Linear Algebra
Seminar 3: Determinants

Problem 1 (2 pts)**.** (a) Evaluate all minors and cofactors of the following matrices:

$$(1)\ \begin{pmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{pmatrix} \qquad (2)\ \begin{pmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{pmatrix}$$

(b) Evaluate the determinants of matrices in part (a) by a cofactor expansion along a row or column of your choice, and then using the arrow technique.

Problem 2 (2 pts)**.** (a) Calculate the inverses of the matrices in Problem 1(a)

(b) Use Cramer's rule to solve the equation $A\mathbf{x} = \mathbf{b} := (1 \quad -2 \quad 1)^\top$ for matrices A from Problem 1, and then by calculating $A^{-1}\mathbf{b}$ with A^{-1} found in part (a).

Problem 3 (3pt)**.** Use determinants to answer the following questions.

(a) For what numbers c is this matrix not invertible, and why not?

$$A = \begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix}$$

(b) Find all numbers c such that the system of vectors is linearly independent:

$$\text{(i)}\quad \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ c \\ 2 \end{pmatrix} \qquad \text{(ii)}\quad \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}, \begin{pmatrix} c \\ 3 \\ -3 \end{pmatrix}$$

Problem 4 (3 pts)**.** (a) Using the determinants, find the equation of the plane π through the points $A(1,0,1)$, $B(3,1,2)$, $C(2,1,3)$.

(b) Find the normal vector to the above plane π and then its equation.

(c) Find the distance from the point $D(0,2,3)$ to the plane π using the geometric meaning of the determinant as the volume or area.

Problem 5 (2 pts)**.** Assume that 3×3 matrices A, B and C are as follows

$$A = \begin{pmatrix} \text{row } 1 \\ \text{row } 2 \\ \text{row } 3 \end{pmatrix}, \qquad B = \begin{pmatrix} \text{row } 1 + 2\cdot \text{ row } 2 \\ \text{row } 2 + 2\cdot \text{ row } 3 \\ \text{row } 3 + 2\cdot \text{ row } 1 \end{pmatrix}, \qquad C = \begin{pmatrix} k\cdot \text{ row } 1 - \text{row } 2 \\ k\cdot \text{ row } 2 - \text{row } 3 \\ k\cdot \text{ row } 3 - \text{row } 1 \end{pmatrix}$$

Given that $\det(A) = -3$,

(a) find $\det(B)$;

(b) find the value of k for which the matrix C is singular

Hint: Be smart! Do you see how B and C can be obtained from A in one step?

Problem 6 (2 pts)**.** Prove in two ways that the matrix A below is always singular ($*$ stand for any numbers) by answering to (a) and (b):

$$\begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

(a) Explain why the last three columns are linearly dependent.

(b) Explain why each of $5! = 120$ terms in the formula for $\det(A)$ is zero.

Problem 7 (3 pts)**.** (a) Prove that a square matrix A is singular if and only if $A\,\mathrm{adj}(A) = \mathbf{0}$.

(b) `(Cauchy)` Given an $n \times n$ matrix A, calculate $\det(\mathrm{adj}(A))$.

(c) For an $n \times n$ matrix A, what rank can its adjugate matrix $\mathrm{adj}(A)$ have?

Problem 8 (3 pts)**.** Find the value of the Vandermonde determinant,

$$V(x_0, x_1, \dots, x_n) := \begin{vmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ \dots & \dots & \dots & \dots \\ 1 & x_n & \dots & x_n^n \end{vmatrix}.$$

To this end,

(a) consider the function $f(x) := V(x, x_1, \dots, x_n)$ and argue that it is a polynomial (of what degree?); then find all its roots $a_1, \dots, a_m$ and conclude that $f(x) = C(a_1 - x) \cdots (a_m - x)$;

(b) the constant C is the coefficient of $(-1)^m x^m$ in $f(x)$; determine it from the row cofactor expansion of $V(x, x_1, \dots, x_n)$ and then obtain $V(x_0, x_1, \dots, x_n)$ by induction.

