Linear Algebra
Seminar 6: Linear transformations
(make sure you can write on the board)

Problem 1. (3pts) Find a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ which maps vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ into $\mathbf{u}_1, \ldots, \mathbf{u}_n$:

(a) $\mathbf{v}_1 = (1,1)^\top$, $\mathbf{v}_2 = (1,0)^\top$; $\mathbf{u}_1 = (1,0)^\top$, $\mathbf{u}_2 = (1,-1)^\top$

(b) $\mathbf{v}_1 = (1,1)^\top$, $\mathbf{v}_2 = (1,0)^\top$, $\mathbf{v}_3 = (0,1)^\top$; $\mathbf{u}_1 = (1,0)^\top$, $\mathbf{u}_2 = (1,-1)^\top$, $\mathbf{u}_3 = (1,1)^\top$

(c) $\mathbf{v}_1 = (0,1,-1)^\top$, $\mathbf{v}_2 = (1,0,1)^\top$, $\mathbf{v}_3 = (-1,1,0)^\top$; $\mathbf{u}_1 = (1,0)^\top$, $\mathbf{u}_2 = (1,1)^\top$, $\mathbf{u}_3 = (1,-1)^\top$

(d) $\mathbf{v}_1 = (0,1,-1)^\top$, $\mathbf{v}_2 = (1,0,1)^\top$, $\mathbf{v}_3 = (-1,1,0)^\top$;
$\mathbf{u}_1 = (1,0,0)^\top$, $\mathbf{u}_2 = (1,1,0)^\top$, $\mathbf{u}_3 = (1,1,1)^\top$

Problem 2. (3pts) Consider the bases $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$ for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\mathbf{v}'_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}, \qquad \mathbf{v}'_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \mathbf{v}'_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

(a) Find the transition matrix $P_{B \to B'}$ from B to B'.

(b) Compute the coordinate vector $(\mathbf{u})_B$ for $\mathbf{u} = (1,1,2)^\top$.

(c) Use the transition matrix $P_{B \to B'}$ to compute the coordinate vector $(\mathbf{u})_{B'}$.

(d) Check your work by computing $(\mathbf{u})_{B'}$ directly.

Problem 3. (3pts) Let T be the linear tranformation of $\mathbb{R}^3$ defined by $T(x,y,z)^\top = (x-y, x+y, x-z)^\top$ and consider the vector

$$\mathbf{v} = \begin{pmatrix}1\\1\\2\end{pmatrix} \qquad \text{and the basis } B \text{ formed by} \qquad \mathbf{v}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(a) Determine the transition matrix $P_{B_0 \to B}$ from the standard basis to B

(b) Determine the matrix A of T in the basis B

(c) Compute $(\mathbf{v})_B$, $(T\mathbf{v})_B$, and check that $(T\mathbf{v})_B = A(\mathbf{v})_B$

Problem 4. (3pts) Assume that T is a linear mapping from a 2-dimensional vector space V to a 3-dimensional vector space W whose matrix in bases $(\mathbf{v}_1, \mathbf{v}_2)$ of V and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ of W is

$$A = \begin{pmatrix}1 & 1\\0 & 1\\1 & -1\end{pmatrix}$$

(a) Express the vector $T\mathbf{v}_1$ in terms of the basis vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and $T\mathbf{v} = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3$, find the relation between c's and d's

(b) Identify the nullspace $\mathrm{Nul}(T) = \{\mathbf{v} \in V \mid T\mathbf{v} = \mathbf{0}\}$ of T and find its dimension

(c) Describe the range $\mathrm{Ran}(T) = \{\mathbf{w} = T\mathbf{v} \mid \mathbf{v} \in V\}$

(d) Find a basis of $\mathrm{Ran}(T)$

(e) Is the mapping T an isomorphism between the spaces V and W? Explain your answer

Problem 5. (4pts) For a linear transformation $T : V \to V$ in a finite-dimensional linear vector space the *rank* of T (denoted $\mathrm{rank}(T)$) is defined as the dimension of the range $\mathrm{Ran}(T) = \{T\mathbf{v} \mid \mathbf{v} \in V\}$. Assume that T_1 and T_2 are linear transformations in V.

