfer function $C(w)G(w)$, together with the Bode diagram of $G(w)$ is shown in Figure 6-24. The figure also shows the phase and gain margins. By transforming the controller back to the z-plane by means of (6.35) we obtain

$$C(z) = \frac{7940z - 7861}{z-1}$$

The corresponding discretized closed-loop step response is plotted in Figure 6-25 and clearly meets the design specifications.

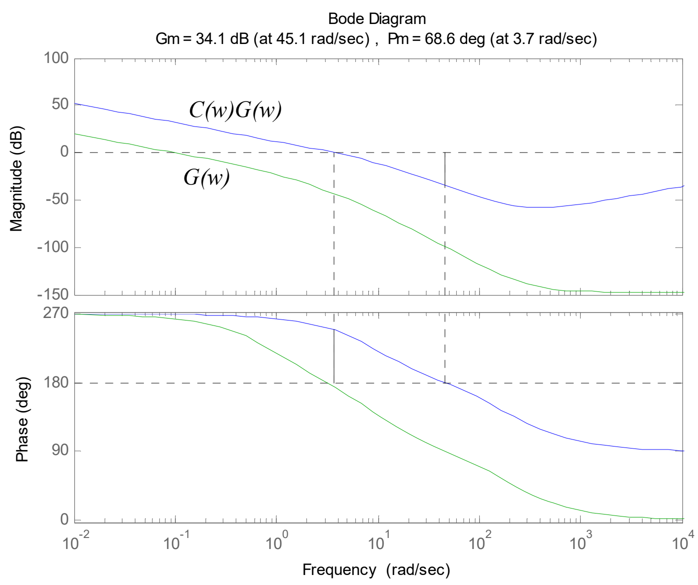


Figure 6-24 Bode plots of $C(w)G(w)$ and $G(w)$ for Problem 6.11

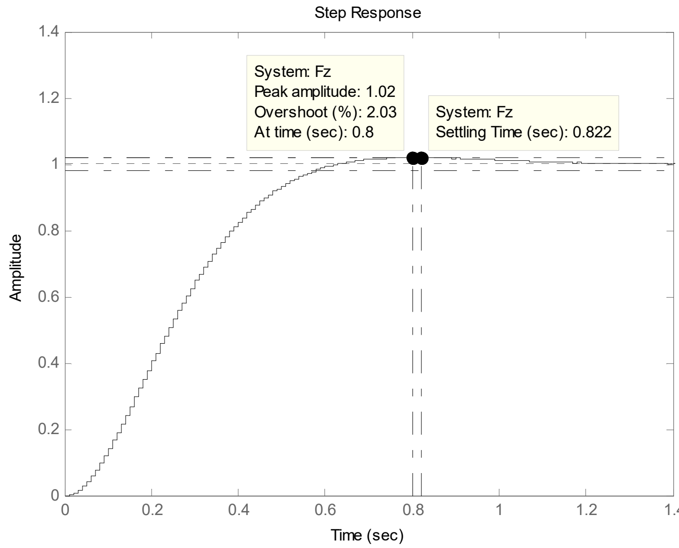


Figure 6-25 Discretized closed-loop step response for Problem 6.11

6.12 Use direct control design for the system of Problem 5.7 (with T=0.1), namely, design a controller for the transfer function

$$G(s)=\frac{1}{(s+1)(s+5)}$$

to obtain (i) zero steady-state error due to step, (ii) a settling time of less than 2 s, and (iii) an undamped natural frequency of 5 rad/s. Obtain the discretized and the analog output. Then, apply the designed controller to the system

$$G(s)=\frac{1}{(s+1)(s+5)(0.1s+1)}$$

and obtain the discretized and the analog output in order to verify the robustness of the control system.

First, we find the discretized process transfer function:

$$G_{ZAS}(z)=\left(1-z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\}=\frac{0.0041172(z+0.8189)}{(z-0.9048)(z-0.6065)}$$

Then, the desired analog characteristic polynomial is

$$s^2-2\zeta\omega_n s+\omega_n^2$$

where, according to the specifications, ω_n=5 and $T_s=\frac{4}{\zeta\omega_n}=2$, which implies that ζ=0.4. Thus, by taking into account that a zero steady-state error is required, the desired closed-loop transfer function is

$$G_{cl}(s) = \frac{25}{s^2 + 4s + 25}$$

Then, the desired closed-loop transfer function is obtained by using $z = e^{sT}$, namely,

$$G_{cl}(z) = 0.1009\frac{z+1}{z^2 - 1.469z + 0.6703}$$

By applying (6.43) we have

$$C(z) = \frac{20.5076(z-0.9048)(z-0.6065)(z+1)}{(z-1)(z+0.8189)(z-0.5694)}$$

The obtained discretized and analog closed-loop system output are shown in Figure **6-**26 and Figure **6-**27 while the corresponding control variable is plotted in Figure 6-28. If the same controller is applied to the system

$$G(s) = \frac{1}{(s+1)(s+5)(0.1s+1)}$$

the process output obtained is that in Figure 6-29. It can be seen that the additional lag causes a more significant overshoot and an increment of the settling time.

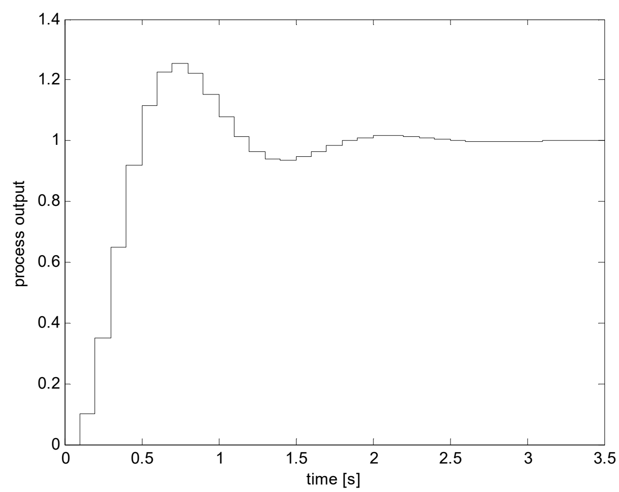


Figure 6-**26 Discretized process output for Problem 6.12**

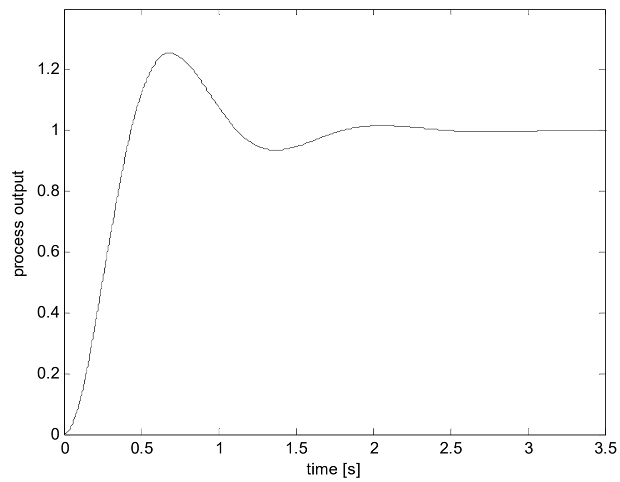


Figure 6-27 Analog output for Problem 6.12

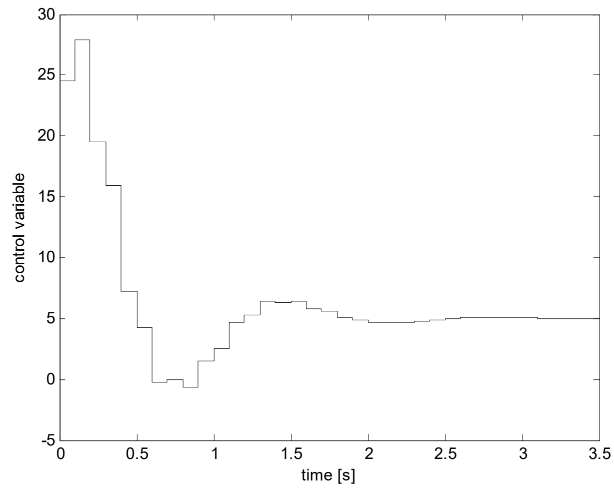


Figure 6-28 Control variable for Problem 6.12

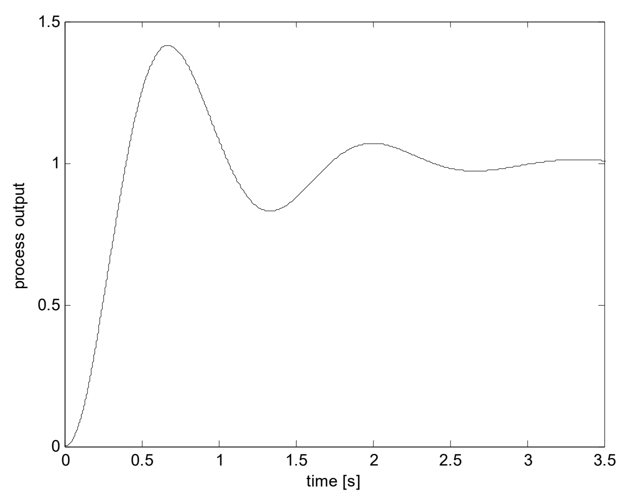


Figure 6-29 Analog output for Problem 6.12 with uncertainty

6.13 Design a deadbeat controller for the system of Problem 5.7 to obtain perfect tracking of a unit step in minimum finite time. Obtain the analog output for the system and compare your design to that obtained in Problem 5.7. Apply then the controller to the process

$$G(s) = \frac{1}{(s+1)(s+5)(0.1s+1)}$$

in order to verify the robustness of the control system.

We consider the same discretized process transfer function of Problem 6.12, but in this case we set

$$G_{cl}(z) = z^{-1}.$$

Thus, by applying (6.43) we have

$$C(z) = \frac{242.8848(z-0.9048)(z-0.6065)}{(z-1)(z+0.8189)}$$

The corresponding discretized and analog output are shown in Figure 6-30, the control variable in Figure 6-31. We observe that the deadbeat controller causes wide intersampling oscillations and requires a much higher control effort. It results in an increased overshoot without significantly decreasing the settling time. When the system

$$G(s) = \frac{1}{(s+1)(s+5)(0.1s+1)}$$

is considered, the process output obtained is plotted in Figure 6-32.

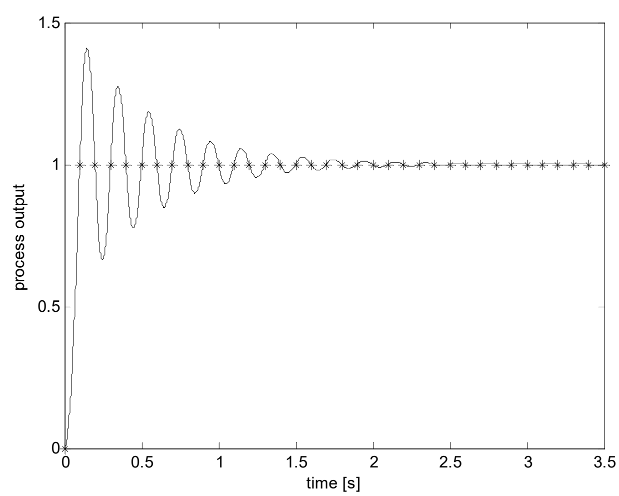


Figure 6-30 Discretized and analog output for Problem 6.13

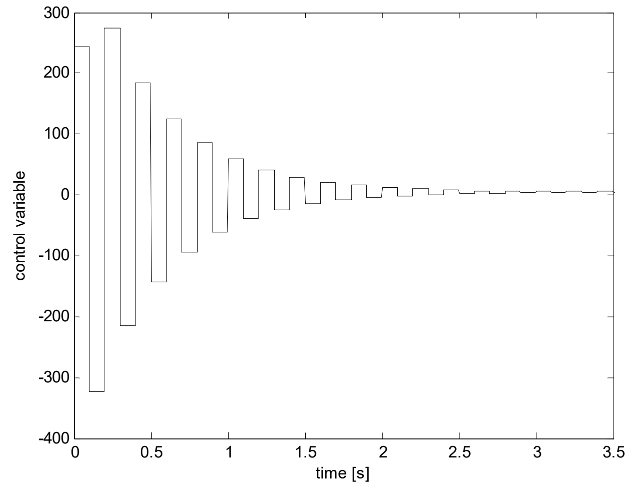


Figure 6-31 Control variable for Problem 6.13

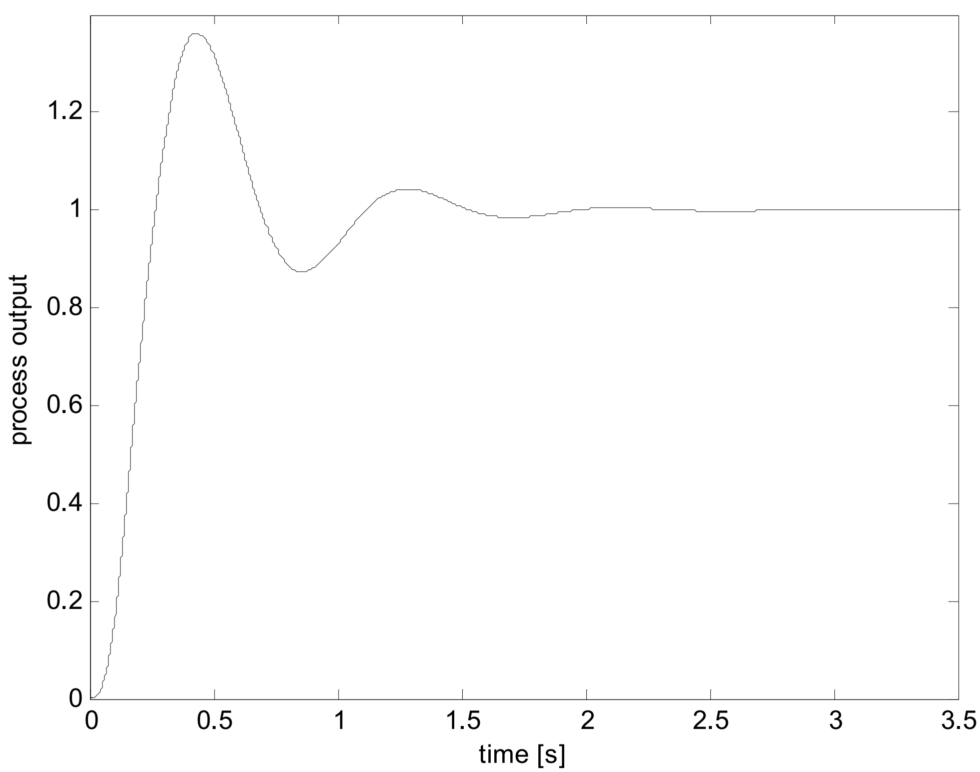


Figure 6-32 Analog output for Problem 6.13 with uncertainty

6.14 Find a solution for Problem 6.13 which avoids intersampling ripple.

By considering the same discretized process transfer function of Problem 6.13:

$$G_{ZAS}(z)=\frac{0.0041172z^{-1}(1+0.8189z^{-1})}{\left(1-0.9048z^{-1}\right)\left(1-0.6065z^{-1}\right)}$$

and

$$R(z)=\frac{1}{1-z^{-1}}$$

we have

$$U(z)=G_{cl}(z)\frac{R(z)}{G_{ZAS}(z)}=G_{cl}(z)\cdot\frac{242.8848\left(1-0.9048z^{-1}\right)\left(1-0.6065z^{-1}\right)}{z^{-1}(1-z^{-1})(1+0.8189z^{-1})}$$

This equation can be verified provided that

$$G_{cl}(z)=K\cdot z^{-1}\ (1+0.8189z^{-1})$$

where the value K=0.55 is found by imposing $G_{cl}(1)=1$. Thus, by applying (6.43) we have

$$C(z)=\frac{133.5339(z-0.9048)(z-0.6065)}{(z-1)(z+0.4502)}$$

The discretized and analog process output is shown in Figure 6-33 , while the control variable is plotted in Figure 6-34. It can be seen that the intersample ripple is avoided and the amplitude of the control variable is decreased with respect to the deadbeat controller of Problem 6.13. When the system

$$G(s)=\frac{1}{(s+1)(s+5)(0.1s+1)}$$

is considered, the process output obtained is that plotted in Figure 6-35.

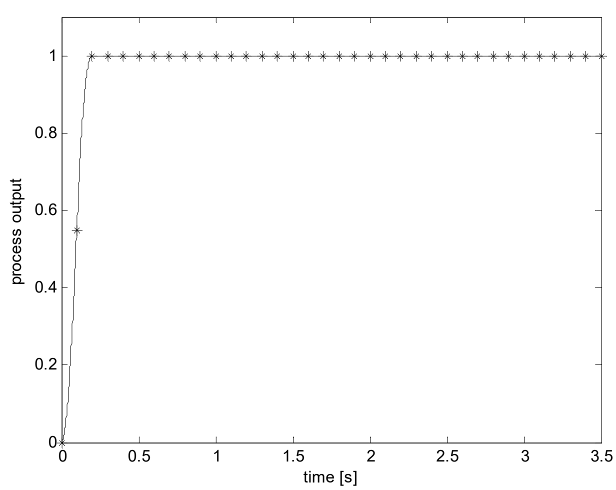


Figure 6-33 Discretized and analog output for Problem 6.14

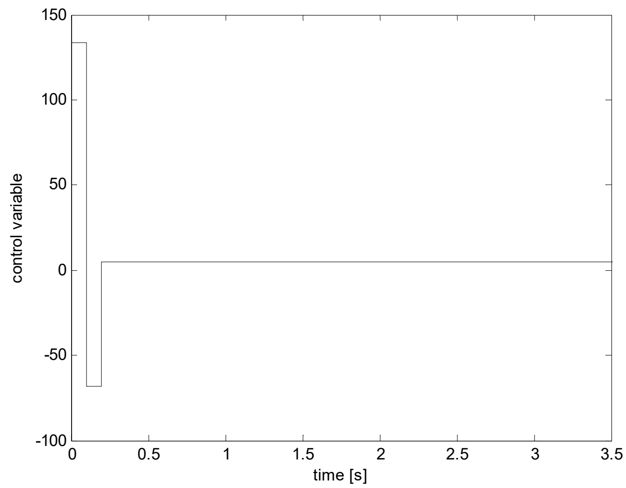


Figure 6-34 Control variable for Problem 6.14

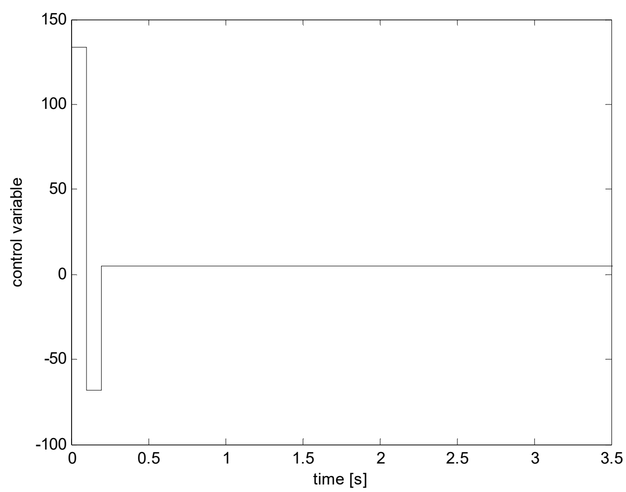


Figure 6-35 Analog output for Problem 6.14 with uncertainty

Computer Exercises

6.15 Write a MATLAB function to plot a constant damped natural frequency contour in the z-plane. For a constant damped natural frequency, we need to plot a radial line.

```
% constantwd
% Program to plot a constant wd line on the z-domain root locus.
% The program plots a radial line at an angle wd T where T is
% the sampling period. The user selects the maximum
% x coordinate for the plot, xmax.
%
function constantwd(wd,T,rmax,step)
theta=tan(wd*T); % Calculate the slope of the radial line
rho=0:step:rmax; % Calculate the magnitude vector
thetaV=theta*ones(1,length(rho)); % Define the angle vector
hold on
```

```
polar(thetaV,rho)
```

6.16 Write a MATLAB function to plot a constant time constant contour in the z-plane.
For a constant time constant, we need to plot a circle.

```
% constant mag
% Program to plot a constant manitude line on the z-domain root locus.
% The program plots a circle of radius exp(-zeta wn T) where T is
% the sampling period. The user selects the step on the x-grid.
%
function constantmag(zeta, wn,T,step)
mag=exp(-zeta*wn*T); % Calculate the pole magnitude
theta=-pi:step:pi; % Define the angle vector from 0 to pi
rho=mag*ones(1,length(theta)); % Define the magnitude vector
hold on
polar(theta,rho) % Plot a circle of radius mag
```

6.17 Write a computer program that estimates a first-order-plus-dead-time transfer function with the tangent method and determines the digital PID parameters according to the Ziegler-Nichols formula. Apply the program to the system

$$G(s) = \frac{1}{(s+1)^8}$$

and simulate the response of the digital control system (with T=0.1) when a set-point step change and a load disturbance step are applied. Compare the result with those of Problem 5.14.

We can slightly modify the computer program of Problem 5.14 in order to estimate the process parameters and to determine the PID parameters with the Ziegler-Nichols formula. Note that here $T/2$ is added to the dead time of the process before applying Table 5.1.

```
function [Cz]=digital_zn(Gs,T)

% Gs: analog process transfer function
% T: sampling period
% Cz: digital PID transfer function

[y,t]=step(Gs); % step response
tfinal=1.5*t(end); % refinement to reduce numerical problems
t=0:tfinal/1000:tfinal;
[y,t]=step(Gs); % step response with 1000 samples

% tangent method
sampling_time=t(2)-t(1);
dy=diff(y); % difference of the process output
[diff_max,max_pos]=max(dy);
t_max=t(max_pos); % time instant of the maximum derivative
dy_max=diff_max/sampling_time; % maximum derivative
L=t_max-y(max_pos)/dy_max; % estimated dead time
tautop=t_max+(y(end)-y(max_pos))/dy_max; % tau+L
tau=Ttop-L; % the time constant is determined by considering
            % the intersection of the tangent line with the
            % straight line representing the final
```

```
             % steady-state value of the process output
% tau=t(find(y>0.632*y(end),1))-L;
% use the second form of tau to determine the time constant
% as the time interval between the application of the
% step input and the time when the process output attains
% the 63.2% of its final value

K=y(end); % process gain
L=L+T/2;  % modified dead time

% Ziegler-Nichols tuning
Kp=1.2*tau/K/L; % proportional gain
Ti=2*L;         % integral time constant
Td=0.5*L;       % derivative time constant

z=tf('z');
```

If we run the MATLAB function with the given process transfer function, two options are available for calculating the time constant. The first is by considering $\tau+L$ as the time interval between the application of the step input and the intersection of the tangent line with the steady-state output level. We obtain K=1, τ=6.7 and L=4.35 (note that with respect to Problem 5.11 only the dead time L is different). Thus, we have K_p=1.85, T_i=8.71, and T_d=2.18 (note that all the three PID paramters are slightly different from those found in Problem 5.11). The digital PID controller transfer function is

$$C(z) = \frac{82.4167(z-0.9773)^2}{(z-1)(z+1)}$$

The set-point step response is shown in Figure 6-36, while the load disturbance response is shown in Figure 6-37. The responses are very similar to those obtained with the analog PID controller of Problem 5.11.

The second approach is to take the value of $\tau+L$ as the time interval between the application of the step input and the time when the output attains 63.2% of its final value. We obtain K=1, τ=4.2 and L=4.35. This yields K_p=1.22, T_i=8.71, and T_d= 2.18. Also in this case the modification of the dead time yields to PID gains that are slightly different from those obtained in Problem 5.11. The digital PID controller transfer function is

$$C(z) = \frac{54.3363(z-0.9773)^2}{(z-1)(z+1)}$$

The resulting set-point step response is shown in Figure 6-38, while the load disturbance response is shown in Figure 6-39. The responses are very similar again to those obtained with the analog PID controller of Problem 5.11.

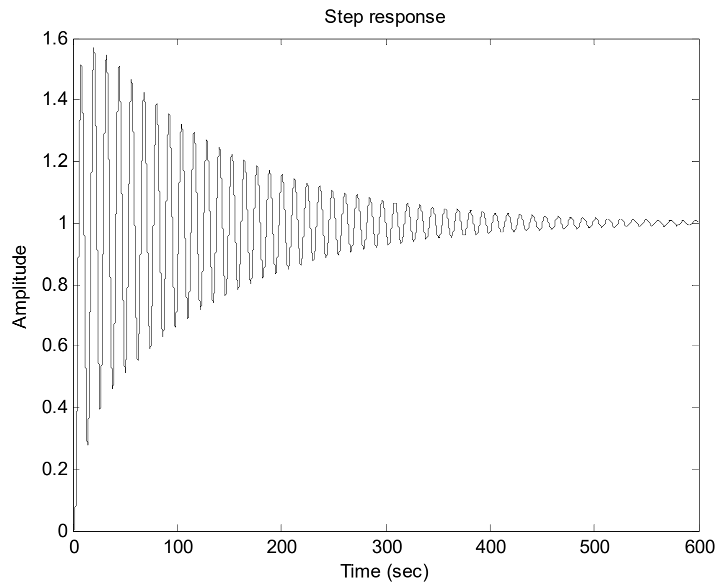


Figure 6-36 Set-point step response for Problem 6.17 with τ=6.7

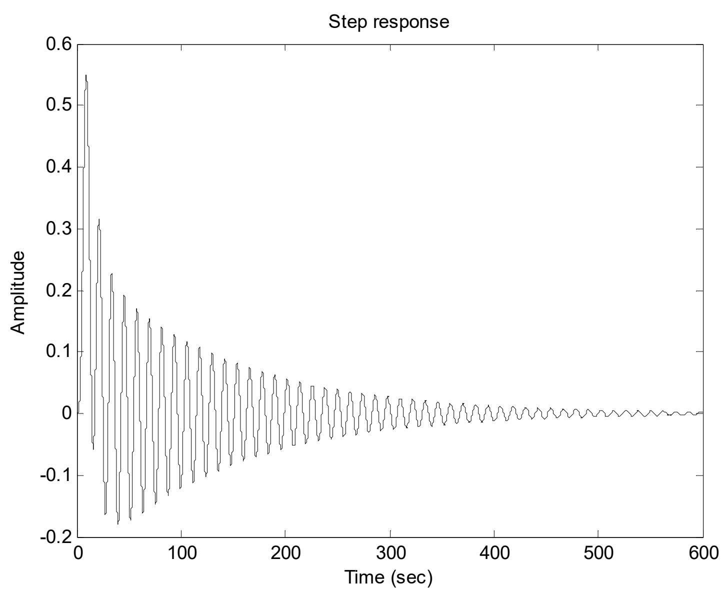


Figure 6-37 Load disturbance step response for Problem 6.17 with τ=6.7

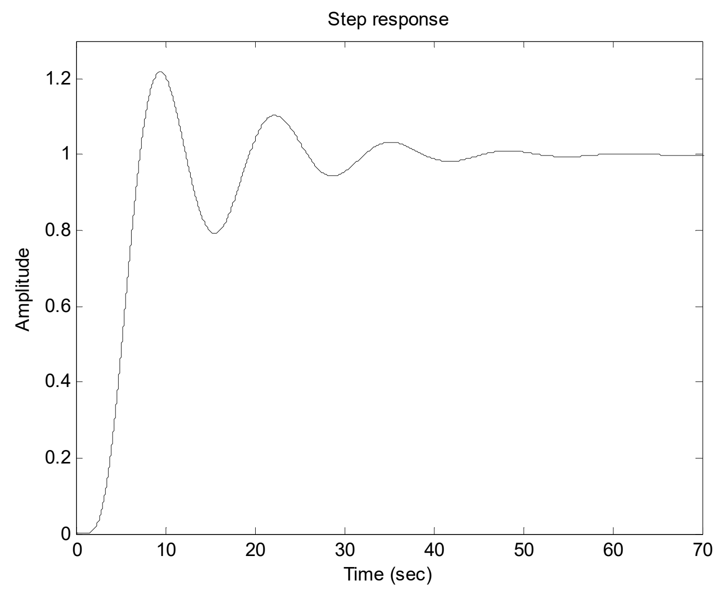


Figure 6-38 Set-point step response for Problem 6.17 with τ=4.2

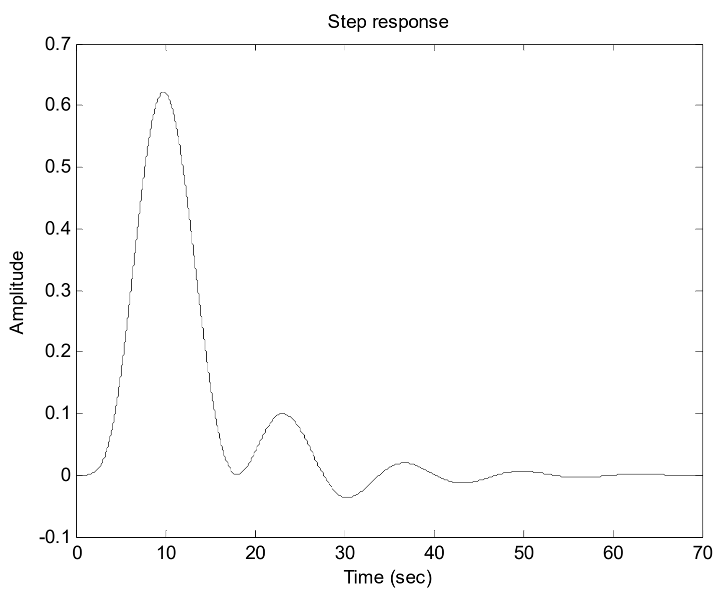


Figure 6-39 Load disturbance step response for Problem 6.17 with τ=4.2

6.18 To examine the effect of the sampling period on the relative stability and transient response of a digital control system, consider the system

$$G(s) = \frac{1}{(s+1)(s+5)}$$

a) Obtain the transfer function of the system, the root locus and the critical gain for T = 0.01, 0.05, 0.1 s.

Using the MATLAB commands **zpk** and **c2d**, we obtain:

```
>> g=zpk([],[-1,-5],1)

Zero/pole/gain:
     1
-----------
(s+1) (s+5)

>> gd1=c2d(g,.01)

Zero/pole/gain:
4.9013e-005 (z+0.9802)
----------------------
 (z-0.99) (z-0.9512)

Sampling time: 0.01
>> gd2=c2d(g,.05)

Zero/pole/gain:
0.0011327 (z+0.9049)
---------------------
(z-0.9512) (z-0.7788)

Sampling time: 0.05
>> gd3=c2d(g,.1)

Zero/pole/gain:
0.0041172 (z+0.8189)
---------------------
(z-0.9048) (z-0.6065)

Sampling time: 0.1
```

Using the command

```
>> [Gm,PM]=margin(gd1)
```

We obtain the gain margin, which is the critical gain value. The values obtain are summarized in the following table.

Sampling Period	Zero	Pole 1	Pole 2	K_{cr}
0.01	–0.9802	0.99	0.9512	1.2124E3
0.05	–0.9049	0.9512	0.7788	252.8872
0.1	–0.8189	0.9048	0.6065	133.8229

b) Obtain the step response for each system at a gain of 50.

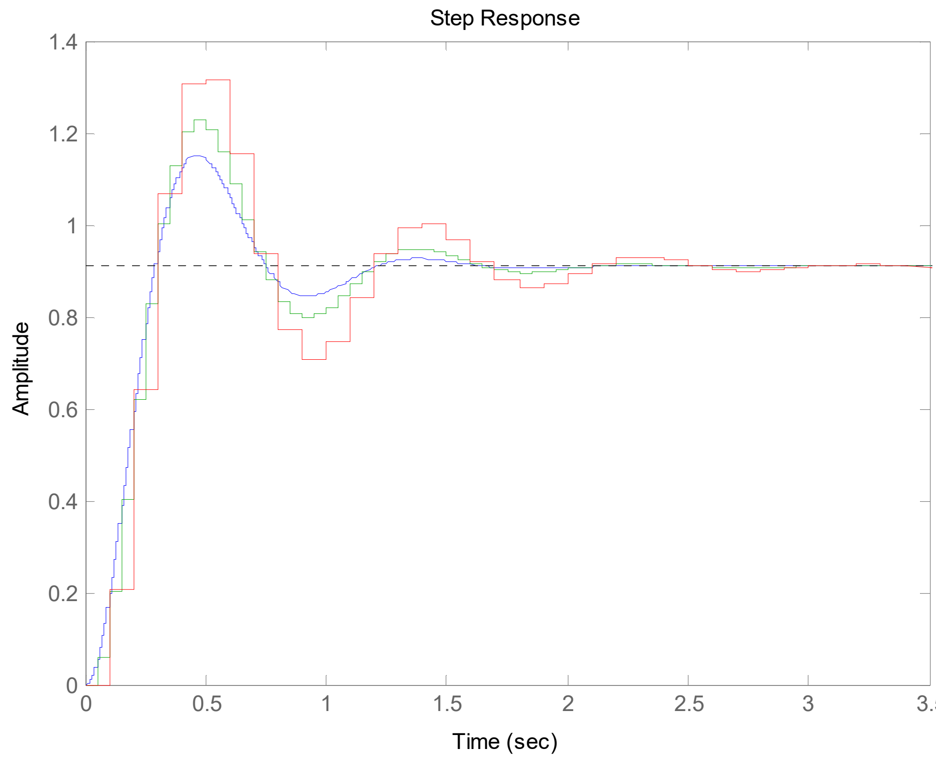


Figure 6- 40 Step response for sampling period *T*= 0.01 (blue), *T*=0.05 (green), and *T*=0.1 s (red).

c) Discuss the effect of the sampling period on the transient response and relative stability of the system based on your results from (a) and (b).

The transfer functions for the different sampling period with generally faster poles for slower sampling. Faster poles are poles that are closer to the origin. The systems with slower sampling does not monitor the process as closely as ones with faster sampling. As a result, the step response of the system is more oscillatory.-

Chapter 7 Solutions

7.1 Classify the state-space equations regarding linearity and time-variance:

a)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \sin(t) & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Linear time-varying: State matrix term a_{11} depends explicitly on time.

b)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Linear time-invariant.

c)
$$\dot{x} = -2x^2 + 7x + x\,u$$
$$y = 3x$$

Nonlinear: Quadratic term and xu term in the state equation.

d)
$$\dot{x} = -7x + u$$
$$y = 3x^2$$

Nonlinear: Quadratic term in the output equation.

7.2 The equations of motion of a 2-D.O.F manipulator are

$$M\ddot{\theta} + \mathbf{d}(\dot{\theta}) + \mathbf{g}(\theta) = \begin{bmatrix} \mathbf{T} \\ f \end{bmatrix}$$
$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \qquad \mathbf{d}(\dot{\theta}) = \begin{bmatrix} 0 \\ D_2\dot{\theta}_2 \end{bmatrix} \qquad \mathbf{g}(\theta) = \begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix}$$
$$M_a = M^{-1} = \begin{bmatrix} m_{a11} & m_{a12} \\ m_{a12} & m_{a22} \end{bmatrix}$$

where $\theta = [\theta_1, \theta_2]^T$ is a vector of joint angles. The entries of the positive definite inertia matrix M depend on the robot coordinates θ. D_2 is a damping constant. The terms g_i, $i = 1, 2$, are gravity related terms that also depend on the coordinates. The right hand side is a vector of generalized forces.

a) Obtain a state-space representation for the manipulator.
b) Obtain the linearized model in the vicinity of arbitrary coordinates with zero inputs in terms of the derivatives of the entries of the matrix M_a and the vectors **d** and **g**.

a) We define the 4 by 1 state and input vectors as

$$\mathbf{x} = col\{\mathbf{x}_1, \mathbf{x}_2\} = col\{\theta, \dot{\theta}\}$$
$$\mathbf{u} = \begin{bmatrix} \mathbf{T} & f \end{bmatrix}^T$$
$$\mathbf{x}_1 = \begin{bmatrix} x_{11} & x_{12} \end{bmatrix}^T$$
$$\mathbf{x}_2 = \begin{bmatrix} x_{21} & x_{22} \end{bmatrix}^T$$

We now have

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = M_a(\mathbf{x}_1)\{\mathbf{u} - \mathbf{d}(\mathbf{x}_2) - \mathbf{g}(\mathbf{x}_1)\}$$
$$M_a = M^{-1} = \begin{bmatrix} m_{a11} & m_{a12} \\ m_{a12} & m_{a22} \end{bmatrix}$$

Multiplying out the second state equation we have

$$\dot{\mathbf{x}}_1 = \mathbf{x}_2$$
$$\dot{\mathbf{x}}_2 = M_a(\mathbf{x}_1)\mathbf{u} - \begin{bmatrix} m_{a12} \\ m_{a22} \end{bmatrix} D_2(\mathbf{x}_{22}) - \begin{bmatrix} m_{a11}g_1 + m_{a12}g_2 \\ m_{a12}g_1 + m_{a22}g_2 \end{bmatrix}$$

b) Obtain the linearized model in the vicinity of zero coordinates, velocities and inputs.

$$\Delta\dot{\mathbf{x}}_1 = \Delta\mathbf{x}_2$$
$$\Delta\dot{\mathbf{x}}_2 = -\left.\begin{bmatrix} \dfrac{\partial^T(m_{a11}g_1 + m_{a12}g_2)}{\partial \mathbf{x}_1} + D_2(\mathbf{x}_{22})\dfrac{\partial m_{a12}}{\partial \mathbf{x}_1} \\ \dfrac{\partial^T(m_{a12}g_1 + m_{a22}g_2)}{\partial \mathbf{x}_1} + D_2(\mathbf{x}_{22})\dfrac{\partial m_{a22}}{\partial \mathbf{x}_1} \end{bmatrix}\right|_{\mathbf{x}_1=\theta_0} \Delta\mathbf{x}_1 - M_a(\mathbf{x}_1)\begin{bmatrix} 0 \\ \dfrac{D_2(\mathbf{x}_{22})}{\partial \mathbf{x}_2} \end{bmatrix}\Delta\mathbf{x}_2$$

7.3 Obtain the matrix exponentials for the state matrices using four different approaches

i) $A = diag\{-3, -5, -7\}$

ii) $$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -6 & 0 & 0 \end{bmatrix}$$

iii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$

iv) A is a block diagonal matrix with the matrices of (ii) and (iii) on its diagonal.

(results for (ii) and (iii) obtained using the MATLAB Symbolic Toolbox or MAPLE)

i) $A = diag\{-3, -5, -7\}$

For a diagonal state matrix, the state-transition matrix can be written by inspection

$$e^{At} = diag\{\exp(-3t), \exp(-5t), \exp(-7t)\}$$

ii) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -6 & 0 & 0 \end{bmatrix}$

$$[sI_3 - A]^{-1} = \begin{bmatrix} s & 0 & -1 \\ 0 & s+1 & 0 \\ 6 & 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s}{s^2+6} & 0 & \frac{1}{s^2+6} \\ 0 & \frac{1}{s+1} & 0 \\ \frac{-6}{s^2+6} & 0 & \frac{s}{s^2+6} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\begin{bmatrix} \frac{s}{s^2+6} & 0 & \frac{1}{s^2+6} \\ 0 & \frac{1}{s+1} & 0 \\ \frac{-6}{s^2+6} & 0 & \frac{s}{s^2+6} \end{bmatrix} = \begin{bmatrix} \cos(\sqrt{6}\,t) & 0 & \frac{1}{\sqrt{6}}\sin(\sqrt{6}\,t) \\ 0 & e^{-t} & 0 \\ -\sqrt{6}\sin(\sqrt{6}\,t) & 0 & \cos(\sqrt{6}\,t) \end{bmatrix}$$

iii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$

$$[sI_3 - A]^{-1} = \begin{bmatrix} s & -1 & -1 \\ 0 & s & -1 \\ 0 & 6 & s+5 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{s-1}{s(s^2+5s+6)} & \frac{s+1}{s(s^2+5s+6)} \\ 0 & \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ 0 & \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s} & \frac{s-1}{s\left(s^2+5s+6\right)} & \frac{s+1}{s\left(s^2+5s+6\right)} \\ 0 & \frac{s+5}{s^2+5s+6} & \frac{1}{s^2+5s+6} \\ 0 & \frac{-6}{s^2+5s+6} & \frac{s}{s^2+5s+6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{-1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 3 & 1 \\ 0 & -6 & -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix} e^{-3t}$$

iv) A is a block diagonal matrix with the matrices of (ii) and (iii) on its diagonal.

$$e^{At} = \left[\begin{array}{c|c} \begin{matrix} \cos\left(\sqrt{6}\,t\right) & 0 & \frac{1}{\sqrt{6}}\sin\left(\sqrt{6}\,t\right) \\ 0 & e^{-t} & 0 \\ -\sqrt{6}\sin\left(\sqrt{6}\,t\right) & 0 & \cos\left(\sqrt{6}\,t\right) \end{matrix} & \mathbf{0}_{3\times 3} \\ \hline \mathbf{0}_{3\times 3} & \begin{bmatrix} 1 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 3 & 1 \\ 0 & -6 & -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix} e^{-3t} \end{array}\right]$$

7.4 Obtain the zero-input responses of the systems of Problem 7.3 due to the initial condition vectors:
(i), (ii), (iii) $[1, 1, 0]^T$ and $[1, 0, 0]^T$.

(iv) $[1, 1, 0, 1, 0, 0]^T$.

$$\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)$$

(i)

$$= diag\{\exp(-3t), \exp(-5t), \exp(-7t)\}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \exp(-3t) \\ \exp(-5t) \\ 0 \end{bmatrix}$$

$$\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)$$

$$= diag\{\exp(-3t), \exp(-5t), \exp(-7t)\}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \exp(-3t) \\ 0 \\ 0 \end{bmatrix}$$

(ii)

$$\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)$$

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 \\ -\sqrt{6} & 0 & 0 \end{bmatrix} \sin\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{-t} \right\} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 \\ 0 \\ -\sqrt{6} \end{bmatrix} \sin\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

$$\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)$$

$$= \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cos\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{6}} \\ 0 & 0 & 0 \\ -\sqrt{6} & 0 & 0 \end{bmatrix} \sin\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{-t} \right\} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 \\ 0 \\ -\sqrt{6} \end{bmatrix} \sin\left(\sqrt{6}\,t\right)$$

$$\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)$$

(iii)

$$= \left\{ \begin{bmatrix} 1 & \frac{-1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 3 & 1 \\ 0 & -6 & -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix} e^{-3t} \right\} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ 3 \\ -6 \end{bmatrix} e^{-2t} + \begin{bmatrix} -\frac{4}{3} \\ -2 \\ 6 \end{bmatrix} e^{-3t}$$

$$\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)$$

$$= \left\{ \begin{bmatrix} 1 & \frac{-1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 3 & 1 \\ 0 & -6 & -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix} e^{-3t} \right\} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

(iv) $[1, 1, 0, 1, 0, 0]^T$.

$$
\mathbf{x}_{ZI}(t) = e^{At}\mathbf{x}(0)
$$

$$
= \left[\begin{array}{ccc:c}
\cos\left(\sqrt{6}\,t\right) & 0 & \frac{1}{\sqrt{6}}\sin\left(\sqrt{6}\,t\right) & \\
0 & e^{-t} & 0 & \mathbf{0}_{3\times 3} \\
-\sqrt{6}\sin\left(\sqrt{6}\,t\right) & 0 & \cos\left(\sqrt{6}\,t\right) & \\
\hdashline
 & \mathbf{0}_{3\times 3} & & \begin{bmatrix} 1 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 3 & 1 \\ 0 & -6 & -2 \end{bmatrix} e^{-2t} + \begin{bmatrix} 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix} e^{-3t}
\end{array}\right]
\left[\begin{array}{c} 1 \\ 1 \\ 0 \\ \hdashline 1 \\ 0 \\ 0 \end{array}\right]
$$

$$
= \left[\begin{array}{c}
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cos\left(\sqrt{6}\,t\right) + \begin{bmatrix} 0 \\ 0 \\ -\sqrt{6} \end{bmatrix}\sin\left(\sqrt{6}\,t\right) \\
\hdashline
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{array}\right]
$$

7.5 Determine the discrete-time state equations for the systems of Problem 7.3(i), (ii), (iii) with $\mathbf{b} = [0, 0, 1]^T$ in terms of the sampling period T.

