Linear Algebra
Seminar 9: Orthogonalization
(make sure you can write on the board)

Problem 1. (3pt)

(a) Find the projection $P_W\mathbf{x}$ of the vector $\mathbf{x} = (1, 2, 0, -1)^T$ onto the subspace W formed by the orthogonal set of vectors $\mathbf{v}_1 = (1, 1, 1, 1)^T$, $\mathbf{v}_2 = (1, 1, -1, -1)^T$, $\mathbf{v}_3 = (1, -1, -1, 1)^T$

(b) Use the Gram–Schmidt process to transform the basis $\mathbf{u}_1 = (1, 0, 1)^T$, $\mathbf{u}_2 = (1, 3, -2)^T$, $\mathbf{u}_3 = (0, 2, 1)^T$ into an orthonormal basis.

Problem 2. (3pt)

(a) Find the projection $P_W\mathbf{x}$ of the vector $\mathbf{x} = (1, 2, 0, -2)^T$ onto the subspace W formed by the orthonormal set of vectors
$\mathbf{v}_1 = (0, 1/\sqrt{18}, -4/\sqrt{18}, -1/\sqrt{18})^T$, $\mathbf{v}_2 = (1/2, 5/6, 1/6, 1/6)^T$, $\mathbf{v}_3 = (1/\sqrt{18}, 0, 1/\sqrt{18}, -4/\sqrt{18})^T$.

(b) Use the Gram–Schmidt orthogonalization procedure to construct an orthonormal set $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of the space $L_2(0, 1)$ starting from $\mathbf{u}_1 = x$, $\mathbf{u}_2 = x^2$, $\mathbf{u}_3 = x^2 + x + 1$.

Problem 3. (3pt)

(a) Find the QR-decomposition of the matrix using the Gram–Schmidt algorithm:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 4 \end{pmatrix}$$

(b) Find the unit vector $\mathbf{u} \in \mathbb{R}^3$ such that the *Householder reflection* $Q_\mathbf{u} := I - 2\mathbf{u}\mathbf{u}^\top$ maps the vector $(1, 0, 1)^\top$ onto a vector collinear to $(1, 0, 0)^\top$

(c) explain how $Q_\mathbf{u}$ helps to derive the QR factorization of the matrix in part (a).

(d) Using the QR-factorization, find the least squares solution to $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (1\,1\,1)^\top$.

Problem 4. (3pt)

(a) Find the QR-decomposition of the matrices using the Gram–Schmidt algorithm:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

(b) Find the QR-factorization of matrices using the Householder reflections approach.

Problem 5. (4pt)

(a) Find the trigonometric polynomial in $1, \cos x, \sin x, \cos(2x), \sin(2x)$ that gives the best approximation of the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi; \\ 1, & \pi < x < 2\pi \end{cases}$$

in the norm of $L_2(0, 2\pi)$.

(b) Find the complex trigonometric polynomial in e^{ikx}, $k = -2, \dots, 2$, that gives the best approximation of the above function f in the norm of $L_2(0, 2\pi)$.

Problem 6. (4pt)

(a) Find the Fourier series of the function f of Problem 5;

(b) Find the complex Fourier series of the above function f.

Extra problems

(to be discussed if time permits)

Problem 7. Assume that the columns of a matrix A are linearly independent and $A = QR$ is the QR -factorization of A factorization. Prove that the diagonal entries of R are nonzero.

Problem 8. A transformation Q in $\mathbb{R}^n$ is called *orthogonal* if $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

(a) Assume that $\mathbf{u} \in \mathbb{R}^n$ is of unit length. Prove that $U := I - 2\mathbf{u}\mathbf{u}^\top$ is orthogonal.

(b) Explain why U is called a *reflection* (it is called the Householder reflection)

(c) For given nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, find the Householder reflection U such that $U\mathbf{a}$ is collinear to $\mathbf{b}$

Problem 9. Prove that an orthogonal matrix that is also upper-triangular must be diagonal.

1. a) $\vec x = (1,2,0,-1)^T$, $\vec v_1 = (1,1,1,1)^T$, $\vec v_2 = (1,1,-1,-1)^T$, $\vec v_3 = (1,-1,-1,1)^T$

the projection $P_W\vec x = \frac{\vec v_1^T\vec x}{\|\vec v_1\|^2}\cdot\vec v_1 + \frac{\vec v_2^T\vec x}{\|\vec v_2\|^2}\cdot\vec v_2 + \frac{\vec v_3^T\vec x}{\|\vec v_3\|^2}\cdot\vec v_3 =$

$$= \frac{(1,1,1,1)\begin{pmatrix}1\\2\\0\\-1\end{pmatrix}}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{(1,1,-1,-1)\begin{pmatrix}1\\2\\0\\-1\end{pmatrix}}{4}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} + \frac{(1,-1,-1,1)\begin{pmatrix}1\\2\\0\\-1\end{pmatrix}}{4}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} =$$

$$= \frac12\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} - \frac12\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = \frac12\begin{pmatrix}1+2-1\\1+2+1\\1-2+1\\1-2-1\end{pmatrix} = \begin{pmatrix}1\\2\\0\\-1\end{pmatrix}$$

