

Linear Algebra

Lecture Notes

Rostyslav Hryniv

Ukrainian Catholic University
Business Analytics and Computer Science Programmes

4th term
Spring 2021



APPLIED
SCIENCES
FACULTY ●

Lecture 1. Systems of linear equations

Outline

- 1 Systems of linear equations
 - Examples. Solutions. Substitution and elimination
 - Solutions in dimension 2
 - Solutions in dimension 3
- 2 Gaussian and Gauss–Jordan elimination
 - Reduction to an echelon form
 - Pivot and free variables
 - A reduced echelon form
- 3 Existence and uniqueness of solutions
 - Homogeneous and non-homogeneous systems
 - Existence and uniqueness of solutions
 - Nonsingular matrices. Rank of a matrix

Outline

1 Systems of linear equations

- **Examples. Solutions. Substitution and elimination**
- Solutions in dimension 2
- Solutions in dimension 3

- Reduction to an echelon form
- Pivot and free variables
- A reduced echelon form

- Homogeneous and non-homogeneous systems
- Existence and uniqueness of solutions
- Nonsingular matrices. Rank of a matrix

Leontief model (Nobel prize in economics in 1973):

- in 1949, divided the American economy in $N = 500$ sectors
- analyzed distribution of the outcomes
- got the $N \times N$ linear system ($N \leq 32$ feasible in 1949)

Production matrix for a simple economy model (N=3):

	Input Required per Dollar Output in		
	Manufacturing	Agriculture	Utilities
Manufacturing	\$ 0.50	\$ 0.10	\$ 0.10
Agriculture	\$ 0.20	\$ 0.50	\$ 0.30
Utilities	\$ 0.10	\$ 0.30	\$ 0.40

x_1, x_2, x_3 : produced output; d_1, d_2, d_3 : demand

$$x_1 - (0.5x_1 + 0.1x_2 + 0.1x_3) = d_1$$

$$x_2 - (0.2x_1 + 0.5x_2 + 0.3x_3) = d_2$$

$$x_3 - (0.1x_1 + 0.3x_2 + 0.4x_3) = d_3$$

Systems of linear equations

Many applied models postulate linear dependence on some **variables** (e.g., economic factors) **x, y, z, \dots** :

$$ax + by + cz + \dots = a$$

Here a, b, c, \dots, α are given numbers, or scalars^a

^aCan be real (\mathbb{R}), complex (\mathbb{C}), quaternionic (\mathbb{H}), binary (\mathbb{Z}_2) or \mathbb{Z}_p ; any *field*

- For many (n) parameters, use notations x_1, x_2, \dots, x_n
- **system of linear equations**, or **linear system**

$$\begin{cases} a_{11}\mathbf{x}_1 + a_{12}\mathbf{x}_2 + \cdots + a_{1n}\mathbf{x}_n = b_1 \\ a_{21}\mathbf{x}_1 + a_{22}\mathbf{x}_2 + \cdots + a_{2n}\mathbf{x}_n = b_2 \\ \vdots \\ a_{m1}\mathbf{x}_1 + a_{m2}\mathbf{x}_2 + \cdots + a_{mn}\mathbf{x}_n = b_m \end{cases} \quad (1)$$

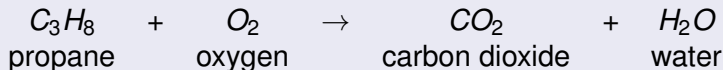
- usually x_1, \dots, x_n are **unknown**; to be determined from (1)

Definition (Solution to a system)

A sequence, or n -tuple (x_1, x_2, \dots, x_n) of numbers turning each equation in (1) into equality is a **solution** of the linear system (1).

Balancing chemical reactions

Combustion of propane produces carbon dioxide and water:



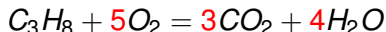
To balance the equation, take x molecules of propane, y of oxygen, z of carbon dioxide and w of water; now equate the number of atoms on both sides to get

$$3x = z \quad (\text{Carbon})$$

$$8x = 2w \quad (\text{Hydrogen})$$

$$2y = 2z + w \quad (\text{Oxygen})$$

Many solutions, e.g. $x = 1$, $y = 5$, $z = 3$, and $w = 4$:



Yes, for the coefficients a_0, a_1, \dots, a_{n-1} !