Extra problems
(to be discussed if time permits)

Problem 9. (a) Find the normal equation for the plane through the points $A(2;1;1)$, $B(1;2;1)$, and $C(1;1;2)$ by writing the corresponding determinant and then expanding it in the first row, and then by finding the normal to the plane as a cross product of the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

(b) A box (parallelepiped) has edges from the origin to the points A, B, and C. Find its volume and also find the area of each parallelogram face.

Problem 10. Assume that $a_1, a_2, \dots, a_n$ are real numbers. Calculate the determinant of the matrix

$$\begin{pmatrix} 1+a_1 & a_2 & \dots & a_n \\ a_1 & 1+a_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & 1+a_n \end{pmatrix}.$$

Hint: induction can be helpful here; try $n = 1$ and $n = 2$ to guess the formula

Problem 11. Assume that $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ vectors. Find $\det(I_n + \mathbf{u}\mathbf{v}^T)$.
Hint: do you see why this is a generalization of the above problem?

1. a) 1) $\begin{pmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{pmatrix} = -3\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix} - 0\cdot\begin{pmatrix} 2 & 1 \\ -1 & 5 \end{pmatrix} + 7\begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$

(1,1)-minor -cofactor; (1,2)-minor (-1)x -cofactor; (1,3)-minor -cofactor

$M_{1,1} = 25$, $C_{1,1} = 25$
$M_{1,2} = 10+1 = 11$, $C_{1,2} = -11$
$M_{1,3} = 5$, $C_{1,3} = 5$

$M_{2,1} = 0$, $C_{2,1} = 0$
$M_{2,2} = -15+7 = -8$, $C_{2,2} = -8$
$M_{2,3} = 0$, $C_{2,3} = 0$

$= 2\begin{pmatrix} 0 & 7 \\ 0 & 5 \end{pmatrix} - 5\begin{pmatrix} -3 & 7 \\ -1 & 5 \end{pmatrix} + 1\begin{pmatrix} -3 & 0 \\ -1 & 0 \end{pmatrix}$

(2,1)-minor (-1)x -cofactor; (2,2)-minor -cofactor; (2,3)-minor (-1)x -cofactor

$M_{3,1} = -35$, $C_{3,1} = -35$
$M_{3,2} = -3-14 = -17$, $C_{3,2} = 17$
$M_{3,3} = -15$, $C_{3,3} = -15$

$= -1\begin{pmatrix} 0 & 7 \\ 5 & 1 \end{pmatrix} - 0\begin{pmatrix} -3 & 7 \\ 2 & 1 \end{pmatrix} + 5\begin{pmatrix} -3 & 0 \\ 2 & 5 \end{pmatrix}$

(3,1)-minor -cofactor; (3,2)-minor (-1)x -cofactor; (3,3)-minor -cofactor

$$M_{i,j} = \begin{pmatrix} 25 & 11 & 5 \\ 0 & -8 & 0 \\ -35 & -17 & -15 \end{pmatrix} \qquad C_{i,j} = \begin{pmatrix} 25 & -11 & 5 \\ 0 & -8 & 0 \\ -35 & 17 & -15 \end{pmatrix}$$

2) $\begin{pmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{pmatrix} = 3\begin{pmatrix} 0 & -4 \\ -3 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix} + 1\begin{pmatrix} 1 & 0 \\ 3 & -3 \end{pmatrix}$:

$M_{1,1} = -12$, $C_{1,1} = -12$; $M_{1,2} = 5+4 = 9$, $C_{1,2} = -9$; $M_{1,3} = -3$, $C_{1,3} = -3$

$= 1\begin{pmatrix} 3 & 1 \\ -3 & 5 \end{pmatrix} - 0\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix} - 4\begin{pmatrix} 3 & 3 \\ 1 & -3 \end{pmatrix}$:

$M_{2,1} = 15+3 = 18$, $C_{2,1} = -18$; $M_{2,2} = 15-1 = 14$, $C_{2,2} = 14$; $M_{2,3} = -9-3 = -12$, $C_{2,3} = 12$

$= 3\begin{pmatrix} 3 & 1 \\ 0 & -4 \end{pmatrix} + 3\begin{pmatrix} 3 & 1 \\ 1 & -4 \end{pmatrix} - 5\begin{pmatrix} 3 & 3 \\ 1 & 0 \end{pmatrix}$