b) $\vec u_1 = (1,0,1)^T$, $\vec u_2 = (1,3,-2)^T$, $\vec u_3 = (0,2,1)^T$

$\vec w_1 = \vec u_1 = (1,0,1)^T$

$$\vec w_2 = \vec u_2 - \frac{\vec u_2^T\vec w_1}{\vec w_1^T\vec w_1}\vec w_1 = \begin{pmatrix}1\\3\\-2\end{pmatrix} - \frac{(1,3,-2)\begin{pmatrix}1\\0\\1\end{pmatrix}}{(1,0,1)\begin{pmatrix}1\\0\\1\end{pmatrix}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + \frac12\begin{pmatrix}1\\0\\1\end{pmatrix} = \frac12\begin{pmatrix}3\\6\\-3\end{pmatrix} = \frac32\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\vec w_3 = \vec u_3 - \frac{\vec u_3^T\vec w_1}{\vec w_1^T\vec w_1}\vec w_1 - \frac{\vec u_3^T\vec w_2}{\vec w_2^T\vec w_2}\vec w_2 = \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac{(0,2,1)\begin{pmatrix}1\\0\\1\end{pmatrix}}{(1,0,1)\begin{pmatrix}1\\0\\1\end{pmatrix}}\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac{(0,2,1)\cdot\cancel{\frac32}\begin{pmatrix}1\\2\\-1\end{pmatrix}}{\cancel{\frac32}(1,2,-1)\cdot\cancel{\frac32}\begin{pmatrix}1\\2\\-1\end{pmatrix}}\cdot\cancel{\frac32}\begin{pmatrix}1\\2\\-1\end{pmatrix} =$$

$$= \begin{pmatrix}0\\2\\1\end{pmatrix} - \frac12\begin{pmatrix}1\\0\\1\end{pmatrix} - \frac12\begin{pmatrix}1\\2\\-1\end{pmatrix} = \frac12\begin{pmatrix}-2\\2\\2\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

Orthogonal Basis $W = \left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \frac32\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$

$\|\vec w_1\| = \sqrt{1+1} = \sqrt2$, $\|\vec w_2\| = \frac32\sqrt{1+4+1} = \frac{3\sqrt6}{2}$, $\|\vec w_3\| = \sqrt{1+1+1} = \sqrt3$

orthonormal Basis $W' = \left\{\frac{1}{\sqrt2}\begin{pmatrix}1\\0\\1\end{pmatrix}, \frac{1}{\sqrt6}\begin{pmatrix}1\\2\\-1\end{pmatrix}, \frac{1}{\sqrt3}\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$

2. a) $\vec x = (1,2,0,-2)^T$

$\vec v_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)^T = \left(0, \frac{\sqrt2}{6}, -4\cdot\frac{\sqrt2}{6}, -\frac{\sqrt2}{6}\right)^T = \frac{\sqrt2}{6}(0,1,-4,-1)^T$

$\vec v_2 = \left(\frac12, \frac56, \frac16, \frac16\right)^T = \frac16(3,5,1,1)^T$

$\vec v_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)^T = \left(\frac{\sqrt2}{6}, 0, \frac{\sqrt2}{6}, -4\cdot\frac{\sqrt2}{6}\right)^T = \frac{\sqrt2}{6}(1,0,1,-4)^T$

$P_W\vec x = \frac{\vec v_1^T\vec x}{\|\vec v_1\|^2}\cdot\vec v_1 + \frac{\vec v_2^T\vec x}{\|\vec v_2\|^2}\cdot\vec v_2 + \frac{\vec v_3^T\vec x}{\|\vec v_3\|^2}\cdot\vec v_3 =$

$$= \frac{\frac{\sqrt2}{6}(0,1,-4,-1)\begin{pmatrix}1\\2\\0\\-2\end{pmatrix}}{1}\cdot\frac{\sqrt2}{6}\begin{pmatrix}0\\1\\-4\\-1\end{pmatrix} + \frac{\frac16(3,5,1,1)\begin{pmatrix}1\\2\\0\\-2\end{pmatrix}}{1}\cdot\frac16\begin{pmatrix}3\\5\\1\\1\end{pmatrix} + \frac{\frac{\sqrt2}{6}(1,0,1,-4)\begin{pmatrix}1\\2\\0\\-2\end{pmatrix}}{1}\cdot\frac{\sqrt2}{6}\begin{pmatrix}1\\0\\1\\-4\end{pmatrix} =$$

$$= \frac{\sqrt2}{6}\cdot4\cdot\frac{\sqrt2}{6}\begin{pmatrix}0\\1\\-4\\-1\end{pmatrix} + \frac16\cdot11\cdot\frac16\begin{pmatrix}3\\5\\1\\1\end{pmatrix} + \frac{\sqrt2}{6}\cdot9\cdot\frac{\sqrt2}{6}\begin{pmatrix}1\\0\\1\\-4\end{pmatrix} = \frac{1}{36}\left(8\begin{pmatrix}0\\1\\-4\\-1\end{pmatrix} + 11\begin{pmatrix}3\\5\\1\\1\end{pmatrix} + 18\begin{pmatrix}1\\0\\1\\-4\end{pmatrix}\right) =$$

$$= \frac{1}{36}\begin{pmatrix}33+18\\8+55\\-32+11+18\\-8+11-72\end{pmatrix} = \frac{1}{36}\begin{pmatrix}51\\63\\-3\\-69\end{pmatrix} = \frac{1}{12}\begin{pmatrix}17\\21\\-1\\-23\end{pmatrix}$$

b) $u_1 = x$, $u_2 = x^2$, $u_3 = x^2+x+1$

$\langle f_1, f_2\rangle = \int_0^1 f_1(t)f_2(t)\,dt$

$w_1 = u_1 = x$

$$w_2 = u_2 - \frac{\langle u_2, w_1\rangle}{\langle w_1, w_1\rangle}\cdot w_1 = x^2 - \frac{\int_0^1 x^2\cdot x\,dx}{\int_0^1 x\cdot x\,dx}\cdot x = x^2 - \frac{1/4}{1/3}x = x^2 - \frac34 x.$$

$$w_3 = u_3 - \frac{\langle u_3, w_1\rangle}{\langle w_1, w_1\rangle}w_1 - \frac{\langle u_3, w_2\rangle}{\langle w_2, w_2\rangle}w_2 =$$

$$= x^2+x+1 - \frac{\int_0^1(x^3+x^2+x)dx}{\int_0^1 x^2dx}\cdot x - \frac{\int_0^1(x^2+x+1)(x^2-\frac34x)dx}{\int_0^1(x^2-\frac34x)^2dx}\cdot\left(x^2-\frac34x\right) =$$