(a) Explain why the range of $T_2 \circ T_1$ is contained in $\mathrm{Ran}(T_2)$ and conclude that $\mathrm{rank}(T_2 \circ T_1) \le \mathrm{rank}\, T_2$.

(b) Prove that $\mathrm{rank}(T_2 \circ T_1) \le \mathrm{rank}(T_1)$.

(c) Prove that if T_1 is onto, then $\mathrm{rank}(T_2 \circ T_1) = \mathrm{rank}(T_2)$.

(d) Prove that if T_2 is one-to-one, then $\mathrm{rank}(T_2 \circ T_1) = \mathrm{rank}(T_1)$.

Comment on the special case of $V = \mathbb{R}^n$.

Problem 6. (4pts) Assume that A and B are (possibly non-square) matrices such that both AB and BA are defined.

(a) Is it always true that $\mathrm{rank}(AB) = \mathrm{rank}(BA)$? Why?

(b) Is it possible that simultaneously $\mathrm{rank}(AB) < \mathrm{rank}(A)$ and $\mathrm{rank}(AB) < \mathrm{rank}(B)$?

(c) Prove that $A^\top A = \mathbf{0}$ if and only if $A = \mathbf{0}$.

(d) Prove that $N(A) = N(A^\top A)$.
Hint: if $A^\top A\mathbf{x} = 0$, then $0 = \mathbf{x}^\top(A^\top A\mathbf{x}) = \dots$

(e) Prove that $\mathrm{rank}(A^\top A) = \mathrm{rank}(A) = \mathrm{rank}(AA^\top)$.

Extra problems
(to be discussed if time permits)

Problem 7. Assume that T is a linear transformation of a linear vector space V. Prove that the following three statements about the nullspaces and ranges of T and $T^2 = T \circ T$ are equivalent:

(a) $\mathrm{Ran}(T^2) = \mathrm{Ran}(T)$;

(b) $\mathrm{Nul}(T^2) = \mathrm{Nul}(T)$;

(c) $\mathrm{Ran}(T) \cap \mathrm{Nul}(T) = \{\mathbf{0}\}$.

Problem 8. Assume that T is a mirror symmetry in $\mathbb{R}^2$ with respect to the line $ax + by = 0$, with $a = -\sin\theta$ and $b = \cos\theta$, for some $\theta \in [0, \pi)$.

(a) Find the matrix A of T is the basis B formed by $\mathbf{u}_1 = (\cos\theta, \sin\theta)^\top$ and $\mathbf{u}_2 = (-\sin\theta, \cos\theta)^\top$.

(b) Find the transition matrix from the standard basis B_0 formed by $\mathbf{e}_1 = (1, 0)^\top$, $\mathbf{e}_2 = (0, 1)^\top$ to B.
Hint: this is a rotation matrix

(c) Find the matrix A_0 of T in the standard basis B_0.

(d) Check that $A\mathbf{u}_1 = \mathbf{u}_1$ and $A\mathbf{u}_2 = -\mathbf{u}_2$.

Problem 9. Obtain the matrix A_0 of the above problem by representing T as $T = R_\phi T_0 R_{-\phi}$, where R_ϕ is the counterclockwise rotation by angle ϕ and T_0 is the mirror symmetry with respect to the x-axis.

Problem 10. Construct a matrix A whose null space consists of all linear combinations of the vectors $\mathbf{v}_1 = (1, 0, 1, 2)^\top$ and $\mathbf{v}_2 = (0, 2, -1, 1)^\top$ and whose column space is the plane $x + y - 2z = 0$ in $\mathbb{R}^3$. To this end,

(a) determine the size of A, the number of pivot and free variables, and the rank of A.

(b) determine the possible RREF R of the matrix A;

(c) find two linearly independent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in the column space of A;

(d) find a non-singular matrix E such that $\mathbf{u}_1$ and $\mathbf{u}_2$ are columns of ER;

(e) explain why the above ER has all the properties that A should have.