$M_{3,1} = -12$, $C_{3,1} = -12$; $M_{3,2} = -12-1 = -13$, $C_{3,2} = 13$; $M_{3,3} = -3$, $C_{3,3} = -3$

$$M_{i,j} = \begin{pmatrix} -12 & 9 & -3 \\ 18 & 14 & -12 \\ -12 & -13 & -3 \end{pmatrix} \qquad C_{ij} = \begin{pmatrix} -12 & -9 & -3 \\ -18 & 14 & 12 \\ -12 & 13 & -3 \end{pmatrix}$$

b) 1) $\det(A) = -3\cdot 25 + 0\cdot(-11) + 7\cdot 5 = -75+35 = -40$ row expansion

$\det(A) = -3\cdot 25 + 2\cdot 0 + (-1)\cdot(-35) = -75+35 = -40$ column expansion

$\begin{matrix} -3 & 0 & 7 & -3 & 0 \\ 2 & 5 & 1 & 2 & 5 \\ -1 & 0 & 5 & -1 & 0 \end{matrix}$; $\det(A) = -75+35 = -40$

2) $\det(A) = 3\cdot(-12) + 3(-9) + 1(-3) = -36-27-3 = -66$ row expansion

$\det(A) = 3(-12) + 1(-18) + 1(-12) = -36-18-12 = -66$ column expansion

$\begin{matrix} 3 & 3 & 1 & 3 & 3 \\ 1 & 0 & -4 & 1 & 0 \\ 1 & -3 & 5 & 1 & -3 \end{matrix}$ $\det(A) = -12-3-36-15 = -66$

2. $A^{-1} = \dfrac{C^T}{\det(A)}$

a) 1) $C^T = \begin{pmatrix} 25 & 0 & -35 \\ -11 & -8 & 17 \\ 5 & 0 & -15 \end{pmatrix}$

$$A^{-1} = \frac{C^T}{\det(A)} = \frac{C^T}{-40} = \begin{pmatrix} -\frac{25}{40} & 0 & \frac{35}{40} \\ \frac{11}{40} & \frac{8}{40} & -\frac{17}{40} \\ -\frac{5}{40} & 0 & \frac{15}{40} \end{pmatrix} = \begin{pmatrix} -\frac{5}{8} & 0 & \frac{7}{8} \\ \frac{11}{40} & \frac{1}{5} & -\frac{17}{40} \\ -\frac{1}{8} & 0 & \frac{3}{8} \end{pmatrix}$$

2) $C^T = \begin{pmatrix} -12 & -18 & -12 \\ -9 & 14 & 13 \\ -3 & 12 & -3 \end{pmatrix}$

$$A^{-1} = \frac{C^T}{\det(A)} = \frac{C^T}{-66} = \begin{pmatrix} \frac{12}{66} & \frac{18}{66} & \frac{12}{66} \\ \frac{9}{66} & -\frac{14}{66} & -\frac{13}{66} \\ \frac{3}{66} & -\frac{12}{66} & \frac{3}{66} \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{2}{11} \\ \frac{3}{22} & -\frac{7}{33} & -\frac{13}{66} \\ \frac{1}{22} & -\frac{2}{11} & \frac{1}{22} \end{pmatrix}$$

b) $Ax = b := (1\ -2\ 1)^T$

1) $\left(\begin{array}{ccc|c} -3 & 0 & 7 & 1 \\ 2 & 5 & 1 & -2 \\ -1 & 0 & 5 & 1 \end{array}\right)$

$$x = \frac{\begin{bmatrix} 1 & 0 & 7 \\ 2 & 5 & 1 \\ 1 & 0 & 5 \end{bmatrix}}{\begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}} = \frac{1\cdot\begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} - 0\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} + 7\begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix}}{\det(A)} = \frac{25-35}{-40} = \frac{1}{4}$$