$= \int_0^1\left(x^4+\frac14x^3+\frac14x^2-\frac34x\right)dx$

$= \int_0^1\left(x^4-\frac32x^3+\frac{9}{16}x^2\right)dx$

$$= x^2+x+1 - \frac{13/12}{1/3}x + \frac{7/240}{1/80}\left(x^2-\frac34x\right) = x^2+x+1 - \frac{13}{4}x + \frac73\left(x^2-\frac34x\right) =$$

$$= x^2+\frac73x^2 + x - \frac{13}{4}x - \frac74x + 1 = \frac{10}{3}x^2 - 4x + 1$$

$\|w_1\| = \sqrt{\int_0^1 w_1w_1dx} = \sqrt{\int_0^1 x^2dx} = \frac{1}{\sqrt3}$

$\|w_2\| = \sqrt{\int_0^1 w_2w_2dx} = \sqrt{\int_0^1\left(x^2-\frac34x\right)^2dx} = \sqrt{\int_0^1\left(x^4-\frac32x^3+\frac{9}{16}x^2\right)dx} = \frac{1}{\sqrt{80}} = \frac{1}{4\sqrt5}$

$\|w_3\| = \sqrt{\int_0^1\left(\frac{10}{3}x^2-4x-1\right)^2dx} = \sqrt{\int_0^1\left(\frac{100}{9}x^4-\frac{10}{3}x^3+\frac{61}{3}x^2-8x+1\right)dx} = \sqrt{\frac{20}{9}-\frac{20}{3}+\frac{61}{9}-4+1} = \sqrt{\frac13}$

orthogonal set: $\left\{\begin{matrix}x,\\ x^2-\frac34x\\ \frac{10}{3}x^2-4x+1\end{matrix}\right\}$ $\Rightarrow$ orthonormal set: $\left\{\begin{matrix}\sqrt3x\\ 4\sqrt5\left(x^2-\frac34x\right)\\ \sqrt3\left(\frac{10}{3}x^2-4x-1\right)\end{matrix}\right\}$

3. a)
$$A = \begin{array}{c} \begin{matrix}\vec a_1 & \vec a_2\end{matrix}\\ \begin{pmatrix}1&2\\0&1\\1&4\end{pmatrix}\end{array}$$

$\vec w_1 = \vec a_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

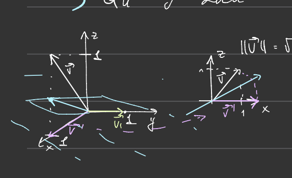
$$\vec w_2 = \vec a_2 - \frac{\vec a_2^T\vec w_1}{\vec w_1^T\vec w_1}\cdot\vec w_1 = \begin{pmatrix}2\\1\\4\end{pmatrix} - \frac{(2,1,4)\begin{pmatrix}1\\0\\1\end{pmatrix}}{(1,0,1)\begin{pmatrix}1\\0\\1\end{pmatrix}}\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix} - \frac62\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2-3\\1\\4-3\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$W = \begin{pmatrix}1&-1\\0&1\\1&1\end{pmatrix} \Rightarrow Q = \begin{pmatrix}\sqrt2/2 & -\sqrt3/3\\ 0 & \sqrt3/3\\ \sqrt2/2 & \sqrt3/3\end{pmatrix}$

$R = Q^TA = \begin{pmatrix}\sqrt2/2 & 0 & \sqrt2/2\\ -\sqrt3/3 & \sqrt3/3 & \sqrt3/3\end{pmatrix}\cdot\begin{pmatrix}1&2\\0&1\\1&4\end{pmatrix} = \begin{pmatrix}\sqrt2 & 3\sqrt2\\ 0 & \sqrt3\end{pmatrix}$

check: $A = QR = \begin{pmatrix}\sqrt2/2 & -\sqrt3/3\\ 0 & \sqrt3/3\\ \sqrt2/2 & \sqrt3/3\end{pmatrix}\begin{pmatrix}\sqrt2 & 3\sqrt2\\ 0 & \sqrt3\end{pmatrix} = \begin{pmatrix}1&2\\0&1\\1&4\end{pmatrix}$ ✓

b) $Q_{\vec u} = I - 2\vec u\vec u^T$



$\|\vec v\| = \sqrt2$ (Pythagorean theorem). An orthogonal matrix Q does not change length of vectors:

$\vec v \mapsto \vec v'$: $(1,0,1)^T \mapsto (\sqrt2,0,0)^T$

$\vec v' \mapsto \vec v$: $(\sqrt2,0,0)^T \mapsto (1,0,1)^T$

$\vec v \mapsto \vec v$: $(0,1,0)^T \mapsto (0,1,0)^T$

$Q\begin{pmatrix}1&\sqrt2&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}\sqrt2&1&0\\0&0&1\\0&1&0\end{pmatrix} \Rightarrow Q = \begin{pmatrix}\sqrt2&1&0\\0&0&1\\0&1&0\end{pmatrix}\cdot\begin{pmatrix}1&\sqrt2&0\\0&0&1\\1&0&0\end{pmatrix}^{-1} = $ [illegible]