Outline

1 Systems of linear equations

- Examples. Solutions. Substitution and elimination
- **Solutions in dimension 2**
- Solutions in dimension 3

2 Gaussian and Gauss–Jordan elimination

- Reduction to an echelon form
- Pivot and free variables
- A reduced echelon form

3 Existence and uniqueness of solutions

- Homogeneous and non-homogeneous systems
- Existence and uniqueness of solutions
- Nonsingular matrices. Rank of a matrix

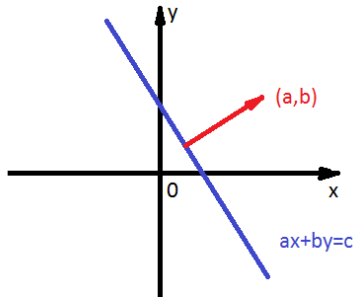
Lines in \mathbb{R}^2

- In the xy -plane, equation $y = kx + m$ gives a line through $(0, m)$ with the slope k
- If $|a| + |b| > 0$, then

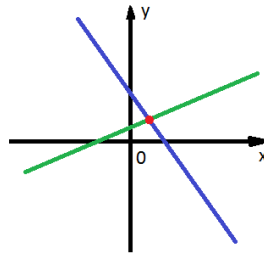
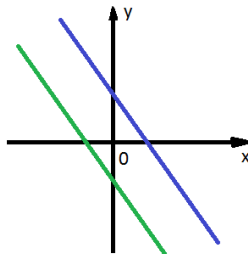
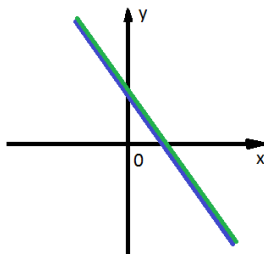
$$ax + by = c$$

describes the line with slope $k = -a/b$ through $(0, c/b)$ if $b \neq 0$, or the line $x \equiv c/a$ otherwise

- this line is orthogonal to the **normal** vector (a, b) (**gradient**)



Possible situations:



- ① Lines coincide \implies infinitely many solutions
- ② Lines parallel \implies no solutions
- ③ Lines intersect \implies single solution

Solving linear systems in dimension 2:

By substitution:

$$\begin{array}{rcl} 2x + y & = & 3 \\ 3x - 2y & = & 8 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x & = & \frac{1}{2}(3 - y) \\ 3 \cdot \frac{1}{2}(3 - y) - 2y & = & 8 \end{array} \quad \times 3$$

From the second equation $y = -1$ and then the first gives $x = 2$

By elimination:

$$\begin{array}{rcl} 2x + y & = & 3 \\ 3x - 2y & = & 8 \end{array} \quad -\frac{3}{2} \times (1) \quad \Rightarrow \quad \begin{array}{rcl} 2x + y & = & 3 \\ -\frac{7}{2}y & = & \frac{7}{2} \end{array} \quad \Rightarrow$$

$$\begin{array}{rcl} 2x + y & = & 3 \\ y & = & -1 \end{array} \quad -1 \times (2) \quad \Rightarrow \quad \begin{array}{rcl} 2x & = & 4 \\ y & = & -1 \end{array}$$

The two methods are only formally different

Outline

1 Systems of linear equations

- Examples. Solutions. Substitution and elimination
- Solutions in dimension 2
- Solutions in dimension 3

2 Gaussian and Gauss–Jordan elimination

- Reduction to an echelon form
- Pivot and free variables
- A reduced echelon form

3 Existence and uniqueness of solutions

- Homogeneous and non-homogeneous systems
- Existence and uniqueness of solutions
- Nonsingular matrices. Rank of a matrix

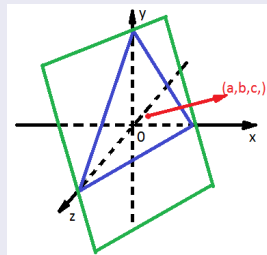
Linear systems in dimension 3:

Planes in \mathbb{R}^3

If $|a| + |b| + |c| > 0$, then the equation

$$ax + by + cz = d$$

describes in the xyz -space the plane
orthogonal to the direction vector (a, b, c)
(also called **normal**, or **gradient** vector)