(i)

$$
A_d = e^{AT} = diag\{\exp(-3T), \exp(-5T), \exp(-7T)\}
$$

$$
B_d = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{1-\exp(-7T)}{7}
$$

(ii)

$$
A_d = e^{AT} = \begin{bmatrix}
\cos\left(\sqrt{6}\,T\right) & 0 & \frac{1}{\sqrt{6}}\sin\left(\sqrt{6}\,T\right) \\
0 & e^{-t} & 0 \\
-\sqrt{6}\sin\left(\sqrt{6}T\right) & 0 & \cos\left(\sqrt{6}\,T\right)
\end{bmatrix}
$$

$$
B_d = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{1-\cos\left(\sqrt{6}\,T\right)}{6} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{\sin\left(\sqrt{6}\,T\right)}{\sqrt{6}}
$$

(iii)

$$A_d = e^{AT} = \begin{bmatrix} 1 & \frac{-1}{6} & \frac{1}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 3 & 1 \\ 0 & -6 & -2 \end{bmatrix} e^{-2T} + \begin{bmatrix} 0 & -\frac{4}{3} & -\frac{2}{3} \\ 0 & -2 & -1 \\ 0 & 6 & 3 \end{bmatrix} e^{-3T}$$

$$B_d = \begin{bmatrix} \frac{1}{6} \\ 0 \\ 0 \end{bmatrix} T + \begin{bmatrix} \frac{1}{2} \\ 1 \\ -2 \end{bmatrix} \frac{1-e^{-2T}}{2} + \begin{bmatrix} -\frac{2}{3} \\ -1 \\ 3 \end{bmatrix} \frac{1-e^{-3T}}{3}$$

7.6 Prove that (right) eigenvectors of the matrix A^T are the left eigenvectors of A and that its eigenvalues are the eigenvalues of A using

$$A = V\Lambda V^{-1} = V\Lambda W$$

For linearly independent eigenvectors, we have $AV = V\Lambda$
Transpose the expression then postmultiply by the inverse of V^T

$$A^T = (V\Lambda W)^T = W^T \Lambda V^T$$

$$A^T \left(V^T\right)^{-1} = A^T \left(V^{-1}\right)^T = A^T W^T = W^T \Lambda$$

The last expression is equivalent shows that the transpose of the rose of the matrix of left eigenvectors are the right eigenvectors (columns) of A^T and that the eigenvalues of A^T are those of A.

7.7 Prove the properties of the constituent matrices given in Section 7.4.c using (7.44).

Properties of Constituent Matrices

1- Constituent matrices have rank 1.

From equation (7.44), we have the expression

$$Z_i = \mathbf{v}_i \mathbf{w}_i^T, i = 1, \cdots, n$$

Since the constituent matrices are the product of a row and a column, all its columns are the same vector scaled by the entries of the row and the matrices have rank 1.

2- The product of two constituent matrices is $$Z_i Z_j = \begin{cases} Z_i, & i = j \\ \mathbf{0}, & i \neq j \end{cases}$$

Raising Z_i to any power gives the matrix Z_i. Z_i is said to be *idempotent*.

Recall that the product of the right eigenvector matrix and the left eigenvector matrix (with appropriate scaling of the vectors is the identity. Hence, we have

$$W V = \begin{bmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \vdots \\ \mathbf{w}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{w}_1^T\mathbf{v}_1 & \mathbf{w}_1^T\mathbf{v}_2 & \cdots & \mathbf{w}_1^T\mathbf{v}_n \\ \mathbf{w}_2^T\mathbf{v}_1 & \mathbf{w}_2^T\mathbf{v}_2 & \cdots & \mathbf{w}_2^T\mathbf{v}_n \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_n^T\mathbf{v}_1 & \mathbf{w}_n^T\mathbf{v}_2 & \cdots & \mathbf{w}_n^T\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{w}_i^T\mathbf{v}_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

We now examine the product of any constituent matrices and observe that

$$Z_i Z_i = \mathbf{v}_i \mathbf{w}_i^T \mathbf{v}_j \mathbf{w}_j^T = \begin{cases} Z_i, & i = j \\ \mathbf{0}, & i \neq j \end{cases}$$

$$i, j = 1, \cdots, n$$

Since the square of any constituent matrix is the matrix itself, all powers of the matrix are equal and the matrix is idempotent.

3- The sum of the n constituent matrices of a $n \times n$ matrix is equal to the identity matrix

$$\sum_{i=1}^{n} Z_i = I_n$$

Equation (7.44) with $t = 0$ gives

$$I_n = e^{At}\Big|_{t=0} = \sum_{i=1}^{\mathbf{n}} Z_i e^0 = \sum_{i=1}^{\mathbf{n}} \mathbf{v}_i \mathbf{w}_i^T$$

7.8 (a) Derive the expressions for the terms of the adjoint matrix used in the Leverrier algorithm.
Hint: Multiply both sides of (7.26) by the matrix $[sI - A]$ and equate coefficients.
(b) Derive the expressions for the coefficients of the characteristic equations used in the Leverrier algorithm.
Hint: Laplace transform the derivative expression for the matrix exponential, take the trace, then use the identity

$$tr\left(\mathsf{L} \left[sI_n - A \right] \right) = \frac{a_1 + 2a_2 s + \ldots + (n-1)a_{n-1}s^{n-2} + ns^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}$$

Take the trace, then use the identity

$$tr\left(\left[sI_n - A \right]^{-1} \right) = \frac{a_1 + 2a_2 s + \ldots + (n-1)a_{n-1}s^{n-2} + ns^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}$$

$$\left(a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n\right)I_n = \left(P_0 + P_1 s + \ldots + P_{n-1}s^{n-1}\right)\left(sI_n - A\right)$$

Equating coefficients gives

$$\left(a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n\right)I_n = \left(P_0 + P_1 s + \ldots + P_{n-1}s^{n-1}\right)\left(sI_n - A\right)$$

$$\begin{array}{ll} s^n & P_{n-1} = I_n \\ s^{n-1} & a_{n-1}I_n = P_{n-2} - P_{n-1}A \\ \vdots & \\ s & a_1 I_n = P_0 - P_1 A \\ s^0 & a_0 I_n = -P_0 A \end{array}$$

(c) Derive the expressions for the coefficients of the characteristic equations used in the Leverrier algorithm.

Hint: Laplace transform the derivative expression for the matrix exponential, take the trace, then use the identity

$$s\mathscr{L}\left\{e^{At}\right\} - I = \mathscr{L}\left\{e^{At}\right\}A$$

Take the trace, then use the identity

$$tr\left(\left[sI_n - A\right]^{-1}\right) = \frac{dD(s)/ds}{D(s)} = \frac{a_1 + 2a_2 s + \ldots + (n-1)a_{n-1}s^{n-2} + ns^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}$$

Using the hint, we have

$$s\mathscr{L}\left\{e^{At}\right\} - I = \mathscr{L}\left\{e^{At}\right\}A$$

where we use the fact that the time function at t=0 is the identity matrix.

The trace of the LHS is

$$s\frac{dD(s)/ds}{D(s)} - n = \frac{sdD(s)/ds - nD(s)}{D(s)}$$

$$= \frac{a_1 s + 2a_2 s^2 + \ldots + (n-1)a_{n-1}s^{n-1} + ns^n - n\left(a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n\right)}{D(s)}$$

$$= \frac{-na_0 - (n-1)a_1 s - (n-2)a_2 s^2 + \ldots - a_{n-1}s^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}$$

The trace of the RHS is

$$\frac{tr\left\{\left(P_0 + P_1 s + \ldots + P_{n-1}s^{n-1}\right)A\right\}}{D(s)} = \frac{tr\{P_0 A\}}{D(s)} + \frac{tr\{P_1 A\}s}{D(s)} + \cdots + \frac{tr\{P_{n-1}A\}s^{n-1}}{D(s)}$$

Equating the coefficients of equal powers of s on both sides gives

$$\left(a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + s^n\right) I_n = \left(P_0 + P_1 s + \ldots + P_{n-1} s^{n-1}\right)\left(s I_n - A\right)$$

$$\begin{array}{ll} s^{n-1} & a_{n-1} = -tr\{P_{n-1}A\} = -tr\{A\} \\ s^{n-2} & 2a_{n-1} = -tr\{P_{n-2}A\} \\ \vdots & \\ s & (n-1)a_1 = -tr\{P_1 A\} \\ s^0 & na_0 = -tr\{P_0 A\} \end{array}$$

7.9 The biological component of the fishery system is assumed to be governed by the population dynamics equation[1]

$$\frac{dx(t)}{dt} = r\, x(t)\,(1 - x(t)/K) - h(t)$$

where r is the intrinsic growth rate per unit time, K is the environment carrying capacity, $x(t)$ is the stock biomass and $h(t)$ is the harvest rate in weight[1]

a) Determine the harvest rate for a sustainable fish population $x_0 < K$.
b) Linearize the system in the vicinity of the fish population x_0.
c) Obtain a discrete-time model for the linearized model with a fixed average yearly harvest rate $h(k)$ in the k^{th} year.
d) Obtain a condition for the stability of the fish population from your discrete-time model and comment on the significance of the condition.

The equilibrium condition is given by

$$0 = r\, x_0\, (1 - x_0 / K) - h_0$$

The harvest rate is given by

$$h_0 = r\, x_0\, (1 - x_0 / K)$$

The first order approximation of the function of x on the RHS is

$$r\, x_0\, (1 - x_0/K) + \left[r\left(1 - 2x_0/K\right)\right]\Delta x(t) = h_0 - a\Delta x(t)$$

$$a = r\left(2x_0 / K - 1\right)$$

The linearized model in the vicinity of x_0 is

$$\frac{d\Delta x(t)}{dt} = -a\Delta x(t) - \Delta h(t)$$

To obtain a discrete-time model, we examine the solution of the equation for fixed harvest rate h

$$x(t) = e^{-at}x(0) - \int_o^t e^{-a(t-\tau)} h\, d\tau$$

$$= e^{-at}x(0) - \left(\frac{1 - e^{-at}}{a}\right)h$$

The discrete model for $h(t)$ fixed over a sampling period of one year is

[1] C. W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources,* J. Wiley, N. Y., 1990.

$$x(k+1) = e^{-a}x(k) - \left(\frac{1-e^{-a}}{a}\right)h(k)$$

The stability condition for the population is $a>0$. Hence, the population must not drop below half the environment carrying capacity to be sustainable for any intrinsic growth rate r.

7.10 The following differential equations represent a simplified model of an overhead crane[2]:

$$(m_L + m_C)\ddot{x}_1(t) + m_L l(\ddot{x}_3(t)\cos x_3(t) - \dot{x}_3^2(t)\sin x_3(t)) = u$$
$$m_L\ddot{x}_1(t)\cos x_3(t) + m_L l\ddot{x}_3(t) = -m_L g\sin x_3(t)$$

where m_C is the mass of the trolley, m_L is the mass of the hook/load, l is the rope length, g is the gravity acceleration, u is the force applied to the trolley, x_1 is the position of the trolley and x_3 is the rope angle. Consider the position of the load $y=x_1+l\sin x_3$ as the output.

(a) Determine a linearized state- space model of the system about the equilibrium point $\mathbf{x}=0$ with state variables x_1, x_3, the first derivative of x_1 and the first derivative of x_3.

(b) Determine a second state-space model when the sum of the trolley position and of the rope angle is substituted for the rope angle as a third state variable.

(a) The nonlinear model can be expressed in state-space form by defining $x_2 = \dot{x}_1$ and $x_4 = \dot{x}_3$. The dynamic model can be linearized about the equilibrium point $\mathbf{x}=0$ using the approximations

$$\cos x_3 \cong 1,\ \sin x_3 \cong x_3,\ \sin^2 x_3 \cong 0,\ x_4^2 \cong 0$$

The following linear state-space model results:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu$$
$$y = C\mathbf{x} + Du$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m_L}{m_C}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m_L+m_C)g}{m_L l} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{m_C} \\ 0 \\ -\frac{1}{m_C l} \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \qquad D = 0$$

(b) The new state-space model is obtained by considering

$$z_1(t) = x_1(t)$$
$$z_2(t) = x_2(t)$$
$$z_3(t) = x_1(t) + x_3(t)$$
$$z_4(t) = x_4(t)$$

We therefore have the similarity transformation

[2] A. Piazzi, A. Visioli, "Optimal dynamic-inversion-based control of an overhead crane", *IEE Proceedings – Control Theory and Applications*, Vol. 149, No. 5, pp. 405-411, 2002.

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding state-space matrices are

$$\text{A}=T^{-1}AT=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{m_L}{m_C}g & 0 & \dfrac{m_L}{m_C}g & 0 \\ 0 & 1 & 0 & 1 \\ \dfrac{m_L+m_C}{m_C l}g & 0 & -\dfrac{m_L+m_C}{m_C l}g & 0 \end{bmatrix} \text{B}=T^{-1}B=\begin{bmatrix} 0 \\ \dfrac{1}{m_C} \\ 0 \\ -\dfrac{1}{m_C l} \end{bmatrix}$$

$$\text{C}=CT=\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{D}=0$$

7.11 Consider the discretized armature-controlled DC motor system obtained in Example 7.15. Obtain the diagonal form for the system (note that the angular position of the motor is measured).

The considered discretized model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.9995 & 0.0095 \\ 0.0 & -0.0947 & 0.8954 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1.622\times10^{-6} \\ 4.821\times10^{-4} \\ 9.468\times10^{-2} \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The **eig** command of MATLAB yields the eigenvalues and the modal matrix

$$\Lambda = diag\{1, 0.990, 0.9049\} \qquad V = \begin{bmatrix} 1 & -0.9901 & 0.1045 \\ 0 & 0.0991 & -0.0994 \\ 0 & -0.0992 & 0.9895 \end{bmatrix}$$

Thus, we have

$$\text{A}=V^{-1}A_dV=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.990 & 0 \\ 0 & 0 & 0.9049 \end{bmatrix} \qquad \text{B}=V^{-1}B=\begin{bmatrix} 0.0998 \\ 0.1120 \\ 0.1069 \end{bmatrix}$$

$$\text{C}=CV=\begin{bmatrix} 1 & -0.9901 & 0.1045 \end{bmatrix} \qquad \text{D}=0$$

7.12 A system whose state and output responses are always nonnegative for any nonnegative initial conditions and any nonnegative input is called a **positive system**[3]. Positive systems arise in many applications where the system variable can never be negative, including: chemical processes, biological systems, economics, among others. Show that the single-input-single-output discrete-time system (A, $\mathbf{b}$, $\mathbf{c}^T$) is positive if and only if all the entries of the state, input, and output matrix are positive.

Since the system is linear, we can consider the zero-state and zero-input responses separately.

Necessity

For zero initial conditions at any time k, the state at time k+1 is

$$\mathbf{x}_{ZS}(k+1) = \mathbf{b}\,u(k)$$

If the input matrix has one or more negative entries, then the corresponding entries of the state vector will also be negative for any positive input. Similarly, for zero input at time k and nonzero state, the state at time k+1 is

$$\mathbf{x}_{ZI}(k+1) = A\,\mathbf{x}(k)$$

Let the ij^{th} entry of the matrix a_{ij} be negative, then for the state $\mathbf{x}(k)$ with 1 as the i^{th} entry and all other entries zero will yield a state equal to the j^{th} column of A and will include the negative term a_{ij}.

The output of the system is given by

$$y(k) = \mathbf{c}^T\,\mathbf{x}(k)$$

Let the i^{th} entry of the output matrix (row vector) $\mathbf{c}_i$ be negative, then for the state $\mathbf{x}(k)$ with 1 as the i^{th} entry and all other entries zero will yield a negative output.

Sufficiency

We examine the three equations considered earlier when the assumptions hold true. Let the entries of the input matrix be nonnegative, then for any nonnegative input the zero-state response is nonnegative. Similarly, if all the entries of the state matrix are positive, then for any nonnegative state at time k, the state at time k+1 is nonnegative. Finally, for an output matrix with nonnegative entries the output is nonnegative for any nonnegative state at time k.

7.13 To monitor river pollution, we need to model the concentration of biodegradable matter contained in the water in terms of biochemical oxygen demand for its degradation. We also need to model the dissolved oxygen deficit defined as the difference between the highest concentration of dissolved oxygen and the actual concentration in mg/l. If the two variables of interest are the state variable x_1 and x_2, respectively, then an appropriate model is given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -k_1 & 0 \\ k_1 & -k_2 \end{bmatrix} \mathbf{x}(t)$$

where k_1 is a biodegradation constant and k_2 is a reaeration constant and both are positive. Assume that the two positive constants are unequal. Obtain a discrete-time model for the system with sampling period T and show that the system is positive.

The resolvent matrix is

[3] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory & Applications*, Wiley-Interscience, NY, 2000.

$$\begin{bmatrix} s+k_1 & 0 \\ -k_1 & s+k_2 \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{s+k_1} & 0 \\ \dfrac{k_1}{(s+k_1)(s+k_2)} & \dfrac{1}{s+k_2} \end{bmatrix}$$

The state transition matrix is

$$A_d = e^{AT} = \begin{bmatrix} e^{-k_1T} & 0 \\ \dfrac{k_1}{k_1-k_2}\left(e^{-k_2T} - e^{-k_1T}\right) & e^{-k_2T} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \dfrac{-k_1}{k_1-k_2} & 0 \end{bmatrix} e^{-k_1T} + \begin{bmatrix} 0 & 0 \\ \dfrac{k_1}{k_1-k_2} & 1 \end{bmatrix} e^{-k_2T}$$

The diagonal entries of the discrete state matrix are clearly positive. The third nonzero term is also positive since

$$e^{-k_1T} < e^{-k_2T} \text{ if } k_1 > k_2$$
$$e^{-k_1T} > e^{-k_2T} \text{ if } k_1 < k_2$$

7.14 Autonomous underwater vehicle (AUV) are robotic submarines that can be used for a variety of studies of the underwater environment. The vertical and horizontal dynamics of the vehicle must be controlled to remotely operate the AUV. The INFANTE is a research AUV operated by the Instituto Superior Tecnico of Lisbon, Portugal[4]. The variables of interest in horizontal motion are the sway speed and the yaw angle. A linearized model of the horizontal plane motion of the vehicle is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.14 & -0.69 & 0.0 \\ -0.19 & -0.048 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.056 \\ -0.23 \\ 0.0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where x_1 is the sway speed, x_2 is the yaw angle, x_3 is the yaw rate and u is the rudder deflection. Obtain the discrete state-space model for the system with a sampling period of 50 ms.

The state transition matrix is the matrix exponential given by the inverse Laplace transform of the matrix

$$[sI_3 - A]^{-1} = \begin{bmatrix} s+0.14 & 0.69 & 0.0 \\ 0.19 & s+0.048 & 0.0 \\ 0.0 & -1.0 & s \end{bmatrix}^{-1}$$

[4] C. Silvestre and A. Pascoal, "Control of the INFANTE AUV using gain scheduled static output feedback," *Contrl Engineering Practice*, Vol. 12, pp. 1501-1509, 2004.

$$= \frac{\begin{bmatrix} s(s+0.048) & -0.69 & 0 \\ -0.19s & s(s+0.14) & 0 \\ -0.19 & s+0.14 & s^2+0.188s-0.12438 \end{bmatrix}}{s(s-0.271)(s+0.459)}$$

$$= \frac{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.5276 & -1.1256 & 1 \end{bmatrix}}{s} + \frac{\begin{bmatrix} 0.437 & -0.9452 & 0 \\ -0.2603 & 0.5630 & 0 \\ -0.9605 & 2.0776 & 0 \end{bmatrix}}{s-0.271} + \frac{\begin{bmatrix} 0.5630 & 0.9452 & 0 \\ 0.2603 & 0.437 & 0 \\ -0.5671 & -0.9521 & 0 \end{bmatrix}}{s+0.459}$$

Next, we inverse Laplace transform to obtain the state-transition matrix

$$e^{At} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.5276 & -1.1256 & 1 \end{bmatrix} + \begin{bmatrix} 0.437 & -0.9452 & 0 \\ -0.2603 & 0.5630 & 0 \\ -0.9605 & 2.0776 & 0 \end{bmatrix} e^{-0.271t} + \begin{bmatrix} 0.5630 & 0.9452 & 0 \\ 0.2603 & 0.437 & 0 \\ -0.5671 & -0.9521 & 0 \end{bmatrix} e^{-0.459t}$$

The discrete state matrix is

$$A_d = e^{A(0.05)} = \begin{bmatrix} 0.9932 & -0.03434 & 0 \\ -0.009456 & 0.9978 & 0 \\ -0.0002368 & 0.04994 & 1 \end{bmatrix}$$

$$B_d = Z_1 BT + Z_2 B\left[\frac{e^{0.271T}-1}{0.271}\right] + Z_2 B\left[\frac{1-e^{-0.459T}}{0.459}\right]$$

$$= \begin{bmatrix} 0.002988 \\ -0.0115 \\ -0.0002875 \end{bmatrix}$$

The discrete state-space equations of the system are

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.9932 & -0.03434 & 0 \\ -0.009456 & 0.9978 & 0 \\ -0.0002368 & 0.04994 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.002988 \\ -0.0115 \\ -0.0002875 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

7.15 A typical assumption in most mathematical models is that the system differential equations or transfer functions have real coefficients. In a few applications, this assumption is not valid. In models of rotating machines, the evolution of the vectors governing the system with time depends on their space orientation relative to fixed inertial axes. For an induction motor, assuming symmetry, two stator fixed axes are used as the reference frame: the direct axis (d) in the horizontal direction and the quadrature axis (q) in the vertical direction. The terms in the quadrature direction are identified with a (j) coefficient which is absent from the direct axis terms. The two axes are shown in Figure P7.1. We write the equations for the electrical subsystem of the motor in terms of the stator and rotor currents and voltages. Each current is decomposed into a direct axis component and quadrature component, with the latter identified with the term (j). The s-domain equations of the motor are obtained from its equivalent circuit using Kirchhoff's laws. The equations relative to the stator axes and including complex terms are

$$\begin{bmatrix} v_s \\ v_r \end{bmatrix} = \begin{bmatrix} R_s + sL_s & sL_m \\ (s - j\omega_r)L_m & R_r + (s - j\omega_r)L_r \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix}$$

where R_s (R_r) = stator (rotor) resistance

L_s, L_r, L_m = stator, rotor, mutual inductance, respectively,

ω_r = rotor angular velocity.

(a) Write the state equations for the induction motor.

(b) Without obtaining the eigenvalues of the state matrix, show that the two eigenvalues are not complex conjugate.

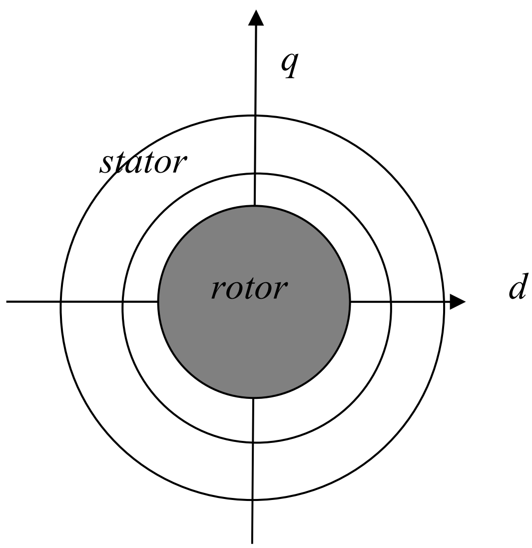


Figure P7.1. Stator frame for the induction motor.

Solution

(a) We rewrite the equation as

$$s\begin{bmatrix} L_s & L_m \\ L_m & L_r \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix} = \begin{bmatrix} R_s & 0 \\ j\omega_r L_m & R_r - j\omega_r L_r \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix} - \begin{bmatrix} v_s \\ v_r \end{bmatrix}$$

We multiply by the inverse of the coefficient matrix of the LHS to obtain

$$s\begin{bmatrix} i_s \\ i_r \end{bmatrix} = \frac{1}{L_sL_r - L_m^2} \begin{bmatrix} L_s & L_m \\ -L_m & L_r \end{bmatrix} \left\{ \begin{bmatrix} R_s & 0 \\ j\omega_r L_m & R_r - j\omega_r L_r \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix} - \begin{bmatrix} v_s \\ v_r \end{bmatrix} \right\}$$

We expand and inverse Laplace transform to obtain the state equation

$$\begin{bmatrix} \frac{di_s}{dt} \\ \frac{di_r}{dt} \end{bmatrix} = \frac{1}{L_sL_r - L_m^2} \begin{bmatrix} L_sR_s - j\omega_r L_m^2 & -L_mR_r + j\omega_r L_m L_r \\ -L_mR_s + j\omega_r L_m L_r & L_rR_r - j\omega_r L_r^2 \end{bmatrix} \begin{bmatrix} i_s \\ i_r \end{bmatrix} - \frac{1}{L_sL_r - L_m^2} \begin{bmatrix} L_s & -L_m \\ -L_m & L_r \end{bmatrix} \begin{bmatrix} v_s \\ v_r \end{bmatrix}$$

(b) The characteristic equation is in the form

$$s^2 + a_1 s + a_0 = 0$$

where

$$a_1 = \frac{1}{L_sL_r - L_m^2}\left(L_sR_s - j\omega_r L_m^2 + L_rR_r - j\omega_r L_r^2\right)$$

$$a_0 = \frac{1}{(L_sL_r - L_m^2)^2}\left\{(L_sR_s - j\omega_r L_m^2)(L_rR_r - j\omega_r L_r^2) - (L_rR_r - j\omega_r L_r^2)(-L_mR_s + j\omega_r L_m L_r)\right\}$$

The eigenvalues of the system are

$$s_{1,2} = \frac{1}{2}\left\{-a_1 \pm \sqrt{a_1^2 - 4a_0}\right\}$$

Clearly, both eigenvalues include the same complex terms in a_i and cannot be complex conjugate.

7.16 In many practical applications, the output sampling in a digital control system is not exactly synchronized with the input transition. Show that the output equation corresponding to output sampling at

$t_k = kT + \Delta_k, k = 0,1,2, \ldots$ is
$y(k) = C(k)x(k) + D(k)u(k)$
where
$C(k) = Ce^{A\Delta_k}$
$D(k) = CB_d(\Delta_k) + D$
$B_d(\Delta_k) = \int_0^{\Delta_k} e^{A\tau} B\, d\tau$

- If the direct transmission matrix D is zero, does the input directly influence the sampled output?
- When is the resulting input-output model time-invariant?

Solution
The state of the continuous time system at the output sampling points is
$$x(t_k) = e^{A\Delta_k}x(k) + \left[\int_0^{\Delta_k} e^{A\tau} B\, d\tau\right]u(k)$$

Substituting in the output equation gives the expression for the sampled output.
Evaluating the matrix $D(k)$ when D is zero gives $D(k) = CB_d(\Delta_k)$, which is nonzero. Thus, the input influences the sampled output even if the direct transmission matrix is zero.
The system is time invariant if the output matrix $C(k)$ and the direct transmission matrix $D(k)$ are both constant. This occurs if the output sampling delay Δ_k is constant and the two matrices reduce to
$$C = Ce^{A\Delta}$$
$$D = CB_d(\Delta) + D$$
$$B_d(\Delta) = \int_0^{\Delta} e^{A\tau} B\, d\tau$$

Computer Exercises

7.17 Write computer programs to simulate the systems of Problem 7.1 for various initial conditions with zero input and discuss your results referring to the solutions of Example 7.4. Obtain plots of the phase trajectories for any second order system.

The following function is for the system of part (i) but can be easily modified to simulate other systems

```
% Function for use with ode45 to solve a differential equation
% [T,Y] = ode45('state',[0 3],[0 1]);% time span 0-3, x(0)=[0,1]

function dx = state(t,x)
          dx = zeros(2,1);    % a column vector
     dx(1) = sin(t) + x(2);
     dx(2) = -2*x(2);
```

To obtain a state-plane plot use the commands

```
>> [T,Y] = ode45('state',[0 3],[0 1]);
>> plot(Y(:,1),Y(:,2))
```

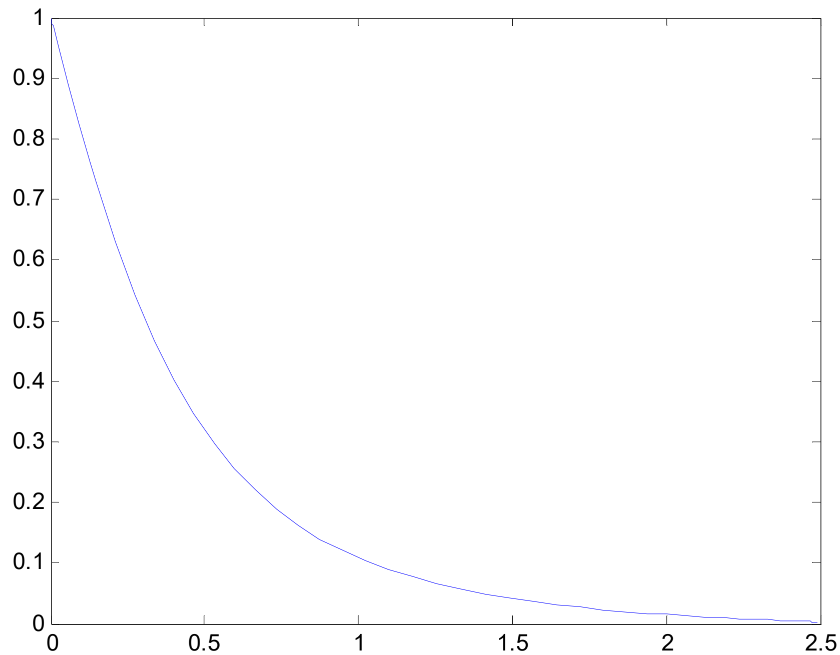


Figure P7.2 State-space trajectories for Problem 7.12.

7.18 Write a program to obtain the state-transition matrix using the Leverrier algorithm.

```
% constit:  program to calculate the constituent matrices
% Calls the function leverrier: executes the Leverrier algorithm
% Z= constitutent matrices
% c_e = coefficients of characteristic polynomial
% characteristic equation
% lambda = vector of eigenvalues of A
% P = cell containing the matrix coefficient of the numerator of the
resolvent matrix
% a = vector of coefficients of the characteristic equation (up to n-1)
% e = measure of error
%
function [Z,lambda,P,a,norme]=constit(A)
%-------check if the matrix is square-------------------
[P,a,norme]=leverrier(A); % Execute the Leverrier algorithm
[n,n]=size(A);
%--------Partial fraction expansion---------------------
%--------check for repeated and complex roots-----------
c_e=[1,a]; % Append coefficient of s^n = 1
%--------Partial fraction ------------------------------
num=zeros(1,n); q={};
for i=1:n
    num1=num; num1(i)=1; % num is s^(i-1)
    [q{i},lambda,k]=residue(num1,c_e);
```

```
end
%-------Computing the Z matrices ----------------------
for i=1:n
    Z{i}=0;
    for j=1:n
        Z{i}=Z{i}+q{j}(i)*P{j};
    end
end

% leverrier: executes the Leverrier algorithm to compute the resolvent
% matrix
% P = cell containing the matrix coefficient of the numerator of the
resolvent matrix
% a = vector of coefficients of the characteristic equation (up to n-1)
% e = measure of error
%
function [P,a,norme]=leverrier(A)

%-------Check if the matrix is square-------------------
[n,m]=size(A);
if n~=m
    error('The matrix must be square')
end
%-------Leverrier algorithm intialization--------------
In=eye(n);P{1}=In; a(1)=-trace(A);

for i=1:n-1
   P{i+1}=P{i}*A+a(i)*In;
   a(i+1)=-trace(P{i+1}*A)/(i+1);
end
% %-------Checking for accuracy-------------------------
e=P{n}*A+a(n)*In; % Ideally zero
norme=norm(e,1);
```

7.19 Simulate the systems of Problem 7.3(i-iii) with the initial conditions of 7.4 and obtain state-trajectory plots with one state variable fixed for each system.

The simplest approach to getting the simulation results is to use the MATLAB command **initial** with a state-space model. Any choice of input matrix is acceptable, with output matrix $C = I$ and d=0. The following commands will yield the step response:

```
>> p=ss(a,b,c,0); % Define the steady-space model
>> x0=[1;1;0]; % Initial conditions
>> [x,t]=initial(p,x0); % Zero-input response
>> x1=x(:,1);x2=x(:,2);
>> plot(x1,x2)
```

Initial conditions: $[1, 1, 0]^T$ and $[1, 0, 0]^T$.

(i) $A = diag\{-3, -5, -7\}$ (the second initial condition vector is an eigenvector).

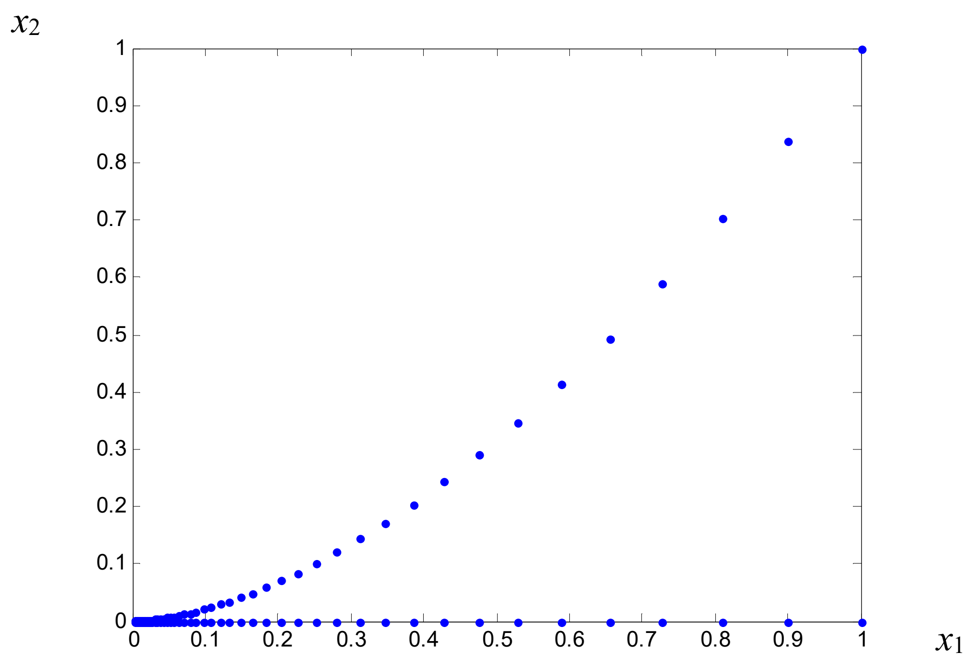


Figure P7.3 State-space trajectories for Problem 7.13(i).

(ii) $$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -6 & 0 & 0 \end{bmatrix}$$ (the second initial condition vector is an eigenvector).

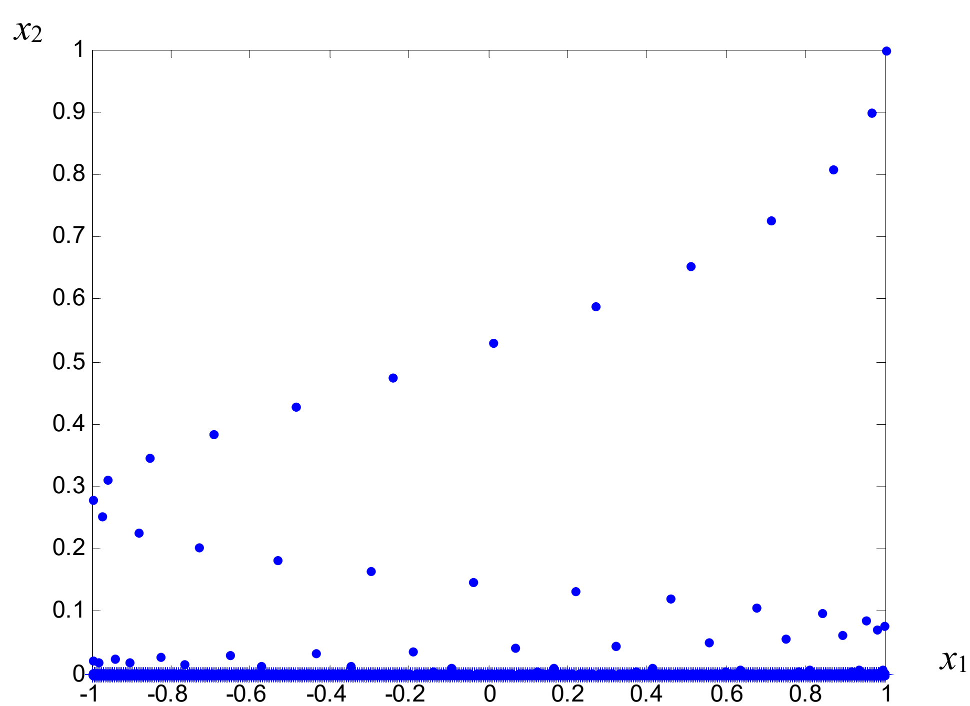


Figure P7.4 State-space trajectories for Problem 7.13(ii).

(iii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$ (the second initial condition vector is an equilibrium point).

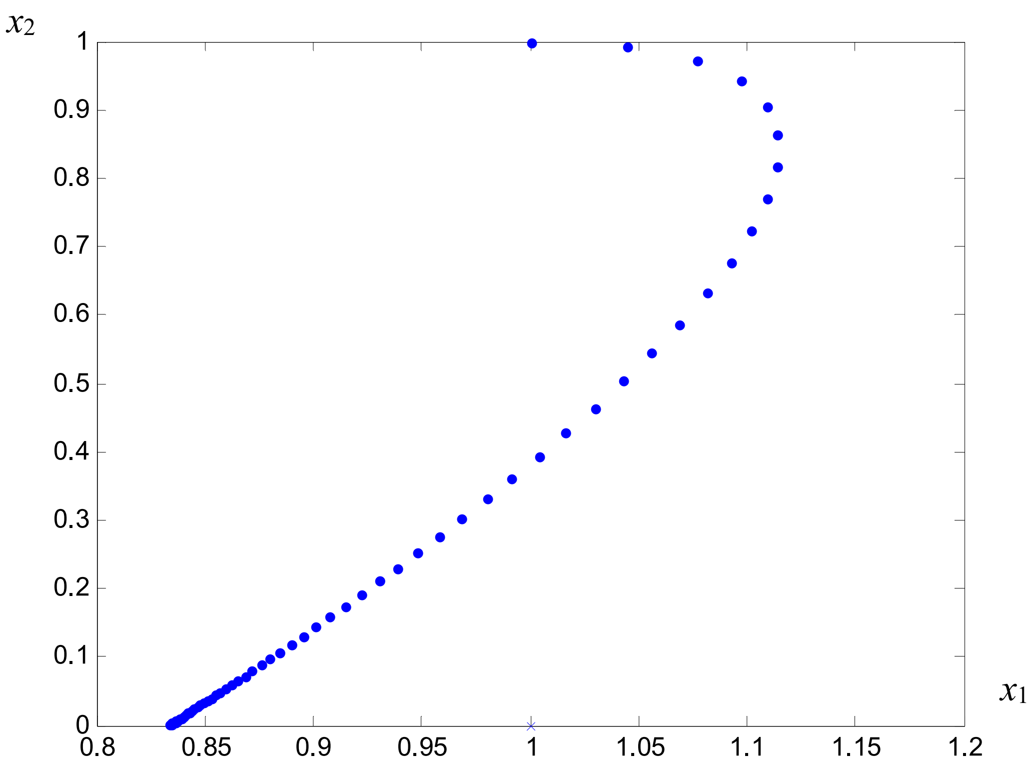


Figure P7.5 State-space trajectories for Problem 7.13(iii).

7.20 Repeat 7.5 using a CAD package for two acceptable choices of the sampling period and compare the resulting systems.

Since we only need the state equation, any choice of output equation is acceptable. For all three systems, $T = 0.01$ s is acceptable.

(i) $A = diag\{-3, -5, -7\}$ Here 0.01 is much smaller than the smallest time constant 1/7.

```
>> pd=c2d(p,.01)

a =
          x1      x2      x3
   x1  0.9704       0       0
   x2       0  0.9512       0
   x3       0       0  0.9324

b =
           u1
   x1       0
```

```
  x2        0
  x3  0.009658

c =
      x1  x2  x3
  y1   1   0   0

d =
      u1
  y1   0

Sampling time: 0.01
Discrete-time model.
```

(ii) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -6 & 0 & 0 \end{bmatrix}$

The state matrix has eigenvalues at $\{-1, \pm j2.4495\}$ and T=0.01 is sufficiently fast sampling.

```
>> pd=c2d(p,.01)

a =
           x1        x2        x3
  x1    0.9997         0  0.009999
  x2         0      0.99         0
  x3  -0.05999         0    0.9997

b =
           u1
  x1    5e-005
  x2         0
  x3  0.009999

c =
      x1  x2  x3
  y1   1   0   0

d =
      u1
  y1   0

Sampling time: 0.01
Discrete-time model.
```

(iii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}$

The state matrix has eigenvalues at {0,-2,-3}and T=0.01 is sufficiently fast sampling for the two nonzero eigenvalues. The first eigenvalue is zero and yields a constant term for any sampling period for both the state and input matrices.

```
>> pd=c2d(p,.01)
```

```
a =
            x1         x2         x3
   x1        1   0.009999  4.917e-005
   x2        0     0.9997   0.009753
   x3        0   -0.05852     0.9509

b =
            u1
   x1  1.646e-007
   x2  4.917e-005
   x3    0.009753

c =
       x1  x2  x3
   y1   1   0   0

d =
       u1
   y1   0

Sampling time: 0.01
Discrete-time model.
```

7.21 Simulate the river pollution system of Problem 7.13 for the normalized parameter values of $k_1 = 1$, $k_2 = 2$, with a sampling period T=0.01 s for the initial conditions $\mathbf{x}^T(0) = [1, 0], [0,1], [1, 1]$, and plot all the results together.

Using the program used for the solution of Problem 7.15, we obtain the trajectories of Figure P7.. all of which are in the first quadrant of the state plane since the system is positive.