$Q = I - 2\vec u\vec u^T$ [illegible]

$\vec u\vec u^T = \frac{I-Q}{2} = $ [illegible] $= \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}(u_1,u_2,u_3) = \begin{pmatrix}u_1^2 & u_1u_2 & u_1u_3\\ u_2u_1 & u_2^2 & u_2u_3\\ u_3u_1 & u_3u_2 & u_3^2\end{pmatrix} \Rightarrow u_2 = 0$

$\vec u\cdot u_1 = \frac14(2-\sqrt2, 0, -\sqrt2)$, $u_1^2 = \frac{2-\sqrt2}{4} \Rightarrow u_1 = \pm\frac12\sqrt{2-\sqrt2}$ [illegible]

$\vec u\cdot u_3 = \frac14(-\sqrt2, 0, 2+\sqrt2)$, $u_3^2 = \frac{2+\sqrt2}{4} \Rightarrow u_3 = \pm\frac12\sqrt{2+\sqrt2}$ [illegible]

$$\vec u = \pm\frac12\left(\sqrt{2-\sqrt2}, 0, -\sqrt{2+\sqrt2}\right).$$

Check whether $\vec u$ is unit: $\|\vec u\| = \sqrt{\frac14\left(2-\sqrt2+2+\sqrt2\right)} = 1.$

c) $Q_uA = Q_u(\vec a_1\ \vec a_2) = (Q_u\vec a_1, Q_u\vec a_2) = \begin{pmatrix}\sqrt2 & *\\ 0 & *\\ 0 & *\end{pmatrix}$ → apply transformation to change any $\begin{pmatrix}*\\ *\end{pmatrix}$: $\begin{pmatrix}1&0&0\\0&\tilde Q_2&\\0&&\end{pmatrix}$, $\tilde a_2 \to \begin{pmatrix}*\\0\end{pmatrix}$

$A_1 = Q_uA = $ [illegible] make $=0$ for R-form → apply [illegible]

$Q_2Q_uA = $ [illegible] $= R$

$R\ R^TA = QR \Rightarrow A = QR$

d) $\vec b = (1,1,1)^T$

$$\hat x = (A^TA)^{-1}A^T\vec b = \left[(QR)^T\cdot(QR)\right]^{-1}\cdot(QR)^{-1}\cdot\vec b = (R^TQ^TQR)^{-1}R^TQ^T\vec b = \left|R^TQ = I_n\right| = (R^TR)^{-1}\cdot R^TQ^T\vec b =$$

$$= R^{-1}(R^T)^{-1}R^TQ^T\vec b = R^{-1}Q^T\vec b$$

$$\left[R\,|\,I_n\right] = \left[\begin{array}{cc|cc}\sqrt2 & 3\sqrt2 & 1 & 0\\ 0 & \sqrt3 & 0 & 1\end{array}\right]\begin{matrix}\cdot\frac{1}{\sqrt2}\\ \cdot\sqrt3\end{matrix} \sim \left[\begin{array}{cc|cc}1 & 3 & \frac{1}{\sqrt2} & 0\\ 0 & 3 & 0 & \sqrt3\end{array}\right]\begin{matrix}-r_2\\ \cdot\frac13\end{matrix} \sim \left[\begin{array}{cc|cc}1 & 0 & \frac{1}{\sqrt2} & -\sqrt3\\ 0 & 1 & 0 & \frac{\sqrt3}{3}\end{array}\right]$$

R^{-1}

$Q = \begin{pmatrix}\sqrt2/2 & -\sqrt3/3\\ 0 & \sqrt3/3\\ \sqrt2/2 & \sqrt3/3\end{pmatrix} \Rightarrow Q^T = \begin{pmatrix}\frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2}\\ -\frac{\sqrt3}{3} & \frac{\sqrt3}{3} & \frac{\sqrt3}{3}\end{pmatrix}$

$$\hat x = \begin{pmatrix}1/\sqrt2 & -\sqrt3\\ 0 & \frac{\sqrt3}{3}\end{pmatrix}\begin{pmatrix}\frac{\sqrt2}{2} & 0 & \frac{\sqrt2}{2}\\ -\frac{\sqrt3}{3} & \frac{\sqrt3}{3} & \frac{\sqrt3}{3}\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3/2 & -1 & -1/2\\ -1/3 & 1/3 & 1/3\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1/3\end{pmatrix}$$

4. a)
$$A = \begin{array}{c}\begin{matrix}\vec a_1 & \vec a_2 & \vec a_3\end{matrix}\\ \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix}\end{array}$$

$\vec w_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$$\vec w_2 = \vec a_2 - \frac{\vec a_2^T\cdot\vec w_1}{\vec w_1^T\cdot\vec w_1}\cdot\vec w_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{(0,1,0)\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}{(1,0,2)\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\vec w_3 = \vec a_3 - \frac{\vec a_3^T\cdot\vec w_1}{\vec w_1^T\cdot\vec w_1}\cdot\vec w_1 - \frac{\vec a_3^T\cdot\vec w_2}{\vec w_2^T\vec w_2}\cdot\vec w_2 = \begin{pmatrix}2\\1\\1\end{pmatrix} - \frac{(2,1,1)\begin{pmatrix}1\\0\\2\end{pmatrix}}{(1,0,2)\begin{pmatrix}1\\0\\2\end{pmatrix}}\cdot\begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{(2,1,1)\begin{pmatrix}0\\1\\0\end{pmatrix}}{(0,1,0)\begin{pmatrix}0\\1\\0\end{pmatrix}}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix} - \frac45\begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \frac35\begin{pmatrix}2\\0\\-1\end{pmatrix}$$