Systems in \mathbb{R}^3 :

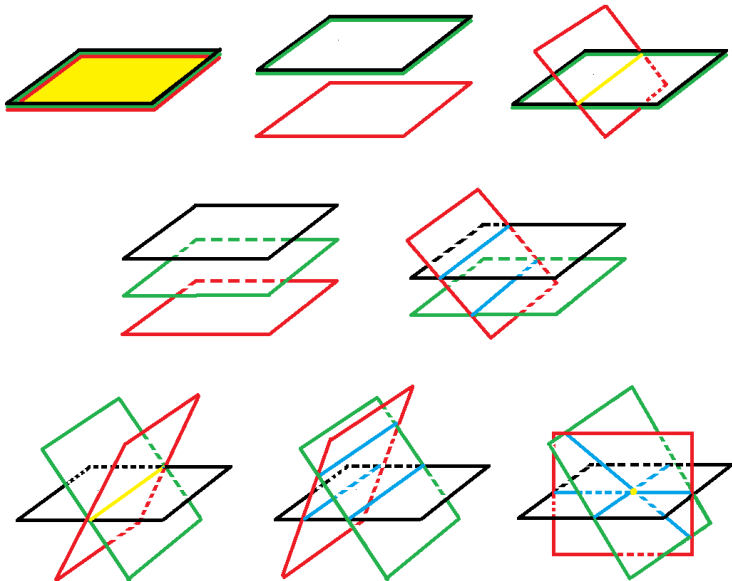
$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

describe the common (intersection) points of three planes in xyz -space

Possible situations



Solving linear systems in dimension 3:

By elimination:

$$\begin{array}{rclcl}
 2x & + & y & & = & 3 \\
 \textcolor{red}{4x} & -2y & +3z & = & 4 & \textcolor{red}{-2 \times (r_1)} \\
 & 4y & -2z & = & 0
 \end{array}$$

$$\begin{array}{rclcl}
 2x & + & y & & = & 3 \\
 & -4y & +3z & = & -2 \\
 & \textcolor{red}{4y} & -2z & = & 0 & \textcolor{red}{+1 \times (r_2)}
 \end{array}$$

$$\begin{array}{rclcl}
 2x & + & y & & = & 3 \\
 & -4y & +\textcolor{blue}{3z} & = & -2 & \textcolor{blue}{-3 \times (r_3); \quad /(-4)} \\
 & & z & = & -2
 \end{array}$$

$$\begin{array}{rclcl}
 2x & + & \textcolor{blue}{y} & & = & 3 & \textcolor{blue}{-1 \times (r_2); \quad /2} \\
 & & y & & = & -1 \\
 & & z & = & -2
 \end{array}$$

Solution: $(2, -1, -2)$

Outline

- 1 Systems of linear equations
 - Examples. Solutions. Substitution and elimination
 - Solutions in dimension 2
 - Solutions in dimension 3
- 2 Gaussian and Gauss–Jordan elimination
 - Reduction to an echelon form
 - Pivot and free variables
 - A reduced echelon form
- 3 Existence and uniqueness of solutions
 - Homogeneous and non-homogeneous systems
 - Existence and uniqueness of solutions
 - Nonsingular matrices. Rank of a matrix

How to solve linear systems for large m, n ?

In real-world applications $m, n > 1000$:

- PageRank algorithm: $m = n \approx 10^9$ for adjacency matrix
- English Wikipedia: $m = n \approx 4 \times 10^6$ for adjacency matrix

Main questions:

- 1 Is there any solution to such a linear system?
- 2 How many?
- 3 How to find all/some special?

Idea:

Transform a system to simpler form without changing the set of solutions

Definition

- A system is called **consistent** if it possesses at least one solution
- Two linear systems are called **equivalent** if they possess **the same** set of solutions

Shorthand matrix notations for linear systems:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \mathbf{Ax} = \mathbf{b} \quad (2)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad \text{is an } m \times n \text{ coefficient matrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \dots, x_n)^\top, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = (b_1, b_2, \dots, b_m)^\top$$

are the **vector of unknowns** and the (given) **RHS** vector resp.