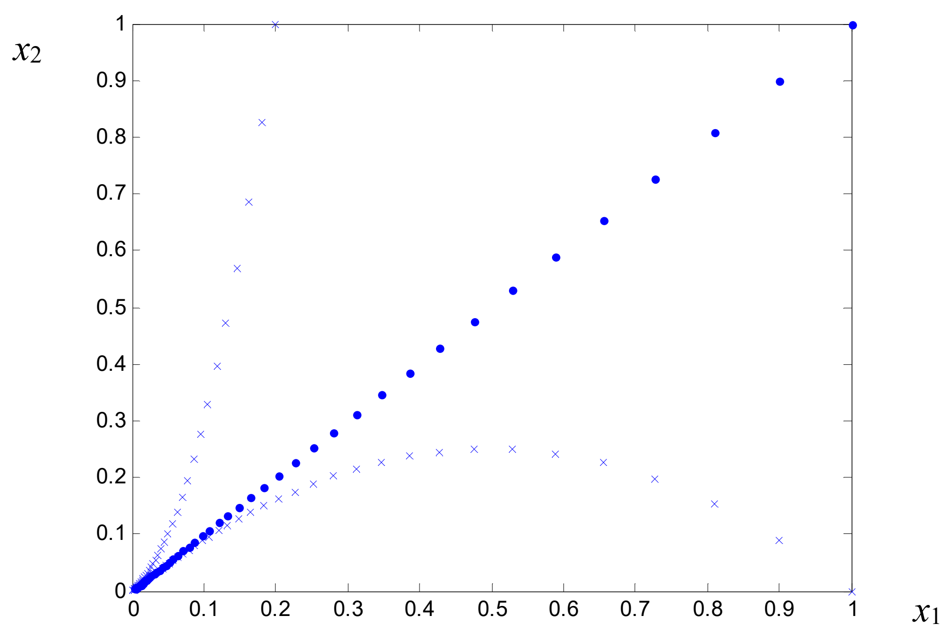


Figure P7.6 Zero-input responses of river pollution system.

7.22 Repeat Problem 7.14 using a CAD package.

```
>> a=[-1.4,-6.9,0;-1.9,-.48,0;0,10,0]/10

a =

   -0.1400   -0.6900         0
   -0.1900   -0.0480         0
         0    1.0000         0

>> b=[.56;-2.3;0]/10

b =

    0.0560
   -0.2300
         0

>> c=[1,0,0;0,1,0]

c =

     1     0     0
     0     1     0

>> p=ss(a,b,c,0)

a =
           x1      x2      x3
   x1   -0.14   -0.69       0
   x2   -0.19  -0.048       0
   x3       0       1       0

b =
          u1
   x1  0.056
   x2  -0.23
   x3      0

c =
       x1  x2  x3
   y1   1   0   0
   y2   0   1   0

d =
       u1
   y1   0
   y2   0

Continuous-time model.
>> pd=c2d(p,.05)

a =
              x1          x2          x3
   x1      0.9932    -0.03434           0
```

```
   x2   -0.009456      0.9978           0
   x3  -0.0002368     0.04994           1

b =
              u1
   x1    0.002988
   x2     -0.0115
   x3  -0.0002875

c =
       x1  x2  x3
   y1   1   0   0
   y2   0   1   0

d =
       u1
   y1   0
   y2   0

Sampling time: 0.05
Discrete-time model.
```

Chapter 8 Solutions

8.1 Find the equilibrium state and the corresponding output for the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

when

(i) $u(k)=0$
(ii) $u(k)=1$

(i) The equilibrium state is trivially at the origin and the corresponding output is $y(k)=0$.
(ii) The equilibrium state can be determined by solving the equation

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which reduces to

$$\mathbf{x_e} = \left(I_2 - \begin{bmatrix} 0 & 1 \\ -0.5 & -0.1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 0.625 \end{bmatrix}$$

The corresponding output is $y(k)=0.625+0.625=1.25$

8.2 A mechanical system has the state-space equations

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.5 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathbf{u}(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

where a_1 is dependent on viscous friction.

(a) Using the results of Chapter 4, determine the range of the parameter a_1 for which the system is internally stable.
(b) Predict the dependence of the parameter a_1 on viscous friction and use physical arguments to justify your prediction (**Hint:** friction dissipates energy and helps the system reach its equilibrium).

The state matrix is in companion form and the characteristic polynomial is

$$\lambda^2 + a_1 \lambda + 0.5$$

The stability condition of Chapter 4 for this polynomial are

i) $1 - a_1 + 0.5 > 0$
ii) $1 - 0.5 > 0$

iii) $1 + a_1 + 0.5 > 0$

Condition (ii) is always met. Combining conditions (i) and (iii) gives

$$1.5 < a_1 < 1.5$$

As the magnitude of the friction increases, the system dissipates more energy and becomes more stable. We predict that as friction increases, the magnitude of a_1 increases.

8.3 Determine the internal stability and the input-output stability of the following linear systems:

i)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 0.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \mathbf{u}(k)$$

$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The state matrix is triangular and its eigenvalues are {0.1, 0.2}, both of which are inside the unit circle. The system is internally stable. Any internally stable system is also input-output stable.

ii)

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -0.2 & .2 & 0 \\ 0 & 1 & 0.1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}(k)$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

The state matrix is triangular and its eigenvalues are {–0.2, −1, 1}, two of which are on the unit circle. The system is not internally stable.

To determine internal stability we find the transfer function

$$G(z) = C\left(zI_3 - A\right)^{-1} B$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z+0.2 & -0.2 & 0 \\ 0 & z-1 & -0.1 \\ 0 & 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} z^2 - 0.98 \\ 0.02 \end{bmatrix}}{(z+0.2)(z-1)(z+1)}$$

The unstable poles are present in the transfer function and the system is not input-output stable.

iii)
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 1 & 0.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \mathbf{u}(k)$$
$$y(k) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The state matrix has the eigenvalues {–0.4, 0.7}, both of which are inside the unit circle. The system is internally stable and is therefore input-output stable.

iv)
$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -0.1 & -0.3 & 0 \\ 0.1 & 1 & 0.1 \\ 0.3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}(k)$$
$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

The state matrix has the eigenvalues {–0.0627, 0.9676, –1.0049}, one of which is outside the unit circle. The system is not internally stable.

To determine internal stability we find the transfer function

$$\begin{aligned} G(z) &= C(zI_3 - A)^{-1} B \\ &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} z+0.1 & 0.3 & 0 \\ -0.1 & z-1 & -0.1 \\ -0.3 & 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \frac{\begin{bmatrix} (z+1.3)(z-1.3) \\ (z+0.1)(z-1) \end{bmatrix}}{(z+1.005)(z-0.9676)(z+0.0627)} \end{aligned}$$

The unstable poles are present in the transfer function and the system is not input-output stable.

8.4 Determine the stable, marginally stable, and unstable modes for each of the unstable systems of Problem 8.3.

i) The stable modes are $(0.1)^k$ and $(0.2)^k$ and there are no unstable modes.
ii) The stable mode is $(-0.2)^k$ and the marginally stable modes are $\{(-1)^k, 1\}$.
iii) The stable modes are $(-0.4)^k$ and $(0.7)^k$ and there are no unstable modes.
iv) The stable modes are $\{(-0.0627)^k, (0.9676)^k\}$ and the unstable mode is $(-1.0049)^k$.

8.5 Determine the controllability and stabilizability of the systems of Problem 8.3.

i) The system is internally stable and therefore stabilizable. The first state equation is decoupled and has no control input term and the mode $(0.1)^k$ is uncontrollable. Clearly, the rank test will yield the same result. The controllability matrix is

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} B_d & A_d B_d \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0.2 & 0.04 \end{bmatrix} \end{aligned}$$

The controllability matrix has rank 1 and the system is not controllable.

ii) The controllability matrix of the system is

$$\mathcal{C} = \left[B_d \;\vdots\; A_d\, B_d \;\vdots\; A_d^2\, B_d\right]$$
$$= \begin{bmatrix} 1 & 0 & -0.2 & 0 & 0.06 & 0.02 \\ 0 & 0 & 0.1 & 0.1 & 0 & 0 \\ 1 & 1 & -0.1 & -0.1 & 1 & 1 \end{bmatrix} T$$

The controllability matrix has rank 3 and the system is controllable and therefore stabilizable.

iii) The system is internally stable and therefore stabilizable. The controllability matrix is

$$\mathcal{C} = \left[B_d \quad A_d\, B_d\right]$$
$$= \begin{bmatrix} 0 & 0.06 \\ 0.2 & 0.04 \end{bmatrix}$$

The controllability matrix has rank 2 and the system is controllable.

iv) The controllability matrix of the system is

$$\mathcal{C} = \left[B_d \;\vdots\; A_d\, B_d \;\vdots\; A_d^2\, B_d\right]$$
$$= \begin{bmatrix} 1 & 0 & -0.4 & 0 & -0.29 & -0.03 \\ 1 & 0 & 1.1 & 0.1 & 1.09 & 0 \\ 0 & 1 & 0.3 & -1 & -0.42 & 1 \end{bmatrix}$$

The controllability matrix has rank 3 and the system is controllable and therefore stabilizable.

8.6 Transform the following system to standard form for uncontrollable systems and use the transformed system to determine if it is stabilizable:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.05 & 0.09 & 0.1 \\ 0.05 & 1.1 & -1 \\ 0.05 & -0.9 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}(k)$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

The controllability matrix of the system is

$$\mathcal{C} = \left[B_d \;\vdots\; A_d\, B_d \;\vdots\; A_d^2\, B_d\right]$$
$$= \begin{bmatrix} 1 & 0 & 0.05 & 0.19 & 0.012 & 0.0285 \\ 0 & 1 & 0.05 & 0.1 & 0.0075 & 0.0195 \\ 0 & 1 & 0.05 & 0.1 & 0.0075 & 0.0195 \end{bmatrix}$$

The second and third rows are identical and the matrix has rank 2. The left eigenvector $\mathbf{w}^T = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ satisfies the condition $\mathbf{w}^T \mathcal{C} = \mathbf{0}^T$. We use it to form the nonsingular transformation matrix whose inverse is

$$T_c^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

This transforms the system to

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.05 & 0.19 & -0.1 \\ 0.05 & 0.1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}(k)$$

$$y = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

8.7 Transform the system to the standard form for unobservable systems and use the transformed system to determine if it is detectable:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.2 & -0.08 \\ 0.125 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mathbf{u}(k)$$

$$y(k) = \begin{bmatrix} 1 & 0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \end{bmatrix} = \begin{bmatrix} 1 & 0.8 \\ -0.1 & -0.08 \end{bmatrix}$$

The second row is equal to the first scaled by –0.1 and the matrix has rank 1. The system has one unobservable mode. The eigenvector

$$\mathbf{v} = \begin{bmatrix} -0.8 & 1 \end{bmatrix}^T$$

satisfies the condition $\mathcal{O}\,\mathbf{v} = \mathbf{0}$. We use it to form the nonsingular transformation matrix

$$T_o = \begin{bmatrix} -0.8 & 0 \\ 1 & 1 \end{bmatrix}$$

then transform the system to

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} -0.1 & 0.1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} -1.25 \\ 1.25 \end{bmatrix} \mathbf{u}(k)$$

$$y(k) = \begin{bmatrix} 0 & 0.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The system has an unobservable eigenvalue at –0.1, inside the unit circle. The system is detectable. The second system eigenvalue, also at –0.1, is observable.

8.8 Determine the controllability and stabilizability of the systems of Problem 8.3 with the input matrices changed to:

i) $B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$

The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B_d & A_d\, B_d \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$$

The controllability matrix has rank 2 and the system is controllable and therefore stabilizable.

ii) $B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$

To use the eigenvector test we determine the left eigenvectors and evaluate the product

$$WB = \begin{bmatrix} 0 & 0 & 1 \\ \hline 0 & 1 & 0.05 \\ \hline 0.986 & -0.164 & -0.021 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \hline 1 \\ \hline 0.822 \end{bmatrix}$$

The first left eigenvector corresponds to the eigenvalues -1 and it yields a zero product. Hence the mode $(-1)^k$ is not controllable and the system is not stabilizable since the eigenvalues is not inside the unit circle.

iii) $B = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$

The controllability matrix is

$$\mathcal{C} = \begin{bmatrix} B_d & A_d\, B_d \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}$$

The controllability matrix has rank 2 and the system is controllable and therefore stabilizable.

iv) $B = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$

The controllability matrix of the system is

$$\mathcal{C} = \left[B_d \;\vdots\; A_d\, B_d \;\vdots\; A_d^2\, B_d \right] = \begin{bmatrix} 1 & -0.1 & -0.05 \\ 0 & 0.2 & 0.12 \\ 1 & -0.7 & 0.67 \end{bmatrix}$$

The controllability matrix has rank 3 and the system is controllable and therefore stabilizable.

8.9 An engineer is designing a control system for a chemical process with reagent concentration as the sole control variable. After determining that the system is not controllable, why is it impossible for him to control all the modes of the system by an innovative control scheme using the same control variables? Explain and suggest an alternative solution to the engineer's problem.

Controllability is a structural property and no control will affect all the modes if the system is not controllable. The engineer must select an additional control variable to make the system controllable. The choice of the control variable will depend on the process.

8.10 The engineer of Problem 8.9 examined the chemical process more carefully and discovered that all the uncontrollable modes with concentration as control variable are asymptotically stable with sufficiently fast dynamics. Why is it possible for the engineer to design an acceptable controller with reagent concentration as the only control variable? If such a design is possible, give reasons for the engineer to prefer it over a design requiring additional control variables.

The process is not controllable but is stabilizable. Furthermore, the uncontrollable modes are sufficiently fast and do not require control action to speed up their dynamics. The engineer can therefore design a satisfactory controller with reagent concentration as the only control variable. Because using one control variable requires less hardware, the cost of the controller with one control variable is less. Engineers always prefer the least costly solution to achieve their design objectives.

8.11 Determine the observability and detectability of the systems of Problem 8.3.

i) The system is internally stable and therefore detectable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.1 & 0.2 \end{bmatrix}$$

The matrix has rank 2 and the system is observable.

ii) We determine the eigenvectors of the state matrix and use the eigenvector test. The system is observable and therefore detectable.

$$CV = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.1644 & 0.0125 \\ 0 & 0.9864 & -0.0499 \\ 0 & 0 & 0.9987 \end{bmatrix} = \begin{bmatrix} 1 & 0.1644 & 0.0125 \end{bmatrix}$$

There are no zero products for an eigenvector with the input matrix and the system is observable. Note that the zeros in the output matrix do not indicate loss of observability since the system is not in diagonal form. The system is detectable since it is observable.

iii) The system is internally stable and therefore detectable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.1 & 0.5 \end{bmatrix}$$

The matrix has rank 2 and the system is observable.

iv) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline CA_d^2 \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0.2 & -0.3 & -1 \\ -0.35 & -0.36 & 0.97 \end{bmatrix}$$

The matrix has rank 3 and the system is observable and therefore detectable.

8.12 Repeat Problem 8.11 with the output matrices

i) $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

The system is internally stable and therefore detectable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0.2 \end{bmatrix}$$

The matrix has rank 2 and the system is observable.

ii) $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline CA_d^2 \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0.1 \\ 0 & 1 & 0 \end{bmatrix}$$

The matrix has rank 2 and the system is not observable. The product of the matrix with the eigenvector $[1\ 0\ 0]^T$ corresponding to the eigenvalue –0.2 is the only one that is equal to zero. The eigenvalues is inside the unit circle and the system is detectable.

iii) $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

The system is internally stable and therefore detectable. The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA_d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.3 \end{bmatrix}$$

The matrix has rank 2 and the system is observable.

iv) $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline CA_d^2 \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & -0.3 & 0 \\ -0.02 & -0.27 & -0.03 \end{bmatrix}$$

The matrix has rank 3 and the system is observable and therefore detectable.

8.13 Consider the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.005 & -0.11 & -0.7 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix} \quad d = 0$$

(a) Can we design a controller for the system that can influence all its modes?
(b) Can we design a controller for the system that can observe all its modes?

Justify your answers using the properties of the system.

(a) The system is in controllable form and is therefore controllable. Hence, we can design a controller for the system that can influence all its modes.

(b) The observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline CA_d^2 \vdots \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 0.5 & 1 \\ -0.005 & -0.11 & -0.2 \end{bmatrix}$$

The matrix has rank 2 and the system is not observable. We cannot design an observer that observes all the system modes.

8.14 Consider the system (A, B, C) and the family of systems $(\alpha A, \beta B, \gamma C)$ with each of (α, β, γ) nonzero.

a) Show that if λ is an eigenvalue of A with right eigenvector $\mathbf{v}$ and left eigenvector $\mathbf{w}^T$ then $\alpha\lambda$ is an eigenvalue of αA with right eigenvector $\mathbf{v}/\alpha$ and left eigenvector $\mathbf{w}^T/\alpha$.

b) Show that (A, B) is controllable if and only if $(\alpha A, \beta B)$ is controllable for any nonzero constants (α,β).

c) Show that system (A, C) is observable if and only if $(\alpha A, \gamma C)$ is observable for any nonzero constants (α,γ).

a) For any eigenvalue λ we have

$$A\mathbf{v} = \lambda\mathbf{v} = \alpha A(\mathbf{v}/\alpha) = \alpha\lambda(\mathbf{v}/\alpha)$$
$$\mathbf{w}^T A = \lambda\mathbf{w}^T = (\mathbf{w}^T/\alpha)\alpha A = \alpha\lambda(\mathbf{w}^T/\alpha)$$

b) Using the eigenvector test for controllability we have

$$\mathbf{w}^T B \neq 0 \Leftrightarrow (\mathbf{w}^T/\alpha)\beta B \neq 0$$

c) Using the eigenvector test for observability we have

$$C\mathbf{v} \neq 0 \Leftrightarrow \gamma C(\mathbf{v}/\alpha) \neq 0$$

8.15 Show that any system in controllable form is controllable.

The controllable realization is given by

$$\mathbf{x}(k+1) = \left[\begin{array}{c|cccc} \mathbf{0}_{n-1\times 1} & & I_{n-1} & & \\ \hline -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{array}\right]\mathbf{x}(k) + \left[\begin{array}{c} \mathbf{0}_{n-1\times 1} \\ \hline 1 \end{array}\right]u(k)$$

We substitute in the controllability matrix

$$\mathcal{C} = \left[B_d \;\vdots\; A_d B_d \;\vdots\; \ldots \;\vdots\; A_d^{n-1} B_d\right]$$
$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & -a_{n-1} \\ \vdots & \vdots & ⋰ & \vdots & x \\ 0 & 1 & x & x & x \\ 1 & -a_{n-1} & x & x & x \end{bmatrix}$$

where x denotes terms that depend on the entries of the last row of the state matrix. The matrix is clearly full-rank and the system is controllable.

8.16 Show that any system in observable form is observable.

Recall that the observable form can be obtained using the state-space matrices of the controllable form by (i) transposing the state matrix, and (ii) transposing and interchanging the input and output matrices. Hence the observability matrix is given by

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA_d \\ \hline \vdots \\ \hline CA_d^{n-1} \end{bmatrix} = \begin{bmatrix} B_c^T \\ \hline B_c^T A_c^T \\ \hline \vdots \\ \hline B_c^T \left(A_c^T\right)^{n-1} \end{bmatrix} = \mathcal{C}_c^T$$

where the subscript c denotes the matrices for controllable form. The transpose of a full-rank matrix is full-rank and the system is observable. The result can be obtained directly using the input and state matrices of the observable form.

8.17 Obtain state-space representations for the following linear systems:

a) In controllable form.
b) In observable form.
c) In diagonal form.

i) $$G(z) = 3\frac{z+0.5}{(z-0.1)(z+0.1)}$$

a) We rewrite the transfer function in the form $G(z) = \dfrac{3z+1.5}{z^2-0.01}$

The matrices for the realization are

$$A = \begin{bmatrix} 0 & 1 \\ 0.01 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 1.5 & 3 \end{bmatrix} \quad \mathbf{d} = 0$$

b) Transpose A, interchange and transpose **b**and $\mathbf{c}^T$ to obtain

$$A = \begin{bmatrix} 0 & 0.01 \\ 1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{d} = 0$$

c) The partial fraction expansion of the transfer function is

$$G(z) = \frac{9}{z-0.1} + \frac{(-6)}{z+0.1}$$

The matrices for the parallel realization are

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 9 & -6 \end{bmatrix} \quad \mathbf{d} = 0$$

ii) $$G(z) = 5\frac{z(z+0.5)}{(z-0.1)(z+0.1)(z+0.8)}$$

a) We rewrite the transfer function in the form

$$G(z) = \frac{5z^2 + 2.5z + 0}{z^3 + 0.8z^2 - 0.01z - 0.008}$$

The controllable realization is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.008 & 0.01 & -0.8 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 2.5 & 5 \end{bmatrix} \qquad \mathbf{d} = 0$$

b) Transpose A, interchange and transpose **b**and $\mathbf{c}^T$ to obtain

$$A = \begin{bmatrix} 0 & 0 & 0.008 \\ 1 & 0 & 0.01 \\ 0 & 1 & -0.8 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 2.5 \\ 5 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{d} = 0$$

c) The partial fraction expansion of the transfer function is

$$G(z) = \frac{1.6667}{z-0.1} + \frac{1.4286}{z+0.1} + \frac{1.9048}{z+0.8}$$

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.8 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 1.6667 & 1.4286 & 1.9048 \end{bmatrix} \qquad \mathbf{d} = 0$$

iii) $$G(z) = \frac{z^2(z+0.5)}{(z-0.4)(z-0.2)(z+0.8)}$$

a) We rewrite the transfer function in the form

$$G(z) = \frac{z^3 + 0.5z^2}{z^3 + 0.2z^2 - 0.4z + 0.064} = 1 + \frac{0.3z^2 + 0.4z - 0.064}{z^3 + 0.2z^2 - 0.4z + 0.064}$$

The controllable realization is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.064 & 0.4 & -0.2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} -0.064 & 0.4 & 0.3 \end{bmatrix} \quad \mathbf{d} = 1$$

b) Transpose A, interchange and transpose **b**and $\mathbf{c}^T$ to obtain

$$A = \begin{bmatrix} 0 & 0 & -0.064 \\ 1 & 0 & 0.4 \\ 0 & 1 & -0.2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -0.064 \\ 0.4 \\ 0.3 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{d} = 1$$

c) The partial fraction expansion of the transfer function is

$$G(z) = \frac{0.6}{z-0.4} + \frac{(-0.14)}{z-0.2} + \frac{(-0.16)}{z+0.8}$$

$$A=\begin{bmatrix}0.4 & 0 & 0\\ 0 & 0.2 & 0\\ 0 & 0 & -0.8\end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \mathbf{c}^T=\begin{bmatrix}0.6 & -0.14 & -0.16\end{bmatrix} \qquad \mathbf{d}=1$$

iv) $$G(z)=\frac{z(z-0.1)}{z^2-0.9z+0.8}$$

a) We rewrite the transfer function in the form

$$G(z)=\frac{z^2-0.1z}{z^2-0.9z+0.8}=1+\frac{0.8z-0.8}{z^2-0.9z+0.8}$$

The controllable realization is

$$A=\begin{bmatrix}0 & 1\\ -0.8 & 0.9\end{bmatrix} \quad \mathbf{b}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \mathbf{c}^T=\begin{bmatrix}-0.8 & 0.8\end{bmatrix} \qquad \mathbf{d}=1$$

b) Transpose A, interchange and transpose **b**and $\mathbf{c}^T$ to obtain

$$A=\begin{bmatrix}0 & -0.8\\ 1 & 0.9\end{bmatrix} \qquad \mathbf{b}=\begin{bmatrix}-0.8\\ 0.8\end{bmatrix} \qquad \mathbf{c}^T=\begin{bmatrix}0 & 1\end{bmatrix} \qquad \mathbf{d}=1$$

c) The partial fraction expansion of the transfer function is

$$G(z)=\frac{0.4+j0.2846}{z-0..45-j0.773}+\frac{0.4-j0.2846}{z-0..45+j0.773}+1$$

The matrices for the parallel realization are

$$A=\begin{bmatrix}0.45+j0.773 & 0\\ 0 & 0..45-j0.773\end{bmatrix} \quad \mathbf{b}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\mathbf{c}^T=\begin{bmatrix}0.4+j0.2846 & 0.4-j0.2846\end{bmatrix} \qquad \mathbf{d}=1$$

8.18 Obtain the controller form that corresponds to a renumbering of the state variables of the controllable realization (also known as phase variable form) from basic principles.

Follow the same step as phase variable form. We start with the denominator of the transfer function

$$\frac{P(z)}{U(z)}=\frac{1}{z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0}$$

Define the state vector as

$$\begin{aligned}\mathbf{x}(k)&=\begin{bmatrix}x_1(k) & x_2(k) & \ldots & x_{n-1}(k) & x_n(k)\end{bmatrix}^T\\ &=\begin{bmatrix}p(k+n-1) & p(k+n-2) & \ldots & p(k+1) & p(k)\end{bmatrix}^T\end{aligned}$$

We have

$x_i(k+1)=x_{i-1}(k), i=2,\cdots,n$

The first state equation is obtained from the transfer function

$$x_1(k+1) = -a_{n-1}x_1(k)\cdots - a_1 x_{n-1}(k) - a_0 x_n(k) + u(k)$$

In matrix form, we have the state equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ . \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \left[\begin{array}{cccc|c} -a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0 \\ \hline & & I_{n-1} & & \mathbf{0}_{n-1\times 1} \end{array}\right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} \left[\begin{array}{c} 1 \\ \hline \mathbf{0}_{n-1\times 1} \end{array}\right] u(k)$$

The output equation is obtained using the numerator

$$\begin{aligned} Y(z) &= c_n U(z) + G_d(z) U(z) \\ &= c_n U(z) + \frac{c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_1 z + c_0}{z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0} U(z) \\ &= c_n U(z) + \left[c_{n-1}z^{n-1} + c_{n-2}z^{n-2} + \ldots + c_1 z + c_0\right] P(z) \end{aligned}$$

Inverse z-transform to obtain

$$\begin{aligned} y(k) &= \begin{bmatrix} c_{n-1} & c_{n-2} & \ldots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} p(k+n-1) \\ p(k+n-2) \\ . \\ . \\ p(k+1) \\ p(k) \end{bmatrix} + du(k) \\ &= \begin{bmatrix} c_{n-1} & c_{n-2} & \ldots & c_1 & c_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + du(k) \end{aligned}$$

8.19 Obtain the transformation matrix to transform a system in phase variable form to controller form. Prove that the transformation matrix will also perform the reverse transformation.

The transformation to renumber the state variables is

$$\mathbf{x}(k) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & & 0 & 0 \end{bmatrix} \mathbf{z}(k) \Leftrightarrow \mathbf{z}(k) = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & & 0 & 0 \end{bmatrix} \mathbf{x}(k)$$

The transformation matrix is identical to its inverse and hence the same matrix performs the reverse transformation.

8.20 Use the two steps of Section 8.5 to obtain a second observable realization from controller form. What is the transformation that will take this form to the first observable realization of Section 8.5.d.

We transpose the state matrix and we transpose and interchange the input and output matrices

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ . \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} -a_{n-1} & \\ -a_{n-2} & \\ \vdots & I_{n-1} \\ -a_1 & \mathbf{0}_{n-1\times 1} \\ -a_0 & \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ . \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix} + \begin{bmatrix} c_{n-1} \\ c_{n-2} \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & \mathbf{0}_{1\times n-1} \end{bmatrix} \mathbf{x}(k) + du(k)$$

8.21 Show that the observable realization obtained from the phase variable form realizes the same transfer function.

The transfer function of the observable form is the scalar

$$G_o(s) = B^T\left[sI_n - A^T\right]^{-1} C^T + d$$

Using the identities

$$(ABC)^T = C^T B^T A^T \qquad \left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$

we transpose the scalar to obtain the transfer function of the controllable form

$$\begin{aligned} G_o(s) &= B^T\left[sI_n - A^T\right]^{-1} C^T + d \\ &= \left\{B^T\left[sI_n - A^T\right]^{-1} C^T + d\right\}^T \\ &= C\left[sI_n - A\right]^{-1} B + d = G_c(s) \end{aligned}$$

8.22 Show that the transfer functions of the following systems are identical and give a detailed explanation.

$$A = \begin{bmatrix} 0 & 1 \\ -0.02 & 0.3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{d} = 0$$

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} -1 & 2 \end{bmatrix} \quad \mathbf{d} = 0$$

For both systems the transfer function is

$$G(z) = C\left[zI - A\right]^{-1} B + D = \frac{z}{z^2 - 0.3z + 0.02}$$

The partial fraction expansion of the transfer function is

$$G(z) = \frac{-1}{z-0.1} + \frac{2}{z-0.2}$$

This yields the parallel realization of the second system. The transformation matrix relating the two systems can be obtained from their controllability matrices. Using the subscript "c" to denote the first realization (controllable form), the transformation matrix to take the system to the parallel realization is

$$T_c = [\mathbf{b}_c \quad A_c\mathbf{b}_c][\mathbf{b} \quad A\mathbf{b}]^{-1} = \begin{bmatrix} -10 & 10 \\ -1 & 2 \end{bmatrix}$$

$$T_c^{-1}A_cT_c = diag\{0.1, 0.2\} \qquad T_c^{-1}\mathbf{b}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Recall that the MATLAB command **ss2ss** uses the inverse of T_c for similarity transformation.

8.23 Obtain a parallel realization for the transfer function matrix of Example 8-15.

The minimal polynomial of the system is $(z-0.2466)(z-0.2479\cdot 10^{-2})(z-0.3012)$ and has no repeated roots. The partial fraction expansion of the matrix is

$$G(z) = diag\left\{\frac{0.4075(z+0.1067)}{(z-0.2466)(z-0.2479\cdot 10^{-2})}, \frac{0.38607(z+0.4182)}{(z-0.2466)(z-0.3012)}\right\}$$

$$= \frac{\begin{bmatrix} 0.5896 & 0 \\ 0 & -4.7012 \end{bmatrix}}{z-0.2466} + \frac{\begin{bmatrix} -0.1822 & 0 \\ 0 & 0 \end{bmatrix}}{z-0.2479\cdot 10^{-2}} + \frac{\begin{bmatrix} 0 & 0 \\ 0 & 5.0873 \end{bmatrix}}{z-0.3012}$$

The ranks of the partial fraction coefficient matrices, given with the corresponding poles are (2, 0.2466), (1, $0.2479\cdot 10^{-2}$) and (1, 0.3012). The matrices can be factorized as

$$\begin{bmatrix} 0.5896 & 0 \\ 0 & -4.7012 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5896 & 0 \\ 0 & -4.7012 \end{bmatrix}$$

$$\begin{bmatrix} -0.1822 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} -0.1822 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 5.0873 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 5.0873 \end{bmatrix}$$

The parallel realization is given by the quadruple

$$A = \begin{bmatrix} 0.2466 & 0 & 0 & 0 \\ 0 & 0.2466 & 0 & 0 \\ 0 & 0 & 0.2479\cdot 10^{-2} & 0 \\ 0 & 0 & 0 & 0.3012 \end{bmatrix} \qquad B = \begin{bmatrix} 0.5896 & 0 \\ 0 & -4.7012 \\ -0.1822 & 0 \\ 0 & 5.0873 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \qquad D = \mathbf{0}_{2\times 2}$$

Note that the realization is minimal since it is fourth order and the system was determined to have four poles in Example 8-15.

8.24 Find the poles and zeros of the following transfer function matrices

(i) $$G(z)=\begin{bmatrix}\dfrac{z-0.1}{(z+0.1)^2} & \dfrac{1}{z+0.1}\\ 0 & \dfrac{1}{z-0.1}\end{bmatrix}$$ (ii) $$G(z)=\begin{bmatrix}0 & \dfrac{1}{z-0.1}\\ \dfrac{z-0.2}{(z-0.1)(z-0.3)} & \dfrac{1}{z-0.3}\\ 0 & \dfrac{2}{z-0.3}\end{bmatrix}$$

(i) The least common denominator of the entries of the matrix is $(z-0.1)(z+0.1)^2$

The determinant of the matrix is
$$\det[G(z)]=\frac{z-0.1}{(z-0.1)(z+0.1)^2}$$

The least common denominator of all the minors is $(z-0.1)(z+0.1)^2$ and the system has poles at $\{0.1, -0.1, -0.1\}$.

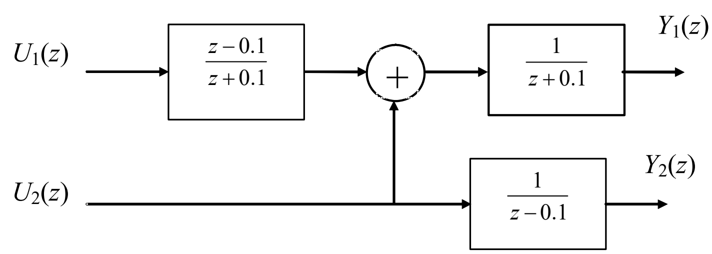


Figure P8.1 Block diagram for the system of Problem 8.24(i).

To obtain the zeros, we rewrite the transfer function matrix with the pole polynomial as the denominator of every entry

$$G(z)=\frac{\begin{bmatrix}(z-0.1)^2 & (z-0.1)(z+0.1)\\ 0 & (z+0.1)^2\end{bmatrix}}{(z-0.1)(z+0.1)^2}$$

Zeros are the roots of the determinant of the transfer function matrix. The determinant of the matrix with cancellation to reduce the denominator to the characteristic polynomial is
$$\det[G(z)]=\frac{(z-0.1)^2(z+0.1)^2}{(z-0.1)^2(z+0.1)^4}=\frac{z-0.1}{(z-0.1)(z+0.1)^2}$$

The system has a zero at 0.1, which is obvious since the transfer function matrix has a zero column at this value. The poles and zeros of the system can also be seen in the block diagram of Figure P8.1

(ii) The least common denominator of the entries of the matrix is $(z-0.1)(z-0.3)$

The nonzero minors of order 2 for the matrix are
$$-\frac{z-0.2}{(z-0.1)^2(z-0.3)},\frac{z-0.2}{(z-0.1)(z-0.3)^2}$$

The least common denominator of all the minors is $(z-0.1)^2(z-0.3)$ and the system has poles at {0.1, 0.1, 0.3, 0.3}.

To obtain the zeros, we rewrite the transfer function matrix with the pole polynomial as the denominator of every entry

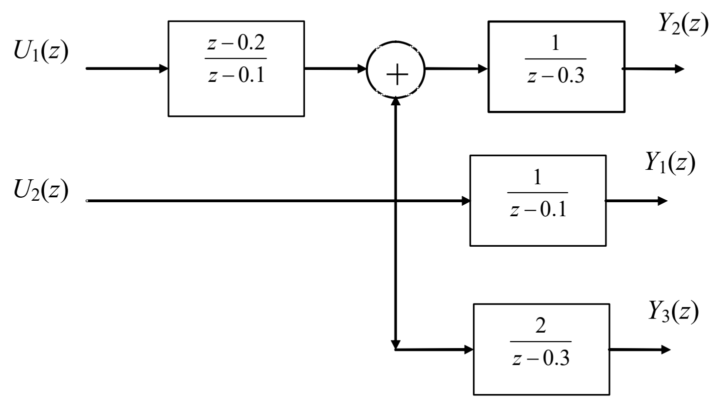


Figure P8.2 Block diagram for the system of Problem 8.24(ii).

$$G(z)=\frac{\begin{bmatrix} 0 & (z-0.1)(z-0.3)^2 \\ (z-0.1)(z-0.2)(z-0.3) & (z-0.1)^2(z-0.3) \\ 0 & 2(z-0.1)^2(z-0.3) \end{bmatrix}}{(z-0.1)^2(z-0.3)^2}$$

Zeros are the roots of the greatest common divisor of all of all nonzero minors of order equal to two. The nonzero minors of order 2 for the matrix are

$$-\frac{(z-0.2)(z-0.3)}{(z-0.1)^2(z-0.3)^2},\ \frac{2(z-0.1)(z-0.2)}{(z-0.1)^2(z-0.3)^2}$$

The system has a zero at 0.2

The poles and zeros of the system can also be seen in the block diagram of Figure P8.1

8.25 Autonomous underwater vehicle (AUV) are robotic submarines that can be used for a variety of studies of the underwater environment. The vertical and horizontal dynamics of the vehicle must be controlled to remotely operate the AUV. The INFANTE is a research AUV operated by the Instituto Superior Tecnico of Lisbon, Portugal. The variables of interest in horizontal motion are the sway speed and the yaw angle. A linearized model of the horizontal plane motion of the vehicle is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -0.14 & -0.69 & 0.0 \\ -0.19 & -0.048 & 0.0 \\ 0.0 & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0.056 \\ -0.23 \\ 0.0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where x_1 is the sway speed, x_2 is the yaw angle, x_3 is the yaw rate and u is the rudder deflection

i) Obtain the zeros of the system using Rosenbrock's system matrix.
ii) Determine the system decoupling zeros by testing its controllability and observability.

Rosenbrock's system matrix for the system gives the equation

$$\begin{bmatrix} -(z_0 I_n - A) & B \\ C & D \end{bmatrix}\begin{bmatrix} \mathbf{x}_w \\ \mathbf{w} \end{bmatrix} = \left[\begin{array}{ccc|c} -0.14 - z_0 & -0.69 & 0 & 0.056 \\ -0.19 & -0.048 - z_0 & 0 & -0.23 \\ 0 & 1 & z_0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]\begin{bmatrix} x_{w1} \\ x_{w2} \\ x_{w3} \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix has rank 3 < 4 at $z_0 = 0$. The controllability matrix of the system is

$$\mathcal{C} = \left[B \mid AB \mid A^2 B\right] = \begin{bmatrix} 0.056 & 0.1509 & -0.0214 \\ -0.23 & 0.0004 & -0.0287 \\ 0 & -0.23 & 0.0004 \end{bmatrix}$$

The matrix is full rank and the system is controllable and therefore has no input-decoupling zeros.

The observability matrix of the system is

$$\mathcal{O} = \begin{bmatrix} C \\ \hline CA \\ \hline CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.14 & -0.690 & 0 \\ -0.19 & -0.048 & 0 \\ 0.1507 & 0.1297 & 0 \\ 0.0357 & 0.1334 & 0 \end{bmatrix}$$

The matrix is has rank 2 < 3 and the system is not observable. The system has an output decoupling zero. Furthermore, the product of the observability matrix and the eigenvector corresponding the eigenvalue zero is zero

$$\mathcal{O}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

Hence, the system has an output-decoupling zero at zero.

The MATLAB commands **zero** and **tzero** also yield the zero value at zero.

8.26 The terminal composition of a binary distillation column uses reflux and steam flow as the control variables. The 2-input-2-output system is governed by the transfer function

$$G(s)=\begin{bmatrix}\dfrac{12.8e^{-s}}{16.7s+1} & \dfrac{-18.9e^{-3s}}{21.0s+1}\\ \dfrac{6.6e^{-7s}}{10.9s+1} & \dfrac{-19.4e^{-3s}}{14.4s+1}\end{bmatrix}$$

Find the discrete transfer function of the system with DAC, ADC, and a sampling period of one time unit then determine the poles and zeros of the discrete-time system.

The time delays are integer multiples of the sampling period and the transfer function can first be obtained with them, then appropriate powers of z can be added to the denominator of each term. Recall that for a single real pole, we have

$$\begin{aligned}G(s)&=\frac{K/\tau}{s+1/\tau}\\ G_{ZAS}(z)&=\left(1-z^{-1}\right)\mathcal{Z}\left\{K\left[\frac{1}{\mathrm{s}}--\frac{1}{s+1/\tau}\right]\right\}\\ &=K\left[1-\frac{z-1}{z-e^{-1/\tau}}\right]\\ &=K\,\frac{1-e^{-1/\tau}}{z-e^{-1/\tau}}\end{aligned}$$

Hence, the transfer function is given by

$$G(z)=\begin{bmatrix}\dfrac{0.744}{z(z-0.9419)} & \dfrac{-0.8789}{z^{3}(z-0.9535)}\\ \dfrac{0.5786}{z^{7}(z-0.9123)} & \dfrac{-1.3015}{z^{3}(z-0.9329)}\end{bmatrix}$$

The l.c.m. of the elements is

$$z^{7}(z-0.9419)(z-0.9536)(z-0.9123)(z-0.9329)$$

The determinant of the transfer function matrix has the denominator polynomial

$$z^{14}(z-0.9419)(z-0.9536)(z-0.9123)(z-0.9329)$$

The system has a pole at the origin with multiplicity 14 and four poles on the positive real axis inside the unit circle at { 0.9123, 0.9329, 0.9419, 0.9535}.

We rewrite the transfer function matrix with the pole polynomial as its denominator

$$G(z)=\frac{\begin{bmatrix}0.744z^{13}(z-0.9536)(z-0.9123)(z-0.9329) & -0.8769z^{11}(z-0.9419)(z-0.9123)(z-0.9329)\\ 0.5786z^{7}(z-0.9419)(z-0.9536)(z-0.9329) & -1.302z^{11}(z-0.9419)(z-0.9536)(z-0.9123)\end{bmatrix}}{z^{14}(z-0.9419)(z-0.9536)(z-0.9123)(z-0.9329)}$$

The determinant of the system is

$$\det[G(z)] = \frac{-0.9683z^{10}(z-0.9536)(z-0.9123)+0.5073z^{4}(z-0.9419)(z-0.9329)}{z^{14}(z-0.9419)(z-0.9536)(z-0.9123)(z-0.9329)}$$

$$= \frac{-0.9683z^{4}(z+0.8986)(z-0.9609)(z^{2}-1.807z+0.8177)(z^{2}+0.8994z+0.8075)(z^{2}-0.8964z+0.8075)}{z^{14}(z-0.9419)(z-0.9536)(z-0.9123)(z-0.9329)}$$

The system has zeros at
$\{0,0,0,0, -0.8986, 0.9609, -0.4497 \pm j\,0.7780,\ 0.4482 \pm j\,0.7789,\ 0.9033 \pm j\,0.0413\}$

The same answer is obtained with the command

>> **zero(gd)**

```
ans =

  -0.8986
  -0.4497 + 0.7780i
  -0.4497 - 0.7780i
   0.4482 + 0.7789i
   0.4482 - 0.7789i
  -0.0000 + 0.0000i
  -0.0000 - 0.0000i
   0.9609
   0.9033 + 0.0413i
   0.9033 - 0.0413i
   0.0000
        0
```

Computer Exercises

8.27 Write computer programs to simulate the second order systems of Problem 8.3 for various initial conditions. Obtain state plane plots and discuss your results referring to the solutions of Examples 8.1 and 8.2.

```
clf
hold on
a=[0.1,0;1,0.2];b=[0;0.2];c=[1,1];%state-space matrices
for ix1=-1:1
    for ix2=-1:1
    x=[ix1;ix2]%ICs
    xa={x}; % Save initial state
        for i=1:30 % Compute states
        x=a*x;
        xa{i+1}=x; % Save state
        end
      for i=1:length(xa)
      x1(i)=xa{i}(1); % First coordinate
      x2(i)=xa{i}(2); % Second coordinate
      end
      plot(x1,x2) % Plot state trajectory
    end
end
```

The computer program yields the phase plots of Figure P8.3 and Figure P8.4.

The phase plots show that all trajectories converge to the origin as expected since the systems are internally stable. The plot for the system of Example 8.1(iii) shows oscillations prior to convergence to the origin because the system has a negative eigenvalues.