$W = \left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \frac35\begin{bmatrix}2\\0\\-1\end{bmatrix}\right\}$

$\|\vec w_1\| = \sqrt5$, $\|\vec w_2\| = 1$, $\|\vec w_3\| = \frac{3\sqrt5}{5}$

$Q = W' = \left\{\frac{\sqrt5}{5}\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \frac{\sqrt5}{5}\begin{bmatrix}2\\0\\-1\end{bmatrix}\right\}$

$Q = \frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}$

$R = Q^TA = \frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix} = \frac{\sqrt5}{5}\begin{pmatrix}5&0&4\\0&\sqrt5&\sqrt5\\0&0&3\end{pmatrix}$

check: $A = QR = \frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}\cdot\frac{\sqrt5}{5}\begin{pmatrix}5&0&4\\0&\sqrt5&\sqrt5\\0&0&3\end{pmatrix} = \frac15\begin{vmatrix}5&0&10\\0&5&5\\10&0&5\end{vmatrix} = \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix}$ ✓

$$B = \begin{array}{c}\begin{matrix}\vec b_1 & \vec b_2 & \vec b_3\end{matrix}\\ \begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix}\end{array}$$

$\vec w_1 = \vec b_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$$\vec w_2 = \vec b_2 - \frac{\vec b_2^T\cdot\vec w_1}{\vec w_1^T\vec w_1}\vec w_1 = \begin{pmatrix}0\\1\\0\end{pmatrix} - \frac{(0,1,0)\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}{(1,0,2)\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}}\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\vec w_3 = \vec b_3 - \frac{\vec b_3^T\cdot\vec w_1}{\vec w_1^T\cdot\vec w_1}\cdot\vec w_1 - \frac{\vec b_3^T\cdot\vec w_2}{\vec w_2^T\cdot\vec w_2}\cdot\vec w_2 = \begin{pmatrix}1\\1\\2\end{pmatrix} - \frac{(1,1,2)\begin{pmatrix}1\\0\\2\end{pmatrix}}{(1,0,2)\begin{pmatrix}1\\0\\2\end{pmatrix}}\begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{(1,1,2)\begin{pmatrix}0\\1\\0\end{pmatrix}}{(0,1,0)\begin{pmatrix}0\\1\\0\end{pmatrix}}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$W = \left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$W' = \left\{\frac{1}{\sqrt5}\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$Q = \frac{\sqrt5}{5}\begin{pmatrix}1&0\\0&\sqrt5\\2&0\end{pmatrix}$

$R = Q^TB = \frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix} = \frac{\sqrt5}{5}\begin{pmatrix}5&0&5\\0&\sqrt5&\sqrt5\end{pmatrix} = \begin{pmatrix}\sqrt5&0&\sqrt5\\0&1&1\end{pmatrix} = \sqrt5\begin{pmatrix}1&0&1\\0&\frac{1}{\sqrt5}&\frac{1}{\sqrt5}\end{pmatrix}$

check: $B = QR = \frac{\sqrt5}{5}\begin{pmatrix}1&0\\0&\sqrt5\\2&0\end{pmatrix}\cdot\sqrt5\begin{pmatrix}1&0&1\\0&\frac{1}{\sqrt5}&\frac{1}{\sqrt5}\end{pmatrix} = \begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix}$ ✓

b)
$$A = \begin{array}{c}\begin{matrix}\vec a_1 & \vec a_2 & \vec a_3\end{matrix}\\ \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix}\end{array}$$

$A_1 = \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix}$, $\vec a_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\vec v_1 = \vec a_1 + \text{sign}(a_{11})\|\vec a_1\|\vec e_1 = \begin{pmatrix}1\\0\\2\end{pmatrix} + \sqrt5\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1+\sqrt5\\0\\2\end{pmatrix}$

$$H_1 = I_n - 2\frac{\vec v_1\cdot\vec v_1^T}{\vec v_1^T\cdot\vec v_1} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\cdot\frac{\begin{pmatrix}1+\sqrt5\\0\\2\end{pmatrix}(1+\sqrt5,0,2)}{(1+\sqrt5,0,2)\begin{pmatrix}1+\sqrt5\\0\\2\end{pmatrix}} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{\cancel{2}}{\cancel{2}(\sqrt5+5)}\begin{pmatrix}2\sqrt5+6&0&2+2\sqrt5\\0&0&0\\2+2\sqrt5&0&4\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}\frac{2\sqrt5+6}{\sqrt5+5}&0&\frac{2\sqrt5+2}{\sqrt5+5}\\0&0&0\\\frac{2\sqrt5+2}{\sqrt5+5}&0&\frac{4}{\sqrt5+5}\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{\sqrt5+5-2\sqrt5-6}{\sqrt5+5}&0&-\frac{2(\sqrt5+1)}{\sqrt5+5}\\0&1&0\\-\frac{2(\sqrt5+1)}{\sqrt5+5}&0&\frac{\sqrt5+5-4}{\sqrt5+5}\end{pmatrix} = \begin{pmatrix}\frac{-(\sqrt5+1)}{\sqrt5+5}&0&\frac{-2(\sqrt5+1)}{\sqrt5+5}\\0&1&0\\\frac{-2(\sqrt5+1)}{\sqrt5+5}&0&\frac{\sqrt5+1}{\sqrt5+5}\end{pmatrix} = -\frac{\sqrt5+1}{\sqrt5+5}\begin{pmatrix}1&0&2\\0&-\frac{\sqrt5+5}{\sqrt5+1}&0\\2&0&-1\end{pmatrix} = \left|\frac{\sqrt5+1}{\sqrt5+5}\cdot\frac{\sqrt5-1}{\sqrt5-1} = \frac{5-1}{5-\sqrt5+5\sqrt5-5} = \frac{4}{4\sqrt5} = \frac{\sqrt5}{5}\right| =$$