Augmented coefficient matrix

Often it is more convenient to combine the matrix notation for the coefficient matrix A and the RHS \mathbf{b} and to encode system (2) via the **augmented coefficient matrix**

$$(A \mid \mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

Elementary row operations that simplify linear systems

Elementary row operations:

1. Multiply an equation/a row through by a nonzero constant
2. Add a constant times one equation/row to another
3. Interchange two equations/rows

Definition

Matrices A and B are **row equivalent** if B can be obtained from A via elementary row operations

Properties

- 1 Elementary row operations are reversible
- 2 Lead to equivalent systems/augmented matrices

Example: elementary row operations

$$\begin{array}{rclcl} x + 2y - z & = & 2 & & \\ 2x + y - z & = & 1 & -2 \times (r_1) & \\ x + y - z & = & 0 & -1 \times (r_1) & \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \end{array} \right)$$

$$\begin{array}{rclcl} x + 2y - z & = & 2 & & \\ -3y + z & = & -3 & (r_2) \leftrightarrow (r_3) & \\ -y & = & -2 & (r_2) \leftrightarrow (r_3) & \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 1 & -3 \\ 0 & -1 & 0 & -2 \end{array} \right)$$

$$\begin{array}{rclcl} x + 2y - z & = & 2 & & \\ -y & = & -2 & \times (-1) & \\ -3y + z & = & -3 & +3 \times (r_2) & \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & -3 & 1 & -3 \end{array} \right)$$

$$\begin{array}{rclcl} x + 2y - z & = & 2 & -2 \times (r_2) + (r_3) & \\ y & = & 2 & & \\ z & = & 3 & & \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

$$\begin{array}{rcl} x & = & 1 \\ y & = & 2 \\ z & = & 3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array} \right)$$

An echelon form

Definition

A rectangular matrix is in a **row echelon form (REF)** if it has the following three properties:

- 1 All nonzero rows are above any zero row
- 2 Each leading entry of a row (i.e., first nonzero entry in that row) is in the column to the right of the leading entry of the row above it
- 3 All entries in a column below a leading entry are zeros

Example (● stands for a nonzero number, * for any number)

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

The first matrix is in echelon form, the second is not unless...

Outline

- 1 Systems of linear equations
 - Examples. Solutions. Substitution and elimination
 - Solutions in dimension 2
 - Solutions in dimension 3
- 2 Gaussian and Gauss–Jordan elimination
 - Reduction to an echelon form
 - **Pivot and free variables**
 - A reduced echelon form
- 3 Existence and uniqueness of solutions
 - Homogeneous and non-homogeneous systems
 - Existence and uniqueness of solutions
 - Nonsingular matrices. Rank of a matrix

Pivot and free variables

Question

Is an echelon form of a matrix unique?

- **No**, because it depends on what elementary row operations and in what order were applied
- However, the positions of the leading entries are fixed!

Definition

Let A be a matrix of a linear system. A **pivot column** of A is a column with a leading entry in an echelon form of A . The corresponding variable is called a **pivot variable**. Otherwise a variable is called **free**.

Example (Columns corresponding to free variables are in red)

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

Why are free variables called free?

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \iff \begin{array}{rrrrr} x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = & 0 \\ & & & & x_3 & - & x_4 & = & 2 \end{array}$$

Therefore,

- 1 $x_3 = 2 + x_4$
- 2 $x_1 = -2x_2 + 2x_3 - x_4 = 4 - 2x_2 + x_4$
- 3 by introducing **parameters** s and t via $x_2 := s$ and $x_4 := t$, we get the following **parametric representation** of general solutions of the system for every values of s and t :

$$x_1 = 4 - 2s + t$$

$$x_2 = s$$

$$x_3 = 2 + t$$

$$x_4 = t$$

Outline

- 1 Systems of linear equations
 - Examples. Solutions. Substitution and elimination
 - Solutions in dimension 2
 - Solutions in dimension 3
- 2 Gaussian and Gauss–Jordan elimination
 - Reduction to an echelon form
 - Pivot and free variables
 - A reduced echelon form
- 3 Existence and uniqueness of solutions
 - Homogeneous and non-homogeneous systems
 - Existence and uniqueness of solutions
 - Nonsingular matrices. Rank of a matrix

A reduced row echelon form

Definition

A rectangular matrix in a row echelon form is in a

reduced row echelon form (RREF)

if in addition:

- 1 The leading entry in each nonzero row is 1
- 2 Each leading 1 is the only nonzero element in its column

Example (RREF; * stands for any number)

$$\begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- 1 Matrix in echelon form can be transformed to a reduced echelon form using elementary row operations
- 2 A reduced echelon form of a matrix is **unique**
- 3 If the augmented matrix is in the reduced echelon form, solutions are easy to write in terms of free variables

Gaussian and Gauss–Jordan elimination

Forward substitution \rightsquigarrow an echelon form

Example (Gaussian elimination \iff substitution)

$$\begin{array}{rcl} x + 2y = 1 \\ 3x + 4y = 1 \end{array} \iff \begin{array}{rcl} x + 2y = 1 \\ 3(1 - 2y) + 4y = 1 \end{array} \iff \begin{array}{rcl} x + 2y = 1 \\ -2y = -2 \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array} \right)$$

Backward substitution \rightsquigarrow reduced echelon form

Example (Gauss–Jordan elimination \iff back substitution)

$$\begin{array}{rcl} x + 2y = 1 \\ -2y = -2 \end{array} \iff \begin{array}{rcl} x + 2 \cdot \textcolor{red}{1} = 1 \\ \textcolor{red}{y} = 1 \end{array} \iff \begin{array}{rcl} x & = & -1 \\ y & = & 1 \end{array}$$
$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 1 \end{array} \right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right)$$

Outline

- 1 Systems of linear equations
 - Examples. Solutions. Substitution and elimination
 - Solutions in dimension 2
 - Solutions in dimension 3
- 2 Gaussian and Gauss–Jordan elimination
 - Reduction to an echelon form
 - Pivot and free variables
 - A reduced echelon form
- 3 Existence and uniqueness of solutions
 - Homogeneous and non-homogeneous systems
 - Existence and uniqueness of solutions
 - Nonsingular matrices. Rank of a matrix

Solution set of homogeneous systems

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has a **trivial solution** $\mathbf{x} = \mathbf{0}$ (is always consistent)
- If $\mathbf{x} = (x_1, \dots, x_n)^\top$ is a solution, then for every scalar k , $k\mathbf{x} := (kx_1, \dots, kx_n)^\top$ is a solution as well:

$$a_{i1}(kx_1) + \dots + a_{in}(kx_n) = k(a_{i1}x_1 + \dots + a_{in}x_n) = k \cdot 0 = 0$$

- If $\mathbf{x} = (x_1, \dots, x_n)^\top$ and $\mathbf{y} = (y_1, \dots, y_n)^\top$ are solutions, then so is $\mathbf{x} + \mathbf{y} := (x_1 + y_1, \dots, x_n + y_n)^\top$:

$$\begin{aligned} a_{i1}(x_1 + y_1) + \dots + a_{in}(x_n + y_n) \\ = (a_{i1}x_1 + \dots + a_{in}x_n) + (a_{i1}y_1 + \dots + a_{in}y_n) = 0 + 0 = 0 \end{aligned}$$

- The **solution set** of a homogeneous linear system is a linear space

Theorem (Solutions to a homogeneous system)

A homogeneous system $A\mathbf{x} = \mathbf{0}$ may have

- **one solution** (the trivial solution $\mathbf{x} = \mathbf{0}$) or
- **infinitely many solutions**

Moreover, its solution set is linear.

Solution set of non-homogeneous systems

- A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ need not be consistent (may not have any solutions)
- If $\mathbf{x} = (x_1, \dots, x_n)^\top$ and $\mathbf{y} = (y_1, \dots, y_n)^\top$ are solutions, then $\mathbf{x} - \mathbf{y} = (x_1 - y_1, \dots, x_n - y_n)^\top$ solves the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} a_{i1}(x_1 - y_1) + \dots + a_{in}(x_n - y_n) \\ = (a_{i1}x_1 + \dots + a_{in}x_n) - (a_{i1}y_1 + \dots + a_{in}y_n) \\ = b_i - b_i = 0 \end{aligned}$$

- The **solution set** of a non-homogeneous linear system is an affine set (shifted linear set)

Theorem (Solutions to a non-homogeneous system)

A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ may have **no** solutions, **one** solution, or **infinitely many** solutions.