$$y = \frac{\begin{bmatrix} -3 & 1 & 7 \\ 2 & -2 & 1 \\ -1 & 1 & 5 \end{bmatrix}}{\det(A)} = \frac{-3(-10-1) - 1(10+1) + 7(2-2)}{-40} = \frac{-3(-11)-1(11)}{-40} = \frac{2\cdot 11}{-40} = -\frac{11}{20}$$

$$z = \frac{\begin{bmatrix} -3 & 0 & 1 \\ 2 & 5 & -2 \\ 1 & 0 & 1 \end{bmatrix}}{\det(A)} = \frac{-3(-5)+1(-5)}{-40} = \frac{-2(-5)}{40} = \frac{1}{4}$$

$$A^{-1}b = \begin{pmatrix} -\frac{5}{8} & 0 & \frac{7}{8} \\ \frac{11}{40} & \frac{1}{5} & -\frac{17}{40} \\ -\frac{1}{8} & 0 & \frac{3}{8} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -\frac{5}{8} \\ \frac{11}{40} \\ -\frac{1}{8} \end{pmatrix} - 2\begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \end{pmatrix} + 1\begin{pmatrix} \frac{7}{8} \\ -\frac{17}{40} \\ \frac{3}{8} \end{pmatrix} = \begin{pmatrix} -\frac{5}{8}+\frac{7}{8} \\ \frac{11}{40}-\frac{2}{5}-\frac{17}{40} \\ -\frac{1}{8}+\frac{3}{8} \end{pmatrix} = \begin{pmatrix} \frac{2}{8} \\ -\frac{22}{40} \\ \frac{2}{8} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -\frac{11}{20} \\ \frac{1}{4} \end{pmatrix}$$

2) $\left(\begin{array}{ccc|c} 3 & 3 & 1 & 1 \\ 1 & 0 & -4 & -2 \\ 1 & -3 & 5 & 1 \end{array}\right)$

$$x = \frac{\begin{bmatrix} 1 & 3 & 1 \\ -2 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}}{\det(A)} = \frac{1\cdot(-12) - 3(-10+4) + 1\cdot 6}{-66} = \frac{12}{-66} = -\frac{2}{33}$$

$$y = \frac{\begin{bmatrix} 3 & 1 & 1 \\ 1 & -2 & -4 \\ 1 & 1 & 5 \end{bmatrix}}{\det(A)} = \frac{3\cdot(-10+4) - 1(5+4) + 1(1+2)}{-66} = \frac{-24}{-66} = \frac{4}{11}$$

$$z = \frac{\begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -2 \\ 1 & -3 & 1 \end{bmatrix}}{\det(A)} = \frac{3(-6) - 3(1+2) + 1(-3)}{-66} = \frac{-30}{-66} = \frac{5}{11}$$

$$A^{-1}b = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{2}{11} \\ \frac{3}{22} & -\frac{7}{33} & -\frac{13}{66} \\ \frac{1}{22} & -\frac{2}{11} & \frac{1}{22} \end{pmatrix}\cdot\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1\begin{pmatrix} \frac{2}{11} \\ \frac{3}{22} \\ \frac{1}{22} \end{pmatrix} - 2\begin{pmatrix} \frac{3}{11} \\ -\frac{7}{33} \\ -\frac{2}{11} \end{pmatrix} + 1\begin{pmatrix} \frac{2}{11} \\ -\frac{13}{66} \\ \frac{1}{22} \end{pmatrix} = \begin{pmatrix} \frac{2}{11}-\frac{6}{11}+\frac{2}{11} \\ \frac{3}{22}+\frac{14}{33}-\frac{13}{66} \\ \frac{1}{22}+\frac{4}{11}+\frac{1}{22} \end{pmatrix} = \begin{pmatrix} -\frac{2}{11} \\ \frac{24}{66} \\ \frac{10}{22} \end{pmatrix} = \begin{pmatrix} -\frac{2}{11} \\ \frac{4}{11} \\ \frac{5}{11} \end{pmatrix}$$