$$= -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}$$

$A_2 = H_1\cdot A_1 = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}5&0&4\\0&-\sqrt5&-\sqrt5\\0&0&3\end{pmatrix} \Rightarrow A_2' = -\frac{\sqrt5}{5}\begin{pmatrix}-\sqrt5&-\sqrt5\\0&3\end{pmatrix} = \begin{pmatrix}1&1\\0&-\frac{3\sqrt5}{5}\end{pmatrix}$, $\vec a_2 = \begin{pmatrix}1\\0\end{pmatrix}$

$\vec v_2 = \vec a_2 + \text{sign}(a_{11})\|\vec a_2\|\vec e_1 = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$H_2 = I_n - 2\frac{\vec v_2\vec v_2^T}{\vec v_2^T\vec v_2} = \begin{pmatrix}1&0\\0&1\end{pmatrix} - 2\cdot\frac{\begin{pmatrix}2\\0\end{pmatrix}(2,0)}{(2,0)\begin{pmatrix}2\\0\end{pmatrix}} = \begin{pmatrix}1&0\\0&1\end{pmatrix} - \frac12\begin{pmatrix}4&0\\0&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$

$A_3 = H_2\cdot A_2 = \begin{pmatrix}-1&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&-\frac{3\sqrt5}{5}\end{pmatrix} = \begin{pmatrix}-1&-1\\0&-\frac{3\sqrt5}{5}\end{pmatrix} \Rightarrow A_3' = \left(-\frac{3\sqrt5}{5}\right)$, $\vec a_3 = \left(-\frac{3\sqrt5}{5}\right)$

$\vec v_3 = \vec a_3 + \text{sign}(a_{11})\|\vec a_3\|\vec e_1 = \left(-\frac{3\sqrt5}{5}\right) - \frac{3}{\sqrt5}(1) = \frac{-3\sqrt5-3\sqrt5}{5} = \left(-\frac{\sqrt5}{5}\cdot6\right)$

$H_3 = I_n - 2\frac{\vec v_3\vec v_3^T}{\vec v_3^T\vec v_3} = (1) - 2 = (-1)$

$Q = H_1H_2H_3 = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&-2\\0&\sqrt5&0\\2&0&1\end{pmatrix}$

$R = H_3H_2H_1A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\cdot\left(-\frac{\sqrt5}{5}\right)\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\-2&0&1\end{pmatrix}\begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix} =$

$= -\frac{\sqrt5}{5}\begin{pmatrix}5&0&4\\0&\sqrt5&\sqrt5\\0&0&-3\end{pmatrix}$

check: $A = QR = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&-2\\0&\sqrt5&0\\2&0&1\end{pmatrix}\cdot\left(-\frac{\sqrt5}{5}\right)\begin{pmatrix}5&0&4\\0&\sqrt5&\sqrt5\\0&0&-3\end{pmatrix} = \frac15\begin{pmatrix}5&0&10\\0&5&5\\10&0&5\end{pmatrix} = \begin{pmatrix}1&0&2\\0&1&1\\2&0&1\end{pmatrix}$ ✓

$B = \begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix}$

$B_1 = \begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix}$, $\vec b_1 = \begin{pmatrix}1\\0\\2\end{pmatrix}$

$\vec v_1 = \vec b_1 + \text{sign}(b_{11})\|\vec b_1\|\vec e_1 = \begin{pmatrix}1\\0\\2\end{pmatrix} + \sqrt5\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1+\sqrt5\\0\\2\end{pmatrix}$

$$H_1 = I_n - 2\frac{\vec v_1\vec v_1^T}{\vec v_1^T\vec v_1} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - 2\cdot\frac{\begin{pmatrix}1+\sqrt5\\0\\2\end{pmatrix}(1+\sqrt5,0,2)}{(1+\sqrt5,0,2)\begin{pmatrix}1+\sqrt5\\0\\2\end{pmatrix}} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{\cancel{2}}{\cancel{2}(\sqrt5+5)}\begin{pmatrix}2\sqrt5+6&0&2+2\sqrt5\\0&0&0\\2+2\sqrt5&0&4\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \begin{pmatrix}\frac{2\sqrt5+6}{\sqrt5+5}&0&\frac{2\sqrt5+2}{\sqrt5+5}\\0&0&0\\\frac{2\sqrt5+2}{\sqrt5+5}&0&\frac{4}{\sqrt5+5}\end{pmatrix} =$$

$$= \begin{pmatrix}\frac{\sqrt5+5-2\sqrt5-6}{\sqrt5+5}&0&-\frac{2(\sqrt5+1)}{\sqrt5+5}\\0&1&0\\-\frac{2(\sqrt5+1)}{\sqrt5+5}&0&\frac{\sqrt5+5-4}{\sqrt5+5}\end{pmatrix} = \begin{pmatrix}\frac{-(\sqrt5+1)}{\sqrt5+5}&0&\frac{-2(\sqrt5+1)}{\sqrt5+5}\\0&1&0\\\frac{-2(\sqrt5+1)}{\sqrt5+5}&0&\frac{\sqrt5+1}{\sqrt5+5}\end{pmatrix} = -\frac{\sqrt5+1}{\sqrt5+5}\begin{pmatrix}1&0&2\\0&-\frac{\sqrt5+5}{\sqrt5+1}&0\\2&0&-1\end{pmatrix} = \left|\frac{\sqrt5+1}{\sqrt5+5}\cdot\frac{\sqrt5-1}{\sqrt5-1} = \frac{5-1}{5-\sqrt5+5\sqrt5-5} = \frac{4}{4\sqrt5} = \frac{\sqrt5}{5}\right| =$$