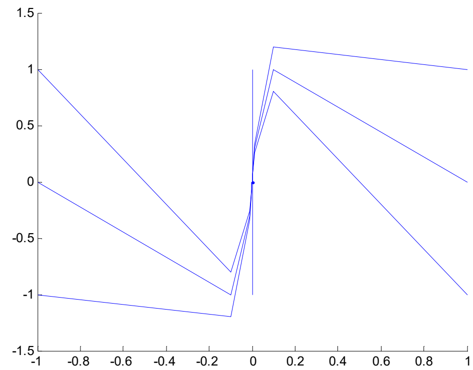


Figure P8.3 Phase plot for Example 8.1(i).

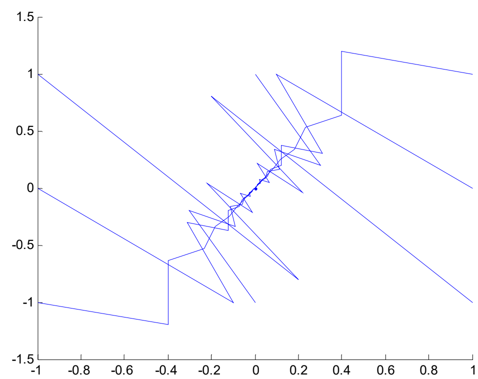


Figure P8.4 Phase plot for Example 8.1(iii).

8.28 Repeat Problem 8.22 using a CAD package. Comment on any discrepancies between CAD results and solution by hand.

$$A = \begin{bmatrix} 0 & 1 \\ -0.02 & 0.3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad \mathbf{d} = 0$$

$$A = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{c}^T = \begin{bmatrix} -1 & 2 \end{bmatrix} \quad \mathbf{d} = 0$$

For both systems the transfer function is

$$G(z) = C[zI - A]^{-1}B + D = \frac{z}{z^2 - 0.3z + 0.02}$$

```
>> a=[0,1;-0.02,0.3];b=[0;1];c=[0,1];
```

```
>> p=ss(a,b,c,0,.01);
>> tf(p)

Transfer function:
  z - 2.776e-017
------------------
z^2 - 0.3 z + 0.02

Sampling time: 0.01

>> a1=[0.1,0;0,0.2];b1=[1;1];c1=[-1,2];
>> p1=ss(a,b,c,0,.01);
>> tf(p1)

Transfer function:
  z - 2.776e-017
------------------
z^2 - 0.3 z + 0.02

Sampling time: 0.01
```

Note that the computer results include the term 2.776e-17 which is the result of computational zeros and is clearly negligible.

The partial fraction expansion of the transfer function is obtained using

```
>> [k,poles,rem]=residue([1,0],[1,-.3,.02])

k =

    2.0000
   -1.0000


poles =

    0.2000
    0.1000


rem =

     []
```

The partial fraction expansion of the transfer function is

$$G(z) = \frac{-1}{z-0.1} + \frac{2}{z-0.2}$$

This yields the parallel realization of the second system. The transformation matrix relating the two systems can be obtained from their controllability matrices. Using the subscript "c" to denote the first realization (controllable form), the transformation matrix to take the system to the parallel realization is T_c and is obtained using the commands

```
>> con=ctrb(a,b)
```

```
con =

     0    1.0000
  1.0000    0.3000

>> con1=ctrb(a1,b1)

con1 =

  1.0000    0.1000
  1.0000    0.2000

>> Tc=con/con1

Tc =

 -10.0000   10.0000
  -1.0000    2.0000
```

$$T_c = \begin{bmatrix} \mathbf{b}_c & A_c\mathbf{b}_c \end{bmatrix}\begin{bmatrix} \mathbf{b} & A\mathbf{b} \end{bmatrix}^{-1} = \begin{bmatrix} -10 & 10 \\ -1 & 2 \end{bmatrix}$$

$$T_c^{-1}A_cT_c = diag\{0.1, 0.2\} \qquad T_c^{-1}\mathbf{b}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Recall that the MATLAB command **ss2ss** uses the inverse of T_c for similarity transformation.

```
>> pa=ss2ss(p,inv(Tc))

a =
            x1          x2
  x1         0.1           0
  x2  6.939e-017         0.2

b =
      u1
  x1   1
  x2   1

c =
      x1  x2
  y1  -1   2

d =
      u1
  y1   0

Sampling time: 0.01
Discrete-time model.
```

We again observe the term 6.939e-017, which is due to computational errors and is actually zero.

8.29 Write a MATLAB function that determines the equilibrium state of the system with the state matrix A, the input matrix B and a constant input u as input parameters.

```
function equilibriumstate=equilibrium(A,B,u);

order=length(A); % order of the system
equilibriumstate=inv(eye(order)-A)*B*u; % equilibrium state
```

8.30 Select a second order state equation in diagonal form for which some trajectories converge to the origin and others diverge. Simulate the system using SIMULINK and obtain plots for one diverging trajectory and one converging trajectory with suitable initial states.

Solution

We choose the unforced state-space model

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

and assign any values to the remainder of the system matrices in the state-space SIMULINK block. The simulation diagram is shown in Figure P8.24.1

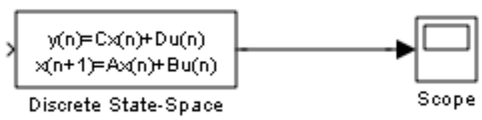


Figure P8.24.1 Simulation diagram for the state-space model.

We simulate the system with no input and with the initial condition vector $[3 \quad 0]^T$ then $[0 \quad 3]^T$. The responses show that the system has an unstable response for the first initial vector and a convergent response for the second. The lesson is that unstable systems may have stable trajectories.

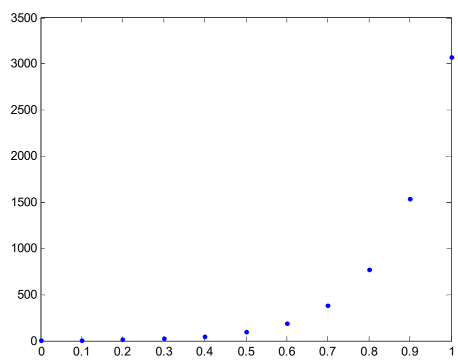


Figure P8.24.1 Response with initial condition vector $[3 \quad 0]^T$

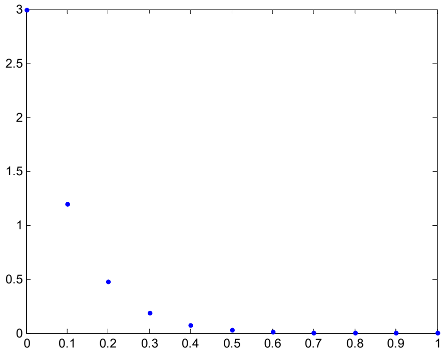


Figure P8.24.1 Response with initial condition vector $[0 \quad 3]^T$

Chapter 9 Problem Solutions

9.1 Show that with the closed-loop quadruple for (A, B, C, D) with the state feedback $\mathbf{u}(k) = -K\mathbf{x}(k)+\mathbf{v}(k)$ is $(A - BK, B, C - DK, D)$

Considering the state equations

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k) + D\mathbf{u}(k)$$

and substituting $-K\mathbf{x}(k)+\mathbf{v}(k)$ to $\mathbf{u}(k)$ we trivially obtain

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\left(-K\mathbf{x}(k) + \mathbf{v}(k)\right) = \left(A - BK\right)\mathbf{x}(k) + B\mathbf{v}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k) + D\left(-K\mathbf{x}(k) + \mathbf{v}(k)\right) = \left(C - DK\right)\mathbf{x}(k) + D\mathbf{u}(k)$$

9.2 Show that a necessary condition for the pair (A, B) with state feedback gain matrix K to have the closed-loop state matrix $A_{cl} = A - BK$ is that for any vector $\mathbf{w}^T$ satisfying $\mathbf{w}^T B = \mathbf{0}^T$, and $\mathbf{w}^T A = \lambda\, \mathbf{w}^T$, A_{cl} must satisfy $\mathbf{w}^T A_{cl} = \lambda\, \mathbf{w}^T$. Explain the significance of this necessary condition (Note that the condition is also sufficient).

The vector $\mathbf{w}^T$ is a left eigenvector of the state matrix A corresponding to an uncontrollable mode. The result follows from premultiplying A_{cl} by $\mathbf{w}^T$ to obtain

$$\mathbf{w}^T A_{cl} = \mathbf{w}^T (A - BK) = \mathbf{w}^T A = \lambda\, \mathbf{w}^T$$

The condition implies that left eigenvectors corresponding to uncontrollable modes are invariant under state feedback.

9.3 Show that a sufficient condition for the pair (A, B) with m by n state feedback gain matrix K to have the closed-loop state matrix $A_{cl} = A - BK$ is

$$\text{rank}\{B\} = \text{rank}\{[A - A_{cl} | B]\} = m$$

Is the matrix K unique for given matrices A and A_{cl}? Explain.

We rewrite the expression for the closed-loop state matrix as

$$A - A_{cl} = BK$$

This is now a linear equation to be solved for the matrix K. The RHS is a linear combination of the columns of the matrix B. Hence, for a solution to the linear system to exist the columns of the matrix on the LHS must also be linear combinations of the columns of B, i.e. must not include any columns that are linearly independent of the columns of B. Since the rank of a matrix is the number of linearly independent columns, we have the condition

$$\text{rank}\{B\} = \text{rank}\{[A - A_{cl} | B]\}$$

Assuming B is full rank and n by m, we have m linearly independent columns and the condition follows. Since K has n columns, there are more unknowns than equations and the solution is not unique except in the rare case where $m = n$, where the matrix K is given by

$$K = B^{-1} \{A - A_{cl}\}$$

9.4 Using the results of Problem 9.3, determine if the closed-loop matrix can be obtained using state feedback for the pair

$$A = \begin{bmatrix} 1.0 & 0.1 & 0 \\ 0 & 1 & 0.01 \\ 0 & -0.1 & 0.9 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(a) $A_{cl} = \begin{bmatrix} 1.0 & 0.1 & 0 \\ -12.2 & -1.2 & 0 \\ 0.01 & 0.01 & 0 \end{bmatrix}$ (b) $A_{cl} = \begin{bmatrix} 0 & 0.1 & 0 \\ -12.2 & -1.2 & 0 \\ 0.01 & 0.01 & 0 \end{bmatrix}$

(a) $$A - A_{cl} = \begin{bmatrix} 1.0 & 0.1 & 0 \\ 0 & 1 & 0.01 \\ 0 & -0.1 & 0.9 \end{bmatrix} - \begin{bmatrix} 1.0 & 0.1 & 0 \\ -12.2 & -1.2 & 0 \\ 0.01 & 0.01 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 12.2 & 2.2 & 0.01 \\ -0.01 & -0.11 & 0.9 \end{bmatrix}$$

The matrix has rank 2 and its columns all have a zero first entry and are linearly dependent on the columns of B. The closed-loop matrix can be obtained using state feedback.

(b) $$A - A_{cl} = \begin{bmatrix} 1.0 & 0.1 & 0 \\ 0 & 1 & 0.01 \\ 0 & -0.1 & 0.9 \end{bmatrix} - \begin{bmatrix} 0 & 0.1 & 0 \\ -12.2 & -1.2 & 0 \\ 0.01 & 0.01 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 12.2 & 2.2 & 0.01 \\ -0.01 & -0.11 & 0.9 \end{bmatrix}$$

The first column of the matrix is linearly independent of the columns of B. The closed-loop matrix cannot be obtained using state feedback.

9.5 Show that a necessary condition for the pair (A, C) with observer gain matrix L to have the observer matrix $A_o = A - LC$ is that for any vector $\mathbf{v}$ satisfying $C\,\mathbf{v} = \mathbf{0}$, and $A\,\mathbf{v} = \lambda\,\mathbf{v}$, A_o must satisfy $A_o\mathbf{v} = \lambda\,\mathbf{v}$. Explain the significance of this necessary condition (Note that the condition is also sufficient).

The vector $\mathbf{v}$ is a right eigenvector of the state matrix A corresponding to an unobservable mode. The result follows from postmultiplying A_o by $\mathbf{v}$ to obtain

$$A_o\,\mathbf{v} = (A - BK)\,\mathbf{v} = A\,\mathbf{v} = \lambda\,\mathbf{v}$$

The condition implies that right eigenvectors corresponding to unobservable modes are invariant in observer design.

9.6 Show that a sufficient condition for the pair (A, C) with n by l observer gain matrix L to have the observer matrix $A_o = A - LC$ is

$$rank\{C\} = rank\left\{\begin{bmatrix} C \\ \hdashline A - A_o \end{bmatrix}\right\} = l$$

Is the matrix L unique for given matrices A and A_o? Explain.

We rewrite the expression for the closed-loop state matrix as

$$A - A_o = LC$$

This is now a linear equation to be solved for the matrix L. The RHS is a linear combination of the rows of the matrix B. Hence, for a solution to the linear system to exist the rows of the matrix on the LHS must also be linear combinations of the rows of C, i.e. must not include any rows that are linearly independent of the rows of C. Since the rank of a matrix is the number of linearly independent rows, we have the condition

$$rank\{C\} = rank\left\{\begin{bmatrix} C \\ ----- \\ A - A_o \end{bmatrix}\right\}$$

Assuming C is full rank and l by l, we have l linearly independent rowss and the condition follows. Since L has n rows, there are more unknowns than equations and the solution is not unique except in the rare case where $l = n$, where the matrix L is given by

$$L = \{A - A_o\}\, C^{-1}$$

9.7 Design a state-feedback control law to assign the eigenvalues to the set {0, 0.1,0.2} for the systems with

a) $$A = \begin{bmatrix} 0.1 & 0.5 & 0 \\ 2 & 0 & 0.2 \\ 0.2 & 1 & 0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$$

The characteristic polynomial of the state matrix is

$\lambda^3 - 0.5\,\lambda^2 - 1.16\,\lambda + 0.4$ i.e. $a_2 = -0.5$, $a_1 = -1.16$, $a_0 = 0.4$

The transformation matrix T_c^{-1} is

$$T_c^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0.5 \\ 1 & 0.5 & 1.41 \end{bmatrix} \begin{bmatrix} 0.01 & 0.001 & 0.0106 \\ 0 & 0.021 & 0.0028 \\ 0.005 & 0.004 & 0.0228 \end{bmatrix}^{-1} = \begin{bmatrix} -29.3542 & -9.785 & 58.708 \\ -10.763 & 44.031 & 21.526 \\ 91.292 & 16.145 & 17.417 \end{bmatrix}$$

The desired characteristic polynomial is

$\lambda^3 - 0.3\lambda^2 + 0.02\,\lambda$ i.e. $a^d_2 = -0.3$, $a^d_1 = 0.02$, $a^d_0 = 0$

Hence, we have the feedback gain vector

$$\mathbf{k}^T = \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & a_2^d - a_2 \end{bmatrix} T_c^{-1}$$

$$= \begin{bmatrix} -0.4 & 0.02 + 1.16 & -0.3 + 0.5 \end{bmatrix} \begin{bmatrix} \mathbf{-29.3542} & -9.785 & 58.708 \\ -\mathbf{1}0.763 & 44.031 & 21.526 \\ 91.292 & 16.145 & 17.417 \end{bmatrix}$$

$$= \begin{bmatrix} 17.299 & 59.100 & 5.401 \end{bmatrix}$$

b) $$A = \begin{bmatrix} -0.2 & -0.2 & 0.4 \\ 0.5 & 0 & 1 \\ 0 & -0.4 & -0.4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}$$

The characteristic polynomial of the state matrix is

$\lambda^3 - 0.5\,\lambda^2 - 1.16\,\lambda + 0.4$ i.e. $a_2 = -0.5$, $a_1 = -1.16$, $a_0 = 0.4$

The transformation matrix T_c^{-1} is

$$T_c^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -0.6 \\ 1 & -0.6 & -0.62 \end{bmatrix} \begin{bmatrix} 0.01 & -0.002 & -0.0006 \\ 0 & 0.005 & -0.001 \\ 0 & 0 & -0.002 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & -500 \\ 0 & 200 & 200 \\ 100 & -80 & 320 \end{bmatrix}$$

The desired characteristic polynomial is

$\lambda^3 - 0.3\lambda^2 + 0.02\,\lambda$ i.e. $a^d_2 = -0.3$, $a^d_1 = 0.02$, $a^d_0 = 0$

Hence, we have the feedback gain vector

$$\begin{aligned} \mathbf{k}^T &= \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & a_2^d - a_2 \end{bmatrix} T_c^{-1} \\ &= \begin{bmatrix} -0.28 & 0.02-0.98 & -0.3-0.6 \end{bmatrix} \begin{bmatrix} 0 & 0 & -500 \\ 0 & 200 & 200 \\ 100 & -80 & 320 \end{bmatrix} \\ &= \begin{bmatrix} -90 & -120 & -340 \end{bmatrix} \end{aligned}$$

9.8 Using eigenvalues that are four times as fast as those of the plant, design a state estimator for the system

a) $$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0 & 0 & 0.3 \\ 0.3 & 0 & 0.3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

We choose the eigenvalue set {0, 0.025, 0.05}

The characteristic polynomial of the state matrix is

$\lambda^3 - 0.5\,\lambda^2 - 0.027$ i.e. $a_2 = -0.5$, $a_1 = 0$, $a_0 = -0.027$

The transformation matrix T_c^{-1} is

$$T_c^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0.5 \\ 1 & 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 1 & 0.2 & 0.19 \\ 1 & 0.3 & 0.06 \\ 0 & 0.5 & 0.28 \end{bmatrix}^{-1} = \begin{bmatrix} 5.3763 & -5.3763 & 1.07553 \\ -0.3226 & 0.3226 & 1.9355 \\ 0.4194 & 0.5806 & 0.4839 \end{bmatrix}$$

The desired characteristic polynomial is

$\lambda^3 - 0.075\lambda^2 + 0.0013\,\lambda$ i.e. $a^d_2 = -0.075$, $a^d_1 = 0.0013$, $a^d_0 = 0$

Hence, we have the observer gain vector

$$\mathbf{l} = T_o^{-1}\begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & a_2^d - a_2 \end{bmatrix}^T$$
$$= \begin{bmatrix} 5.3763 & -0.3226 & 0.4194 \\ -5.3763 & 0.3226 & 0.5806 \\ 1.07553 & 1.9355 & 0.4839 \end{bmatrix} \begin{bmatrix} 0.027 \\ 0.0013 \\ -0.075 + 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.323 & 0.102 & 0.237 \end{bmatrix}^T$$

b) $$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0 & 0 & 0.3 \\ 0.3 & 0 & 0.3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

We choose the eigenvalue set $\{0, 0.025, 0.05\}$

The characteristic polynomial of the state matrix is

$\lambda^3 - 0.5\,\lambda^2 - 0.027$ i.e. $a_2 = -0.5$, $a_1 = 0$, $a_0 = -0.027$

The transformation matrix T_o^{-1} is

$$T_o^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0.5 \\ 1 & 0.5 & 0.25 \end{bmatrix} \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0 & 0.3 & 0.06 \\ 0 & 0.2 & 0.19 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -4.444 & 6.667 \\ 0 & 2 & 2 \\ 1 & 0.6 & 0.6 \end{bmatrix}$$

The desired characteristic polynomial is

$\lambda^3 - 0.075\lambda^2 + 0.0013\,\lambda$ i.e. $a^d_2 = -0.075$, $a^d_1 = 0.0013$, $a^d_0 = 0$

Hence, we have the observer gain vector

$$\mathbf{l} = T_c^{-1}\begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & a_2^d - a_2 \end{bmatrix}^T$$
$$= \begin{bmatrix} 0 & 0 & 1 \\ -4.444 & 2 & 0.6 \\ 6.667 & 2 & 0.6 \end{bmatrix} \begin{bmatrix} 0.027 \\ 0.0013 \\ -0.075 + 0.5 \end{bmatrix}$$
$$= \begin{bmatrix} 0.425 & 0.1375 & 0.4375 \end{bmatrix}^T$$

9.9 Consider the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.005 & -0.11 & -0.7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix} \qquad d = 0$$

(a) Design a controller that assigns the eigenvalues $\{-0.8, -0.3 \pm j\,0.3\}$. Why is the controller guaranteed to exist?

(b) Why can we design an observer for the system with the eigenvalues $\{-0.5, -0.1 \pm j\, 0.1\}$. Explain why the value (-0.5) must be assigned [**Hint**: $(s+0.1)^2\,(s+0.5) = s^3 + 0.7s^2 + 0.11s + 0.005$].

(c) Obtain a similar system with a 2nd order observable subsystem for which an observer can be easily designed as in Section 8.3.b. Design an observer for the transformed system with two eigenvalues shifted as in (b) and check your design using the MATLAB commands **place** or **acker**. Use the result to obtain the observer for the original system.

Hint: Obtain an observer gain $\mathbf{l}_r$ for the similar third order system from your design by setting the first element equal to zero, then obtain the observer gain for the original system using $\mathbf{l} = T_r\, \mathbf{l}_r$ where T_r is the similarity transformation matrix.

(d) Design an observer-based feedback controller for the system with the controller and observer eigenvalues selected as in (a) and (b) respectively.

(a) The system is in controllable form and is therefore controllable. Hence, a controller that arbitrarily assigns its eigenvalues is guaranteed to exist. The desired characteristic polynomial is

$$\lambda^3 + 1.4\,\lambda^2 + 0.66\,\lambda + 0.144$$

Since the system is in controllable form, we can write its characteristic polynomial by inspection

$$\lambda^3 + 0.7\,\lambda^2 + 0.11\,\lambda + 0.005$$

and the feedback gain vector is

$$\begin{aligned}\mathbf{k}^T &= \begin{bmatrix} a_0^d - a_0 & a_1^d - a_1 & a_2^d - a_2 \end{bmatrix} \\ &= \begin{bmatrix} 0.144 - 0.005 & 0.66 - 0.11 & 1.4 - 0.7 \end{bmatrix} \\ &= \begin{bmatrix} 0.139 & 0.55 & 0.7 \end{bmatrix}\end{aligned}$$

The MATLAB commands **place** or **acker** give the same answer.

(b) The state matrix has the eigenvalues $\{-0.5, -0.1, -0.1\}$ and the eigenvector corresponding to the first eigenvalue satisfies

$$C\mathbf{v} = \begin{bmatrix} 0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} = 0$$

The mode $(-0.5)^k$ is unobservable and cannot be shifted in observer design. However, the system is detectable and we can still design a stable observer if the eigenvalue at -0.5 is included but the usual procedure for observer design will not work because the observability matrix is singular.

(c) The desired characteristic polynomial is

$$\lambda^3 + 0.7\,\lambda^2 + 0.12\,\lambda + 0.01$$

We use the eigenvector of the unobservable mode to form the nonsingular transformation matrix

$$T_r = \left[\begin{array}{c|cc} 4 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

This transforms the state and output matrix to

$$A_n = \left[\begin{array}{c|cc} 0.5 & -0.005 & -0.11 \\ \hline 0 & 0.02 & 1.44 \\ 0 & -0.01 & -0.22 \end{array}\right] \qquad C_n = \left[\begin{array}{c|cc} 0 & 0.5 & 1 \end{array}\right]$$

The observer gain matrix for the observable subsystem is

$$\begin{aligned} \mathbf{l} &= T_o^{-1}\left[a_0^d - a_0 \quad a_1^d - a_1\right]^T \\ &= \left[-0.04 \quad 0.02\right]^T \end{aligned}$$

The MATLAB commands **place** or **acker** give the same answer.

(d) By the separation principle, we simply use the controller gain of (b) and the observer gain of (c) for our design. The state feedback is given by

$$\mathbf{u}(k) = -K\,\hat{\mathbf{x}}(k) + \mathbf{v}(k)$$

with the observer dynamics

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + \mathbf{l}\left[y(k) - C\,\hat{\mathbf{x}}(k)\right]$$

9.10 Design a reduced-order estimator state feedback controller for the discretized system

$$A = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix} \qquad \mathbf{c}^T = [1,\ 1,\ 0]$$

to obtain the eigenvalues $\{0.1, 0.4 \pm j\,0.4\}$.

The feedback gain matrix is determined by using the command **place** of MATLAB as

$$\mathbf{k}^T = \left[-10 \quad 85 \quad 40\right]$$

Then, the second and third state variables must be estimated. Let the desired observer eigenvalues be $\{0.1 \pm j\,0.1\}$. The state matrix is partitioned as

$$A = \left[\begin{array}{c|c} a_1 & \mathbf{a}_2^T \\ \hline \mathbf{a}_3 & A_4 \end{array}\right] = \left[\begin{array}{c|cc} 0.1 & 0 & 0.1 \\ \hline 0 & 0.5 & 0.2 \\ 0.2 & 0 & 0.4 \end{array}\right]$$

The similarity transformation is given by

$$Q_0 = \left[\begin{array}{c|cc} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The system is transformed to

$$A_t = \left[\begin{array}{c:cc} 0.1 & 0.4 & 0.3 \\ \hdashline 0 & 0.5 & 0.2 \\ 0.2 & -0.2 & 0.4 \end{array}\right] \qquad \mathbf{b}_t = \begin{bmatrix} 0.01 \\ 0 \\ 0.005 \end{bmatrix}$$

We now need to solve the linear equation

$$A_o = \begin{bmatrix} 0.5 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} - \mathbf{l}\begin{bmatrix} 0.4 & 0.3 \end{bmatrix}$$

to obtain

$$\bar{\mathbf{l}} = \begin{bmatrix} 0.4474 & 1.7368 \end{bmatrix}^T \qquad A_o = \begin{bmatrix} 0.3211 & 0.0658 \\ -0.8947 & -0.1211 \end{bmatrix}$$

$$\begin{aligned} \mathbf{b}_o &= \bar{\mathbf{b}}_2 - \mathbf{l}\bar{\mathbf{b}}_1 \\ &= \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} \times 10^{-2} - 0.01\begin{bmatrix} 0.4474 \\ 1.7368 \end{bmatrix} = \begin{bmatrix} -0.4474 \\ -1.2368 \end{bmatrix} \times 10^{-2} \end{aligned}$$

$$\begin{aligned} \mathbf{a}_y &= A_o\mathbf{l} + \mathbf{a}_3 - \mathbf{l}a_1 \\ &= \begin{bmatrix} 0.3211 & 0.0658 \\ -0.8947 & -0.1211 \end{bmatrix}\begin{bmatrix} 0.4474 \\ 1.7368 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} - 0.1\begin{bmatrix} 0.4474 \\ 1.7368 \end{bmatrix} \\ &= \begin{bmatrix} 0.2132 \\ -0.5842 \end{bmatrix} \end{aligned}$$

The estimator state equation is

$$\begin{aligned} \bar{\mathbf{x}}(k+1) &= A_o\bar{\mathbf{x}}(k) + A_y\mathbf{y}(k) + B_o\mathbf{u}(k) \\ &= \begin{bmatrix} 0.3211 & 0.0658 \\ -0.8947 & -0.1211 \end{bmatrix}\bar{\mathbf{x}}(k) + \begin{bmatrix} 0.2132 \\ -0.5842 \end{bmatrix}\mathbf{y}(k) + \begin{bmatrix} -0.4474 \\ -1.2368 \end{bmatrix} \times 10^{-2}\,\mathbf{u}(k) \end{aligned}$$

Thus, we have

$$\begin{aligned} \hat{\mathbf{x}}(k) &= Q_o\begin{bmatrix} 1 & \mathbf{0}_{1\times 2}^T \\ \mathbf{l} & I_2 \end{bmatrix}\begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \\ &= \left[\begin{array}{c:cc} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c:c} 1 & \mathbf{0}_{1\times 2}^T \\ \hdashline 0.4474 & \\ 1.7368 & I_2 \end{array}\right]\begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \\ &= \left[\begin{array}{c:cc} 0.5526 & -1 & 0 \\ 0.4474 & 1 & 0 \\ 1.7368 & 0 & 1 \end{array}\right]\begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \end{aligned}$$

Finally, we have the estimator state feedback

$$
\begin{aligned}
u(k) &= -\mathbf{k}^T \hat{\mathbf{x}}(k) \\
&= \begin{bmatrix} -10 & 85 & 40 \end{bmatrix} \left[\begin{array}{c|cc} 0.5526 & -1 & 0 \\ 0.4474 & 1 & 0 \\ 1.7368 & 0 & 1 \end{array}\right] \begin{bmatrix} \mathbf{y}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix} \\
&= \begin{bmatrix} 101.9737 & 95 & 40 \end{bmatrix} \begin{bmatrix} \mathbf{y}(k) \\ \overline{\mathbf{x}}(k) \end{bmatrix}
\end{aligned}
$$

9.11 Consider the following model of an armature controlled DC motor, which is slightly different from that of Example 7.15

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -11 & -11.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

For digital control with T=0.02, apply the state feedback controllers determined in Example 9.4 in order to verify their robustness.

The discretized system is

$$
A_d = \begin{bmatrix} 1.0 & 0.1 & 0.0 \\ 0.0 & 0.9995 & 0.0095 \\ 0.0 & -0.10417 & 0.8944 \end{bmatrix} \qquad B_d = \begin{bmatrix} 1.621\times 10^{-6} \\ 4.820\times 10^{-4} \\ 9.463\times 10^{-2} \end{bmatrix} \qquad C_d = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
$$

Then, the eigenvalues for the vector gains of Example 9.4 obtained using MATLAB are respectively

(i) eig(Ad-Bd*K)

```
ans =

  0.964979897460521
 -0.032562329563822 + 0.515050949150576i
 -0.032562329563822 - 0.515050949150576i
```

(ii) eig(Ad-Bd*K)

```
ans =

  0.975131078267913
  0.312223251280780 + 0.416340814288603i
  0.312223251280780 - 0.416340814288603i
```

(iii) eig(Ad-Bd*K)

```
ans =
```

0.959539232727393
-0.479676083698749 + 0.522496615587290i
-0.479676083698749 - 0.522496615587290i

Clearly, the eigenvalues obtained are somewhat different from the desired ones. The lack of robustness of the control system can also be evaluated by considering the discretized zero-input response for the three states and the corresponding control variable *u* for the initial condition [1,1,1]. The response for case (i) is shown in Figure P9.1, for case (ii) in Figure P9.2, and for case (iii) in Figure P9.3.

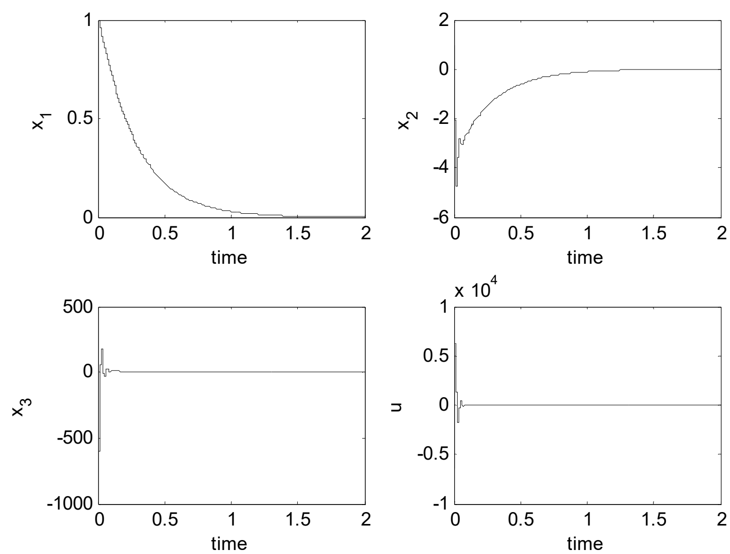


Figure P9.1 Zero input state response and control variable for case (i) of Problem 9.11

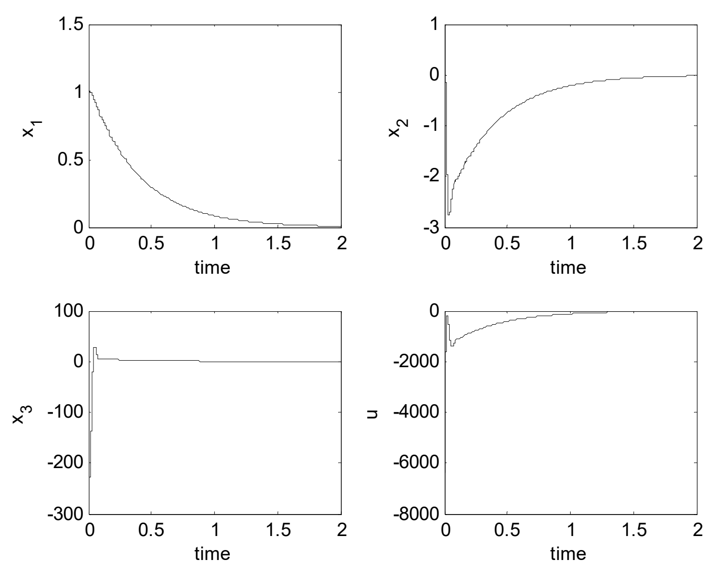


Figure P9.2 Zero input state response and control variable for case (ii) of Problem 9.11

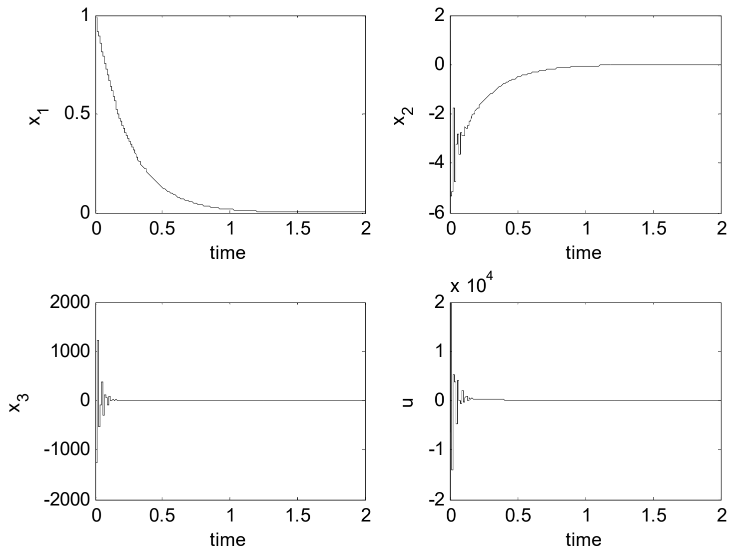


Figure P9.3 Zero input state response and control variable for case (iii) of Problem 9.11

9.12 Consider the following model of a DC motor speed control system, which is slightly different from that of Example 6.8

$$G(s) = \frac{1}{(1.2s+1)(s+10)}$$

For a sampling period T=0.02, obtain a state-space representation corresponding to the discrete-time system with DAC and ADC, then use it to verify the robustness of the state controller of Example 9.5.

The discretized transfer function of the system with DAC and ADC is

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = 1.5521 \times 10^{-4} \frac{z + 0.9303}{(z - 0.8187)(z - 0.9835)}$$

The corresponding state-space model, computed with MATLAB, is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.802 & -0.8052 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.009934 & 0.009242 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The feedback gain vector determined in Example 9.5 is

$$K = \begin{bmatrix} -0.068517 & 0.997197 \end{bmatrix}$$

The gain vector yields the closed-loop state matrix

$$A_{cl} = A - BK = \begin{bmatrix} 1.8033 & -0.8208 \\ 1 & 0 \end{bmatrix}$$

whose eigenvalues are {0.9+j 0.088, 0.9–j 0.088} are very close to the desired values. The closed-loop state-space model is therefore

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.8033 & -0.8208 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.01563 \\ 0 \end{bmatrix} v(k)$$

$$y(k) = \begin{bmatrix} 0.009934 & 0.009242 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

Note that the DC gain of the closed-loop system is

$$\lim_{z \to 1} C\left[zI_n - A_{cl}\right]^{-1} B = C\left[I_n - A_{cl}\right]^{-1} B = 0.017114$$

Thus, if the feedforward gain F =50.42666 is employed with $v(k)$=$Fr(k)$, we obtain the discretized step response of Figure P9.4 which has a nonzero steady-state error (unlike the response for Example 9.11). This occurs despite the fact that the modeling error is unrelated to the gain of the plant.

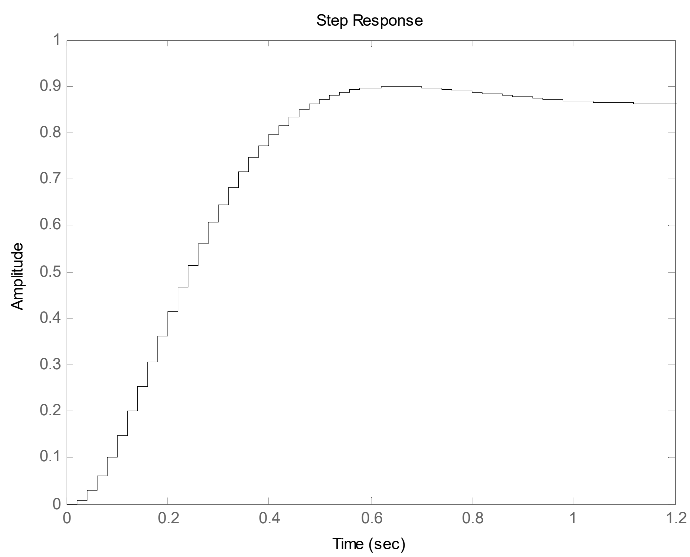


Figure P9.4 Step response of the closed-loop system of Example 9.12

9.13 Verify the robustness of the state controller determined in Example 9.6 by applying it to the model of Problem 9.12.

As in Problem 9.12 the discretized transfer function of the system with DAC and ADC is

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = 1.5521 \times 10^{-4} \frac{z + 0.9303}{(z - 0.8187)(z - 0.9835)}$$

Then, adding integral control, we obtain

$$\widetilde{A}=\begin{bmatrix} A & \mathbf{0} \\ -C & 1 \end{bmatrix}=\begin{bmatrix} 1.8033 & -0.8208 & 0 \\ 1 & 0 & 0 \\ -0.009934 & -0.009242 & 1 \end{bmatrix} \qquad \widetilde{B}=\begin{bmatrix} B \\ 0 \end{bmatrix}=\begin{bmatrix} 0.01563 \\ 0 \\ 0 \end{bmatrix}$$

If we employ the feedback gain vector

$$\widetilde{K}=\begin{bmatrix} 51.1315 & -40.4431 & -40.3413 \end{bmatrix}$$

The closed-loop system state matrix is

$$A_{cl}=A-B\widetilde{K}=\begin{bmatrix} 1.0033 & -0.1733 & 0.6303 \\ 1 & 0 & 0 \\ -0.0099 & -0.0092 & 1 \end{bmatrix}$$

with eigenvalues are {0.9+j 0.07, 0.9–j 0.07, 0.206} almost equal to those selected in Example 9.6. The closed-loop state-space model is therefore

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix}=\begin{bmatrix} 1.0033 & -0.1733 & 0.6303 \\ 1 & 0 & 0 \\ -0.0099 & -0.0092 & 1 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}r(k)$$

$$y(k)=\begin{bmatrix} 0.009934 & 0.009934 & 0 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

The discretized step response of the system is plotted in Figure P9.5. Note that the presence of integral control yields zero steady-state error despite the presence of modeling uncertainties.

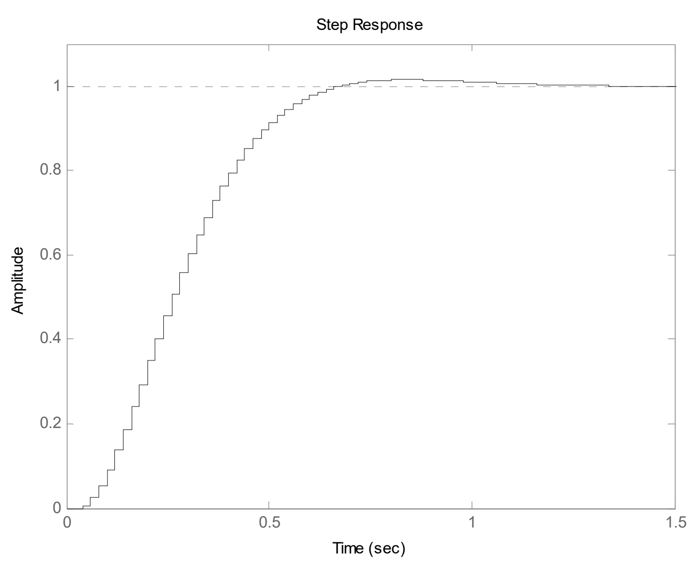


Figure P9.5 Step response of the closed-loop system of Problem 9.13

9.14 Consider the DC motor position control system of Example 3.6, where the (type 1) analog plant has the transfer function

$$G(s) = \frac{1}{s(s+1)(s+10)}$$

For the digital control system with T=0.02, design a state-feedback controller to obtain a step response with null steady-state error, zero overshoot, and with a settling time of less than 0.5 s.

The discretized transfer function of the system with DAC and ADC is

$$G_{ZAS}(z) = \left(1-z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = 1.2629 \times 10^{-6} \frac{(z+3.535)(z+0.2534)}{(z-1)(z-0.8187)(z-0.9802)}$$

The corresponding state-space model, computed with MATLAB, is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 2.799 & -1.301 & 0.4013 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.001953 \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0.0006466 & 0.001225 & 0.0002896 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Since zero steady-state error is required, we need integral control. We therefore obtain

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ -C & 1 \end{bmatrix} = \begin{bmatrix} 2.700 & -1.301 & 0.4013 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.0006466 & -0.001225 & -0.0002896 & 1 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0.01563 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Real closed-loop poles are required to eliminate overshoot in the step response. For the desired settling time of 0.5 s, a suitable choice of closed-loop poles is $\{0.2, 0.4, 0.6, 0.8\}$. The feedback gain vector determined with the MATLAB command **place** is

$$\tilde{K} = 10^3 \begin{bmatrix} 0.9211 & -0.5653 & 0.1972 & -5.3491 \end{bmatrix}$$

The corresponding closed-loop state-space model is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -0.1966 & 0.0162 & 10.4475 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.0006 & -0.0012 & -0.0003 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(k)$$

$$y(k) = \begin{bmatrix} 0.0006466 & 0.001225 & 0.0002896 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{bmatrix}$$

The discretized step response of the system given in Figure P9.6 shows that the design specifications are met.

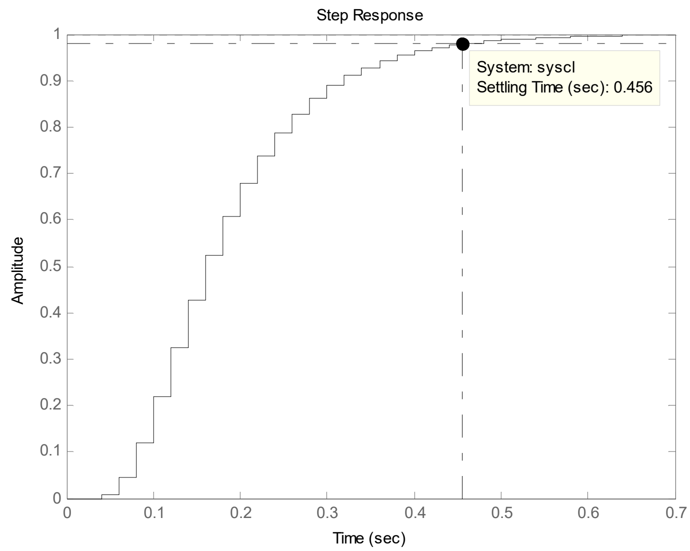


Figure P9.6 Step response of the closed-loop system of Problem 9.14

9.15 Design a digital state feedback controller for the analog system

$$G(s) = \frac{-s+1}{(5s+1)(10s+1)}$$

with T=0.1 to place the closed-loop poles at {0.4, 0.6}. Show that the zero of the closed-loop system is the same as the zero of the open-loop system.

