j. Detectable	A partition of the state space that illuminates the states that are controllable and unobservable, uncontrollable and unobservable, controllable and observable, and uncontrollable and observable.	______
k. Controllable system	An optimal controller designed to minimize a quadratic performance index.	______
l. Pole placement	A linear system is (completely) observable if and only if this matrix has full rank.	______
m. Estimation error	A dynamic system used to estimate the state of another dynamic system given knowledge of the system inputs and measurements of the system outputs.	______
n. Kalman state-space decomposition	A design methodology wherein the objective is to place the eigenvalues of the closed-loop system in desired regions of the complex plane.	______
o. Observable system	The principle that states that the full-state feedback law and the observer can be designed independently and when connected will function as an integrated control system in the desired manner (that is, stable).	______
p. Separation principle	A system in which the states that are not controllable are naturally stable.	______
q. Observability matrix	A controller that stabilizes the closed-loop system.	______

EXERCISES

E11.1 The ability to balance actively is a key ingredient in the mobility of a device that hops and runs on one springy leg, as shown in Figure E11.1 [8]. The control of the attitude of the device uses a gyroscope and a feedback such that $u(t) = \mathbf{K}\mathbf{x}(t)$, where

$$\mathbf{K} = \begin{bmatrix} -k & 0 \\ 0 & -2k \end{bmatrix},$$

and

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{B} = \mathbf{I}.$$

Determine a value for k so that the response of each hop is critically damped.

E11.2 A magnetically suspended steel ball can be described by the linear equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t).$$

The state variables are $x_1(t)$ = position and $x_2(t)$ = velocity, and both are measurable. Select a feedback so that the system is critically damped and the settling time (with a 2% criterion) is $T_s = 4$ s. Choose the feedback in the form

$$u(t) = -k_1x_1(t) - k_2x_2(t) + r(t)$$

where $r(t)$ is the reference input and the gains k_1 and k_2 are to be determined.

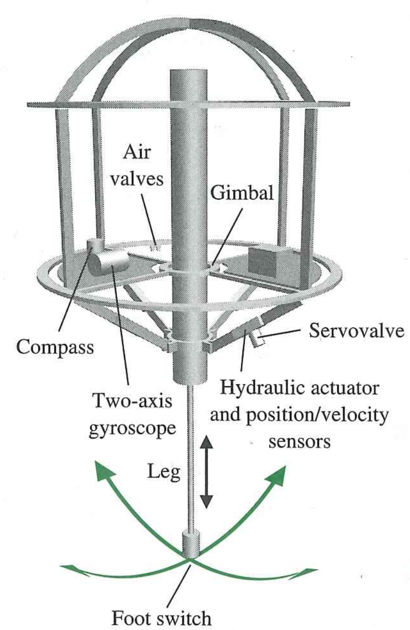

FIGURE E11.1 Single-leg control.

E11.3 A system is described by the matrix equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 2]\mathbf{x}(t).$$

Determine whether the system is controllable and observable.

Answer: controllable, not observable

E11.4 A system is described by the matrix equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -8 & 0 \\ 0 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t).$$

Determine whether the system is controllable and observable.

E11.5 A system is described by the matrix equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ -3 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t).$$

Determine whether the system is controllable and observable.

E11.6 A system is described by the matrix equations

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t).$$

Determine whether the system is controllable and observable.

Answer: controllable and observable

E11.7 Consider the system represented in state variable form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t),$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 10 \end{bmatrix},$$

$$\mathbf{C} = [2 \quad -4], \quad \text{and} \quad \mathbf{D} = [0].$$

Sketch a block diagram model of the system.

E11.8 Consider the third-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -3 & -1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} -1 \\ 2 \\ -6 \end{bmatrix}u(t)$$

$$y(t) = [2 \quad 8 \quad 10]\mathbf{x}(t) + [1]u(t).$$

Sketch a block diagram model of the system.

E11.9 Consider the second-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t) + [0]u(t).$$

For what values of k_1 and k_2 is the system completely controllable?