$$= -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}$$

$B_2 = H_1\cdot B_1 = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}5&0&5\\0&-\sqrt5&-\sqrt5\\0&0&0\end{pmatrix} = \begin{pmatrix}-\sqrt5&0&-\sqrt5\\0&1&1\\0&0&0\end{pmatrix} \Rightarrow B_2' = \begin{pmatrix}1&1\\0&0\end{pmatrix}$, $\vec b_2 = \begin{pmatrix}1\\0\end{pmatrix}$

$\vec v_2 = \vec b_2 + \text{sign}(b_{11})\|\vec b_2\|\vec e_1 = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

$H_2 = I_n - 2\frac{\vec v_2\vec v_2^T}{\vec v_2^T\vec v_2} = \begin{pmatrix}1&0\\0&1\end{pmatrix} - 2\cdot\frac{\begin{pmatrix}2\\0\end{pmatrix}(2,0)}{(2,0)\begin{pmatrix}2\\0\end{pmatrix}} = \begin{pmatrix}1&0\\0&1\end{pmatrix} - \frac12\begin{pmatrix}4&0\\0&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$

$B_3 = H_2\cdot B_2 = \begin{pmatrix}-1&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix} = \begin{pmatrix}-1&-1\\0&0\end{pmatrix} \Rightarrow B_3' = (0)$, $\vec b_3 = (0)$

$\vec v_3 = \vec b_3 + \text{sign}(b_{11})\|\vec b_3\|\vec e_1 = (0)$

$H_3 = I_n - 2\frac{\vec v_3\vec v_3^T}{\vec v_3^T\vec v_3} = (1) - (0) = (1)$

$Q = H_1H_2H_3 = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}$

$R = H_1H_2H_3B = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\cdot\left(-\frac{\sqrt5}{5}\right)\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&2\\0&-\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&-1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix} = -\frac{\sqrt5}{5}\begin{pmatrix}5&0&5\\0&\sqrt5&\sqrt5\\0&0&0\end{pmatrix} =$

$= \begin{pmatrix}\sqrt5&0&\sqrt5\\0&1&1\\0&0&0\end{pmatrix}$

Check: $A = QR = -\frac{\sqrt5}{5}\begin{pmatrix}1&0&2\\0&\sqrt5&0\\2&0&1\end{pmatrix}\cdot\begin{pmatrix}\sqrt5&0&\sqrt5\\0&1&1\\0&0&0\end{pmatrix} = \frac{\sqrt5}{5}\begin{pmatrix}\sqrt5&0&\sqrt5\\0&\sqrt5&\sqrt5\\2\sqrt5&0&2\sqrt5\end{pmatrix} = \begin{pmatrix}1&0&1\\0&1&1\\2&0&2\end{pmatrix}$ ✓

$\boxed{5.}$ a) $\langle f,g\rangle = \int_0^{2\pi} f(t)\,g(t)\,dt$, $\|f\| = \sqrt{\langle f,f\rangle}$

$f(x) = \begin{cases} 0, & 0<x<\pi \\ 1, & \pi<x<2\pi \end{cases}$ – f

$f_1 = 1,\quad f_2 = \cos x,\quad f_3 = \sin x,\quad f_4 = \cos(2x),\quad f_5 = \sin(2x)$

the best approximation (the projection $\vec{p}$):

$$\vec{p} = \frac{\langle f_1,f\rangle}{\|f_1\|^2}\cdot f_1 + \frac{\langle f_2,f\rangle}{\|f_2\|^2}\cdot f_2 + \frac{\langle f_3,f\rangle}{\|f_3\|^2}\cdot f_3 + \frac{\langle f_4,f\rangle}{\|f_4\|^2}\cdot f_4 + \frac{\langle f_5,f\rangle}{\|f_5\|^2}\cdot f_5 = ① + ② + ③ + ④ + ⑤ = \frac{1}{2}\cdot 1 - \frac{2}{\pi}\cdot\sin x$$

① $\frac{\langle f_1,f\rangle}{\|f_1\|^2} = \frac{\int_0^{2\pi} 1\cdot f(x)\,dx}{\int_0^{2\pi} 1\,dx} = \left[\ \right] = \frac{\pi}{x\big|_0^{2\pi}} = \frac{\pi}{2\pi} = \frac{1}{2}$

② $\frac{\langle f_2,f\rangle}{\|f_2\|^2} = \frac{\int_0^{2\pi}\cos x\cdot f(x)\,dx}{\int_0^{2\pi}\cos^2 x\,dx} = \left[\begin{matrix} f_2\cdot f = \begin{cases}0, & 0<x<\pi\\ \cos x, & \pi<x<2\pi\end{cases} \\ \langle f_2,f\rangle = \int_0^{2\pi} f_2 f\,dx = \int_\pi^{2\pi}\cos x\,dx \end{matrix}\right] = 0$