The discretized transfer function of the system with DAC and ADC is

$$G_{ZAS}(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} = -18.712 \times 10^{-4} \frac{z - 1.1053}{(z-0.99)(z-0.9802)}$$

Thus, the system has a zero at z_0=1.1053. The corresponding state-space model, computed with MATLAB, is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1.97 & -0.9704 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0625 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0.02994 & 0.03309 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

For the desired closed-loop poles at {0.4, 0.6} the feedback gain vector is determined with the MATLAB command **place** as

$$K = \begin{bmatrix} 15.524 & -11.687 \end{bmatrix}$$

The closed-loop state-space model is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & -0.24 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.0625 \\ 0 \end{bmatrix} v(k)$$

$$y(k) = \begin{bmatrix} -0.030 & 0.033 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

To check the location of the closed-loop zero, we compute the closed-loop transfer function

$$G_{cl}(z) = \begin{bmatrix} -0.030 & 0.033 \end{bmatrix} \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -0.24 \\ 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0.0625 \\ 0 \end{bmatrix}$$
$$= -18.712 \times 10^{-4} \frac{z - 1.1053}{(z-0.4)(z-0.6)}$$

As expected, the zero of the closed-loop system is the same as that of the open-loop system.

9.16 Write the closed-loop system state space equations of a full-observer state feedback system with integral action.

By combining (9.25) and (9.30) we have

$$\begin{aligned}
\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\
\overline{\mathbf{x}}(k+1) &= \overline{\mathbf{x}}(k) + \mathbf{r}(k) - \mathbf{y}(k) \\
\hat{\mathbf{x}}(k+1) &= A\hat{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k) + \mathbf{L}\left[\mathbf{y}(k) - C\hat{\mathbf{x}}(k)\right] \\
\mathbf{y}(k) &= C\mathbf{x}(k) \\
\mathbf{u}(k) &= -K\mathbf{x}(k) - \overline{K}\overline{\mathbf{x}}(k)
\end{aligned}$$

By substituting the expression of $\mathbf{y}(k)$ and $\mathbf{u}(k)$ in the first three equations, after a few trivial passages we obtain

$$\begin{aligned}
\mathbf{x}(k+1) &= A\mathbf{x}(k) - B\overline{K}\overline{\mathbf{x}}(k) - BK\hat{\mathbf{x}}(k) \\
\overline{\mathbf{x}}(k+1) &= -C\mathbf{x}(k) + \overline{\mathbf{x}}(k) + \mathbf{r}(k) \\
\hat{\mathbf{x}}(k+1) &= LC\mathbf{x}(k) - B\overline{K}\overline{\mathbf{x}}(k) + (A - BK - LC)\hat{\mathbf{x}}(k) \\
\mathbf{y}(k) &= C\mathbf{x}(k)
\end{aligned}$$

which can be rewritten in matrix form as

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \hline \overline{\mathbf{x}}(k+1) \\ \hline \hat{\mathbf{x}}(k+1) \end{bmatrix} = \left[\begin{array}{c|c|c} A & -B\overline{K} & -BK \\ \hline -C & I & 0 \\ \hline LC & -B\overline{K} & A - BK - LC \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \hline \overline{\mathbf{x}}(k) \\ \hline \hat{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \hline I \\ \hline 0 \end{bmatrix} \mathbf{r}(k)$$

$$\mathbf{y}(k) = \left[\begin{array}{c|c|c} C & 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \hline \overline{\mathbf{x}}(k) \\ \hline \hat{\mathbf{x}}(k) \end{bmatrix}$$

Alternatively, we can consider the estimation error vector as a component of state vector, namely, we can combine (9.25) and (9.31):

$$\mathbf{x}(k+1) = A\mathbf{x}(k) - B\overline{K}\overline{\mathbf{x}}(k) - BK\hat{\mathbf{x}}(k)$$
$$\overline{\mathbf{x}}(k+1) = -C\mathbf{x}(k) + \overline{\mathbf{x}}(k) + \mathbf{r}(k)$$
$$\tilde{\mathbf{x}}(k+1) = (A - LC)\tilde{\mathbf{x}}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k)$$

Adding and subtracting the term $BK\mathbf{x}(k)$ from the first equation, we obtain after trivial calculations:

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \hline \overline{\mathbf{x}}(k+1) \\ \hline \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \left[\begin{array}{c|c|c} A-BK & -B\overline{K} & BK \\ \hline -C & I & 0 \\ \hline 0 & 0 & A-LC \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \hline \overline{\mathbf{x}}(k) \\ \hline \tilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ \hline I \\ \hline 0 \end{bmatrix} \mathbf{r}(k)$$

$$\mathbf{y}(k) = \left[\begin{array}{c|c|c} C & 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \hline \overline{\mathbf{x}}(k) \\ \hline \tilde{\mathbf{x}}(k) \end{bmatrix}$$

9.17 Consider the continuous-time model of the overhead crane proposed in Problem 7.10 with m_c=1000 kg, m_l=1500 kg and l=8 m. Design a discrete full-order observer state feedback in order to provide motions of the load without sway.

As in Problem 7.10, we obtain a linearized state-space model of the system about the equilibrium point $\mathbf{x}$=0 with state variables x_1, x_3, the first derivative of x_1 and the first derivative of x_3. Using the approximations

$$\cos x_3 \cong 1,\ \sin x_3 \cong x_3,\ \sin^2 x_3 \cong 0,\ x_4^2 \cong 0$$

gives the following linear state-space model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{m_L}{m_C} g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{(m_L + m_C)g}{m_L l} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m_C} \\ 0 \\ -\dfrac{1}{m_C l} \end{bmatrix} u(k)$$

Substituting the numerical values of the parameters, we obtain

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 14.7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2.04 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \cdot 10^{-3} \\ 0 \\ -0.125 \cdot 10^{-3} \end{bmatrix} u(t)$$

We examine the possibility of only measuring the state variable x_1 (the position of the trolley). This gives the output equation

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

And the observability matrix

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 14.7 & 0 \\ 0 & 0 & 0 & 14.7 \end{bmatrix}$$

Since the observability matrix is evidently full-rank, we can design an observer state feedback by only measuring the trolley position. The transfer function of the system with one measurement is

$$G(s) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 14.7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2.04 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \cdot 10^{-3} \\ 0 \\ -0.125 \cdot 10^{-3} \end{bmatrix} =$$

$$= 0.001 \frac{s^2 + 0.2042}{s^2 \left(s^2 + 2.042 \right)}$$

The poles of the system are $\{0, 0, \pm j1.43\}$. Because of the systems physical limitations, the controller can only speed up the system to a limited degree and we can use the open-loop dynamics to select a suitable sampling interval. For a choice of T=0.1, discretizing the system with MATLAB gives the discrete state-space model

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 0.1 & 0.07338 & 0.00245 \\ 0 & 1 & 1.465 & 0.07338 \\ 0 & 0 & 0.9898 & 0.09966 \\ 0 & 0 & -0.2035 & 0.9898 \end{bmatrix} \quad B = \begin{bmatrix} 4.992 \cdot 10^{-6} \\ 9.969 \cdot 10^{-5} \\ -6.239 \cdot 10^{-7} \\ -1.246 \cdot 10^{-5} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

For steady-state error, we need integral control and therefore we obtain

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ -C & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & 0.07338 & 0.00245 & 0 \\ 0 & 1 & 1.465 & 0.07338 & 0 \\ 0 & 0 & 0.9898 & 0.09966 & 0 \\ 0 & 0 & -0.2035 & 0.9898 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 4.992 \cdot 10^{-6} \\ 9.969 \cdot 10^{-5} \\ -6.239 \cdot 10^{-7} \\ -1.246 \cdot 10^{-5} \\ 0 \end{bmatrix}$$

Real closed-loop poles are required to eliminate sway in the step response. Taking the actuator effort into account, a sensible choice for the closed-loop poles is p_c={0.7, 0.75, 0.8, 0.85, 0.9}. The feedback gain vector is therefore determined with the MATLAB command

>> Ktilde=place(Atilde,Btilde,pc)

as

$$\tilde{K} = \begin{bmatrix} K & \bar{K} \end{bmatrix} = 10^3 \begin{bmatrix} 303.577 & -231.168 & 2.14372 & -191.432 & -11.039 \end{bmatrix}$$

Next, we choose the eigenvalues of the full-order observer sufficiently faster than the controller eigenvalues. We select the values p_o={0.1±j0.1, 0.2±j0.2} and use the MATLAB command

>> L=place(A',C',po)'

to obtain

$$L = \begin{bmatrix} 3.3796 \\ 32.540 \\ 13.012 \\ 33.477 \end{bmatrix}$$

The closed-loop system state-space equations can be therefore derived as (see Problem 9.16)

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) \\ \hat{\mathbf{x}}(k+1) \end{bmatrix} = \left[\begin{array}{c|c|c} A & -B\bar{K} & -BK \\ \hline -C & 1 & 0 \\ \hline LC & -B\bar{K} & A-BK-LC \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} + \left[\begin{array}{c} 0 \\ \hline 1 \\ \hline 0 \end{array}\right] r(k)$$

$$y(k) = \left[\begin{array}{c|c|c} C & 0 & 0 \end{array}\right] \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix}$$

The closed-loop unit step response (namely, the motion of the trolley) is plotted in Figure P9.7

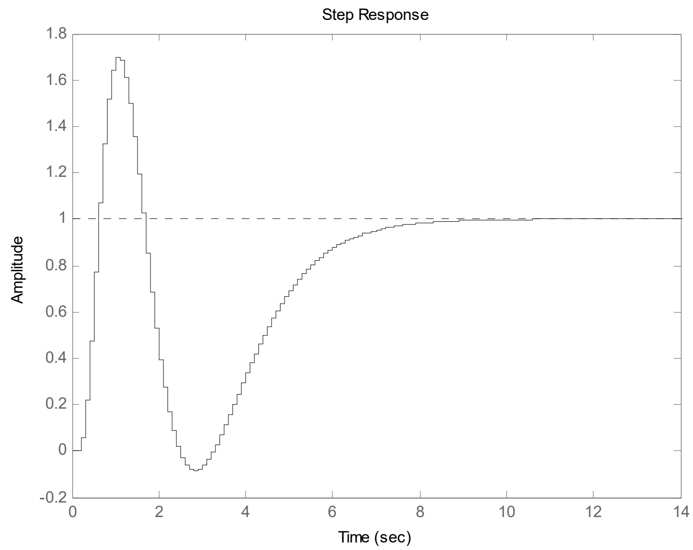


Figure P9.7 Step response of the closed-loop system of Problem 9.17

9.18 Consider the continuous-time model of the overhead crane proposed in Problem 7.10 with m_c=1000 kg, m_l=1500 kg and l=8 m. Design a control system based on pole assignment using transfer functions in order to provide motions of the load without sway.

As in problem 9.17 we obtain the transfer function of the system as

$$G(s) = 0.001\frac{s^2 + 0.2042}{s^2\left(s^2 + 2.042\right)}$$

Discretizing the system with T=0.1 we obtain

$$G_{ZAS}(z) = \left(1 - z^{-1}\right)\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{4.992\cdot 10^{-6}z^3 - 4.982\cdot 10^{-6}z^2 - 4.982\cdot 10^{-6}z + 4.992\cdot 10^{-6}}{z^4 - 3.98z^3 + 5.959z^2 - 3.98z + 1}$$

Thus, we have

$$P(z) = 4.992\cdot 10^{-6}z^3 - 4.982\cdot 10^{-6}z^2 - 4.982\cdot 10^{-6}z + 4.992\cdot 10^{-6}$$

i.e. p_3=4.992×10^{-6}, p_2=-4.982×10^{-6}, p_1=-4.982×10^{-6}, p_0=4.992×10^{-6}

$$Q(z) = z^4 - 3.98z^3 + 5.959z^2 - 3.98z + 1$$

i.e. q_3=-3.98, q_2=5.959, q_1=-3.98, q_0=1

We observe that the plant is fourth order i.e. n=4, and the solvability condition of the Diophantine equation is $m = n - 1 = 3$. The order of the desired closed-loop characteristic polynomial is $m+n$=7. We can therefore select the controller poles as p_c={0.75, 0.8, 0.85, 0.9} and the observer poles as {0.1, 0.2±j0.2} with the corresponding polynomials

$$\Delta_c^d(z) = z^4 - 3.3z^3 + 4.0775z^2 - 2.23575z + 0.459$$

$$\Delta_o^d(z) = z^3 - 0.5z^2 + 0.12z - 0.008$$

$$\Delta_{cl}^d(z) = \Delta_c^d(z)\Delta_o^d(z) =$$
$$= z^7 - 3.8z^6 + 5.8475z^5 - 4.6785z^4 + 2.092575z^3 - 0.53041z^2 + 0.072966z - 0.003672$$

i.e., δ_6=-3.8, δ_5=5.8475, δ_4=-4.6785, δ_3=2.092575, δ_2=-0.53041, δ_1=0.072966, δ_0=-0.003672.

Thus, we have the following matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3.98 & 1 & 0 & 0 & 4.992\cdot 10^{-6} & 0 & 0 & 0 \\ 5.959 & -3.98 & 1 & 0 & -4.982\cdot 10^{-6} & 4.992\cdot 10^{-6} & 0 & 0 \\ -3.98 & 5.959 & -3.98 & 1 & -4.982\cdot 10^{-6} & -4.982\cdot 10^{-6} & 4.992\cdot 10^{-6} & 0 \\ 1 & -3.98 & 5.959 & -3.969 & 4.992\cdot 10^{-6} & -4.982\cdot 10^{-6} & -4.982\cdot 10^{-6} & 4.992\cdot 10^{-6} \\ 0 & 1 & -3.98 & 5.939 & 0 & 4.992\cdot 10^{-6} & -4.982\cdot 10^{-6} & -4.982\cdot 10^{-6} \\ 0 & 0 & 1 & -3.969 & 0 & 0 & 4.992\cdot 10^{-6} & -4.982\cdot 10^{-6} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4.992\cdot 10^{-6} \end{bmatrix} \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \\ s_3 \\ s_2 \\ s_1 \\ s_0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3.8 \\ 5.8475 \\ -4.6785 \\ 2.092575 \\ -0.53041 \\ 0.072966 \\ -0.003672 \end{bmatrix}$$

Using the MATLAB command **linsolve**, we obtain the solution

d_3=1, d_2=-146.99, d_1=-3.379, d_0=143.48, s_3=$2.9480\cdot 10^7$, s_2=$-8.7101\cdot 10^7$, s_1=$8.6385\cdot 10^7$, s_0=$-2.8741\cdot 10^7$

and the polynomials

$$D(z) = z^3 - 146.99z^2 - 3.379z + 143.48$$

$$S(z) = 2.9480\cdot 10^7 z^3 - 8.7101\cdot 10^7 z^2 + 8.6385\cdot 10^7 z + -2.8741\cdot 10^7$$

We compute the gain

$$k_{ff} = \frac{\Delta_c^d(1)}{P(1)} = 3.6797\cdot 10^4$$

and the polynomial

$$N(z) = 3.6797\cdot 10^4 \left(z^3 - 0.5z^2 + 0.12z - 0.008\right)$$

The step response (namely, the motion of the trolley) of the control system is shown in Figure P9.8.

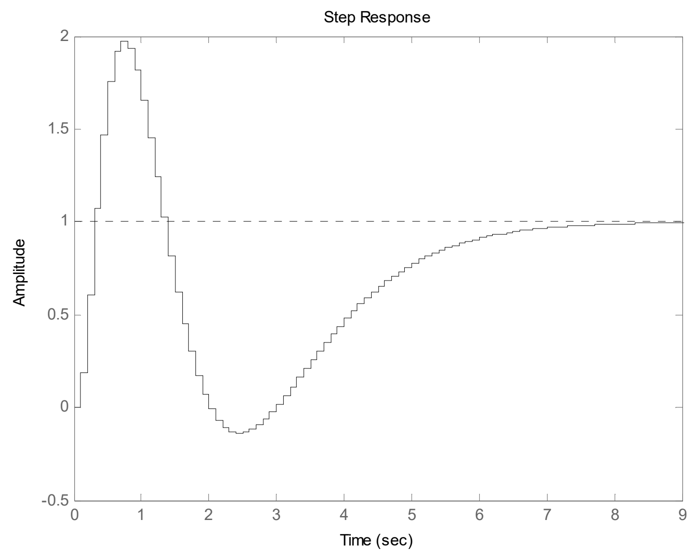


Figure P9.8 Step response of the closed-loop system of Problem 9.18

Computer Exercises

9.19 Write a MATLAB script to evaluate the feedback gains using Ackermann's formula for any pair (A,B) and any desired poles $\{\lambda_1,\ldots, \lambda_n\}$.

```
% Calculations for Ackermann's formula using basic principles
[n,m]=size(B) % Order of input matrix
polyA=poly(A); % Characteristic polynomial in descending
      % order
polyc=fliplr(polyA); % Characteristic polynomial in ascending
       % order
polyc=polyc(1,1:n)% Last row of state matrix
desired= poly(poles) % Desired characteristic polynomial
Delta=polyvalm(desired, A); % Evaluate desired polynomial for
   % A
cons=ctrb(A,B); % Controllability matrix for given form
ac=[zeros(n-1,1),eye(n-1);-polyc]; % State matrix of
      % controllable form
bc=[zeros(n-1,1);1]; % Input matrix of controllable form
conc=ctrb(ac,bc); % Controllability matrix of controllable
   % form
Tinv=conc/cons; % Inverse of transformation matrix
K=Tinv(1,:)*Delta % % Feedback gain matrix
% K=acker(A,B,poles) % Same answer
```

9.20 Write a MATLAB function that, given the system state-space matrices A, B, and C, the desired closed-loop poles, and the observer poles, determines the closed-loop system state-space matrices of a full-observer state feedback system with integral action.

```
function [Acl,Bcl,Ccl]=closedloop(A,B,C,pc,po);

% Acl: closed-loop system state matrix
```

```
% Bcl: closed-loop system input matrix
% Ccl: closed-loop system output matrix

nos=length(A);       % number of states
noi=length(B(1,:)); % number of inputs
noo=length(C(:,1)); % number of outputs
Atilde=[A zeros(nos,noo); -C eye(noo)]; % see (9.27)
Btilde=[B; zeros(noo,noi)];
Ktilde=place(Atilde,Btilde,pc); % see (9.26)
L=place(A',C',po)'; % observer pole assignment
K=Ktilde(1:nos,1:nos) % see the solution of Problem 9.16
Kbar=Ktilde(:,nos+1:end);
Acl=[A -B*Kbar -B*K; -C eye(noo) zeros(noo,nos); L*C -B*Kbar A-B*K-L*C];
Bcl=[zeros(nos,noo); eye(noo); zeros(nos,noo)];
Ccl=[C zeros(noo) zeros(noo,nos)];
```

9.21 Write a MATLAB function that uses the transfer function approach to determine the closed-loop system transfer function for a given plant transfer function *Gz*, desired closed-loop system poles, and observer poles.

```
function [Gcl]=transfer(Gz,pc,po);

% Gz: process transfer function
% (n: order of the numerator, m: order of the denominator)
% pc: desired closed-loop poles (nx1 array)
% po: desired observer poles (mx1 array)
% Gcl: control system transfer function

A=[]; % initialization

T=Gz.Ts; % sampling period
[num,den]=tfdata(Gz,'v'); % numerator and denominator of the system
transfer function
Pz=num/den(1); % numerator of the system transfer function with a monic
denominator
Qz=den/den(1); % monic denominator of the system transfer function
Delta_c=poly(pc); % desired controller polynomial
Delta_o=poly(po); % desired observer polynomial
Delta_cl=conv(Delta_c,Delta_o); % desired control system polynomial
r=length(Delta_cl); % order of the desired control system polynomial
for i=1:r/2, % construction of the matrix for solving the Diophantine
equation
    A=[A [zeros(i-1,1); Qz'; zeros(r/2-i,1)]];
end
for i=1:r/2,
    A=[A [zeros(i-1,1); Pz'; zeros(r/2-i,1)]];
end
X=linsolve(A,Delta_cl'); % solution of the Diophantine equation
Dz=X(1:r/2)';      % D(z) polynomial
Sz=X(r/2+1:end)'; % S(z) polynomial
kff=polyval(Delta_c,1)/polyval(Pz,1); % coefficient for null steady-state
error
Nz=kff*Delta_o;    % N(z) polynomial
Hz=tf(Sz,Dz,T);    % S(z)/N(z)
Fz=tf(Nz,Dz,T);    % N(z)/D(z)
Gcl=minreal(Fz*(Gz/(1+Gz*Hz))); % control system transfer function
```

Chapter 10 Solutions

10.1 Show that for a voltage source v_s with source resistance R_s connected to a resistive load R_L, the maximum power transfer to the load occurs when $R_L = R_s$.

The current in the load is given by

$$i = \frac{v_s}{R_s + R_L}$$

The power dissipated in the resistive load is

$$P_L = i^2 R_L = \frac{R_L}{(R_s + R_L)^2} v_s^2$$

The maximum power transfer condition is obtained by differentiating and equating to zero

$$\frac{dP_L}{dR_L} = \frac{(R_s + R_L) - 2R_L}{(R_s + R_L)^3} v_s^2 = \frac{R_s - R_L}{(R_s + R_L)^3} v_s^2 = 0$$

This gives the condition $R_L = R_s$. The second derivative is

$$\left.\frac{d^2 P_L}{dR_L^2}\right|_{R_s = R_L} = -\left.\frac{(R_s + R_L) + 3(R_s - R_L)}{(R_s + R_L)^4} v_s^2\right|_{R_s = R_L} = -\left|\frac{v_s^2}{(R_s + R_L)^3}\right| < 0$$

Note that the power is zero for zero and infinite load and positive otherwise. Hence, the necessary condition clearly yields a maximum between the two zero limiting values.

10.2 Let $\mathbf{x}$ be a n by 1 vector whose entries are the quantities produced by a manufacturer. The profit of the manufacturer is given by the quadratic form

$$J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T P \mathbf{x} + \mathbf{q}^T \mathbf{x} + \mathbf{r}$$

where P is a negative definite symmetric matrix, and $\mathbf{q}$ and $\mathbf{r}$ are constant vectors. Find the vector $\mathbf{x}$ to maximize profit

a. With no constraints on the quantity produced.

The necessary condition for a minimum is

$$\frac{\partial J(\mathbf{x})}{\partial \mathbf{x}} = P\mathbf{x} + \mathbf{q} = \mathbf{0}$$
$$\mathbf{x}^* = P^{-1}\mathbf{q}$$

The sufficient condition for a maximum is satisfied since

$$\frac{\partial^2 J(\mathbf{x})}{\partial \mathbf{x}^2} = P > 0$$

b. If the quantity produced is constrained by

$$B\mathbf{x} = \mathbf{c}$$

where B is a m by n matrix, $m < n$, and **c** is a constant vector.

We introduce the constraints using Lagrange multipliers to obtain

$$J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T P\mathbf{x} + \mathbf{q}^T\mathbf{x} + \mathbf{r} + \lambda^T\left(B\mathbf{x} - \mathbf{c}\right)$$

The necessary conditions for a minimum are

$$\frac{\partial J(\mathbf{x})}{\partial \mathbf{x}} = P\mathbf{x} + \mathbf{q} + B^T\lambda = \mathbf{0}$$
$$\frac{\partial J(\mathbf{x})}{\partial \lambda} = B^T\mathbf{x} - \mathbf{c} = \mathbf{0}$$
$$\mathbf{x}^* = -P^{-1}\left(B^T\lambda^* + \mathbf{q}\right)$$
$$-B\mathbf{x}^* = BP^{-1}\left(B^T\lambda^* + \mathbf{q}\right) = -\mathbf{c}$$
$$\lambda^* = -\left(BP^{-1}B^T\right)^{-1}\left(BP^{-1}\mathbf{q} + \mathbf{c}\right)$$
$$\mathbf{x}^* = P^{-1}\left(B^T\left(BP^{-1}B^T\right)^{-1}\left(BP^{-1}\mathbf{q} + \mathbf{c}\right) - \mathbf{q}\right)$$

The sufficient condition for a maximum is satisfied since

$$\frac{\partial^2 J(\mathbf{x})}{\partial \mathbf{x}^2} = P > 0$$

10.3 Prove that the rectangle of largest area that fits inside a circle of diameter D is a square of diagonal D.

Consider a rectangle of width W and height H. The largest rectangle must have its four vertices on the circle and must therefore satisfy

$$W^2 + H^2 = D^2$$

To maximize the area subject to the constraint, we have the Lagrangian

$$L = WH + \lambda\left(W^2 + H^2 - D^2\right)$$

The necessary conditions for a maximum are

$$\begin{aligned}
\frac{\partial L}{\partial W} &= H + 2\lambda W = 0 \\
\frac{\partial L}{\partial H} &= W + 2\lambda H = 0 \\
\frac{\partial L}{\partial \lambda} &= W^2 + H^2 - D^2 = 0
\end{aligned}$$

From the first two conditions, we have

$$-\lambda = \frac{H}{2W} = \frac{W}{2H} \Rightarrow H = W$$

From the constraint, the diameter of the square is clearly equal to D.

10.4 With $q = 1$ and $r = 2$, $S(k_f)=1$, write the design equations for digital optimal quadratic regulator for the integrator

$$\dot{x} = u$$

We first obtain the discretized plant

$$x(k+1) = x(k) + u(k)$$

We have $A=B=1$, and the Riccati equation becomes

$$\begin{aligned}
S(k) &= A^T\left\{S(k+1) - S(k+1)B\left(B^T S(k+1)B + R(k)\right)^{-1} B^T S(k+1)\right\}A + Q(k) \\
&= S(k+1) - \frac{S^2(k+1)}{S(k+1)+r} + 1 \\
&= \frac{S(k+1)r}{S(k+1)+r} + 1
\end{aligned}$$

The control is given by

$$\mathbf{u}^*(k) = -K(k)\mathbf{x}^*(k)$$

$$K(k) = \left[R(k) + B^T S(k+1)B\right]^{-1} B^T S(k+1)A$$

$$= \frac{S(k+1)}{S(k+1)+r}$$

10.5 The discretized state-space model of the Infante AUV of Problem 7.14 is given by

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.9932 & -0.03434 & 0 \\ -0.009456 & 0.9978 & 0 \\ -0.0002368 & 0.04994 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.002988 \\ -0.0115 \\ -0.0002875 \end{bmatrix} u(k)$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Design a steady-state linear quadratic regulator for the system using the weight matrices $Q = I_3$ and $r = 2$.

The MATLAB command **dlqr** gives

```
>> [K, P, E] = dlqr(A, B, Q, R)

K =
0.8576   -1.8840   -0.6986

S =
  127.3268 -119.0444    5.4625
 -119.0444  301.4720  124.3046
    5.4625  124.3046  117.5985

E =
   0.9676
   0.9858
   0.9920
```

10.6 A simplified linearized model of a drug delivery system to maintain blood glucose and insulin levels at prescribed values is given by[1]

[1] F. Chee, A. V. Savkin, T. L. Fernando, and S. Nahavandi, "Optimal H^{∞} Insulin Injection Control for Blood Glucose Regulation in Diabetic Patients," *IEEE Trans. Biomed. Eng.* Vol. 52, No. 10, pp. 1625-1631, 2005.

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -0.04 & -4.4 & 0 \\ 0 & -0.025 & 1.3\times10^{-5} \\ 0 & 0.09 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

where all variables are perturbations from the desired steady-state levels. The state variables are the blood glucose concentration x_1 in mg/dl., the blood insulin concentration x_3 in mg/dl., and a variable describing the accumulation of blood insulin x_2. The controls are the rate of glucose infusion u_1 and the rate of insulin infusion u_2, both in mg/dl./min. Discretize the system with a sampling period $T=5$ min and design a steady-state regulator for the system with weight matrices $Q = I_3$ and $R = 2\,I_2$. Simulate the system with the initial state $\mathbf{x}(0) = [6 \;\; 0 \;\; -1]^T$ and plot the trajectory in the x_1-x_3 plane as well as the time evolution of the glucose concentration.

The problem is solved using the following MATLAB script

```
% State-space model
x{1}=[6;0;-1]; % Initial state
A=[-0.04 -04.4 0
    0 -0.025 13E-6
    0 0.09 0];
B=[1,0
    0,0
    0,.1 ];
C=[1,0,0;0,0,1];
p=ss(A,B,C,0); % Analog state-space model
T=5; % Sampling period
pd=c2d(p,T); % Discretize with sampling period T
[Ad,Bd,C,D]=ssdata(pd); % Discrete state-space data
[K, S, e] = dlqr(Ad, Bd, Q, R); % Solve lqr
Acl=Ad-Bd*K; % Closed-loop state matrix
pc=ss(Acl,Bd,C,0);
% time=0:T:20*T;
t(1)=0; % Initial time
for i=1:N
    t(i+1)=t(i)+T;
    x{i+1}=Acl*x{i}-Bd*K*x{i}; % State equation
end
xmat=cell2mat(x); % Change cell to mat to extract data
xx=xmat(1,:); % Glucose concentration
xv=xmat(3,:); % Insulin concentration
hold on
stem(t,xx) % Plot glucose vs. time
hold off
figure
plot(xx,xv)% Plot phase plane trajectory
```

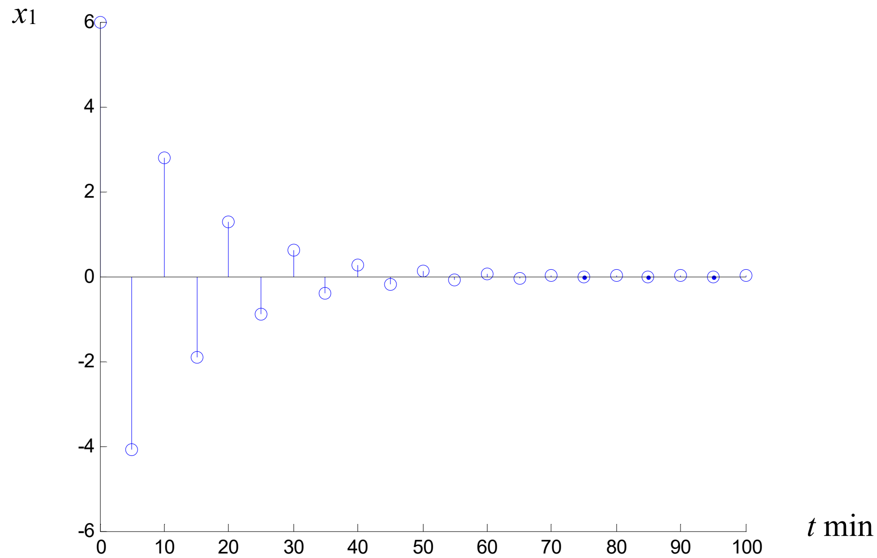


Figure P10.1 Plot of the evolution of glucose concentration.

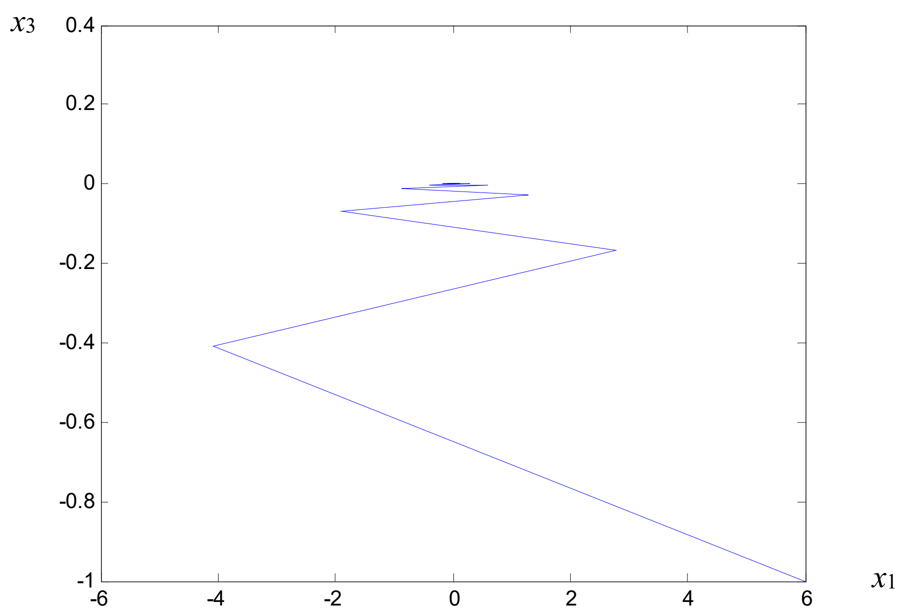


Figure P10.2 Phase plane trajectory in the x_1-x_3 plane.

10.7 Optimal control problems can be solved as unconstrained optimization problem without using Lagrange multipliers. This exercise shows the advantages of using Lagrange multipliers and demonstrates that discrete-time optimal control is equivalent to an optimization problem over the control history.

(i) Substitute the solution of the state equation

$$
\begin{aligned}
\mathbf{x}(k) &= A^k\mathbf{x}(0) + \sum_{i=0}^{k-1} A^{k-i-1}B\mathbf{u}(i) \\
&= A^k\mathbf{x}(0) + \mathcal{C}(k)\boldsymbol{u}(k)
\end{aligned}
$$

$$\mathcal{C}(k) = \left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{k-1}B\right]$$

$$\boldsymbol{u}(k) = col\{\mathbf{u}(k-1), \cdots, \mathbf{u}(0)\}$$

In the performance measure

$$J = \frac{1}{2}\mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f) + \frac{1}{2}\sum_{k=0}^{k_f-1}\left(\mathbf{x}^T(k)Q(k)\mathbf{x}(k) + \mathbf{u}^T(k)R(k)\mathbf{u}(k)\right)$$

to eliminate the state vector and obtain

$$J = \frac{1}{2}\sum_{k=0}^{k_f}\begin{pmatrix} \boldsymbol{u}^T(k)\overline{R}(k)\boldsymbol{u}(k) + 2^T\mathbf{x}(0)\left(A^T\right)^k Q(k)\mathcal{C}(k)\boldsymbol{u}(k) \\ +\mathbf{x}(0)\left(A^T\right)^k Q(k)A^k\mathbf{x}(0) + \mathbf{u}^T(k)R(k)\mathbf{u}(k) \end{pmatrix}$$

with $Q(k_f) = S(k_f),\ R(k_f) = \mathbf{0}_{m\times m}$

(ii) Without the tedious evaluating the matrix R_{eq} and the vector $\mathbf{l}$, explain why it is possible to rewrite the performance measure in the equivalent form

$$J_{eq} = \frac{1}{2}\boldsymbol{u}^T(k_f)R_{eq}\,\boldsymbol{u}(k_f) + \boldsymbol{u}^T(k_f)\mathbf{l}$$

(iii) Show that the solution of the optimal control problem is given by

$$\boldsymbol{u}(k_f) = -R_{eq}^{-1}\mathbf{l}$$

(i) We substitute in the quadratic cost the state $\mathbf{x}(k)$

$$
\begin{aligned}
\mathbf{x}^T(k)Q(k)\mathbf{x}(k) &= \left[A^k\mathbf{x}(0) + \mathcal{C}(k)\boldsymbol{u}(k)\right]^T Q(k)\left[A^k\mathbf{x}(0) + \mathcal{C}(k)\boldsymbol{u}(k)\right] \\
&= \boldsymbol{u}^T(k)\mathcal{C}^T(k)Q(k)\mathcal{C}(k)\boldsymbol{u}(k) + 2^T\mathbf{x}(0)\left(A^T\right)^k Q(k)\mathcal{C}(k)\boldsymbol{u}(k) \\
&\quad + \mathbf{x}(0)\left(A^T\right)^k Q(k)A^k\mathbf{x}(0)
\end{aligned}
$$

Substituting in the performance measure gives

$$
\begin{aligned}
J &= \frac{1}{2}\mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f) \\
&+ \frac{1}{2}\sum_{k=0}^{k_f-1}\begin{pmatrix} \boldsymbol{u}^T(k)\overline{R}(k)\boldsymbol{u}(k) + 2^T\mathbf{x}(0)\left(A^T\right)^k Q(k)\mathcal{C}(k)\boldsymbol{u}(k) \\ +\mathbf{x}(0)\left(A^T\right)^k Q(k)A^k\mathbf{x}(0) + \mathbf{u}^T(k)R(k)\mathbf{u}(k) \end{pmatrix}
\end{aligned}
$$

We define the matrices $Q(k_f) = S(k_f), R(k_f) = \mathbf{0}_{m\times m}$, and rewrite the performance measure in the form

$$J = \frac{1}{2}\sum_{k=0}^{k_f}\begin{pmatrix} \boldsymbol{u}^T(k)\,\overline{R}(k)\,\boldsymbol{u}(k) + 2^T\,\mathbf{x}(0)\left(A^T\right)^k Q(k)\mathcal{C}(k)\,\boldsymbol{u}(k) \\ + \mathbf{x}(0)\left(A^T\right)^k Q(k)A^k\mathbf{x}(0) + \mathbf{u}^T(k)R(k)\mathbf{u}(k) \end{pmatrix}$$

(ii) We use the expression for the cost obtained in Part (i). Since each vector $\boldsymbol{u}(k)$ includes the control inputs $\mathbf{u}(k)$, $k = 0, 1, k-1$, all the terms in the performance measure that involve the control are entries of $\boldsymbol{u}(k_f)$. The terms are either quadratic or linear. We expand the quadratic in $\boldsymbol{u}(k)$

$$\boldsymbol{u}^T(k)\mathcal{C}^T(k)Q(k)\mathcal{C}(k)\,\boldsymbol{u}(k) = \boldsymbol{u}^T(k)\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T\left(A^T\right)^{k-1} \end{bmatrix} Q(k)\left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{k-1}B\right]\boldsymbol{u}(k)$$

$$= \boldsymbol{u}^T(k)\,\overline{R}(k)\,\boldsymbol{u}(k)$$

$$\overline{R}(k) = \begin{bmatrix} B^TQ(k)B & B^TQ(k)AB & \cdots & B^TQ(k)A^{k-1}B \\ B^TA^TQ(k)B & & \cdots & B^TA^TQ(k)A^{k-1}B \\ \vdots & \vdots & \ddots & \vdots \\ B^T\left(A^T\right)^{k-1}Q(k)B & B^T\left(A^T\right)^{k-1}Q(k)AB & \cdots & B^T\left(A^T\right)^{k-1}Q(k)A^{k-1}B \end{bmatrix}$$

The matrix can be written more concisely in terms of its ij^{th} term as

$$\overline{R}(k) = \left[B^T\left(A^T\right)^{i-1}Q(k)A^{j-1}B\right]$$

and the quadratic form can be written as

$$\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\mathbf{u}^T(i)B^T\left(A^T\right)^{i-1}Q(k)A^{j-1}B\,\mathbf{u}(j)$$

The linear term is in the form

$$\mathbf{x}(0)\left(A^T\right)^k Q(k)\mathcal{C}(k)\,\boldsymbol{u}(k) = \mathbf{x}(0)\left(A^T\right)^k\left[B \;\vdots\; AB \;\vdots\; \cdots \;\vdots\; A^{k-1}B\right]\boldsymbol{u}(k)$$

$$= \mathbf{x}(0)\left(A^T\right)^k\sum_{i=0}^{k-1}A^{k-i-1}B\mathbf{u}(i)$$

Returning to the performance measure, we observe that the quadratic in the initial conditions can be dropped since it has not relevance to optimization using the control. The problem is equivalent to minimizing the performance measure

$$J = \frac{1}{2}\sum_{k=0}^{k_f}\left(\begin{array}{l}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\mathbf{u}^T(i)B^T\left(A^T\right)^{i-1}Q(k)A^{j-1}B\mathbf{u}(j) + 2^T\mathbf{x}(0)\left(A^T\right)^k\sum_{i=0}^{k-1}A^{k-i-1}B\mathbf{u}(i)\\ +\mathbf{u}^T(k)R(k)\mathbf{u}(k)\end{array}\right)$$

The quadratic terms can be collected in a quadratic form with weight matrix R_{eq} and the linear terms can be written in terms of a vector $\mathbf{l}$. However, the exact forms of R_{eq} and $\mathbf{l}$ are very difficult to evaluate. The quadratic in the initial conditions can be dropped since it has not relevance to optimization using the control.

Using Lagrange multipliers we can avoid all the tedious algebraic manipulation required to obtain the equivalent performance measure and solve the optimal control problem directly.

(iii) We differentiate partially with respect to $\boldsymbol{u}(k_f)$ and equate to zero

$$\frac{\partial J_{eq}}{\partial \boldsymbol{u}(k_f)} = R_{eq}\,\boldsymbol{u}(k_f) + \mathbf{l} = \mathbf{0}_{(k_f-1)m\times 1}$$

The solution is

$$\boldsymbol{u}(k_f) = -R_{eq}^{-1}\mathbf{l}$$

A sufficient conditions for a minimum is that the R_{eq} be positive definite and a necessary condition is that it be positive semidefinite.

10.8 For (A,B) stabilizable and $(A, Q^{1/2})$ detectable, the linear quadratic regulator yields a closed-loop stable system. To guarantee that the eigenvalues of the closed—loop system will lie inside a circle of radius $1/\alpha$, we solve the regulator problem for the scaled state and control

$$\overline{\mathbf{x}}(k) = \alpha^k\mathbf{x}(k) \qquad\qquad \overline{\mathbf{u}}(k) = \alpha^{k+1}\mathbf{u}(k)$$

(i) Obtain the state equation for the scaled state vector.

(ii) Show that if the scaled closed-loop state matrix with the optimal control $\overline{\mathbf{u}}(k) = -\overline{K}\,\overline{\mathbf{x}}(k)$ has eigenvalues inside the unit circle, then the eigenvalues of the original state matrix with the control $\mathbf{u}(k) = -K\,\mathbf{x}(k), K = \overline{K}/\alpha$, are inside a circle of radius $1/\alpha$.

(i) From the definition of the scaled state and control we have

$$\begin{aligned}\overline{\mathbf{x}}(k+1) &= \alpha^{k+1}\mathbf{x}(k+1)\\ &= \alpha A \alpha^{k}\mathbf{x}(k) + B\alpha^{k+1}\mathbf{u}(k) = \overline{A}\,\overline{\mathbf{x}}(k) + B\overline{\mathbf{u}}(k)\\ \overline{A} &= \alpha A\\ \overline{\mathbf{u}}(k) &= \alpha^{k+1}\mathbf{u}(k)\end{aligned}$$

(ii) The scaled closed-loop system has the dynamics

$$\begin{aligned}\overline{\mathbf{x}}(k+1) &= \overline{A}\,\overline{\mathbf{x}}(k) + B\overline{\mathbf{u}}(k)\\ &= \left(\alpha A - B\overline{K}\right)\overline{\mathbf{x}}(k) = \overline{A}_{cl}\overline{\mathbf{x}}(k)\end{aligned}$$

The associated unscaled dynamics are

$$\begin{aligned}\mathbf{x}(k+1) &= \alpha^{-(k+1)}\overline{\mathbf{x}}(k+1)\\ &= \alpha^{-1}\left(\alpha A - B\overline{K}\right)\alpha^{-k}\overline{\mathbf{x}}(k)\\ &= \left(A - B\alpha^{-1}\overline{K}\right)\mathbf{x}(k)\end{aligned}$$

Equivalently, using the control $\mathbf{u}(k) = -K\,\mathbf{x}(k), K = \overline{K}/\alpha$, gives the closed-loop state matrix $A_{cl} = A - B\alpha^{-1}\overline{K} = \overline{A}_{cl}/\alpha$

Since optimal control stabilizes the closed-lop system, the eigenvalues of the scaled matrix satisfy $\left|\lambda(\alpha\overline{A}_{cl})\right| < 1$. In addition, scaling a matrix also scales its eigenvalues and we have the result $\left|\lambda(\alpha A_{cl})\right| < 1/\alpha$.