E11.10 Consider the block diagram model in Figure E11.10. Write the corresponding state variable model in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t).$$

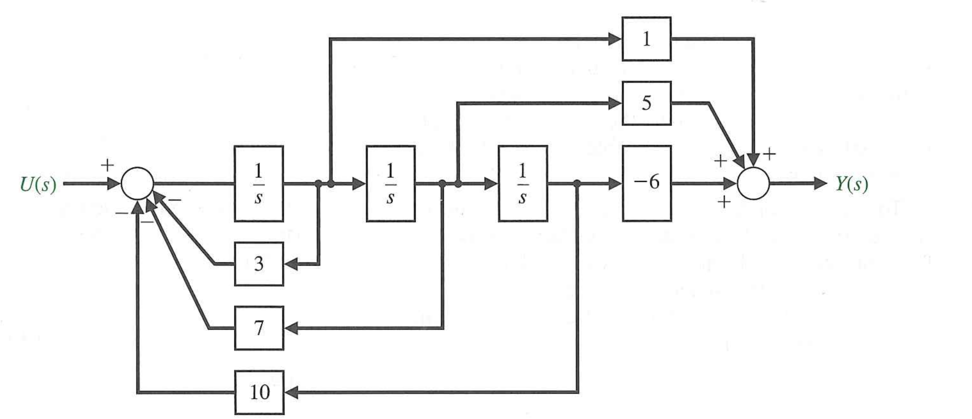

FIGURE E11.10 State variable block diagram.

E11.11 Consider the system shown in block diagram form in Figure E11.11. Obtain a state variable representation of the system. Determine if the system is controllable and observable.

E11.12 Consider the single-input, single-output system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \mathbf{C} = [1 \quad 0].$$

Compute the corresponding transfer function representation of the system. If the initial conditions are zero (i.e., $x_1(0) = 0$ and $x_2(0) = 0$), determine the response when $u(t)$ is a unit step input for $t \geq 0$.

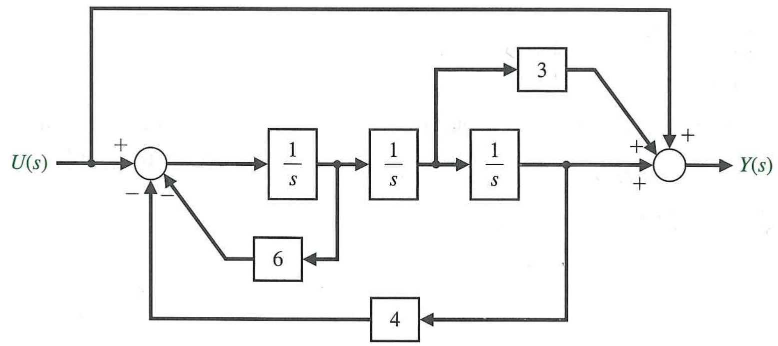

FIGURE E11.11 State variable block diagram with a feedforward term.

PROBLEMS

P11.1 A first-order system is represented by the time-domain differential equation

$$\dot{x}(t) = x(t) + u(t).$$

A feedback controller is to be designed such that

$$u(t) = -kx(t),$$

and the desired equilibrium condition is $x(t) = 0$ as $t \rightarrow \infty$. The performance integral is defined as

$$J = \int_0^\infty x^2(t)\, dt,$$

and the initial value of the state variable is $x(0) = \sqrt{2}$. Obtain the value of k in order to make J a minimum. Is this k physically realizable? Select a practical value for the gain k and evaluate the performance index with that gain. Is the system stable without the feedback due to $u(t)$?

P11.2 To account for the expenditure of energy and resources, the control signal is often included in the performance integral. Then the operation will not involve an unlimited control signal $u(t)$. One suitable performance index, which includes the effect of the magnitude of the control signal, is

$$J = \int_0^\infty (x^2(t) + \lambda u^2(t))\, dt.$$

(a) Repeat Problem P11.1 for the performance index.
(b) If $\lambda = 2$, obtain the value of k that minimizes the performance index. Calculate the resulting minimum value of J.