① a) $T: \overset{\vec{u}_1}{\begin{bmatrix}1\\1\end{bmatrix}} \longmapsto \begin{bmatrix}1\\0\end{bmatrix}$

$T: \overset{\vec{u}_2}{\begin{bmatrix}1\\0\end{bmatrix}} \longmapsto \begin{bmatrix}1\\-1\end{bmatrix}$

Consider augmented matrix $\left[\begin{array}{cc|c}1&1&x\\1&0&y\end{array}\right]\begin{matrix}-r_1\\ \end{matrix} \sim \left[\begin{array}{cc|c}0&1&x-y\\1&0&y\end{array}\right] \sim \left[\begin{array}{cc|c}1&0&y\\0&1&x-y\end{array}\right]$

$w = y\cdot\vec{u}_1 + (x-y)\cdot\vec{u}_2$

$T(w) = yT(\vec{u}_1) + (x-y)\cdot T(\vec{u}_2) = y\begin{bmatrix}1\\0\end{bmatrix} + (x-y)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}y+x-y\\-x+y\end{bmatrix} = \begin{bmatrix}x\\-x+y\end{bmatrix} = \begin{bmatrix}1&0\\-1&1\end{bmatrix}$

Check: $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1+1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\-1+0\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$

b) $T: \left\{\underset{\vec{v}_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\0\end{bmatrix}}\right\} \longmapsto \left\{\underset{\vec{u}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\vec{u}_2}{\begin{bmatrix}1\\-1\end{bmatrix}}\right\}$

$\left[\begin{array}{cc|cc}1&1&1&0\\1&0&0&1\end{array}\right]-r_1 \sim \left[\begin{array}{cc|cc}0&1&1&-1\\1&0&0&1\end{array}\right] \sim \left[\begin{array}{cc|cc}1&0&0&1\\0&1&1&-1\end{array}\right]$

$A\cdot V = U$

$A = U\cdot V^{-1} = \begin{bmatrix}1&1\\0&-1\end{bmatrix}\cdot\begin{bmatrix}0&1\\1&-1\end{bmatrix} = \begin{bmatrix}1&0\\-1&1\end{bmatrix} = \begin{bmatrix}x\\-x+y\end{bmatrix}$

$T(\vec{v}_3) = \begin{bmatrix}0\\1\end{bmatrix} \neq \begin{bmatrix}1\\1\end{bmatrix}$

$T: \left\{\underset{\vec{v}_2}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\end{bmatrix}}\right\} \longmapsto \left\{\underset{\vec{u}_2}{\begin{bmatrix}1\\-1\end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix}1\\1\end{bmatrix}}\right\}$

$A = U\cdot V^{-1} = \begin{bmatrix}1&1\\-1&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&1\\-1&1\end{bmatrix} = \begin{bmatrix}x+y\\-x+y\end{bmatrix}$

$T(\vec{v}_1) = \begin{bmatrix}2\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\end{bmatrix}$

$T: \left\{\underset{\vec{v}_1}{\begin{bmatrix}1\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\1\end{bmatrix}}\right\} \longmapsto \left\{\underset{\vec{u}_1}{\begin{bmatrix}1\\0\end{bmatrix}}, \underset{\vec{u}_3}{\begin{bmatrix}1\\1\end{bmatrix}}\right\}$

$\left[\begin{array}{cc|cc}1&0&1&0\\1&1&0&1\end{array}\right]-r_1 \sim \left[\begin{array}{cc|cc}1&0&1&0\\0&1&-1&1\end{array}\right]$

$A = U\cdot V^{-1} = \begin{bmatrix}1&1\\0&1\end{bmatrix}\cdot\begin{bmatrix}1&0\\-1&1\end{bmatrix} = \begin{bmatrix}0&1\\-1&1\end{bmatrix} = \begin{bmatrix}y\\-x+y\end{bmatrix}$

$T(\vec{v}_2) = \begin{bmatrix}0\\-1\end{bmatrix} \neq u_2$

there is no such a transformation

c) $T: \left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\} \longmapsto \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\}$