10.9 Repeat Problem 10.5 with a design that guarantees that the eigenvalues of the closed-loop system are inside a circle of radius equal to one half.

For the desired eigenvalues we use $\alpha = 2$ and the MATLAB commands

```
>> [Kbar, Sbar, Ebar] = dlqr(A, B, Q, R)

Kbar =
1.0e+005 *
  -2.0214   -0.4948   -1.5833

Sbar =

 1.0e+012 *
   1.6800    0.4095    1.2995
   0.4095    0.0998    0.3168
   1.2995    0.3168    1.0052
```

```
Ebar =
  0.5181
   0.4978
   0.5000
```

The state feedback and the eigenvalues for the final design are

```
>> K=Kbar/alpha, E=ebar/alpha

K =
  1.0e+005 *
  -1.0107   -0.2474   -0.7916

E =
   0.2590
   0.2489
   0.2500
```

Unlike the design of Problem 10.5, the eigenvalues meet are all of magnitude less than half.

10.10 Show that the linear quadratic regulator with cross-product term of the form

$$J = \mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f) + \sum_{k=k_0}^{k_f-1}\left(\mathbf{x}^T(k)Q\mathbf{x}(k) + 2\mathbf{x}^T N\mathbf{u} + \mathbf{u}^T(k)R\mathbf{u}(k)\right)$$

where N is a $n{\times}m$ matrix, is equivalent to a linear quadratic regulator with no cross-product term with the cost

$$J = \mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f) + \sum_{k=k_0}^{k_f-1}\left(\mathbf{x}^T(k)\bar{Q}\mathbf{x}(k) + \bar{\mathbf{u}}^T(k)R\bar{\mathbf{u}}^T(k)\right)$$

$$\bar{Q} = Q - NR^{-1}N^T$$

$$\bar{\mathbf{u}}(k) = \mathbf{u}(k) + R^{-1}N^T\mathbf{x}(k)$$

and the plant dynamics

$$\mathbf{x}(k+1) = \bar{A}\mathbf{x}(k) + B\bar{\mathbf{u}}(k) \qquad k = k_0, \cdots, k_f - 1$$

$$\bar{A} = A - BR^{-1}N^T$$

We only need to consider a general term in the summation of

$$J = \mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f) + \sum_{k=k_0}^{k_f-1}\left(\mathbf{x}^T(k)Q\mathbf{x}(k) + 2\mathbf{x}^T N\mathbf{u} + \mathbf{u}^T(k)R\mathbf{u}(k)\right)$$

to show the equivalence for the performance index. We add and subtract $\mathbf{x}^T(k)NR^{-1}N^T\mathbf{x}(k)$ and use the equality $\mathbf{x}^T(k)N\mathbf{u}(k)=\mathbf{u}^T(k)N^T\mathbf{x}(k)$ to rewrite the general term as

$$\begin{aligned}
&\mathbf{x}^T\left\{Q-NR^{-1}N^T\mathbf{x}\right\}\mathbf{x}+\mathbf{u}^T R\mathbf{u}+2\mathbf{x}^T N\mathbf{u}+\mathbf{x}^T NR^{-1}N^T\mathbf{x}\\
&\qquad=\mathbf{x}^T\overline{Q}\mathbf{x}+\left\{\mathbf{u}+R^{-1}N^T\mathbf{x}\right\}^T R\left\{\mathbf{u}+R^{-1}N^T\mathbf{x}\right\}\\
&\qquad=\mathbf{x}^T\overline{Q}\mathbf{x}+\overline{\mathbf{u}}^T R\,\overline{\mathbf{u}}
\end{aligned}$$

with the argument k dropped for brevity. Substitution gives the equivalent cost function.

Next, we add and subtract the term $R^{-1}N^T\mathbf{x}(k)$ to the plant state equation

$$\begin{aligned}
\mathbf{x}(k+1)&=\left\{A-BR^{-1}N^T\right\}\mathbf{x}(k)+B\left\{\mathbf{u}(k)+R^{-1}N^T\mathbf{x}(k)\right\}\\
&=\overline{A}\mathbf{x}(k)+B\,\overline{\mathbf{u}}(k)\qquad k=k_0,\cdots,k_f-1
\end{aligned}$$

This completes the proof of equivalence of the two problems.

The control for the cross-product problem is given by

$$\begin{aligned}
\mathbf{u}(k)&=\overline{\mathbf{u}}(k)-R^{-1}N^T\\
&=-\left\{\left[R+B^T S(k+1)B\right]^{-1}B^T S(k+1)A+R^{-1}N^T\right\}\mathbf{x}(k)
\end{aligned}$$

10.11 Rewrite the performance measure of Problem 10.10 in terms of a combined input and state vector col{**x** (k), **u**(k)}. Then use the Hamiltonian to show that for the linear quadratic regulator with cross-product term a sufficient condition for a minimum is that the matrix

$$\left[\begin{array}{c|c} Q & N \\ \hline N^T & R \end{array}\right]$$

must be positive definite.

$$\begin{aligned}
J&=\mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f)+\sum_{k=k_0}^{k_f-1}\left(\mathbf{x}^T(k)Q\mathbf{x}(k)+2\mathbf{x}^T(k)N\mathbf{u}(k)+\mathbf{u}^T(k)R\mathbf{u}(k)\right)\\
&=\mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f)+\sum_{k=k_0}^{k_f-1}\left(\mathbf{x}^T(k)Q\mathbf{x}(k)+\mathbf{x}^T N\mathbf{u}+\mathbf{u}^T(k)N^T\mathbf{x}(k)+\mathbf{u}^T(k)R\mathbf{u}(k)\right)\\
&=\mathbf{x}^T(k_f)S(k_f)\mathbf{x}(k_f)+\sum_{k=k_0}^{k_f-1}\left[\mathbf{x}^T(k)\quad \mathbf{u}^T(k)\right]\left[\begin{array}{c|c} Q & N \\ \hline N^T & R \end{array}\right]\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{u}(k)\end{bmatrix}
\end{aligned}$$

The Hamiltonian of the system is given by

$$H(\mathbf{x}(k),\mathbf{u}(k),k)=\left[\mathbf{x}^T(k) \quad \mathbf{u}^T(k)\right]\left[\begin{array}{c:c} Q & N \\ \hdashline N^T & R \end{array}\right]\left[\begin{array}{c} \mathbf{x}(k) \\ \mathbf{u}(k) \end{array}\right]$$
$$+\lambda^T(k+1)\left[A\mathbf{x}(k)+B\mathbf{u}(k)\right], \qquad k=k_0,\cdots,k_f-1$$

For this Hamiltonian, the sufficient condition (10.27) becomes

$$\left[\begin{array}{c:c} Q & N \\ \hdashline N^T & R \end{array}\right]>0$$

Computer Exercise

10.12 Design a steady-state regulator for the Infante AUV of Problem 10.5 with the performance measure modified to include a cross-product term with $S=[1\ \ 0.2\ \ 0.1]^T$

(i) Using the MATLAB command **dlqr** with the cross-product term.

(ii) Using the MATLAB command **dlqr** and the equivalent problem with no cross product as in Problem Problem 10.5.

(i) We use the MATLAB script

```
Abar=Ad-Bd/R*N'; % State matrix for equivalent problem (no cross-product terms)
Qbar=Q-N/R*N';  % State weight matrix for equivalent problem
[Kbar, Sbar, Ebar] = dlqr(Abar, Bd, Qbar, R) % Solve equivalent lqr problem
K=Kbar+R\N'% Gain correction to obtain gain for the cross-product
```

MATLAB gives the following output

```
Kbar =

   0.438   -1.6137   -0.7502

Sbar =
   38.2988  -50.8816   -5.5560
  -50.8816  270.3801  130.3031
   -5.5560  130.3031  102.0525

Ebar =
   0.9768
   0.9815
   0.9914

K =
   0.8438   -1.5137   -0.7002
```

(ii) We need a single MATLAB command

```
>> [K, S, E] = dlqr(Ad, Bd, Q, R,N)  % Solve lqr problem with cross-product
```

The command gives the value obtained earlier for the gain K and the eigenvalues E.

10.13 Design an output quadratic regulator for the Infante UAV of Problem 10.5 with the weights Q_y=1 and r = 100. Plot the output response for the initial condition vector $\mathbf{x}(0) = [1, 0, 1]^T$.

We use the MATLAB command

```
>> [Ky,S,E] = dlqry(A,B,C,D,eye(2),100)

Ky =
   0.978745866207956   -2.114628801206936                  0

S =
  1.0e+004 *

   0.358662942722262  -0.769638465049177                  0
  -0.769638465049177   1.664089378965256                  0
                   0                   0                  0
E =
   1.000000000000000
   0.977397502782181
   0.986359773355693
```

Alternatively, we can use the MATLAB command

```
>> [K,S,E] = dlqry(A,B,C'*eye(2)*C,100)

K =
   0.978745866207956  -2.114628801206936                  0
S =

  1.0e+004 *
   0.358662942722262  -0.769638465049177                  0
  -0.769638465049177   1.664089378965256                  0
                   0                   0                  0
E =
   1.000000000000000
   0.977397502782181
   0.986359773355693
```

The resulting closed-loop system state matrix is

```
>> Acl=A-B*K

Acl =
  0.990275507351771  -0.028021489141994                  0
  0.001799577461391   0.973481768786120                  0
  0.000044589436535   0.049332044219653   1.000000000000000
```

The response of the system to the initial conditions $\mathbf{x}(0) = [1, 0, 1]^T$ is shown in Figure P10.3.

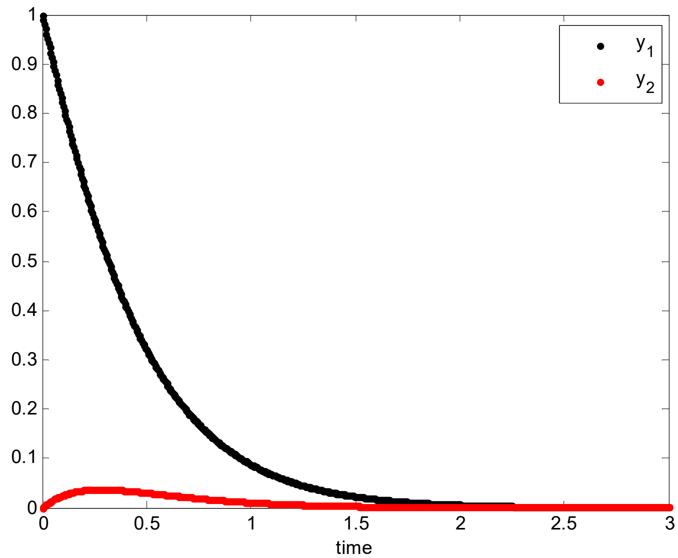


Figure P10.3 System response for Problem 10.13.

10.14 Design an optimal LQ state-space tracking controller for the drug delivery system of Problem 10.6 to obtain zero steady-state error due to a unit step input.

The state-space matrices are

$$A = \begin{bmatrix} -0.04 & -4.4 & 0 \\ 0 & -0.025 & 1.3\times10^{-5} \\ 0 & 0.09 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Adding integral control, we obtain the state-space matrices

$$\tilde{A} = \begin{bmatrix} A & \mathbf{0} \\ -C & I \end{bmatrix} = \begin{bmatrix} -0.04 & -4.4 & 0 & 0 & 0 \\ 0 & -0.025 & 1.3\times 10^{-5} & 0 & 0 \\ 0 & 0.09 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0.1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Choosing the weight matrices $Q = I_5 \;\; R = I_2$, we have the MATLAB output

```
>> [Ktilde,S,E]=dlqr(Atilde,Btilde,Q,R)

Ktilde =

  Columns 1 through 4

   0.471643752376288  -3.280160852094671  -0.000006047152418  -
0.501580435062761
  -0.000000608047822   0.091410094211754   0.946763887535928
0.000000620130781

  Column 5

  -0.000000647595529
  -0.946762916583234
```

The closed-loop system state-space model is

$$\tilde{\mathbf{x}}(k+1) = (\tilde{A} - \tilde{B}\tilde{K})\tilde{\mathbf{x}}(k) + \begin{bmatrix} \mathbf{0} \\ I_l \end{bmatrix} \mathbf{r}(k)$$
$$\mathbf{y}(k) = \begin{bmatrix} C & \mathbf{0} \end{bmatrix} \tilde{\mathbf{x}}(k)$$

which gives the step response of Figure P10.4.

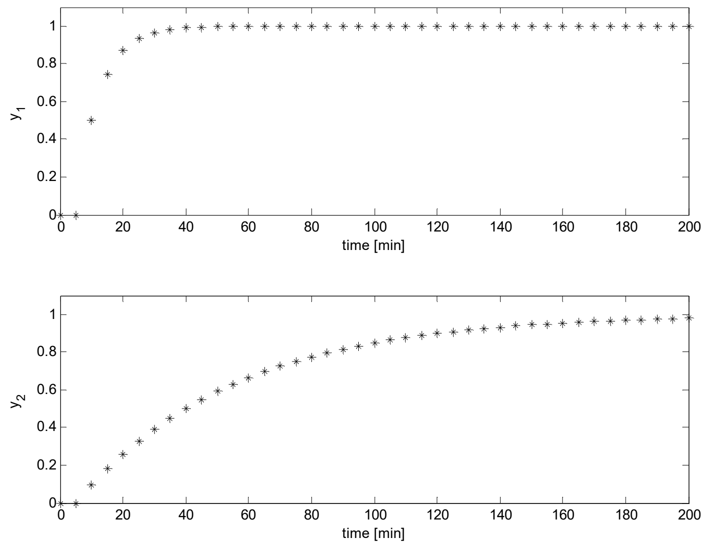


Figure P10.4 Step response for Problem 10.14.

10.15 Write a MATLAB script that determines the steady-state quadratic regulator for the inertial system of Example 10-4 for r=1 and different values of Q. Use the following three matrices and discuss the effect of changing Q on the results of your computer simulation:

$$\text{(i) } Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \quad \text{(ii) } Q=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix} \qquad \text{(iii) } Q=\begin{bmatrix}100 & 0\\ 0 & 100\end{bmatrix}$$

and simulate the closed-loop system state response for the initial condition vector $\mathbf{x}(0) = [1, 1]^T$. Plot the state trajectory and the control variable and evaluate the result.

The MATLAB script is

```
T=0.02;        % Sampling period
A=[1 T; 0 1]; % State matrix
B=[T^2/2; T]; % Input matrix
C=[1 0];       % Output matrix
D=0;           % Direct transmission matrix
r=1;           % Control effort weighting matrix
Q{1}=[1 0; 0 1]; % Regulator error weighting matrices
Q{2}=[10 0; 0 10];
Q{3}=[100 0; 0 100];
for i=1:length(Q)
[K,S,E]=dlqr(A,B,Q{i},r); % Solve the steady-state regulator
                       % problem
Acl=A-B*K; % Closed-loop system state matrix
sys_cl=ss(Acl,B,C,D,0.02); % Closed-loop system
[y,t,x]=initial(sys_cl,[1 1],7); % Response to initial conditions
figure
subplot(2,1,1);
```

```
plot(t,x(:,1),'k*',t,x(:,2),'r*'); % Plot the time response
legend('x_1','x_2')
xlabel('time')
u=-K*x';
subplot(2,1,2);
plot(t,u,'k*') % Plot the control variable
legend('u')
xlabel('time')
end
```

The response for the cases (i)-(iii) are shown in Figures P10.5-7. The response shows that a faster transient response is obtained at the expenses of a higher amplitude of the control variable.

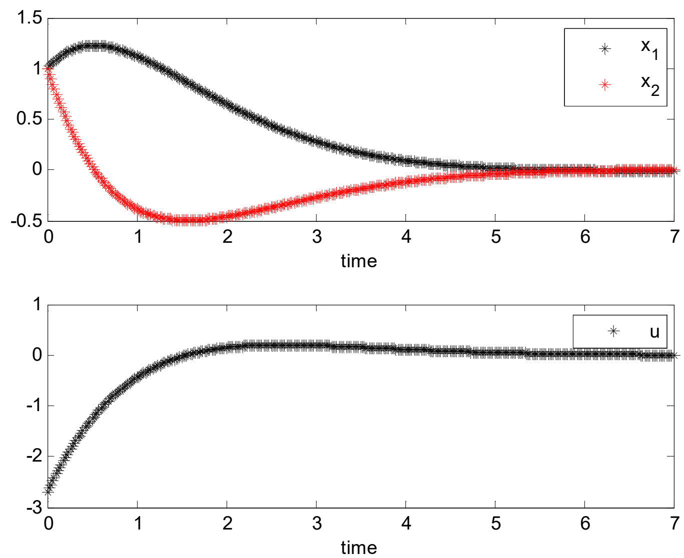


Figure P10.5 State response and control variable for case (i) of Problem 10.15.

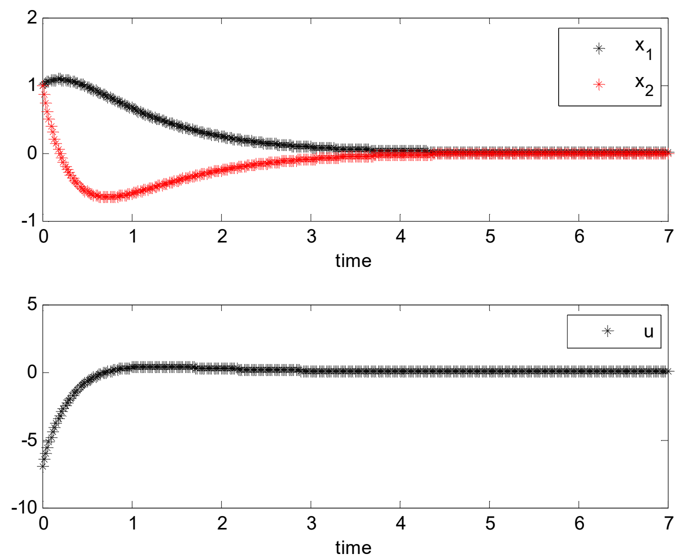


Figure P10.6 State response and control variable for case (ii) of Problem 10.15.

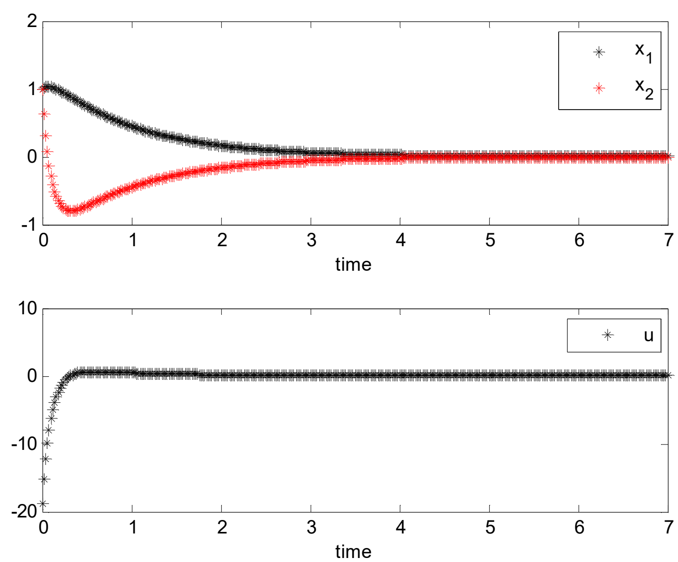


Figure P10.7 State response and control variable for case (iii) of Problem 10.15.

10.16 The linearized analog state-space model of the three tanks system shown in Figure P10.8 can be written as[2]

[2] D. M. Koenig, Practical Control Engineering, McGraw-Hill, 2009.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

where $x_i(t)$ is the level of the i^{th} tank and $u(t)$ is the inlet flow of the first tank. A digital controller is required to control the fluid levels in the tanks. The controller must provide a fast step response without excessive overshoot that could lead to fluid overflow. Using the appropriate MATLAB commands, design a digital LQ tracking controller for the system then simulate the closed-loop dynamics using SIMULINK. simulate the closed-loop system using SIMULINK. Use a sampling period of 0.1 s, a state-weighting matrix Q=0.1 I_4, and an input-weighting matrix R=1.

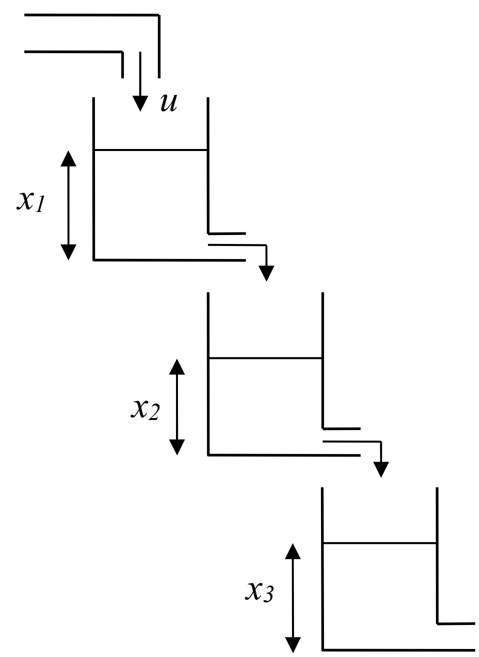


Figure P10.8 Schematic of the three tanks system.

Solution

The discretized model can be obtained by applying the MATLAB commands

```
>> A=[-1 0 0; 1 -1 0; 0 1 -1];
>> B=[1 0 0]';
>> C=[0 0 1];
>> D=0;
>> sys=ss(A,B,C,D);
>> Tc=0.1;
>> sysd=c2d(sys,Tc);
```

The dscrete-time state-space model is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0.9048 & 0 & 0 \\ 0.09048 & 0.9048 & 0 \\ 0.004524 & 0.09048 & 0.9048 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0.09516 \\ 0.004679 \\ 0.0001547 \end{bmatrix} u(k)$$

$$[y(k)] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

Integral control can be introduced by considering the augmented system of (10.55) and by applying the following commands:

```
>> A1=sysd.a;
>> B1=sysd.b;
>> C1=sysd.c;
>> D1=sysd.d;
>> Atilde=[A1 zeros(3,1); -C1 1];
>> Btilde=[B1; 0];
>> Ctilde=[C1 0];
```

We obtain

$$\tilde{A} = \begin{bmatrix} 0.904837418035960 & 0 & 0 & 0 \\ 0.090483741803596 & 0.904837418035960 & 0 & 0 \\ 0.004524187090180 & 0.090483741803596 & 0.904837418035960 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0.095162581964040 \\ 0.004678840160444 \\ 0.000154653070265 \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

The state feedback gain matrix is determined using the MATLAB command `dlqr`:

```
>> Q=0.01*eye(4);
>> R=1;
>> [Ktilde,S,E]=dlqr(Atilde,Btilde,Q,R);
>> K=-Ktilde(1:3);
>> Kbar=-Ktilde(4);
```

By selecting the previous values of the matrices Q and R we obtain

$$\widetilde{K} = [0.597016913804221 \quad 0.767974619370758 \quad 0.953903516058524 \quad -0.097013552768741]$$

We simulate the system using SIMULINK to obtain the closed-loop step response of Figure P10.9. For simplicity, we assume that all state variables are measurable and can be used for state feedback without the need for an observer. The resulting state trajectories are shown in Figure P10.10 while the control variable is shown in Figure P10.11.

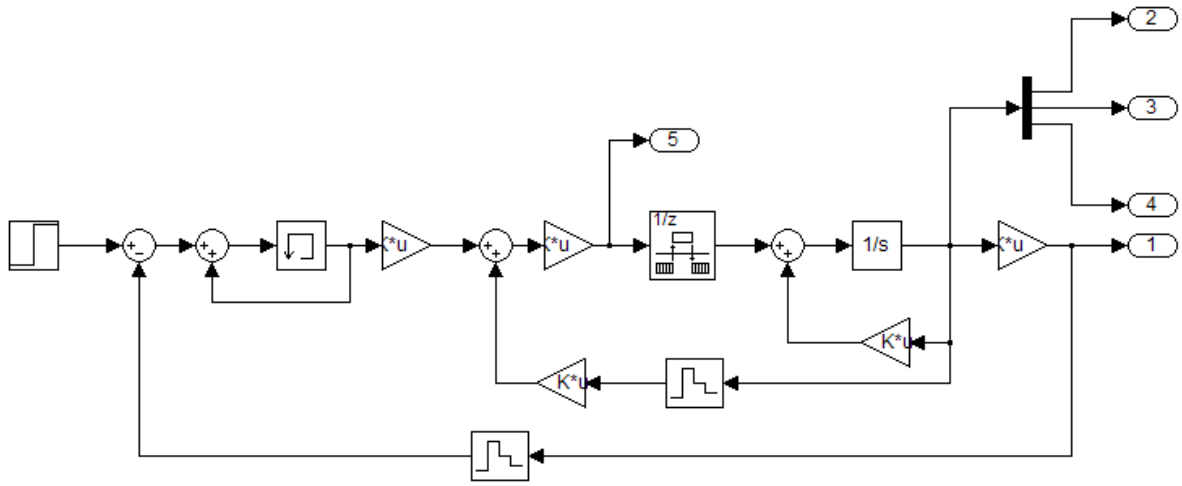


Figure P10.9 SIMULINK file for simulating the step response

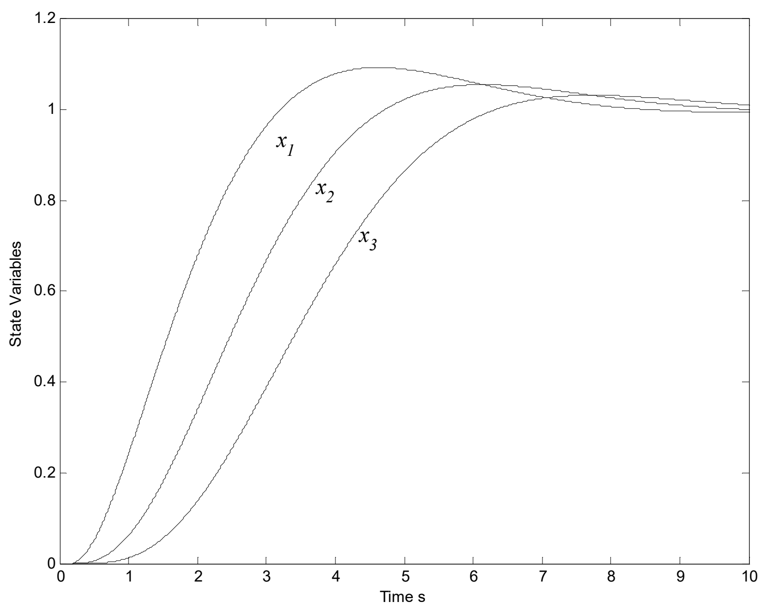


Figure P10.10 State variables resulting for a step reference input and Q=0.1I_4.

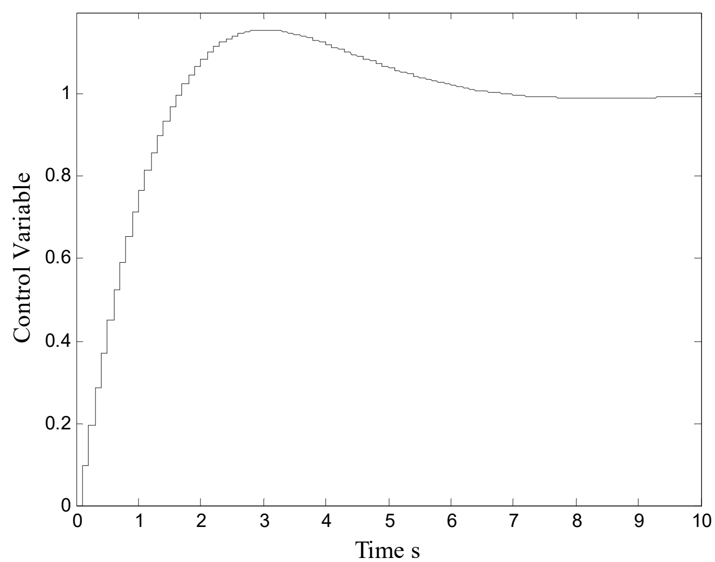


Figure P10.11 Control variable for a step reference input and Q=0.1I_4.

Note that by selecting

```
Q=1*eye(4);
```

we obtain the faster response shown in Figure P10.12 with the larger control peak of Figure P10.13. In addition to the larger control cost, the faster response comes with a large overshoot that can yield an overflow of the fluid in the tanks.

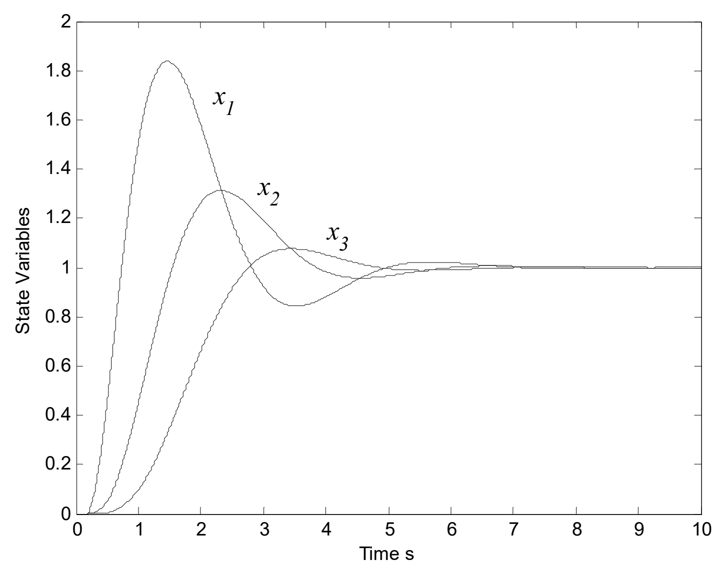


Figure P10.12 State variables resulting for a step reference input and Q=I_4.

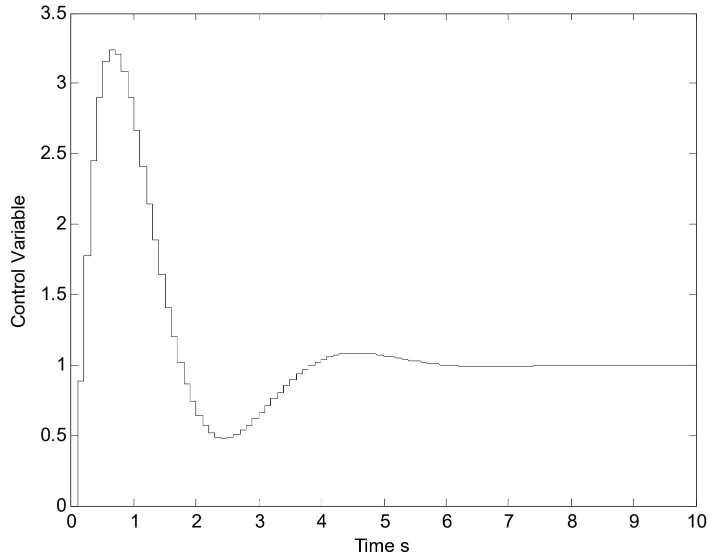


Figure P10.13 Control variable resulting for a step reference input and $Q=I_4$.

10.17. xxx

Chapter 11 Solutions

11.1 Discretize the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3x_1 - x_1^2/\left(2x_2^2\right) \\ x_2^2/x_1 + 1 \end{bmatrix} + \begin{bmatrix} x_1^2 \\ 0 \end{bmatrix} u(t)$$

Rewrite the system equations in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1^2 & 0 \\ 0 & -x_2^2 \end{bmatrix} \left\{ \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -x_1^{-1} \\ x_2^{-2}/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \right\}$$

Thus, we can apply Theorem 11.1. We define the terms

$$B_1(\mathbf{x}) = \begin{bmatrix} x_1^2 & 0 \\ 0 & -x_2^2 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

and note that the Jacobian of the vector is

$$\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}} = \begin{bmatrix} \partial h_1/\partial x_1 & \partial h_1/\partial x_2 \\ \partial h_2/\partial x_1 & \partial h_2/\partial x_2 \end{bmatrix} = \left[B_1(\mathbf{x})\right]^{-1} = \begin{bmatrix} x_1^{-2} & 0 \\ 0 & -x_2^{-2} \end{bmatrix}$$

We integrate to obtain

$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix} -x_1^{-1} \\ x_2^{-1} \end{bmatrix}$$

We then solve the state equation

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

to obtain

$$\begin{bmatrix} w_1(k+1) \\ w_2(k+1) \end{bmatrix} = \left\{ \begin{bmatrix} 0.2764 & 0.4472 \\ 0.4472 & 0.7236 \end{bmatrix} e^{1.382T} + \begin{bmatrix} 0.7236 & -0.4472 \\ -0.4472 & 0.2764 \end{bmatrix} e^{3.618T} \right\} \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix}$$
$$+ \left\{ \begin{bmatrix} 0.2764 \\ 0.4472 \end{bmatrix} \frac{e^{1.382T} - 1}{1.382} + \begin{bmatrix} 0.7236 \\ -0.4472 \end{bmatrix} \frac{e^{3.618T} - 1}{3.618} \right\} u(k)$$

where $\mathbf{w}(k) = \mathbf{h}[\mathbf{x}(k)]$, $k = 0, 1\ 2, \ldots$ Then we have

$$x_1 = -1/w_1 \qquad x_2 = 1/w_2$$

which is clearly unique.

11.2 The equations for rotational maneuvering of a helicopter are given by[1]

$$\ddot{\theta} = -\frac{1}{2}\left(\frac{I_z - I_y}{I_x}\right)\dot{\psi}\sin(2\theta) - mgA\cos(\theta) + 2\left(\frac{I_z - I_y}{I_x}\right)\dot{\psi}\dot{\theta}\frac{\sin(2\theta)}{(I_z + I_y) + (I_z - I_y)\cos(2\theta)}T_p$$

$$\dot{\psi} = \frac{1}{(I_z + I_y) + (I_z - I_y)\cos(2\theta)}T_y$$

where I_x, I_y, I_z are the moments of inertia about the center of gravity
m is the total mass of the system
g is the acceleration due to gravity
θ and ψ are the pitch and yaw angles in radians
$T_p$and T_y are the pitch and yaw input torques.

Obtain an equivalent linear discrete-time model for the system and derive the equations for the torque in terms of the linear system inputs.

We use the state vector

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} \theta & \dot{\theta} & \psi \end{bmatrix}$$

The linear model is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= u_2 \end{aligned}$$

The discrete-time model is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \\ T \end{bmatrix} u(k)$$

An approximate expression for the control inputs is given by

$$T_p = \frac{(I_z + I_y) + (I_z - I_y)\cos(2x_1(k))}{2(I_z - I_y)\sin(2x_1(k))x_2(k)u_2(k)}\left\{I_x u_1(k) + \frac{1}{2}(I_z - I_y)u_2(k)\sin(2x_1(k)) - mgA\cos(x_1(k))\right\}$$

$$T_y(k) = \left[(I_z + I_y) + (I_z - I_y)\cos(2x_1(k))\right]u_2(k)$$

11.3 A single-link manipulator with a flexible link has the equation of motion[2]

$$\begin{aligned} &I\ddot{\theta} + MgL\sin(\theta) - mgA\cos(\theta) + k(\theta - \psi) = 0 \\ &J\ddot{\psi} + k(\psi - \theta) = \tau \end{aligned}$$

where

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[1] A. L. Elshafei and F. Karray, "Variable structure based fuzzy logic identification of a class of nonlinear systems", *IEEE Trans. Control Systems Tech.*, Vol. 13, No. 4, July 2005, pp. 646-653.
[2] M. W. Spong and M. Vidyasagar, *Robot Dynamics and Control*", J. Wiley, NY, 1989, pp. 269-273.

L is the distance from the shaft to the center of gravity of the link
M is the mass of the link
I is the moment of inertia of the link
J is the moment of inertia of the joint
K is the rotational spring constant for the flexible joint
L is the distance between the center of gravity of the link and the flexible joint.
θ and ψ are the link and joint rotational angles in radians
τ is the applied torque

Obtain a discrete-time model of the manipulator.

Using the angular accelerations as the input we have the model

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)\end{aligned}$$

where $\mathbf{x}_1 = [x_{11} \quad x_{12}]^T = [\theta \quad \psi]^T, \mathbf{x}_1 = [x_{21} \quad x_{22}]^T = [\dot{\theta} \quad \dot{\psi}]^T$. The equivalent discrete model is

$$\begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \end{bmatrix} = \begin{bmatrix} I_2 & T\,I_2 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{3\times 1} \\ T \end{bmatrix} u(k)$$

The torque is approximately given by

$$\tau(k) = Ju(k) + k\big(x_{11}(k) - x_{12}(k)\big)$$

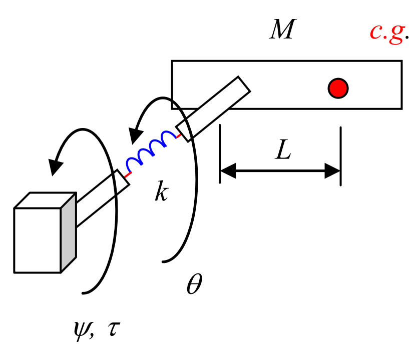

Figure 11.1 Schematic of single

11.4 Solve the nonlinear difference equation

$$[y(k+2)][y(k+1)]^{-2}[y(k)]^{1.25} = u(k)$$

with zero initial conditions and the input

$$u(k) = e^{-k}.$$

Taking the natural log of the equation, we obtain

$$x(k+2)-2x(k+1)+1.25x(k)=-k$$

The z-transform of the equation

$$\left[z^2-2z+1.25\right]X(z)=\frac{-z}{(z-1)^2}$$

yields $X(z)$ as

$$X(z)=\frac{-z}{(z-1)^2\left[(z-1)^2+0.5^2\right]}=4\left\{\frac{z}{(z-1)^2+0.5^2}-\frac{z}{(z-1)^2}\right\}$$

Inverse z-transforming gives the discrete-time function

$$x(k)=-4k+8(1.118)^k\sin(0.464k), k\geq 0$$

Hence, the solution of the nonlinear difference equation is

$$y(k)=e^{-4k+8(1.118)^k\sin(0.464)^k}, k\geq 0$$

11.5 Determine the equilibrium point for the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix}=\begin{bmatrix} -x_1(k)/9+2x_2^2(k) \\ -x_2(k)/9+0.4x_1^2(k) \end{bmatrix}+\begin{bmatrix} 0 \\ x_1(k) \end{bmatrix}u(k)$$

(i) unforced, (ii) with a fixed input u_e=1.

(i) At equilibrium, we have

$$\begin{bmatrix} -x_{1r}/9+2x_{2e}^2 \\ -x_{2e}/9+0.4x_{1e}^2 \end{bmatrix}=\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}$$

$$x_{1e}=1.8x_{2e}^2$$

$$x_{2e}=0.36x_{1e}^2=1.1664x_{2e}^4$$

We solve for the equilibrium points

$$\mathbf{x}_e=\begin{bmatrix}0 & 0\end{bmatrix}^T$$

$$\mathbf{x}_e=\begin{bmatrix}0.165 & 1.624\end{bmatrix}^T$$

(ii) in terms of $u(k)$

$$\begin{bmatrix} -x_{1r}/9+2x_{2e}^2 \\ -x_{2e}/9+0.4x_{1e}^2+x_{1e} \end{bmatrix}=\begin{bmatrix} x_{1e} \\ x_{2e} \end{bmatrix}$$

$$x_{1e}=1.8x_{2e}^2$$

$$x_{2e}=0.36x_{1e}^2+0.9x_{1e}=1.1664x_{2e}^4+1.62x_{2e}^2$$

$$1.1664x_{2e}^4+1.62x_{2e}^2-x_{2e}=0$$

We solve for the equilibrium points

$$\mathbf{x}_e = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$
$$\mathbf{x}_e = \begin{bmatrix} 0.482 & 0.517 \end{bmatrix}^T$$

11.6 Use the Lyapunov approach to show that if the function $\mathbf{f}(\mathbf{x})$ is a contraction then the system $\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k))$ is exponentially stable.

Using a quadratic Lyapunov function

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{x}(k) = \left\|\mathbf{x}(k)\right\|^2$$

The difference is given by

$$\begin{aligned}\Delta V(k) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \left\|\mathbf{f}(\mathbf{x}(k))\right\|^2 - \left\|\mathbf{x}(k)\right\|^2 < (\alpha^2 - 1) < 0\end{aligned}$$

where α is the contraction constant in the relation

$$\left\|\mathbf{f}(\mathbf{x}-\mathbf{y})\right\| \le \alpha\left\|\mathbf{x}-\mathbf{y}\right\|, \quad |\alpha| < 1$$

We conclude that the system is exponentially stable.

11.7 Obtain a general expression for the eigenvalues of a 2 by 2 matrix and use it to characterize the equilibrium points of the second-order system with the given state matrix

(i) $\begin{bmatrix} 0.9997 & 0.0098 \\ -0.0585 & 0.9509 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 0.3 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1.2 & -0.4 \\ 0.4 & 0.8 \end{bmatrix}$

For any 2 by 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The characteristic polynomial is given by

$$\det[\lambda I_2 - A] = (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

And the eigenvalues are

$$\lambda_{1,2} = \left(\frac{a_{11} + a_{22}}{2}\right) \pm \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}a_{21}}$$

(i) The eigenvalues of the matrix are $\{0.9802, 0.9704\}$, which are positive and less than unity. The equilibrium is a stable node.

(ii) The eigenvalues of the matrix are $\{1.382, 3.618\}\}$, which are positive and greater than unity. The equilibrium is an unstable node.

(iii) The eigenvalues of the matrix are $\{0.25 \pm j\, 0.0.0866\}$, which are complex conjugate with positive real part and magnitude 0.2646 less than unity. The equilibrium is a stable focus.

(iv) The eigenvalues of the matrix are $\{1 \pm j\, 0.0.3464\}$, which are complex conjugate with positive real part and magnitude 1.0583 greater than unity. The equilibrium is an unstable focus.

11.8 Determine the stability of the origin using the linear approximation for the system

$$\begin{aligned} x_1(k+1) &= 0.2x_1(k) + 1.1x_2^3(k) \\ x_2(k+1) &= x_1(k) + 0.1x_2(k) + 2x_1(k)x_2^2(k), \quad k = 0,1,2,\cdots \end{aligned}$$

We first rewrite the state equations in the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.2 & 0 \\ 1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1.1x_2^3(k) \\ 2x_1(k)x_2^2(k) \end{bmatrix}, \qquad k = 0,1,2,\cdots$$

The state matrix A of the linear approximation has one eigenvalue = 2 >1. Hence the origin is an unstable equilibrium of the nonlinear system.