P11.3 An unstable robot system is described by the vector differential equation [9]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t),$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))^T$. Both state variables are measurable, and so the control signal is set as $u(t) = -k(x_1(t) + x_2(t))$. Design the gain k so that the performance index

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t)\, dt$$

is minimized. Evaluate the minimum value of the performance index. Determine the sensitivity of the performance to a change in k. Assume that the initial conditions are

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Is the system stable without the feedback signals due to $u(t)$?

P11.4 Consider the system

$$\dot{\mathbf{x}}(t) = [\mathbf{A} - \mathbf{B}\mathbf{K}]\mathbf{x}(t) = \mathbf{H}\mathbf{x}(t),$$

where $\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix}$. Determine the feedback gain k that minimizes the performance index

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t) dt$$

when $\mathbf{x}^T(0) = [1, \quad -1]$. Plot the performance index J versus the gain k.

P11.5 Consider the system described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where $\mathbf{x}(t) = (x_1(t), x_2(t))^T$, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let the feedback be $u(t) = -kx_1(t) - kx_2(t)$. Determine the feedback gain k that minimizes the performance index

$$J = \int_0^\infty (\mathbf{x}^T(t)\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{u}(t)) dt$$

when $\mathbf{x}^T(0) = [1, \quad 1]$. Plot the performance index J versus the gain k.

P11.6 For the solutions of Problems P11.3, P11.4, and P11.5, determine the roots of the closed-loop optimal control system. Note that the resulting closed-loop roots depend on the performance index selected.

P11.7 A system has the vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We want both state variables to be used in the feedback so that $u(t) = -k_1x_1(t) - k_2x_2(t)$. Also, we desire to have a natural frequency $\omega_n = 2$. Find a set of gains k_1 and k_2 in order to achieve an optimal system when J is given by

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t) + u^T(t)u(t)\, dt.$$

Assume $\mathbf{x}^T(0) = [1, \quad 0]$.

P11.8 For the system of P11.7 determine the optimum value for k_2 when $k_1 = 1$ and $\mathbf{x}^T(0) = [1, \quad 0]$.

P11.9 An interesting mechanical system with a challenging control problem is the ball and beam, shown in Figure P11.9(a) [10]. It consists of a rigid beam that is free to rotate in the plane of the paper around a center pivot, with a solid ball rolling along a groove in the top of the beam. The control problem is to position the ball at a desired point on the beam using a torque applied to the beam as a control input at the pivot.

A linear model of the system with a measured value of the angle $\phi(t)$ and its angular velocity $\dot{\phi}(t) = \omega(t)$ is available. Select a feedback scheme so that the response of the closed-loop system has a percent overshoot of $P.O. = 4\%$ and a settling time (with a 2% criterion) of $T_s = 1$ s for a step input.

P11.10 The dynamics of a rocket are represented by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$
$$y(t) = [0 \quad 1]\mathbf{x}(t)$$

and state variable feedback is used, where $u(t) = -10x_1(t) - 25x_2(t) + r(t)$. Determine the roots of the characteristic equation of this system and the response of the system when the initial conditions are $x_1(0) = 1$ and $x_2(0) = -1$. Assume the reference input $r(t) = 0$.

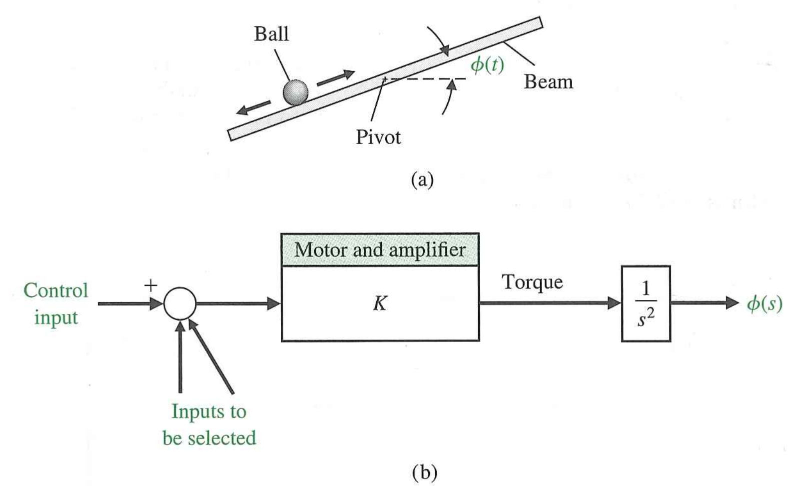

FIGURE P11.9 (a) Ball and beam. (b) Model of the ball and beam.