3. a) $A = \begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix}$

$\det(A) = 2(c^2-7c) - c(c^2-8c) + c(7c-8c) = 2c^2 - 14c - c^3 + 8c^2 - c^2 = -c^3 + 9c^2 - 14c = 0$

$-c(c^2-9c+14) = 0$

$c = 0 \quad c_1 = 2 \quad c_2 = 7$

Matrix is not invertible for $\begin{cases} c = 0 \\ c = 2 \\ c = 7 \end{cases}$

b) 1) $A = \begin{pmatrix} 3 & 6 & 6 \\ -6 & 4 & c \\ 1 & -3 & 2 \end{pmatrix}$

$\det(A) = 3(8+3c) - 6(-12-c) + 6(18-4) = 24 + 9c + 72 + 6c + 84 = 15c + 180$

if $\det(A) \neq 0$ => columns are linearly independent

$15c + 180 = 0$
$15c = -180$
$c = -\frac{180}{15} = -12$ => vectors are linearly independent for $c \in \mathbb{R} \setminus \{-12\}$

2) $A = \begin{pmatrix} 1 & 2 & c \\ 5 & 4 & 3 \\ -3 & -6 & -3 \end{pmatrix}$

$\det(A) = 1(-12+18) - 2(-15+9) + c(-30+12) = 6 + 12 - 18c = 18 - 18c = 0$
$c = 1$

vectors are linearly independent for $c \in \mathbb{R} \setminus \{1\}$

4. a) $A(1,0,1)$, $B(3,1,2)$, $C(2,1,3)$

$\det A(x,y,z) = 0 \iff ax + by + cz = d$ for some a, b, c, d

$$A(x,y,z) := \begin{pmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{pmatrix} = \begin{pmatrix} x-1 & y-0 & z-1 \\ 3-1 & 1-0 & 2-1 \\ 2-1 & 1-0 & 3-1 \end{pmatrix} = \begin{pmatrix} x-1 & y & z-1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$\det(A) = (x-1)(2-1) - y(4-1) + (z-1)(2-1) = x - 1 - 3y + z - 1 = 0 \Rightarrow x - 3y + z = 2$

b) $\vec{n} = (a,b,c)$ is the cross-product $u \times v$, where $u = \overrightarrow{P_1P_2} = \overrightarrow{AB} = (2,1,1)$
$v = \overrightarrow{P_1P_3} = \overrightarrow{AC} = (1,1,2)$

$\vec{u} \times \vec{v} = (C_{11}, C_{12}, C_{13})$

$C_{11} = 2-1 = 1$
$C_{12} = -(4-1) = -3$ => $x - 3y + z = d$ ← plug point A
$C_{13} = 2-1 = 1$ $\quad 1+1 = d \Rightarrow d = 2 \Rightarrow x - 3y + z = 2$

c) $D(0,2,3)$

$V = Sh$, h - distance from D to plane π,
S - area of the figure in the plane π (parallelogram), V - volume of the parallelepiped

$\vec{u} \times \vec{v}$ = area of the parallelogram formed by vectors $\vec{u}$ and $\vec{v}$

(magnitude of the vector achieved by cross-product is the area of the parallelogram)

Determinant in 3D is the volume of the parallelepiped

$h = \frac{V}{S}$.

$S = \sqrt{C_{11}^2 + C_{12}^2 + C_{13}^2} = \sqrt{1+9+1} = \sqrt{11}$

$V = \det\begin{pmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{pmatrix}$, $\vec{w} = (0-x_1,\ 2-y_1,\ 3-z_1) = (-1,2,2)$

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix} = 2(2-4) - 1(2+2) + 1(2+1) = -4 - 4 + 3 = -5$

$h = \frac{|-5|}{\sqrt{11}} = \frac{5}{\sqrt{11}}$

5. $A = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad B = \begin{pmatrix} r_1 + 2r_2 \\ r_2 + 2r_3 \\ r_3 + 2r_1 \end{pmatrix} \quad C = \begin{pmatrix} k\cdot r_1 - r_2 \\ k\cdot r_2 - r_3 \\ k\cdot r_3 - r_1 \end{pmatrix}$