11.9 Verify the stability of the origin using the Lyapunov approach and estimate the rate of convergence to the equilibrium

$$\begin{aligned} x_1(k+1) &= 0.1x_1(k)x_2(k) - 0.05x_2^2(k) \\ x_2(k+1) &= -0.5x_1(k)x_2(k) + 0.05x_2^3(k), \qquad k = 0,1,2,\cdots \end{aligned}$$

The system has an equilibrium point at the origin. We select the quadratic Lyapunov function

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}, P = diag\{p_1, 1\}$$

The corresponding difference is

$$\begin{aligned} \Delta V(k) &= p_1 \left\{ \left[0.1x_1(k)x_2(k) - 0.05x_2^2(k)\right]^2 - x_1^2(k) \right\} \\ &\quad + \left\{ \left[-0.5x_1(k)x_2(k) + 0.05x_2^3(k)\right]^2 - x_2^2(k) \right\} \\ &= -p_1x_1^2(k) + \left\{ \begin{aligned} &(0.25 + 0.01p_1)x_1^2(k) - \left(0.01p_1x_2(k) + 0.05x_2^2(k)\right)x_1(k) \\ &+ 0.0025p_1x_2^2(k) + 0.0025x_2^4 - 1 \end{aligned} \right\} x_2^2(k), \\ k &= 0,1,2,\cdots \end{aligned}$$

The difference remains negative provided that the term between braces is negative. If we select p_1=15, we have

$$\begin{aligned} \Delta V(k) &= -15x_1^2(k) \\ &\quad + \left\{0.4x_1^2(k) - \left(0.15x_2(k) + 0.05x_2^2(k)\right)x_1(k) + 0.0375x_2^2(k) + 0.0025x_2^4 - 1\right\} x_2^2(k) \\ &< -11.4x_1^2(k) - \left[0.46 + 0.9x_1(k)\right]x_2^2(k) \\ &< -11.4x_1^2(k) - 0.19x_2^2(k), \qquad k = 0,1,2,\cdots \end{aligned}$$

If we then restrict the magnitude of $x_1(0)$ and $x_2(0)$ to less than 3, we obtain negative values of the difference as shown in Figure P1. We conclude that the equilibrium is locally stable.

Note that different values of p_1 give different estimates of the stable domain of the equilibrium point. For example, choosing $p_1 = 1$ would lead us to restrict the magnitudes of $x_1(0)$ and $x_2(0)$ to less than 2 for a negative difference.

To assess the rate of convergence, we observe that with $p_1 = 15$ we locally have

$$\begin{aligned}\Delta V(k) &< -11.4x_1^2(k) - 0.19x_2^2(k) \\ &< -0.19\left[15x_1^2(k) + x_2^2(k)\right] = -0.19V(\mathbf{x}(k)), \qquad k = 0,1,2,\cdots\end{aligned}$$

We can obtain the rate of convergence for the weighted vector norm as

$$\begin{aligned}&V(\mathbf{x}(k)) < (1-0.19)^k V(\mathbf{x}(0)) = 0.81^k V(\mathbf{x}(0)) \\ &\|\mathbf{x}(k)\|^2 < 0.81^k \|\mathbf{x}(0)\|^2 \\ &\|\mathbf{x}(k)\| \le 0.09^k \|\mathbf{x}(0)\| \\ &\|\mathbf{x}(k)\|^2 = 15x_1^2(k) + x_2^2(k)\end{aligned}$$

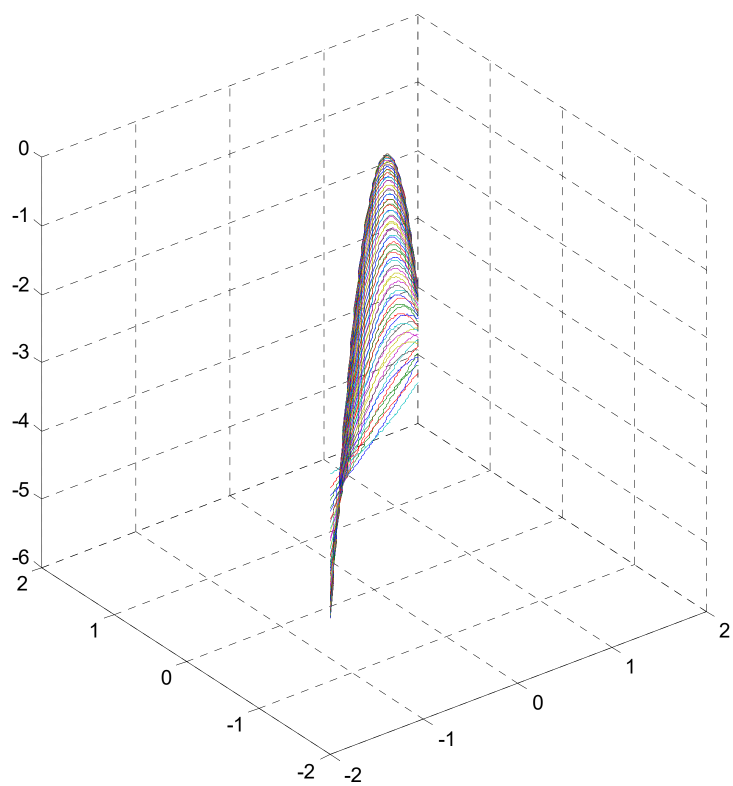

Figure P1 Plot of ΔV in the vicinity of the origin.

11.10 Show that the convergence of the trajectories of a nonlinear discrete-time system $\mathbf{x}(k+1)=\mathbf{f}[\mathbf{x}(k)]$ to a known nominal trajectory $\mathbf{x}^*(k)$ is equivalent to stability of the dynamics of the tracking error $\mathbf{e}(k)=\mathbf{x}(k)-\mathbf{x}^*(k)$.

Using the expression for the tracking error,

$$\mathbf{e}(k)=\mathbf{x}(k)-\mathbf{x}^*(k)$$

we can express the state vector in terms of the error and the nominal state vector as

$$\mathbf{x}(k)=\mathbf{e}(k)+\mathbf{x}^*(k)$$

We obtain the error dynamics by subtracting the nominal state vector from the state equation

$$\mathbf{e}(k+1)=\mathbf{x}(k+1)-\mathbf{x}^*(k+1)=\mathbf{f}\left[\mathbf{e}(k)+\mathbf{x}^*(k)\right]-\mathbf{x}^*(k)$$

Since the nominal trajectory is known, we write the error dynamics as

$$\mathbf{e}(k+1)=\mathbf{f}_{\mathbf{e}}[\mathbf{e}(k)]$$
$$\mathbf{f}_{\mathbf{e}}[\mathbf{e}(k)]=\mathbf{f}\left[\mathbf{e}(k)+\mathbf{x}^*(k)\right]-\mathbf{x}^*(k)$$

If the error converges to zero, i.e. if they are asymptotically stable, then the system trajectories will converge to the nominal trajectory.

11.11 Prove that the scalar system

$$x(k+1)=-ax^3(k)$$

is locally asymptotically stable in the region $|x(k)|\le 1/\sqrt{a}$.

We use the quadratic Lyapunov function

$$V(x(k))=x^2(k)$$

The difference at time k is

$$\begin{aligned}\Delta V(x(k))&=a^2x^6(k)-x^2(k)\\&=-x^2(k)\left[1-a^2x^4(k)\right]\end{aligned}$$

is negative definite in the region $|x(k)|\le 1/\sqrt{a}$. Hence, the system is locally asymptotically stable in the region.

11.12 Use Lyapunov stability theory to investigate the stability of the system

$$x_1(k+1) = \frac{ax_1(k)}{a + bx_2^2(k)}$$

$$x_2(k+1) = \frac{bx_2(k)}{b + ax_1^2(k)}, a > 0, b > 0$$

The system has an equilibrium point at the origin. We select the quadratic Lyapunov function

$$V(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$$

The corresponding difference is

$$\Delta V(k) = \left\{ \frac{a^2}{\left[a + bx_2^2(k)\right]^2} - 1 \right\} x_1^2(k) + \left\{ \frac{b^2}{\left[b + ax_1^2(k)\right]^2} - 1 \right\} x_2^2(k)$$

$$= -\left\{ \frac{b^2 x_2^2(k) + 2ab}{\left[a + bx_2^2(k)\right]^2} + \frac{a^2 x_1^2(k) + 2ab}{\left[b + ax_1^2(k)\right]^2} \right\} x_1^2(k) x_2^2(k) < 0$$

which is globally negative definite. The Lyapunov function is radially unbounded and the system is globally asymptotically stable.

11.13 Use the Lyapunov approach to determine the stability of the discrete-time linear time-invariant systems

(i) $\begin{bmatrix} 0.3 & -0.1 \\ 0.1 & 0.22 \end{bmatrix}$ (ii) $\begin{bmatrix} 0.3 & -0.1 & 0 \\ 0.1 & 0.22 & 0.2 \\ 0.4 & 0.2 & 0.1 \end{bmatrix}$

We use the Lyapunov equation

$$A_d^T P A_d - P = -I_n$$

(i) The solution is

$$P = \begin{bmatrix} 1.1099 & -0.0105 \\ -0.0105 & 1.0630 \end{bmatrix}$$

P is positive definite and the system is exponentially stable.

(ii) The solution is

$$P = \begin{bmatrix} 1.3295 & 0.0831 & 0.0783 \\ 0.0831 & 1.1088 & 0.0720 \\ 0.0783 & 0.0720 & 1.0578 \end{bmatrix}$$

P is positive definite and the system is exponentially stable.

11.14 Show that the origin is an unstable equilibrium for the system

$$x_1(k+1) = -1.4x_1(k) + 0.1x_2^2(k)$$
$$x_2(k+1) = 1.5x_2(k)\left(0.1x_1(k)+1\right), \qquad k = 0,1,2,\cdots$$

Choose the Lyapunov function

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}, P = diag\{p_1, 1\}$$

The corresponding difference is given by

$$\Delta V(k) = p_1\left\{\left[-1.4x_1(k) + 0.1x_2^2(k)\right]^2 - x_1^2(k)\right\} + \left\{\left[1.5x_2(k)\left(0.1x_1(k)+1\right)\right]^2 - x_2^2(k)\right\}$$
$$= 0.96p_1x_1^2(k) + 0.01p_1x_2^4(k) + \left\{0.0225x_1^2(k) + \left(0.45 - 0.28p_1\right)x_1(k) + 1.25\right\}x_2^2(k),$$
$$k = 0,1,2,\cdots$$

We complete the squares for the last term by choosing $p_1 = (15/28)\left(3-\sqrt{5}\right) = 0.4092$ and reduce the difference to

$$\Delta V(k) = 0.3929\,x_1^2(k) + 0.0041x_2^4(k) + \left(0.15x_1(k) + \sqrt{1.25}\right)^2 x_2^2(k)$$
$$\geq 0.3929\,x_1^2(k) + 0.0041x_2^4(k), \qquad k = 0,1,2,\cdots$$

The inequality follows from the fact that the last term in ΔV is positive semidefinite. We conclude that ΔV is positive definite since it is greater than the sum of even powers and that the equilibrium at $\mathbf{x} = \mathbf{0}$ is unstable.

11.15 Estimate the domain of attraction of the system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.2 & 0.3 \\ -0.4 & 0.5 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.3x_2^2(k) \\ 0.36x_1^2(k) \end{bmatrix}$$

The nonlinear vector satisfies

$$\left\|\mathbf{f}\left[\mathbf{x}(k)\right]\right\| =\leq 0.36\left\|\mathbf{x}(k)\right\|, \qquad k = 0,1,2,\cdots$$

We solve the Lyapunov equation with $Q = I_2$ to obtain

$$P = \begin{bmatrix} 1.31 & -0.1996 \\ -0.1996 & 1.4107 \end{bmatrix}$$

The largest eigenvalue of P is equal to 1.5662. The norm $\|PA\|$ is 0.8844 and our estimate of the domain of attraction is

$$B(\mathbf{x}) = \left\{\mathbf{x} : \|\mathbf{x}\| < \frac{1}{0.36\sqrt{1.5662}}\left[-0.8844 + \sqrt{(0.8844)^2 + 0.36}\right]\right\}$$
$$= \left\{\mathbf{x} : \|\mathbf{x}\| < 0.3269\right\}$$

Computer simulation shows that the estimate of the domain of attraction is quite conservative and that the system is stable well outside the estimated region.

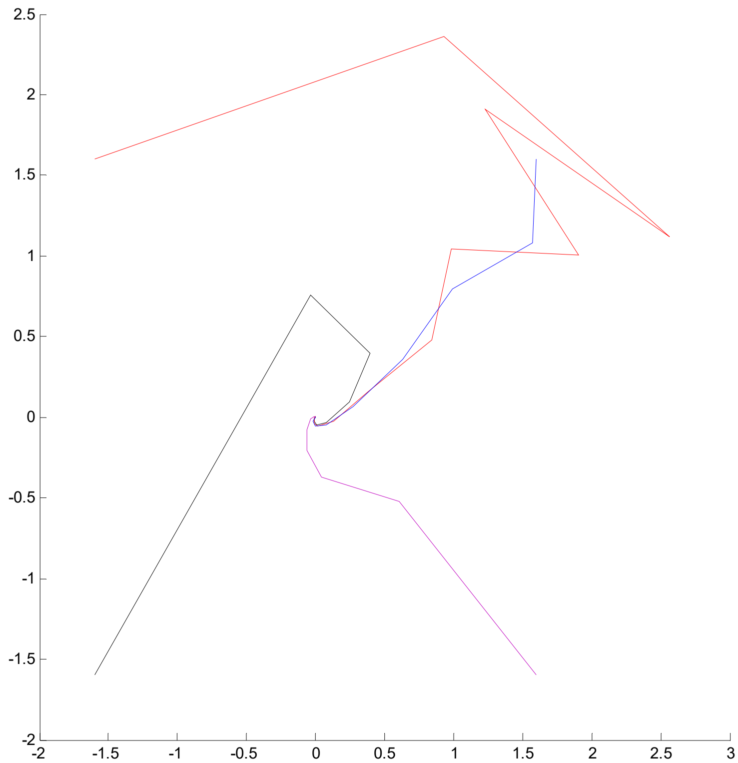

Figure P2. Phase portrait for the nonlinear system of Problem 11.15.

11.16 Design a controller to stabilize the origin for the system

$$x_1(k+1) = 0.4x_1(k) + 0.5x_2(k) + x_2^2(k)u(k)$$
$$x_2(k+1) = 0.1x_1(k) + 0.2x_2(k) + [x_2(k) + x_1(k)]u(k), \quad k = 0,1,2,\cdots$$

We rewrite the system dynamics in the form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} x_2^2(k) \\ x_2(k) + x_1(k) \end{bmatrix} u(k)$$

The Lyapunov equation with $Q=I_2$ has the solution

$$P = \begin{bmatrix} 1.2377 & 0.3174 \\ 0.3174 & 1.4301 \end{bmatrix}$$

The corresponding stabilizing control is given by

$$u(k) = -\left[B^T(\mathbf{x}(k))PB(\mathbf{x}(k))\right]^{-1} B^T(\mathbf{x}(k))PA\mathbf{x}(k)$$
$$= -\frac{\left[0.5268x_2^2(k) + 0.267(x_1(k) + x_2(k)) \quad 0.6823x_2^2(k) + 0.4447(x_1(k) + x_2(k))\right]}{1.2377x_2^4(k) + 0.6348x_2^3(k) + 0.4447(x_1(k) + x_2(k)) + 0.6348x_1(k)x_2^4(k) + 1.4301x_1^2(k)}\mathbf{x}(k)$$

Computer Exercises

11.17 Write a MATLAB program to generate phase plane plots for a discrete-time second order linear time-invariant system. The function should accept the eigenvalues of the state matrix and the initial conditions needed to generate the plots

```
% plotstate: Plot phase trajectories
x1min=-1;x1max=1; % Initial conditions
x2min=-1;x2max=1;
% Complex conjugate on the unit circle: uncomment for complex conjugate
% theta=45*pi/180;
% lambda1=cos(theta)+j*sin(theta); lambda2=cos(theta)-j*sin(theta);
% r1=real(lambda1);i1=imag(lambda1);
% a=[r1,-i1;i1,r1]; % complex conjugate
lambda1=0.1 ;lambda2= 0.3;
a=diag([lambda1,lambda2]); % diagonal form
% Phase variable form: uncomment for phase plots
% po=poly([lambda1,lambda2]); % characteristic polynomial
% po1=-fliplr(po);
% a=[0,1;po1(1:2)]; % phase variable form
b=[0;1];
c=eye(2);
p=ss(a,b,c,0,.01); % Define state-space model to use initial
clf
hold on
for x10=x1min:1:x1max
    for x20=x2min:1:x2max
x0=[x10;x20]; % Define new initial conditions
[x,t]=initial(p,x0); % Obtain response to initial conditions
x1=x(:,1);x2=x(:,2); % Separate date into x1 and x2 vectors
plot(x1,x2)
    end
end
```

11.18 Design a controller for the nonlinear mechanical system of Example 11.15 with the nonlinear damping $b(\dot{x}) = 0.25\dot{x}^5$, the nonlinear spring $c(x) = 0.5x + 0.02x^3$, T =0.02 s, and the desired eigenvalues for the linear design equal to $\{0.2 \pm j0.1\}$. Determine the value of the reference input for a steady state position of unity and simulate the system using SIMULINK.

Using the MATLAB command **place**, we obtain the feedback gain matrix

$$\mathbf{k}^T = [1625 \quad 63.8]$$

For a reference input r, we have the nonlinear control

$$f = u + b(x_2) + c(x_1)$$
$$u(k) = r(k) - \mathbf{k}^T \mathbf{x}(k)$$

The simulation diagram for the system is shown in Figure P.3 and the simulation diagram for the controller block in Figure P.4. We select the amplitude of the step input to obtain a steady state value of unity using the equilibrium condition

$$\mathbf{x}(k+1) = \begin{bmatrix} 1 & 0.02 \\ 0 & 1 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 2\times 10^{-4} \\ 0.02 \end{bmatrix} \left(r - \begin{bmatrix} 1625 & 63.8 \end{bmatrix} \mathbf{x}(k)\right) = \mathbf{x}(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This simplifies to

$$\begin{bmatrix} 2\times 10^{-4} \\ 0.02 \end{bmatrix} r = \begin{bmatrix} 0.41 \\ 41 \end{bmatrix}$$

which gives the amplitude $r = 2050$. The step response for the nonlinear system with digital control of Figure P.5 shows a fast response to a step input at $t = 0.2$ s that quickly settles to the desired steady state value of unity. Figure P.6 shows a plot of the velocity for the same input.

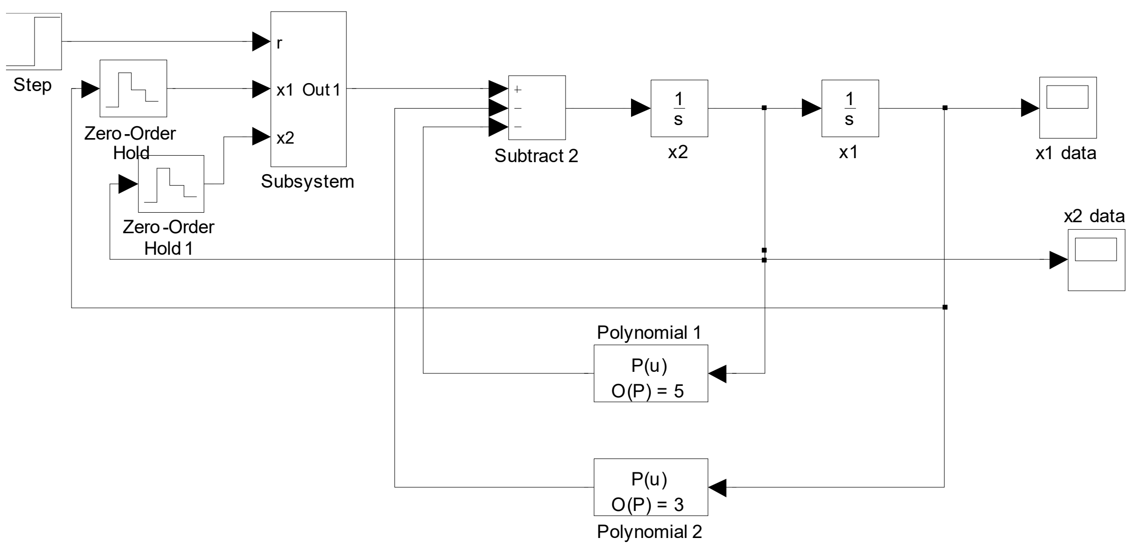

Figure P.3 Simulation diagram for the system of Problem 11.18

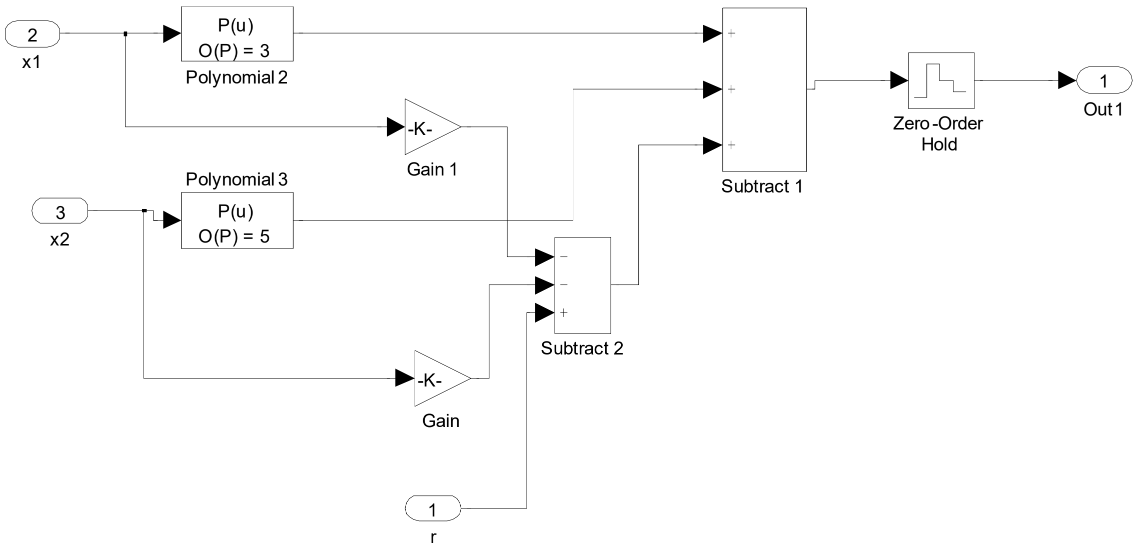

Figure P.4 Controller simulation diagram for Problem 11.18

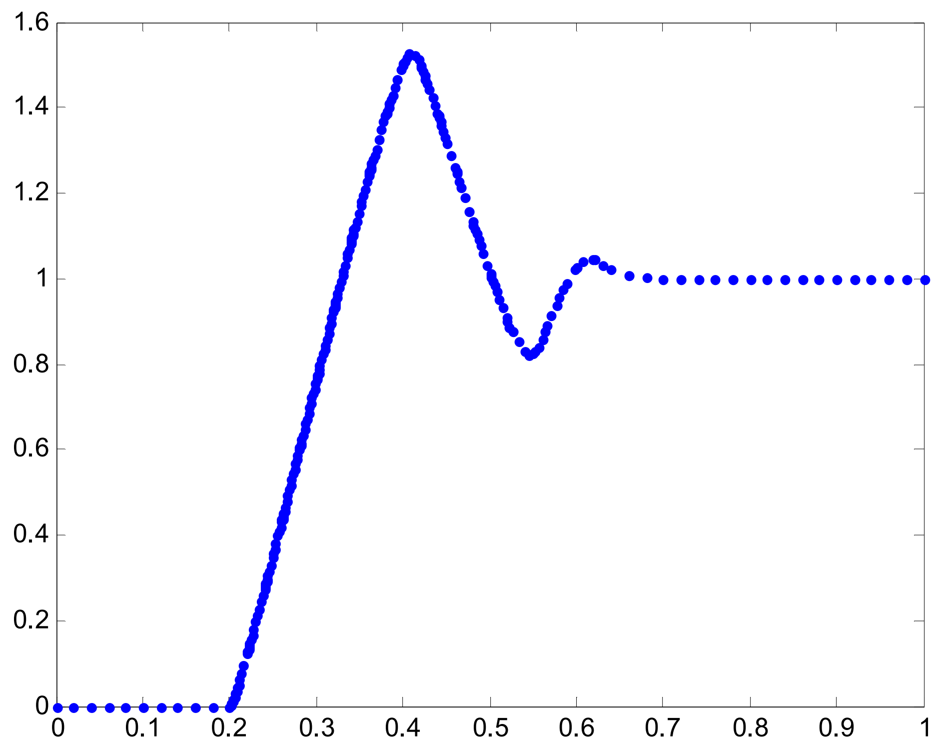

Figure P.5 Step response for the linear design of Problem 11.18

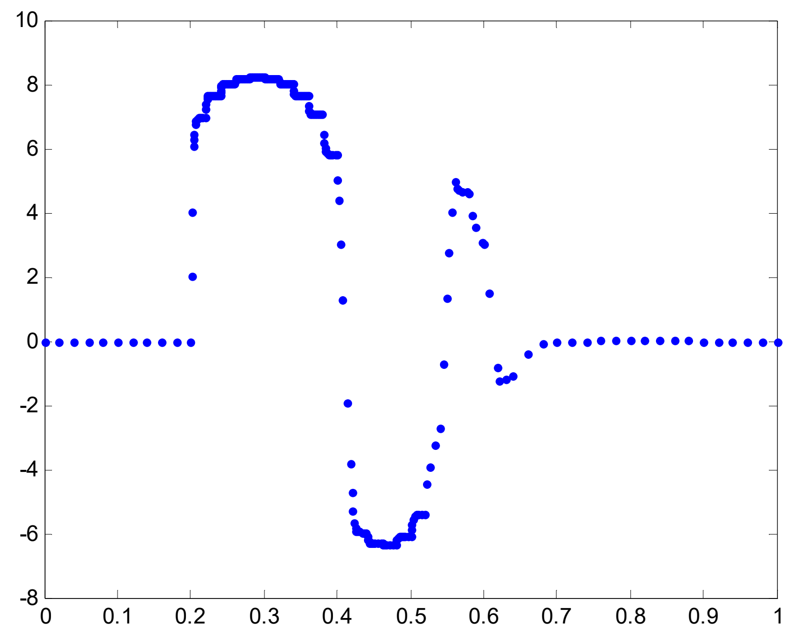

Figure P.6 Velocity plot for the step response for the linear design of Problem 11.18

11.19 Design a stabilizing digital controller with a sampling period $T = 0.01$ s for a single-link manipulator using extended linearization then simulate the system with your design. The equation of motion of the manipulator is given by

$$\ddot{\theta} + 0.01\sin(\theta) + 0.01\theta + 0.001\theta^3 = \tau$$

Assign the eigenvalues of the discrete-time linear system to $\{0.6 \pm j0.3\}$

Hint: Use SIMULINK for your simulation and use a **ZOH block**, or a **Discrete Filter** block with both the numerator and denominator set to 1 for sampling.

The state equation of the manipulator is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -0.01\sin(x_1) - 0.01x_1 - 0.001x_1^3 + \tau = u(t)$$

The discrete linear model is

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ x(k) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} u(k)$$

We use the MATLAB command to design the linear controller

```
> place(ad,bd,.6*[1+.5*j,1-.5*j])
ans =
1.0e+003 *

   2.5000    0.0675
```

$$u(k) = -2500x_1(k) - 67.5x_2(k)$$
$$\tau(k) = u(k) + 0.01\sin(x_1(k)) + 0.01x_1(k) + 0.001x_1^3(k)$$

The simulation diagram for SIMULINK is shown in Figure P11.7. The Sample/Hold block is a sampler. The time responses obtained by saving the scope output to a workspace then plotting them in MATLAB are shown in Figure P11.8 for θ and in Figure p11.9 for $\dot{\theta}$. The response is stable and fast the system quickly converges to its equilibrium.

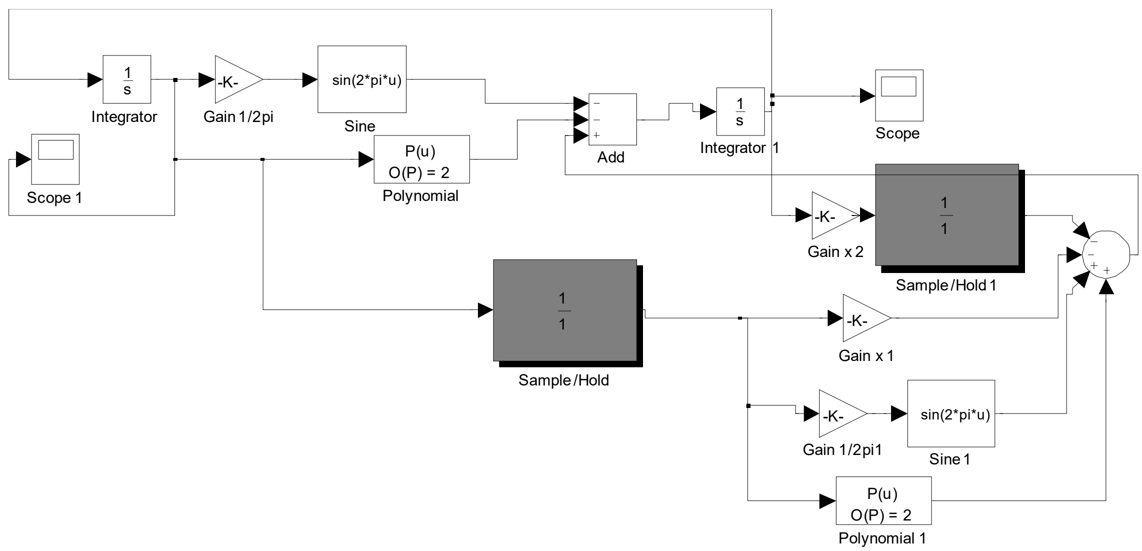

Figure P11.7 Simulation diagram for the single-link manipulator.

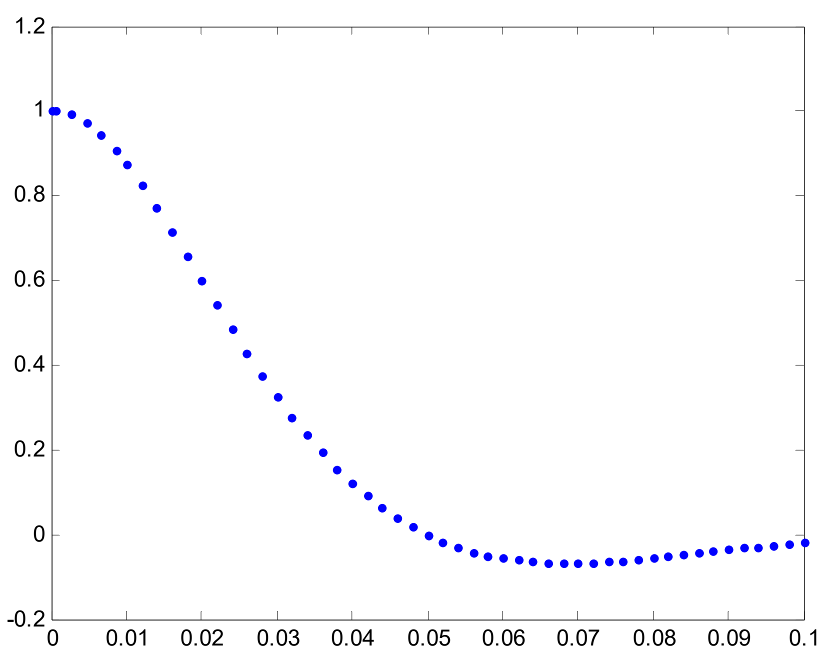

Figure P11.8 Time response for θ of the single-link manipulator with θ= 1 rad.

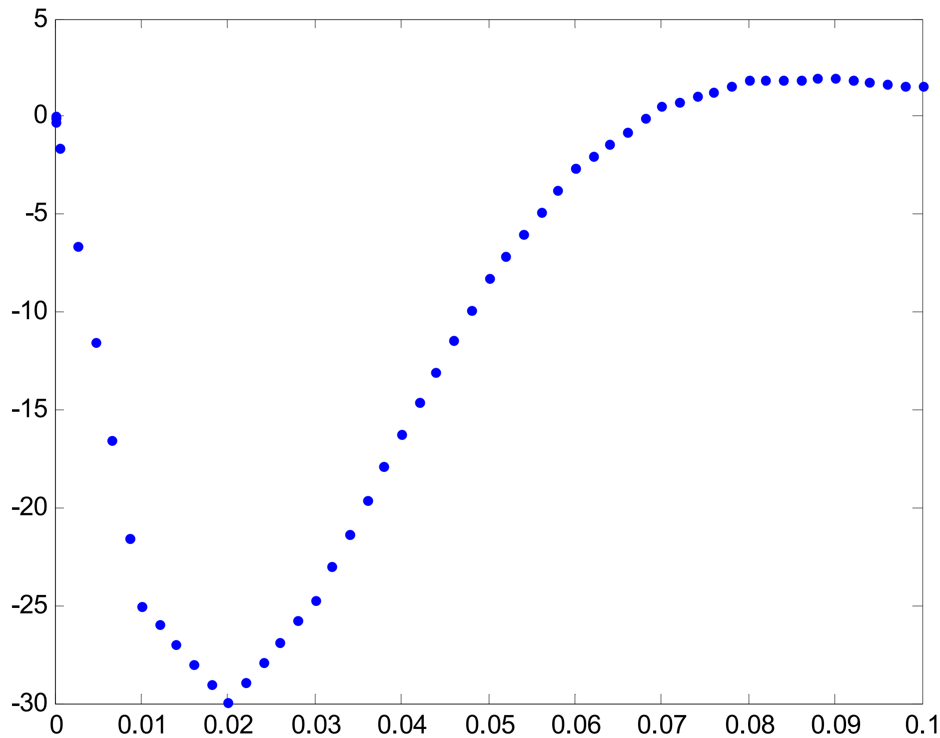

Figure p11.9 Time response for $\dot{\theta}$ of the single-link manipulator with θ= 1 rad.

Chapter 12 Problem Solutions

12.1 Write pseudocode that implements the following controller

$$C(z) = \frac{U(z)}{E(z)} = \frac{2.01z - 1.99}{z - 1}$$

The difference equation corresponding to the controller transfer function is

$$u(k) = u(k-1) + 2.01e(k) - 1.99e(k-1)$$

This control law can be implemented by writing the following code:

```
function controller
% This function must be executed for each sampling period
% r is the value of the reference signal
% u1 and e1 are the values of the control variable and of the tracking
% error, respectively, at the previous sampling instant
y=read_ADC(ch0)      % Read the process output value from ADC channel 0
e=r-y;               % Compute the new tracking error
u=u1+2.01*e-1.99*e1;    % Compute the new control variable
u1=u;            % Save the control variable for the next sampling period
e1=e;            % Save the tracking error for the next sampling period
write_DAC(ch0,u);    % Write the control variable to DAC channel 0
```

12.2 Rewrite the pseudocode for the controller of Problem 12.1 to decrease the execution time by assigning priorities to computational tasks.

In order to decrease the execution time, two tasks can be implemented with different priorities.

Task 1 (maximum priority)

```
y=read_ADC(ch0)      % Read the process output value from ADC channel 0
e=r-y;                  % Compute the tracking error
u=u1+2.01*e-1.99*e1;    % Compute the control variable
write_DAC(ch0,u);       % Write the control variable to DAC channel 0
```

```
Task 2

u1=u;     % Save the previous control value for the next sampling period
e1=e;     % Save the previous tracking error for the next sampling period
```

12.3 Design an antialiasing filter for the position control system

$$G(s) = \frac{1}{s(s+10)}$$

with the analog controller (see Example 5.6)

$$C(s) = 50\frac{s+0.5}{s}$$

Select an appropriate sampling frequency and discretize the controller.

The closed-loop transfer function is

$$G_{cl}(s) = \frac{50s + 25}{s^3 + 10s^2 + 50s + 25}$$

and its bandwidth can be easily determined using the MATLAB commands

```
>> g=tf([50,25], [1,10,50,25])

Transfer function:
      50 s + 25
------------------------
s^3 + 10 s^2 + 50 s + 25

>> bandwidth(g)

ans =
   7.4005
```

i.e. ω_b=7.4 rad/s. A suitable first-order antialiasing filter has a bandwidth of 50 rad/s and the transfer function

$$F(s) = \frac{1}{0.02s + 1}$$

We compute the phase shift introduced by the antialiasing filter at ω_b using the MATLAB commands

```
>> f=tf(1, [0.02,1])
>> [mag,ph]=bode(f,7.400)

mag =

   0.9892
ph =
  -8.4187
```

The phase shift of about −8.42° is not very significant. The sampling frequency can therefore be chosen as 314 rad/s. This is more than 6 times the bandwidth of the antialiasing filter and much larger than the bandwidth of the closed-loop system. The corresponding sampling interval is T 0.02 s. Bilinear transformation gives the digital controller transfer function

$$C(z) = \frac{50.25z - 49.75}{z - 1}$$

12.4 Determine the mean and variance values of the quantization noise when a 12-bit ADC is used to sample a variable in a range 0-10 V for (i) rounding, and (ii) truncation.

(i) The quantization interval is

$$q = \frac{10}{2^{12}}$$

and therefore the mean and variance values are

$$\bar{e} = 0$$

$$\sigma_e^2 = \frac{q^2}{12} = \left(\frac{10}{2^{12}}\right)^2 \frac{1}{12}$$

(ii) Since the value of q is the same as in (i), we have

$$\bar{e} = \frac{q}{2} = \frac{1}{2}\frac{10}{2^{12}}$$

$$\sigma_e^2 = \frac{q^2}{12} = \left(\frac{10}{2^{12}}\right)^2 \frac{1}{12}$$

12.5 For the system and the controller of Problem 12.3 with a sampling period T=0.02 s, determine the decrease in the phase margin due to the presence of the ZOH.

We determine the gain crossover frequency of the system of Problem 12.3 by solving the equation

$$\left|C(j\omega_c)G(j\omega_c)\right| = 1$$

Squaring the equation gives

$$X^2\left(X+100\right) = 2500X + 625, X = \omega_c^2$$

Whose only feasible solution is ω_c=4.574 rad/s. The decrease in the phase margin due to the ZOH is therefore

$$\omega_c \frac{T}{2}\frac{180}{\pi} = 2.62^\circ$$

12.6 Consider an oven control system [4] with transfer function

$$G(s) = \frac{1.1}{1300s+1}e^{-25s}$$

and a PI controller

$$C(s) = 13\frac{200s+1}{200s}$$

Let both the actuator and the sensor signals be in the range 0-5 V and let 1° Celsius of the temperature variable corresponds to 0.02 V. Design the hardware and software architecture of the digital control system.

The critical frequency of the analog control system is ω_c=0.012 rad/s and the phase margin is φ_m=53.9°. Thus, a possible choice is to select a sampling period T=1 s and to employ a second-order Butterworth antialiasing filter with cut-off frequency 0.2 rad/s. The filter transfer function is,

$$F(s) = \frac{0.04}{s^2 + 0.283s + 0.04}$$

The antialiasing filter does not change the crossover frequency, nor the phase margin (which is reduced to φ_m=49.0°), significantly. However, the antialiasing filter decreases the magnitude of the noise at the Nyquist frequency of π/T=3.14 rad/s of 47.8 dB. The phase delay introduced by the zero-order hold is $\omega_c T/2 \cdot 180/\pi = 0.34°$, which is negligible. A 12-bit ADC can be chosen with a quantization level of 1.2 mV, which corresponds a quantization error in the temperature of 0.06°. We also choose a 12-bit DAC. Its conversion time is in the order of microseconds and does not influence the overall design.

12.7 Write the difference equations for the controller in (i) direct form, (ii) parallel form, and (iii) cascade form.

$$C(z) = 50\frac{(z-0.9879)(z-0.9856)}{(z-1)(z-0.45)}$$

(i) The controller transfer function can be rewritten as

$$C(z) = \frac{U(z)}{E(z)} = \frac{50z^2 - 98.68z + 48.68}{z^2 - 1.45z + 0.45}$$

The difference equation corresponding to the direct form of the controller is

$$u(k) = 1.45u(k-1) - 0.45u(k-2) + 50e(k) - 98.68e(k-1) + 48.68e(k-2)$$

(ii) To obtain the controller in parallel form, we use the partial fraction expansion

$$C(z) = \frac{U(z)}{E(z)} = \frac{0.0158}{z-1} - \frac{26.191}{z-0.45} + 50$$

The controller is implemented with the difference equations

$$\begin{aligned}
u_1(k) &= u(k-1) + 0.0158e(k-1) \\
u_2(k) &= 0.45u(k-1) - 26.191e(k-1) \\
u_3(k) &= 50e(k) \\
u(k) &= u_1(k) + u_2(k) + u_3(k)
\end{aligned}$$

(iii) We write the controller transfer function in the cascade form

$$C(z) = \frac{U(z)}{E(z)} = \frac{50(z-0.9879)}{z-1}\frac{z-0.9856}{z-0.45} = \frac{X(z)}{E(z)}\frac{U(z)}{X(z)}$$

The controller is implemented with the difference equations

$$x(k) = x(k-1) + 50e(k) - 50 \cdot 0.9879e(k-1)$$
$$u(k) = 0.45u(k-1) + x(k) - 0.9856x(k-1)$$

12.8 For the PID controller that results by applying the Ziegler-Nichols tuning rules to the process

$$G(s) = \frac{1}{8s+1} e^{-2s}$$

determine the discretized PID controller transfer functions (12.11) and (12.12) with N=10 and T=0.1.

The process transfer function $G(s)$ has a gain $K = 1$, a time constant $\tau = 8$. and a dead time $L = 2$. Using Table 5.1 we calculate K_p=4.8, T_i=4 and T_d=1. Thus, we have

$$K_0 = K_p \left(\frac{T_d}{NT+T_d} - \frac{T}{T_i} \frac{T_d}{NT+T_d} + \frac{NT_d}{NT+T_d} \right) = 26.34$$

$$K_1 = -K_p \left(1 + \frac{T_d}{NT+T_d} - \frac{T}{T_i} + 2\frac{NT_d}{NT+T_d} \right) = -55.08$$

$$K_2 = K_p \left(1 + \frac{NT_d}{NT+T_d} \right) = 28.80$$

$$\gamma = \frac{T_d}{NT+T_d} = 0.5$$

which implies that the controller transfer function can be written as

$$C(z) = \frac{26.34 - 55.08z + 28.80z^2}{(z-1)(z-0.5)}$$

12.9 Design a bumpless manual/automatic mode scheme for the PID controller (T=0.1)

$$C(z) = \frac{252z^2 - 493.4z + 241.6}{(z-1)(z-0.13)}.$$

From the expression of the PID controller (12.11) we compute K_0=241.6, K_1=–493.4, K_2=252, and γ=0.13. The block diagram for the bumpless manual/automatic transfer is that of Figure 12.10 where K=K_2=252 and

$$D(z) = \frac{(K_1 + K + K\gamma)z + K_0 - K\gamma}{K_2 z^2 + K_1 z + K_0} = \frac{-0.8279z + 0.8287}{z^2 - 1.958z + 0.9587}$$

12.10 Design a bumpless manual/automatic mode scheme for the controller obtained in Example 6.18

12.11

$$C(z) = \frac{1.422(z-0.8187)(z-0.9802)(z+1)}{(z-1)(z+0.9293)(z-0.96)}$$

The block diagram for the bumpless manual/automatic transfer is that of Figure 12.10 where

$$D(z) = \frac{C(z)-K}{C(z)}$$

To avoid an algebraic loop, we ensure that the numerator of $D(z)$ must is second order with a third order denominator by selecting K=1.422. Then, we have the transfer function

$$D(z) = \frac{0.2318(z-0.9778)(z+0.3955)}{(z+1)(z-0.9802)(z-0.8187)}$$

12.12 Determine the digital PID controller (with T=0.1) in incremental form for the analog PID controller

$$C(s) = 3\left(1+\frac{1}{8s}+2s\right)$$

From the expression of the analog PID controller it is easy to determine the value of the parameters as K_p=3, T_i=8, and T_d=2. By applying (12.18)-(12.20) we have therefore

$$K_2 = K_p\left(1+\frac{T}{T_i}+\frac{T_d}{T}\right) = 63.04$$

$$K_1 = K_p\left(-1-\frac{2T_d}{T}\right) = -123$$

$$K_0 = K_p\frac{T_d}{T} = 60$$

and the corresponding z-transform transfer function is

$$C(z) = \frac{63.04z^2-123z+60}{z(z-1)}$$

Computer Exercises

12.13 Write a MATLAB script and design a SIMULINK diagram that implements the solution of problem 12.8 with different filter parameter values N and discuss the set-point step responses obtained by considering the effect of measurement noise on the process output.