$\left[\begin{array}{ccc|ccc}0&1&-1&1&0&0\\1&0&1&0&1&0\\-1&1&0&0&0&1\end{array}\right]+r_2 \sim \left[\begin{array}{ccc|ccc}0&1&-1&1&0&0\\1&0&1&0&1&0\\0&1&1&0&1&1\end{array}\right]-r_1 \sim \left[\begin{array}{ccc|ccc}0&1&-1&1&0&0\\1&0&1&0&1&0\\0&0&2&-1&1&1\end{array}\right]\begin{matrix}+\frac{1}{2}r_3\\-\frac{1}{2}r_3\\\cdot\frac{1}{2}\end{matrix} \sim \left[\begin{array}{ccc|ccc}0&1&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\1&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&1&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right]$

$A = U\cdot U^{-1} = \begin{bmatrix}1&1&1\\0&1&-1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&\frac{3}{2}&\frac{1}{2}\\1&0&0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}x+\frac{3}{2}y+\frac{1}{2}z\\x\end{bmatrix}$

d) $T: \left\{\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\} \longmapsto \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

$\left[\begin{array}{ccc|ccc}0&1&-1&1&0&0\\1&0&1&0&1&0\\-1&1&0&0&0&1\end{array}\right]+r_2 \sim \left[\begin{array}{ccc|ccc}0&1&-1&1&0&0\\1&0&1&0&1&0\\0&1&1&0&1&1\end{array}\right]-r_1 \sim \left[\begin{array}{ccc|ccc}0&1&-1&1&0&0\\1&0&1&0&1&0\\0&0&2&-1&1&1\end{array}\right]\begin{matrix}+\frac{1}{2}r_3\\-\frac{1}{2}r_3\\\cdot\frac{1}{2}\end{matrix} \sim \left[\begin{array}{ccc|ccc}0&1&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\1&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\0&1&0&\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right]$

$A = U\cdot U^{-1} = \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&\frac{3}{2}&\frac{1}{2}\\0&1&1\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}x+\frac{3}{2}y+\frac{1}{2}z\\y+z\\-\frac{1}{2}x+\frac{1}{2}y+\frac{1}{2}z\end{bmatrix}$

② a) $B = \begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$, $B' = \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$

augmented matrix $(B'|B) \Rightarrow (I|P^{-1})$

$\left[\begin{array}{ccc|ccc}0&0&1&1&0&0\\0&1&1&0&1&0\\1&1&1&1&1&1\end{array}\right]\begin{matrix}\\-r_1\\-r_2\end{matrix} \sim \left[\begin{array}{ccc|ccc}0&0&1&1&0&0\\0&1&0&-1&1&0\\1&0&0&1&0&1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1&0&0&1&0&1\\0&1&0&-1&1&0\\0&0&1&1&0&0\end{array}\right]$, P^{-1} – transition m from B to B'

(I_n under the left block, P^{-1} under the right block)

b) $\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\1&1&1&2\end{array}\right]-r_1-r_2 \sim \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&0\end{array}\right]$

c) $(\vec{u})_{B'} = P^{-1}\cdot(\vec{u})_B = P_{B\to B'}\cdot(\vec{u})_B = \begin{bmatrix}1&0&1\\-1&1&0\\1&0&0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}$

d) $\left[\begin{array}{ccc|c}0&0&1&1\\0&1&1&1\\1&1&1&2\end{array}\right]\begin{matrix}\\-r_1\\-r_2\end{matrix} \sim \left[\begin{array}{ccc|c}0&0&1&1\\0&1&0&0\\1&0&0&1\end{array}\right]$

③ $T(x,y,z)^T = \begin{bmatrix}x-y\\x+y\\x-z\end{bmatrix} = \begin{bmatrix}1&-1&0\\1&1&0\\1&0&-1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}1\\1\\2\end{bmatrix}$,

$B = \begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}$

a) $[B|B_0] \to [I_n|P^{-1}]$

if B_0 is the standard basis, then $P^{-1} = (B)^{-1}$

$\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&0&1&0\\1&1&1&0&0&1\end{array}\right]-r_1-r_2 \sim \left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&0&1&0\\0&0&1&-1&-1&1\end{array}\right]$