$\det(A) = -3$

a) $\det(B) = \begin{vmatrix} r_1+2r_2 \\ r_2+2r_3 \\ r_3+2r_1 \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2+2r_3 \\ r_3+2r_1 \end{vmatrix} + \begin{vmatrix} 2r_2 \\ r_2+2r_3 \\ r_3+2r_1 \end{vmatrix} = \begin{vmatrix} r_1 \\ r_2 \\ r_3+2r_1 \end{vmatrix} + \begin{vmatrix} r_1 \\ 2r_3 \\ r_3+2r_1 \end{vmatrix} + \begin{vmatrix} 2r_2 \\ r_2 \\ r_3+2r_1 \end{vmatrix} + \begin{vmatrix} 2r_2 \\ 2r_3 \\ 2r_3+2r_1 \end{vmatrix} =$

$= \begin{vmatrix} r_1 \\ r_2 \\ r_3 \end{vmatrix} + \underbrace{\begin{vmatrix} r_1 \\ r_2 \\ 2r_1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} r_1 \\ 2r_3 \\ r_3 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} r_1 \\ 2r_3 \\ 2r_1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} 2r_2 \\ r_2 \\ r_3 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} 2r_2 \\ r_2 \\ 2r_1 \end{vmatrix}}_{0} + \underbrace{\begin{vmatrix} 2r_2 \\ 2r_3 \\ 2r_3 \end{vmatrix}}_{0} + \begin{vmatrix} 2r_2 \\ 2r_3 \\ 2r_1 \end{vmatrix} =$

$= \begin{vmatrix} r_1 \\ r_2 \\ r_3 \end{vmatrix} + (-1)(-1)2^3\begin{vmatrix} r_1 \\ r_2 \\ r_3 \end{vmatrix} = \det(A) + 8\det(A) = -27$

OR:

B can be obtained from A by:
1. interchange r_2 and r_3
2. interchange new r_3 (old r_2) with r_1 } det stays the same ($(-1)\cdot(-1)\det(A)$)
3. multiply by 2 each row -> $\det(B) = 2^3\det(A) = 8\det(A)$
4. add A one time -> $\det(B) = 3\det(A)$

b) C can be obtained from A by:
1. -''-
2. -''- } $\det(C) = \det(A)$
3. multiply by $(-1)^3$ => $\det(C) = -\det(A)$
4. add k times A => $\det(C) = (k^3-1)\det(A)$ => $\det(C) = 0$ for $k = 1$

6. $A = \begin{pmatrix} * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$, $*$ = any number

a)

7. a) $A\cdot\operatorname{adj}(A) = \det(A)\cdot I_n$

1) A is singular => $\det(A) = 0$
$A\cdot\operatorname{adj}(A) = 0\cdot I_n = 0 \Rightarrow A\cdot\operatorname{adj}(A) = 0$ } if singular, then $A\cdot\operatorname{adj}(A) = 0$

2) $A\cdot\operatorname{adj}(A) = 0$:
$A\cdot\operatorname{adj}(A) = \det(A) I_n = 0$,
But $I_n \neq 0$, => $\det(A) = 0$ => A is singular } if $A\cdot\operatorname{adj}(A) = 0$, then A is singular

} => singularity of A $\Updownarrow$ $A\cdot\operatorname{adj}(A) = 0$

b) $\det(\operatorname{adj}(A))$ - ?

1) $\det(A) = 0$: => $A\cdot\operatorname{adj}(A) = 0$.
if $\operatorname{adj}(A) \neq 0$, then $A = 0$. But if $A = 0$, then $\operatorname{adj}(A) = 0 \Rightarrow \det(\operatorname{adj}(A)) = 0$

2) $\det(A) \neq 0$: $A\operatorname{adj}(A) = \det(A)\cdot I_n$, $A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} \Rightarrow A^{-1}\det(A) = \operatorname{adj}(A) \ |\cdot A$ (to left) =>
$A\cdot A^{-1}\det(A) = A\operatorname{adj}(A)$