P11.11 The state variable model of a plant to be controlled is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & -2 \\ 2 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 1]\mathbf{x}(t) + [0]u(t).$$

Use state variable feedback and incorporate a command input $u(t) = -\mathbf{K}\mathbf{x}(t) + \alpha r(t)$. Select the gains $\mathbf{K}$ and α so that the system has a rapid response with a percent overshoot of $P.O. = 1\%$, a settling time (with a 2% criterion) of $T_s \le 1$ s, and a zero steady-state error to a unit step input.

P11.12 A DC motor has the state variable model

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -3 & -2 & -0.8 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 0 \quad 0 \quad 0 \quad 3.2]\mathbf{x}(t).$$

Determine whether this system is controllable and observable.

P11.13 A feedback system has a plant transfer function

$$\frac{Y(s)}{R(s)} = G(s) = \frac{K}{s(s+70)}.$$

We want the velocity error constant K_v to be 35 and the percent overshoot to a step to be $P.O. = 4\%$ so that $\zeta = 1/\sqrt{2}$. The settling time (with a 2% criterion) desired is $T_s = 0.11$ s. Design an appropriate state variable feedback system for $r(t) = -k_1x_1(t) - k_2x_2(t)$.

P11.14 A process has the transfer function

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -10 & 0 \\ 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 1]\mathbf{x}(t) + [0]u(t).$$

Determine the state variable feedback gains to achieve a settling time (with a 2% criterion) of $T_s = 1$ second and a percent overshoot of $P.O. = 10\%$. Also sketch the block diagram of the resulting system. Assume the complete state vector is available for feedback.

P11.15 A telerobot system has the matrix equations [16]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}u(t)$$

and

$$y(t) = [2 \quad 1 \quad 0]\mathbf{x}(t).$$

(a) Determine the transfer function, $G(s) = Y(s)/U(s)$. (b) Draw the block diagram indicating the state variables. (c) Determine whether the system is controllable. (d) Determine whether the system is observable.

P11.16 Hydraulic power actuators were used to drive the dinosaurs of the movie *Jurassic Park* [20]. The motions of the large monsters required high-power actuators requiring 1200 watts.

One specific limb motion has dynamics represented by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -4 & 0 \\ 1 & -1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [0 \quad 1]\mathbf{x}(t) + [0]u(t).$$

We want to place the closed-loop poles at $s = -1 \pm 3j$. Determine the required state variable feedback using Ackermann's formula. Assume that the complete state vector is available for feedback.

P11.17 A system has a transfer function

$$\frac{Y(s)}{R(s)} = \frac{s+a}{s^4 + 13s^3 + 54s^2 + 82s + 60}.$$

Determine a real value of a so that the system is either uncontrollable or unobservable.

P11.18 A system has a plant

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{(s+1)^2}.$$

(a) Find the matrix differential equation to represent this system. Identify the state variables on a block diagram model. (b) Select a state variable feedback structure using $u(t)$, and select the feedback gains so that the response $y(t)$ of the unforced system is critically damped when the initial condition is $x_1(0) = 1$ and $x_2(0) = 0$, where $x_1 = y(t)$. The repeated roots are at $s = -\sqrt{2}$.

P11.19 The block diagram of a system is shown in Figure P11.19. Determine whether the system is controllable and observable.

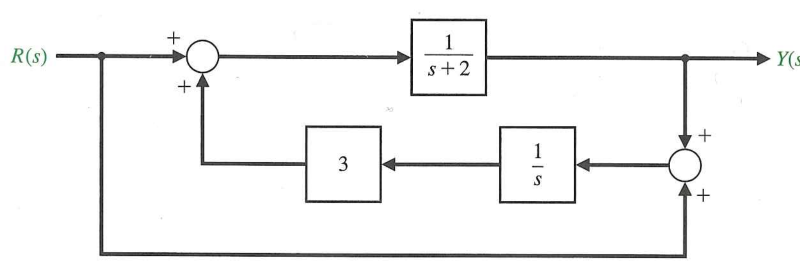

FIGURE P11.19 Multiloop feedback control system.