($P_{B_0\to B}$ under the right block)

b) $A = P_{B_0\to B}\, A_0\, P_{B\to B_0}$

$P_{B\to B_0} = (P_{B_0\to B})^{-1} = B$

$A = \begin{bmatrix}1&0&0\\0&1&0\\-1&-1&1\end{bmatrix}\cdot\begin{bmatrix}1&-1&0\\1&1&0\\1&0&-1\end{bmatrix}\cdot\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix} = \begin{bmatrix}1&-1&0\\1&1&0\\-2&-1&-1\end{bmatrix}$

c) $(\vec{v})_B$:

$\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\1&1&1&2\end{array}\right]-r_1-r_2 \sim \left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&0\end{array}\right]$

$(T\vec{v})_B = \left(\begin{bmatrix}1-1\\1+1\\1-2\end{bmatrix}\right)_B = \left(\begin{bmatrix}0\\2\\-1\end{bmatrix}\right)_B$

$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&2\\1&1&1&-1\end{array}\right]-r_1-r_2 \sim \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&2\\0&0&1&-3\end{array}\right]$

check: $A\cdot(\vec{v})_B = \begin{bmatrix}1&-1&0\\1&1&0\\-2&-1&-1\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}0\\2\\-3\end{bmatrix}$

④ $T: V \mapsto W$, $\dim V = 2$, $\dim W = 3$

$A = \begin{pmatrix}1&1\\0&1\\1&-1\end{pmatrix} = \begin{bmatrix}x+y\\y\\x-y\end{bmatrix}$

a) $\begin{cases}\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2\\ T\vec{v} = d_1\vec{w}_1 + d_2\vec{w}_2 + d_3\vec{w}_3\end{cases}$

$Tv_1 = w_1 + w_3$

$Tv_2 = w_1 + w_2 - w_3$

$T\vec{v} = \begin{bmatrix}c_1+c_2\\c_2\\c_1-c_2\end{bmatrix}[w_1, w_2, w_3]$

$\begin{cases}d_1 = c_1+c_2\\d_2 = c_2\\d_3 = c_1-c_2\end{cases}$

b) $N(T) = \{v \in V : Tv = 0\}$

$\begin{pmatrix}1&1\\0&1\\1&-1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow c_1 = c_2 = 0$

$N(T) = \{0, 0\}$

c) $Ran = \{Tv : v \in V\}$ – the subset of vect.space W, such that every element is an image

$A\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ – we look for b_n, so that system will have solutions

$\begin{cases}c_1+c_2 = w_1 & \Rightarrow c_1 = w_1 - w_2\\c_2 = w_2\\c_1 - c_2 = w_3\end{cases} \Rightarrow$ plug in $w_1 - w_2 - w_2 = w_3 \Rightarrow w_1 - 2w_2 = w_3$, then the system is consistent

d) Range is a plane $x - 2y - z = 0$ with basis $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}$

e) it is not onto. If we take vector, that doesn't satisfy condition in c), there will be no image

⑤ a) image of composition is contained in the image of T_2 $\Rightarrow$ $\operatorname{rank}(T_2\circ T_1) \leq \operatorname{rank} T_2$

b)

c)

d) dimensions remains the same

⑥ a) $\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$ – rank(AB)

But: $\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ = rank(BA) $\neq$

b) ↗ yes ?

c) $A^T = A = 0 \Leftrightarrow A = 0$

if $A = 0 \Rightarrow$ obvious

if $A^T = A = 0 \Rightarrow \begin{pmatrix}c_1 & \dots & c_n\\ \vdots & & \end{pmatrix}\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = c_1^2 + \dots + c_n^2 = 0 \Rightarrow c_{1,\dots,n} = 0$

d) $0 = x^T A^T A x = (Ax)^T \cdot Ax \Rightarrow Ax = 0$ (same as part c.)

$(AB)^T = B^T\cdot A^T$ for every matrix