A script that solves the problem is the following:

```
Gs=tf(1,[8 1],'inputdelay',2); % process transfer function
Kp=4.8; % PID parameters
Ti=4;
Td=1;
T=0.1; % sampling interval
z=tf('z',T);
for N=5:5:20, % different values of N
    % digital PID parameters
    K0=Kp*(Td/(N*T+Td)-T/Ti*Td/(N*T+Td)+N*Td/(N*T+Td));
    K1=-Kp*(1+Td/(N*T+Td)-T/Ti+2*N*Td/(N*T+Td));
    K2=Kp*(1+N*Td/(N*T+Td));
    gam=Td/(N*T+Td);
    Cz=(K2*z^2+K1*z+K0)/(z-1)/(z-gam); % digital PID
transfer
                                          % function
    [t,x,y]=sim('Problem12_12',50); % simulations with
                                     % SIMULINK
    figure                        % plot of the step
responses
    subplot(2,1,1);
    plot(t,y(:,1),'k-');
    title(['process output N=',num2str(N)]);
    subplot(2,1,2);
    plot(t,y(:,2),'k-');
    title('control variable');
    axis([0 50 -10 40])
end
```

The SIMULINK diagram for the simulation of the closed-loop system is shown in Figure P12.1.

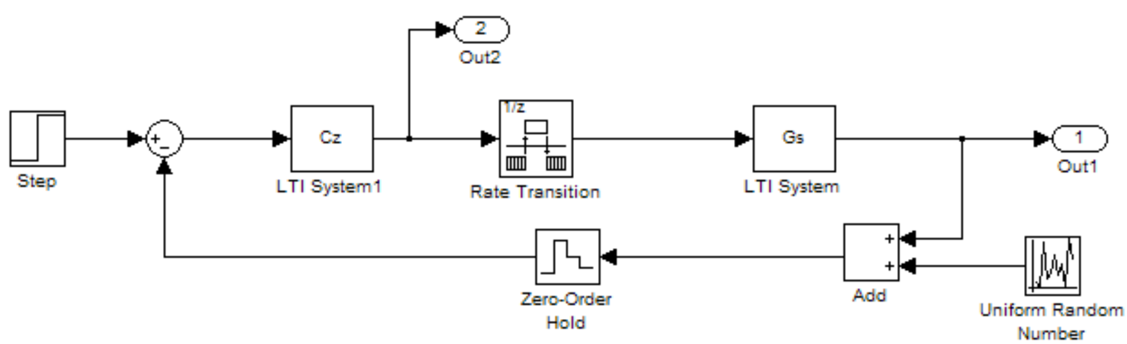

Figure P12.1 SIMULINK block diagram for Problem 12.13

By setting the Uniform Random Number between –0.1 and 0.1, we obtain the step responses shown in Figure P12.2. It appears that the noise effect increases, as expected, by increasing the value of *N*. The detrimental effect of the noise on the actuator is more relevant, as it can damage the actuator itself.

12.14 Consider the analog process

$$G(s) = \frac{1}{8s+1} e^{-2s}$$

and the analog PI controller with K_p=3 and T_i=8. Obtain the set-point step response with a saturation limit of u_{min}=–1.1 and u_{max}=1.1 and with a digital PI controller (T=0.1) with

(i) no antiwindup;
(ii) with a conditional integration antiwindup strategy;
(iii) with a back-calculation antiwindup strategy;
(iv) with a digital PI controller in incremental form.

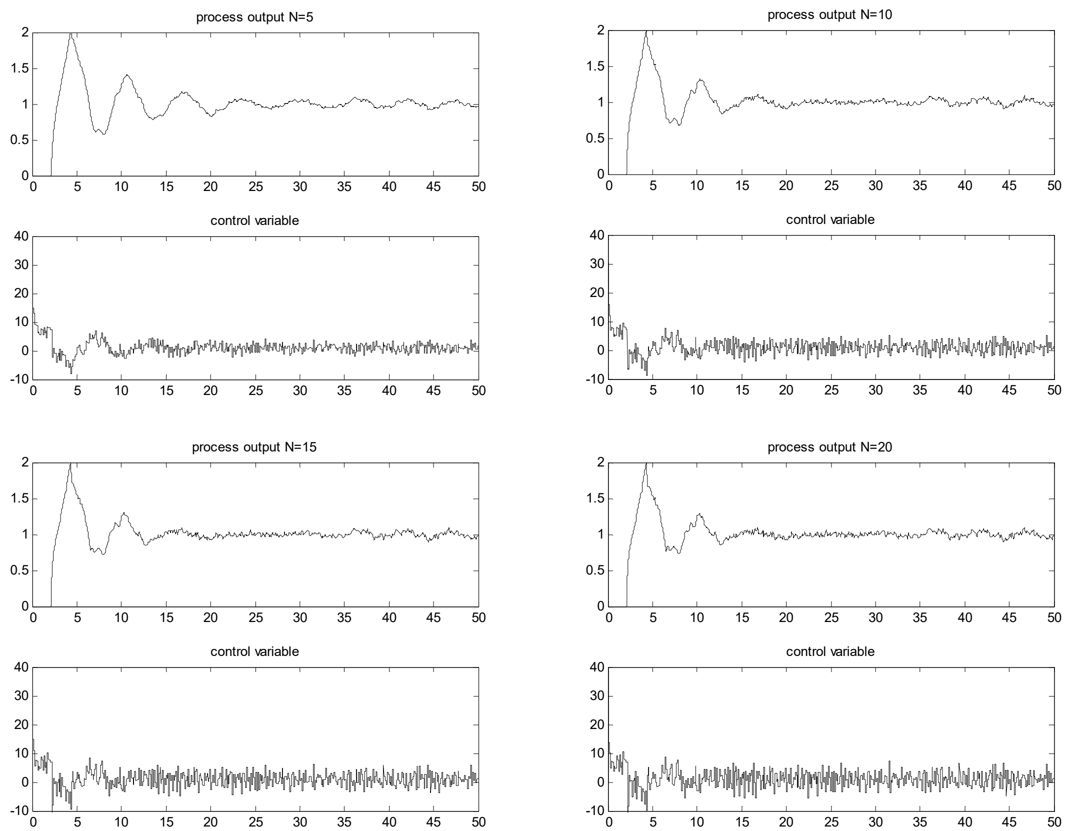

Figure P12.2 Step responses for Problem 12.13

(i) The difference equation corresponding to the digital PI controller is

$$u(k) = K_p e(k) + \frac{K_p T}{T_i} \sum_{i=0}^{k} e(i)$$

and this can be implemented by the SIMULINK diagram shown in Figure P12.3. The step response is shown in Figure P12.4 where the windup effect appears clearly.

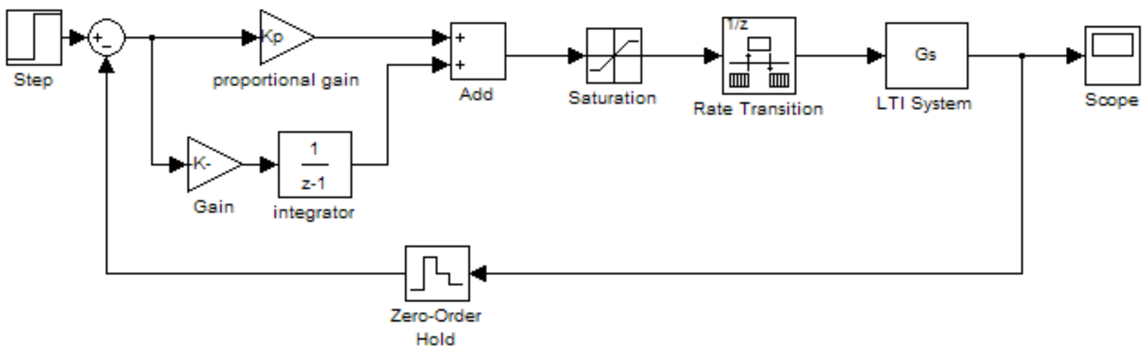

Figure P12.3 SIMULINK block diagram for Problem 12.14 (i)

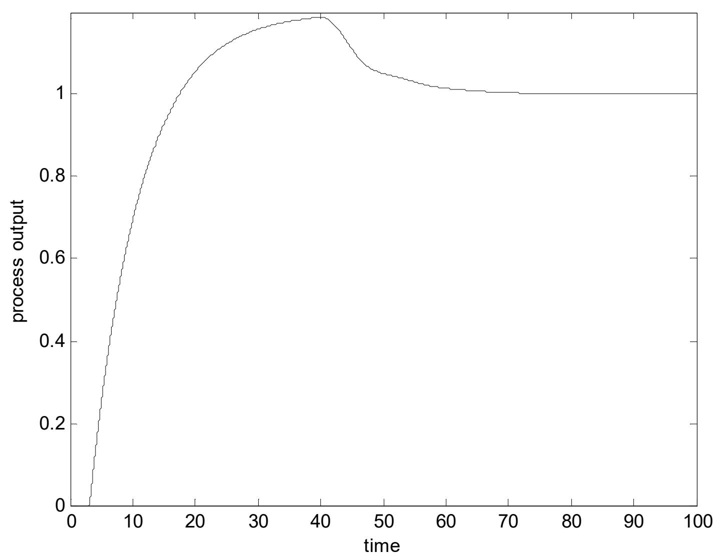

Figure P12.4 Step response for Problem 12.14 (i)

(ii) The conditional integration method consists of freezing the integral action when the actuator saturates and when the control variable and the control error have the same sign (namely, it is $u \times e > 0$). The SIMULINK implementation of this scheme is shown in Figure P12.5 where the error to be integrated is set to zero when the condition is true.

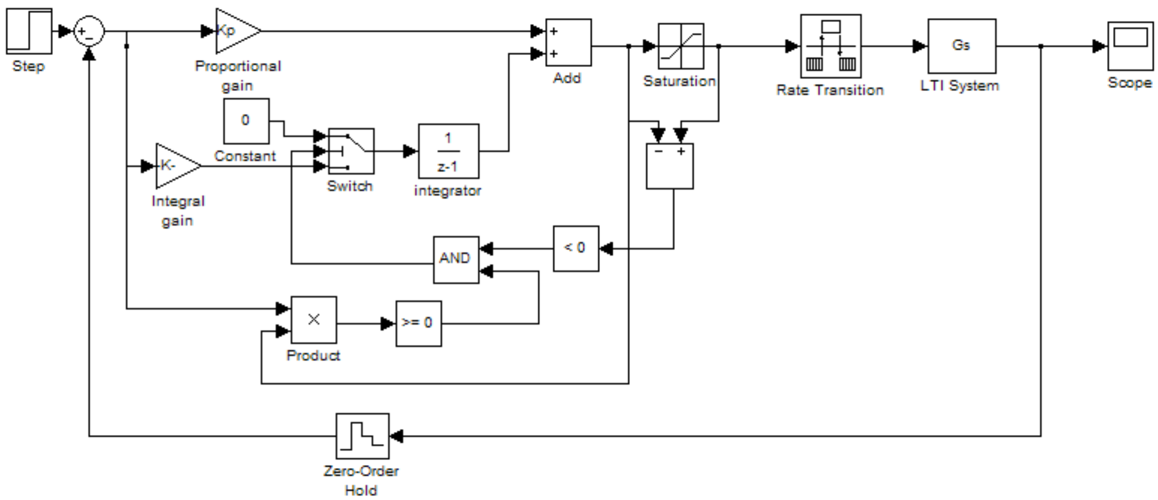

Figure P12.5 SIMULINK block diagram for Problem 12.14 (ii)

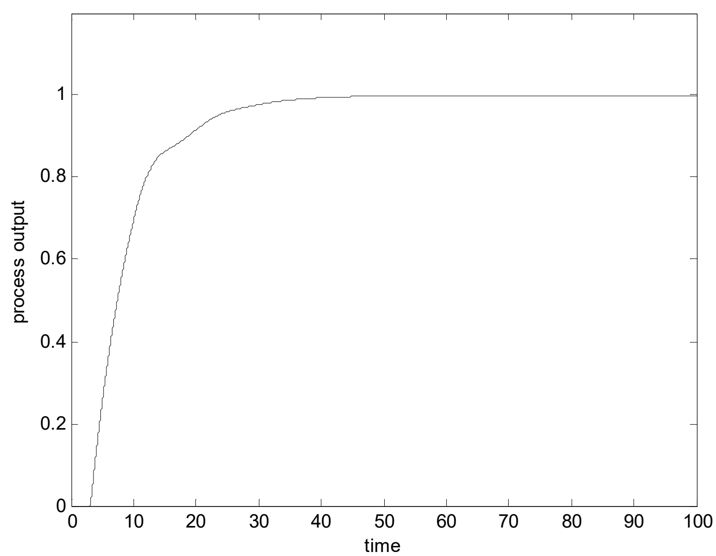

Figure P12.6 Step response for Problem 12.14 (ii)

(iii) The back-calculation strategy is implemented by considering the SIMULINK diagram of Figure P12.7 where T_t=T_i=8. The step response is shown in Figure P12.8.

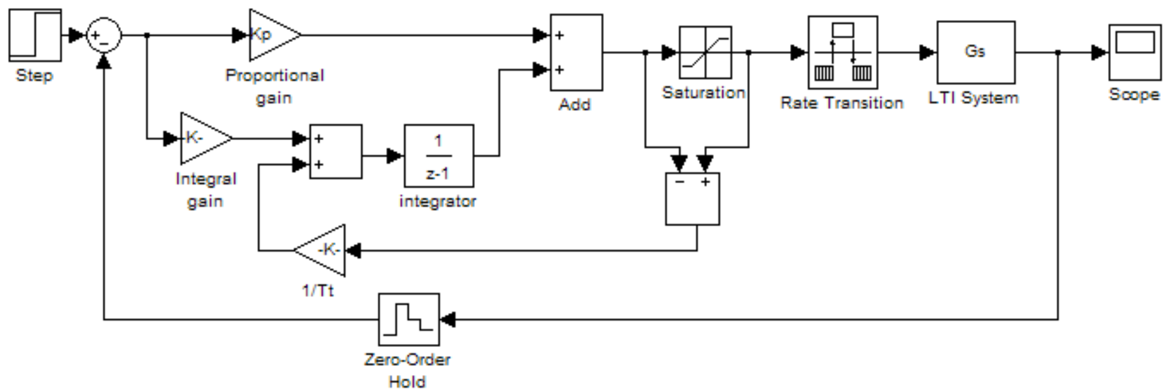

Figure P12.7 SIMULINK block diagram for Problem 12.14 (iii)

(iv) The PI controller in incremental form can be determined by using (12.8)-(12.9) with T_d=0. We obtain

$$K_2 = K_p\left(1+\frac{T}{T_i}\right) = 3.0375$$

$$K_1 = -K_p = -3$$

and therefore the controller transfer function is

$$C(z) = \frac{K_2 z + K_1}{z-1} = \frac{3.0375z - 3}{z-1}$$

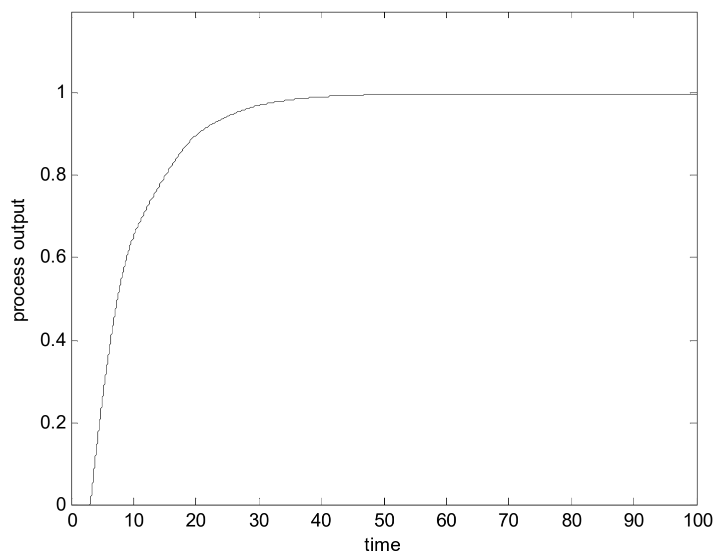

Figure P12.8 Step response for Problem 12.14 (iii)

The controller can be implemented by using the SIMULINK block diagram of Figure P12.9 where the employed MATLAB function is

```
function u=incrementalPI(input_parameters);

e=input_parameters(1);  % Current error
e1=input_parameters(2); % Delayed error
u1=input_parameters(3); % Delayed control

u=u1+3.0375*e-3*e1; % Control variable
if (u>1.2)          % The control variable saturates
    u=1.2;
elseif (u<-1.2)
    u=-1.2;
end
```

The resulting step response is shown in Figure P12.10.

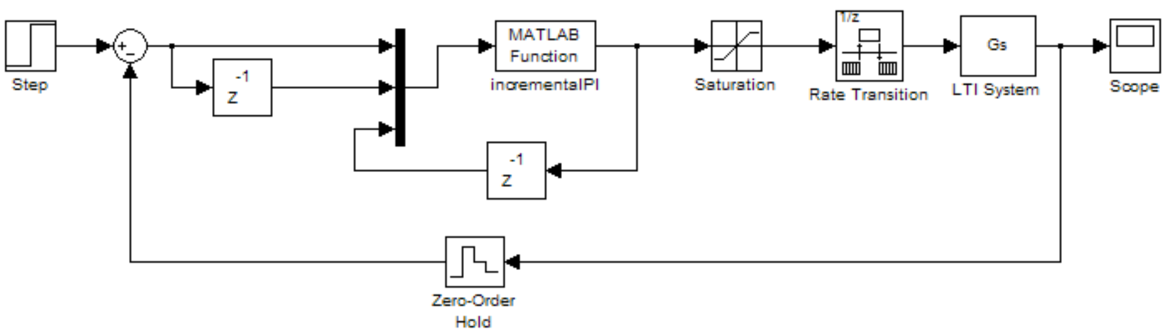

Figure P12.9 SIMULINK block diagram for Problem 12.14 (iv)

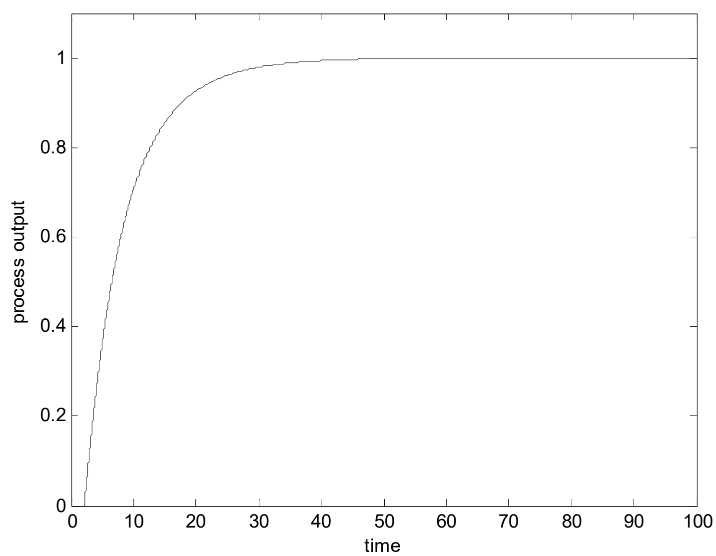

Figure P12.10 Step response for Problem 12.14 (iv)

12.15 Consider the analog process and the PI controller of Problem 12.14. Design a scheme that provides a bumpless transfer between manual and automatic mode and simulate it by applying a step set-point signal and by switching from the manual mode where the control variable is equal to one to the automatic mode at time t=60 s. Compare the results with those obtained by a scheme without bumpless transfer.

The scheme for the bumpless transfer between automatic and manual mode is that of Figure 12.10, where in this case we have (see Problem 12.14)

$$C(z)=\frac{K_2 z+K_1}{z-1}=\frac{3.0375z-3}{z-1}$$

We obtain the filter transfer function

$$D(z)=\frac{C(z)-K}{C(z)}=\frac{(K_2-K)z+K_1+K}{K_2 z+K_1}$$

To avoid an algebraic loop, we set $K=K_2$ and the filter transfer function reduces to

$$D(z)=\frac{K_1+K}{K_2 z+K_1}=\frac{-3+3.0375}{3.0375z-3}$$

The SIMULINK diagram that simulates the bumpless control scheme is shown in Figure P12.11, while the obtained step response is shown in Figure P12.12. The scheme for (bumpy) manual/automatic transfer is shown in Figure P12.13 and the corresponding step response is shown in Figure P12.14, where the bump in the process output is evident.

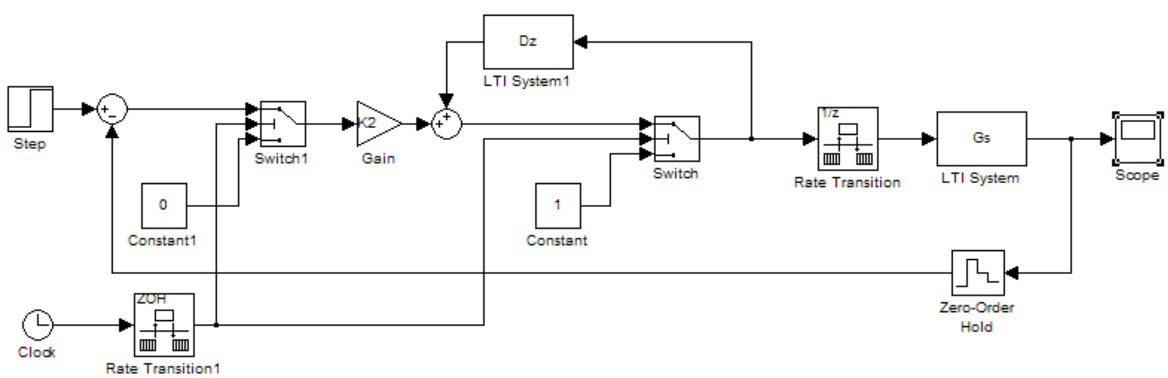

Figure P12.11 SIMULINK block diagram for Problem 12.15 (bumpless transfer)

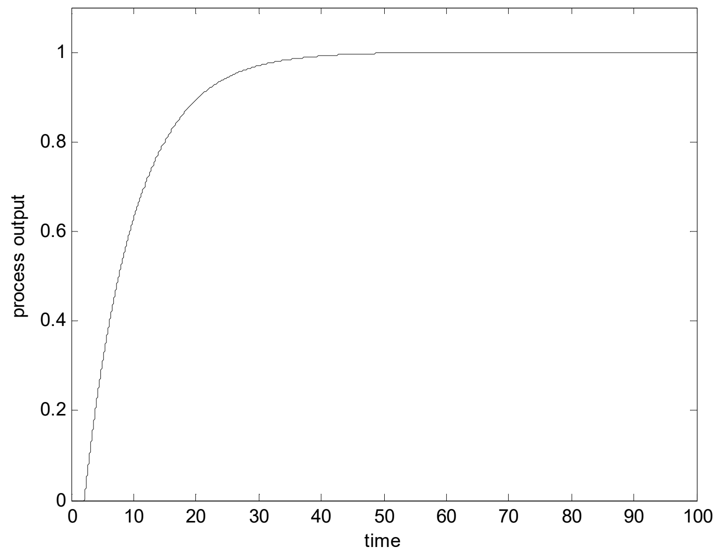

Figure P12.12 Step response for Problem 12.15 (bumpless transfer)

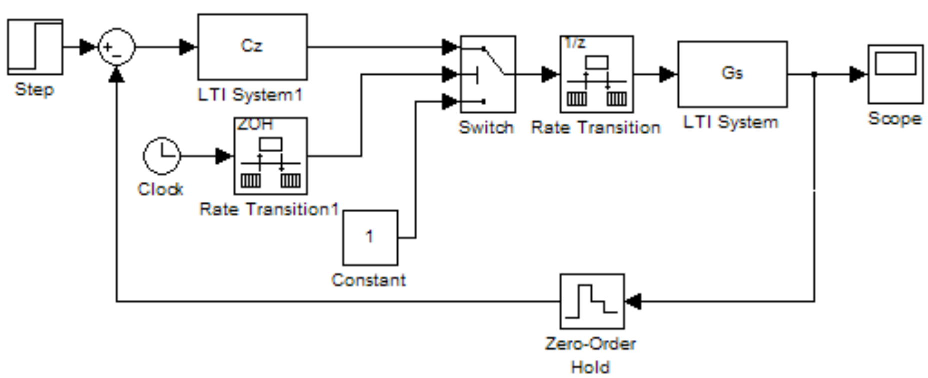

Figure P12.13 SIMULINK block diagram for Problem 12.15 (bumpy transfer)

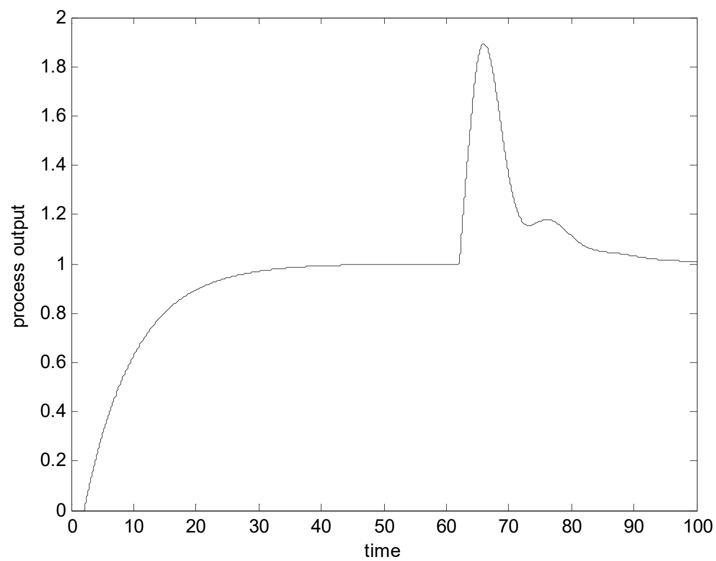

Figure P12.14 Step response for Problem 12.15 (bumpy transfer)

12.16 Design and simulate a dual-rate inferential control scheme with T=0.01 and λ=4 for the plant

$$G(s) = \frac{1}{(s+1)(s+5)}$$

and the analog PI controller (see Problem 5.7)

$$C(s) = 25\frac{s+1}{s}.$$

Apply then the controller to the process

$$\widetilde{G}(s) = \frac{1}{(s+1)(s+5)(0.1s+1)}$$

in order to verify the robustness of the control system.

First, the fast rate process model is determined as (T=0.01):

$$G_{ZAS}(z) = 4.9013 \cdot 10^{-5} \frac{z+0.9802}{(z-0.99)(z-0.9512)}$$

Then, the controller transfer function is obtained, by applying the bilinear transformation with T=0.01, as

$$C(z) = \frac{25.13z - 24.88}{z-1}$$

The control scheme of Figure 12.23 is implemented in SIMULINK as in Figure P12.15, where the lower part is necessary in order to cope with numerical inaccuracies. The resulting step response is shown in Figure P12.16. If $\widetilde{G}(s)$ is used instead of $G(s)$, the step response is shown in Figure P12.17. It can be seen that the overshoot is larger. Thus, in general, the robustness of the controller must be considered when a dual-rate inferential control scheme is employed.

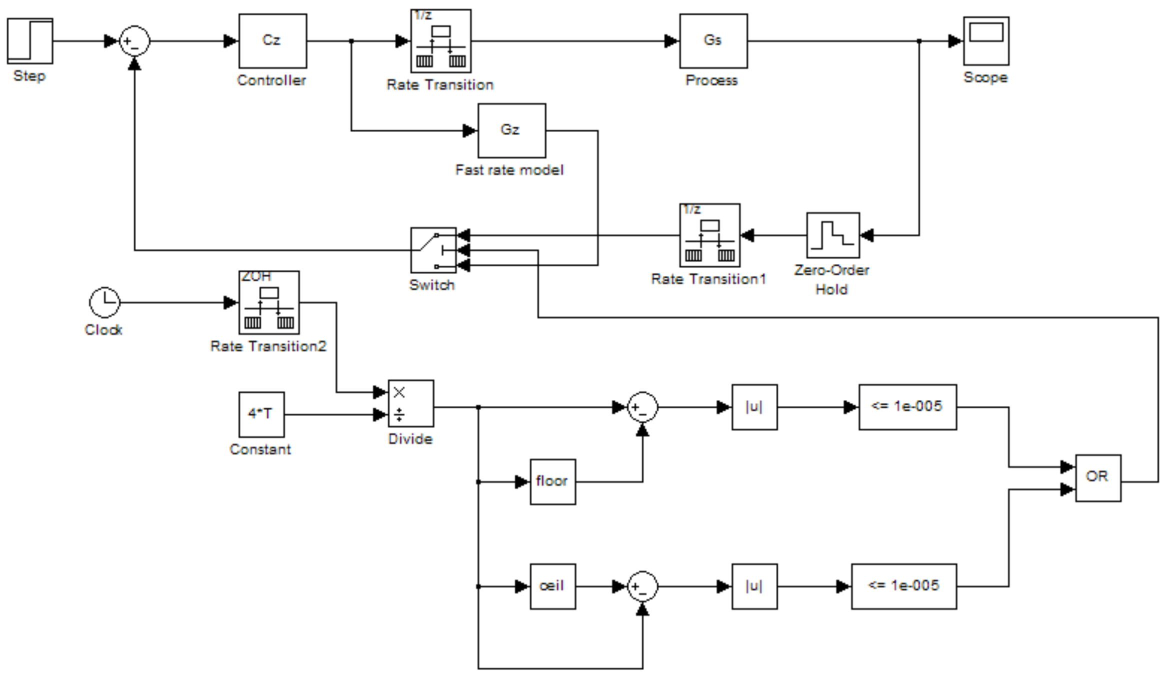

Figure P12.15 SIMULINK block diagram for Problem 12.16

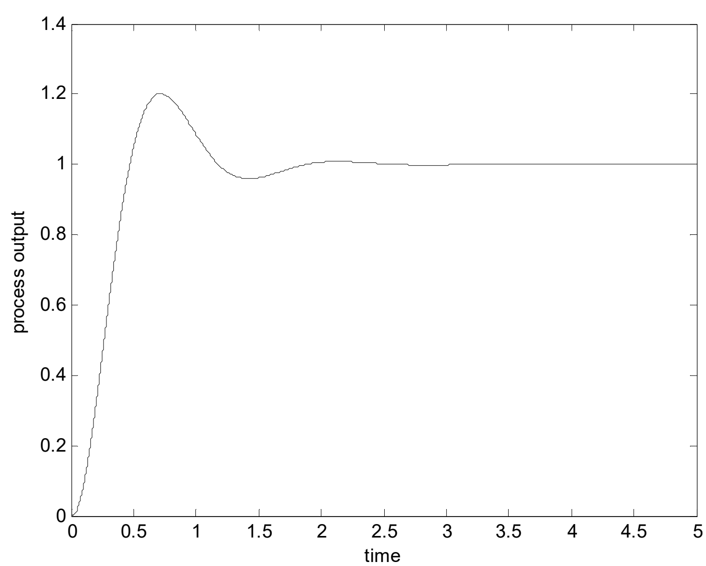

Figure P12.16 Step response for Problem 12.16 with $G(s)$

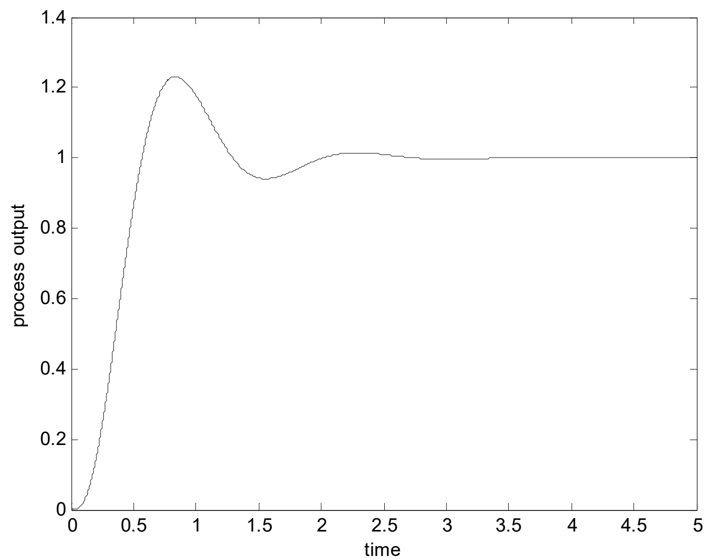

Figure P12.17 Step response for Problem 12.16 with $\widetilde{G}(s)$

12.17 Consider the analog process and the analog PI controller of Problem 12.15. Write a MATLAB script that simulates the step response with a digital controller when the sampling period switches at time t=0.52 from T=0.04 to T=0.01.\

By considering the technique described in Section 12.5, a MATLAB script that solves the problem is the following one.

```
Gs=tf(1,[1 6 5]);        % Process transfer function
Cs=tf(25*[1 1],[1 0]);  % Controller transfer function
final_time=3;            % Simulation duration
output=[];               % Initialization of the output vector
control=[];              % Initialization of the control
vector
error=[0 0 0 0];         % Initialization of the last four
                          % elements of the error vector
swt=0.52;                % Switching time
T1=0.04;                 % Initial sampling interval
Cz=c2d(Cs,T1,'tustin');  % Discretized controller transfer
function
for i=1:swt/T1,     % Step response before the switching time
    t=0:T1/1000:T1-T1/1000;
    if i==1
      e1=0;
      e=1;
      u1=0;
      u=(-
Cz.den{1}(2)*u1+Cz.num{1}(1)*e+Cz.num{1}(2)*e1)*...
         ones(1,length(t)); % Control variable calculation
```

```
        x0=[0 0]; % Zero initial conditions
      else
        u=(-
Cz.den{1}(2)*u1+Cz.num{1}(1)*e+Cz.num{1}(2)*e1)*...
          ones(1,length(t)); % Control variable calculation
      end
   % Simulation of the response over one sampling interval
     [y,t,x]=lsim(ss(Gs),u,t,x0);
      output=[output;y];    % Update the output vector
      control=[control u]; % Update the control vector
      x0=x(end,:);          % Initial conditions for the next
                            % sampling interval
      e1=e;
      e=1-y(end);
      u1=u(end);
      error=[error e];      % Update the error vector
      error=error(2:end);
end
% Time vector before the switching time
time1=linspace(0,swt-T1/1000,length(output));
T2=0.01;                    % Final sampling interval
Cz=c2d(Cs,T2,'tustin');     % Discretized controller transfer
                            % function
% Determinate the previous control error through
nterpolation
e1=interp1([swt-3*T1-T1/1000,swt-2*T1-T1/1000,swt-T1-...
   T1/1000,swt-T1/1000],error,swt-T2,'cubic');
% Step response after the switching time
for i=1:final_time/T2,
   t=0:T2/1000:T2-T2/1000;
   u=(-Cz.den{1}(2)*u1+Cz.num{1}(1)*e+Cz.num{1}(2)*e1)*...
     ones(1,length(t)); % Calculate the control variable
   % Simulation of the response over one sampling interval
   [y,t,x]=lsim(ss(Gs),u,t,x0);
   output=[output;y];       % Update the output vector
   control=[control u];     % Update the control vector
   x0=x(end,:);             % Initial conditions for the next
                            % sampling interval
   e1=e;
   e=1-y(end);
   u1=u(end);
end
time2=linspace(swt,final_time-T2/1000,length(output)-...
      length(time1)); % Time vector after the switching time
time=[time1 time2]; % Overall time vector
subplot(2,1,1);
plot(time,output,'k-');
```

```
xlabel('time');
ylabel('process output');
subplot(2,1,2);
plot(time,control,'k-');
xlabel('time');
ylabel('control variable');
```

The resulting step response is shown in Figure P12.18.

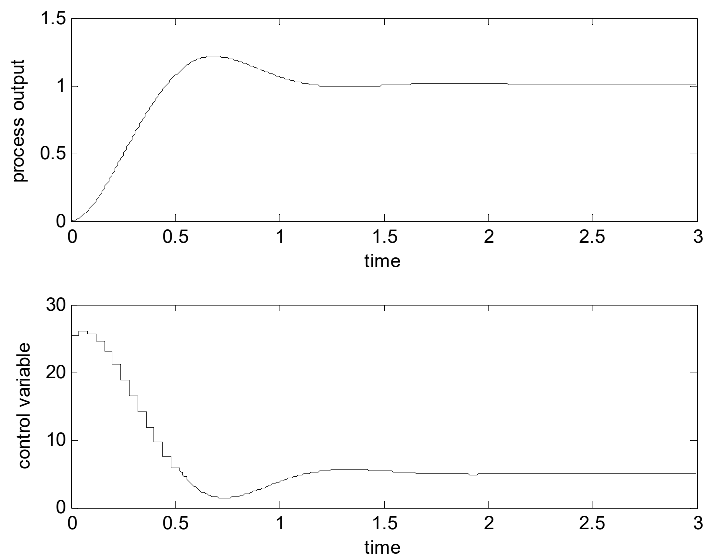

Figure P12.18 Step response for Problem 12.17

12.18 Consider the analog process and the analog PI controller of Problem 12.15. Write a MATLAB script that simulates the step response with a digital controller when the sampling period switches at time t=0.52 from T=0.01 to T=0.04.

By considering the technique described in Section 12.5, a MATLAB script that solves the problem is the following one.

```
Gs=tf(1,[1 6 5]);        % Process transfer function
Cs=tf(25*[1 1],[1 0]); % Controller transfer function
final_time=3;            % Simulation time
output=[];               % Initialization of the output vector
control=[];              % Initialization of the control
vector
po=[0 0];                % Initialization of the last two
                         % elements of the output vector
pu=[0 0];                % Initialization of the last two
                         % elements of the control vector
swt=0.52;                % Switching time
T1=0.01;                 % Initial sampling interval
Cz=c2d(Cs,T1,'tustin'); % Discretized controller transfer
```

```
                                   % function
for i=1:swt/T1,      % Step response before the switching
time
    t=0:T1/1000:T1-T1/1000;
    if i==1
      e1=0;
      e=1;
      u1=0;
      u=(-
Cz.den{1}(2)*u1+Cz.num{1}(1)*e+Cz.num{1}(2)*e1)*...
        ones(1,length(t)); % Calculate the control variable
        x0=[0 0]; % Zero initial conditions
    else
      u=(-
Cz.den{1}(2)*u1+Cz.num{1}(1)*e+Cz.num{1}(2)*e1)*...
        ones(1,length(t)); % Calculate the control variable
    end
    [y,t,x]=lsim(ss(Gs),u,t,x0); % Simulation of the
response
                                 % over one sampling
interval
    output=[output;y];    % Update the output vector
    control=[control u]; % Update the control vector
    x0=x(end,:);          % Initial conditions for the next
                          % sampling interval
    e1=e;
    e=1-y(end);
    u1=u(end);
    po=[po y(end)];       % Update the previous outputs
vector
    po=po(2:end);
    pu=[pu u(end)];       % Update the previous control
vector
    pu=pu(2:end);
end
% Time vector before the switching time
time1=linspace(0,swt-T1/1000,length(output));
T2=0.04;                  % Final sampling interval
Cz=c2d(Cs,T2,'tustin');   % Discretized controller transfer
                          % function
Gz=c2d(Gs,T2,'zoh'); % Discretized process transfer function
% Determination of the previous control error through
% optimization
u1_2=...
   fminsearch(@(un)(abs(output(end)-(-Gz.den{1}(2)*po(2)-...
   Gz.den{1}(3)*po(1)+Gz.num{1}(1)*un(1)+...
   Gz.num{1}(2)*un(2)))),[po(2) po(1)]);
```

```
u1=u1_2(1);
for i=1:final_time/T2,  % Step response after switching time
    t=0:T2/1000:T2-T2/1000;
    u=(-Cz.den{1}(2)*u1+Cz.num{1}(1)*e+Cz.num{1}(2)*e1)*...
      ones(1,length(t)); % Calculate the control variable
% simulation of the response in a sampling interval
    [y,t,x]=lsim(ss(Gs),u,t,x0);
    output=[output;y];             % Update the output vector
    control=[control u];           % Update the control vector
    x0=x(end,:);                   % Initial conditions for
the
                                   % next sampling interval
    e1=e;
    e=1-y(end);
    u1=u(end);
end
time2=linspace(swt,final_time-T2/1000,length(output)-...
      length(time1)); % Time vector after the switching time
time=[time1 time2]; % Time vector for the overall step
                    % response
subplot(2,1,1);
plot(time,output,'k-');
xlabel('time');
ylabel('process output');
subplot(2,1,2);
plot(time,control,'k-');
xlabel('time');
ylabel('control variable');
```

The resulting step response is shown in Figure P12.19.

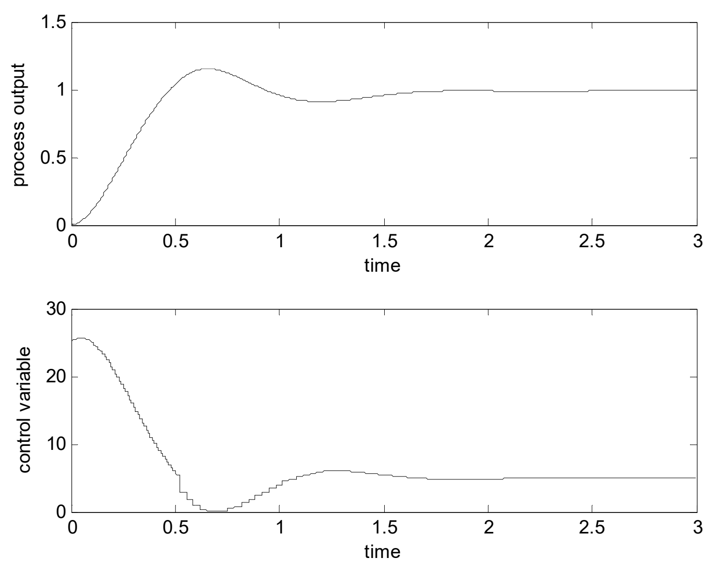

Figure P12.19 Step response for Problem 12.18