P11.20 Consider the automatic ship-steering system. The state variable form of the system differential equation is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.05 & -6 & 0 & 0 \\ -10^{-3} & -0.15 & 0 & 0 \\ 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} -0.2 \\ 0.03 \\ 0 \\ 0 \end{bmatrix}\delta(t),$$

where $\mathbf{x}^T(t) = [v(t) \quad \omega_s(t) \quad y(t) \quad \theta(t)]$. The state variables are $x_1(t) = v(t) =$ the transverse velocity; $x_2(t) = \omega_s(t) =$ angular rate of ship's coordinate frame relative to response frame; $x_3(t) = y(t) =$ deviation distance on an axis perpendicular to the track; $x_4(t) = \theta(t) =$ deviation angle. (a) Determine whether the system is stable. (b) Feedback can be added so that

$$\delta(t) = -k_1x_1(t) - k_3x_3(t).$$

Determine whether this system is stable for suitable values of k_1 and k_3.

P11.21 An *RL* circuit is shown in Figure P11.21. (a) Select the two stable variables and obtain the vector differential equation where the output is $v_0(t)$. (b) Determine whether the state variables are observable when $R_1/L_1 = R_2/L_2$. (c) Find the conditions when the system has two equal roots.

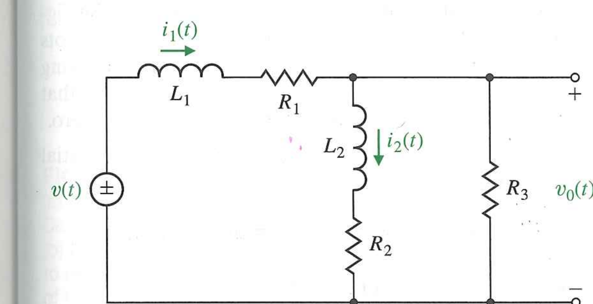

FIGURE P11.21 *RL* circuit.

P11.22 A manipulator control system has a loop transfer function of

$$G(s) = \frac{1}{s(s+0.4)}$$

and negative unity feedback [15]. Represent this system by a state variable signal-flow graph or block diagram and a vector differential equation. (a) Plot the response of the closed-loop system to a step input. (b) Use state variable feedback so that the percent overshoot is $P.O. = 5\%$ and the settling time (with a 2% criterion) is $T_s = 1.35$ s. (c) Plot the response of the state variable feedback system to a step input.

P11.23 Consider again the system of Example 11.7 when we desire that the steady-state error for a step input be zero and the desired roots of the characteristic equation be $s = -2 \pm j1$ and $s = -10$.

P11.24 Consider again the system of Example 11.7 when we desire that the steady-state error for a ramp input be zero and the roots of the characteristic equation be $s = -2 \pm j2$ and $s = -20$.

P11.25 Consider the system represented in state variable form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}u(t),$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ -5 & 10 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\mathbf{C} = [1 \quad -4], \quad \text{and} \quad \mathbf{D} = [0].$$

Verify that the system is observable. Then design a full-state observer by placing the observer poles at $s_{1,2} = -1$. Plot the response of the estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ with an initial estimation error of $\mathbf{e}(0) = [1, \quad 1]^T$.

P11.26 Consider the third-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -5 & -3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}u(t)$$

$$y(t) = [2 \quad -5 \quad 0]\mathbf{x}(t) + [0]u(t).$$

Verify that the system is observable. If so, determine the observer gain matrix required to place the observer poles at $s_{1,2} = -1 \pm j$ and $s_3 = -5$.

P11.27 Consider the second-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 \\ -3 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 10 \\ 0 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t) + [0]u(t).$$

Determine the observer gain matrix required to place the observer poles at $s_{1,2} = -1 \pm j$.

P11.28 Consider the single-input, single-output system is described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{x}(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ K \end{bmatrix}, \mathbf{C} = [1 \quad 0].$$