Any solution has the form $\mathbf{x} = \mathbf{x}_{\text{par}} + \mathbf{x}_{\text{hom}}$, where \mathbf{x}_{par} is a particular solution, and \mathbf{x}_{hom} is any solution to a homogeneous system $A\mathbf{x} = \mathbf{0}$

Outline

- 1 Systems of linear equations
 - Examples. Solutions. Substitution and elimination
 - Solutions in dimension 2
 - Solutions in dimension 3
- 2 Gaussian and Gauss–Jordan elimination
 - Reduction to an echelon form
 - Pivot and free variables
 - A reduced echelon form
- 3 Existence and uniqueness of solutions
 - Homogeneous and non-homogeneous systems
 - **Existence and uniqueness of solutions**
 - Nonsingular matrices. Rank of a matrix

Existence of solutions

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ is **always** consistent
- A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent
 - \iff an echelon form of its augmented matrix $(A|\mathbf{b})$ has no row $(0 \dots 0 | \bullet)$ with a non-zero \bullet
- A system $A\mathbf{x} = \mathbf{b}$ is consistent for **every** \mathbf{b}
 - \iff an echelon form of A has no zero rows
 - \iff A has a pivot position in every its row
- For every A , the system $A\mathbf{x} = \mathbf{b}$ is consistent for **some** choice of \mathbf{b} .

Uniqueness of solutions

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution
 \iff there is **no** free variables
- A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ has a unique solution
 \iff **no** rows $(0 \dots 0 \mid \bullet)$ in echelon form of $(A \mid \mathbf{b})$ **and**
no free variables
- A system $A\mathbf{x} = \mathbf{b}$ has a unique solution for **every** \mathbf{b}
 \iff **no** zero rows in an echelon form of A **and**
no free variables
 \implies A must be a **square matrix** ($m = n$)

Outline

1 Systems of linear equations

- Examples. Solutions. Substitution and elimination
- Solutions in dimension 2
- Solutions in dimension 3

2 Gaussian and Gauss–Jordan elimination

- Reduction to an echelon form
- Pivot and free variables
- A reduced echelon form

3 Existence and uniqueness of solutions

- Homogeneous and non-homogeneous systems
- Existence and uniqueness of solutions
- Nonsingular matrices. Rank of a matrix

Non-singular matrices

Definition

A coefficient matrix A is called **non-singular** if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every choice of \mathbf{b} .

Theorem

An $m \times n$ matrix A is non-singular if and only if $m = n$ and A has m pivot columns

Proof:

- should be no free variables (otherwise nonuniqueness)
- an echelon form of A should contain no zero rows (otherwise nonexistence)



Example

$$\begin{pmatrix} 1 & 2 & a \\ 3 & b & 0 \\ c & 0 & 0 \end{pmatrix} \xrightarrow{c \neq 0} \begin{pmatrix} 1 & 2 & a \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow{b \neq 0} \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

Thus A is non-singular $\iff abc \neq 0$

Rank of a matrix

Definition (Rank; baby version)

The number of nonzero rows in an echelon form of a matrix A is called its **rank** and denoted $\text{rank } A$.

Theorem (Some properties of the rank)

Assume that A is the $m \times n$ coefficient matrix of a linear system and \hat{A} is the corresponding augmented matrix. Then

- ① $\text{rank } A \leq \min\{m, n\}$ and $\text{rank } A \leq \text{rank } \hat{A}$
- ② $A\mathbf{x} = \mathbf{b}$ exists $\iff \text{rank } A = \text{rank } \hat{A}$
- ③ $A\mathbf{x} = \mathbf{b}$ is consistent for every \mathbf{b} $\iff \text{rank } A = m$
- ④ $A\mathbf{x} = \mathbf{b}$ has at most one solution for every \mathbf{b} $\iff \text{rank } A = n$
- ⑤ A is nonsingular $\iff \text{rank } A = m = n$.

Summary:

We discussed

- solutions to linear systems; equivalent systems
- elementary row transformations and equivalent matrices
- (reduced) row echelon form
- free and pivot variables
- consistent systems, solution set to (non-)homogeneous systems
- non-singular systems, rank of a matrix
- ...
- ... (much more to come)