③ $\frac{\langle f_3,f\rangle}{\|f_3\|^2} = \frac{\int_0^{2\pi}\sin x\cdot f(x)\,dx}{\int_0^{2\pi}\sin^2 x\,dx} = \left[\begin{matrix} f_3\cdot f = \begin{cases}0, & 0<x<\pi\\ \sin x, & \pi<x<2\pi\end{cases} \\ \langle f_3,f\rangle = \int_0^{2\pi} f_3 f\,dx = \int_\pi^{2\pi}\sin x\,dx \end{matrix}\right] = \frac{\int_\pi^{2\pi}\sin x\,dx}{\int_0^{2\pi}\frac{1-\cos(2x)}{2}\,dx} = \frac{-\cos x\big|_\pi^{2\pi}}{\left(\frac{x}{2} - \frac{\sin(2x)}{4}\right)\Big|_0^{2\pi}} = \frac{-2}{\pi}$

④ $\frac{\langle f_4,f\rangle}{\|f_4\|^2} = \frac{\int_0^{2\pi}\cos(2x)\cdot f(x)\,dx}{\int_0^{2\pi}\cos^2(2x)\,dx} = \left[\ \right] = 0$

⑤ $\frac{\langle f_5,f\rangle}{\|f_5\|^2} = \frac{\int_0^{2\pi}\sin(2x)\cdot f(x)\,dx}{\int_0^{2\pi}\sin^2(2x)\,dx} = \left[\ \right] = 0$

b) $\langle f,g\rangle = \int_0^{2\pi} f(t)\,\overline{g(t)}\,dt$, $\|f\| = \sqrt{\langle f,\bar f\rangle}$

$f(x) = \begin{cases} 0, & 0<x<\pi \\ 1, & \pi<x<2\pi \end{cases}$ – f

$f_1 = e^{-2ix},\quad f_2 = e^{-ix},\quad f_3 = 1,\quad f_4 = e^{ix},\quad f_5 = e^{2ix}$

$$\vec{p} = \frac{\langle f_1,f\rangle}{\|f_1\|^2}\cdot f_1 + \frac{\langle f_2,f\rangle}{\|f_2\|^2}\cdot f_2 + \frac{\langle f_3,f\rangle}{\|f_3\|^2}\cdot f_3 + \frac{\langle f_4,f\rangle}{\|f_4\|^2}\cdot f_4 + \frac{\langle f_5,f\rangle}{\|f_5\|^2}\cdot f_5 = \frac{① + ② + ③ + ④ + ⑤}{2\pi} \ (=)$$

$\|f_n\|^2 = \int_0^{2\pi} f_n\cdot\bar{f_n}\,dx = \int_0^{2\pi} x = 2\pi$

① $\langle f_1,f\rangle = \int_\pi^{2\pi} e^{-2ix}\,dx = \frac{e^{-2ix}}{-2i}\Big|_\pi^{2\pi} = \frac{i}{2}e^{-2ix}\Big|_\pi^{2\pi} = \left| e^{ix} = \cos x + i\sin x \right| = \frac{i}{2}\left(\cos(-4\pi) + i\sin(-4\pi) - \cos(-2\pi) - i\sin(-2\pi)\right) = 0$

② $\langle f_2,f\rangle = \int_\pi^{2\pi} e^{-ix}\,dx = \frac{e^{-ix}}{-i}\Big|_\pi^{2\pi} = ie^{-ix}\Big|_\pi^{2\pi} = i\left(\cos(-2\pi) + i\sin(-2\pi) - \cos(-\pi) - i\sin(-\pi)\right) = 2i$

③ $\langle f_3,f\rangle = \int_\pi^{2\pi} 1\,dx = x\big|_\pi^{2\pi} = \pi$

④ $\langle f_4,f\rangle = \int_\pi^{2\pi} e^{ix}\,dx = \frac{e^{ix}}{i}\Big|_\pi^{2\pi} = -ie^{ix}\Big|_\pi^{2\pi} = -i\left(\cos(2\pi) + i\sin(2\pi) - \cos(\pi) - i\sin(\pi)\right) = -2i$

⑤ $\langle f_5,f\rangle = \int_\pi^{2\pi} e^{2ix}\,dx = \frac{e^{2ix}}{2i} = \frac{-i}{2}e^{2ix}\Big|_\pi^{2\pi} = \frac{-i}{2}\left(\cos(4\pi) + i\sin(4\pi) - \cos(2\pi) - i\sin(2\pi)\right) = 0$

$(=)\ \frac{2i}{2\pi}\cdot e^{-ix} + \frac{\pi}{2\pi}\cdot 1 + \frac{-2i}{2\pi}\cdot e^{ix} = \frac{1}{2} + \frac{i}{\pi}e^{-ix} - \frac{i}{\pi}e^{ix}$

$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2} \Rightarrow \frac{1}{2} - \frac{2}{\pi}\sin x = \frac{1}{2} - \frac{e^{ix}}{\pi} + \frac{e^{-ix}}{\pi}$

$\boxed{6.}$ a) $f(x) = \begin{cases} 0, & 0<x<\pi \\ 1, & \pi<x<2\pi \end{cases}$

The Fourier series:

1. Looks at functions over an interval as a vector space with an inner product
2. Picks an orthonormal basis for the space
3. Represents an arbitrary function in this basis by projecting it out on the basis

1) We define a vector space V with the scalars $\in\mathbb{R}$ and the vectors are functions over the interval $[0, 2\pi]$ with the inner product defined as: $\langle f,g\rangle := \int_0^{2\pi} f(t)g(t)\,dt$

2) Build an orthonormal basis. Our basis will be a set of functions, and it will be infinite (V is infinite-dimensional)

$B = \left\{ \begin{matrix} 1(x) \\ \sin(x), \sin(2x), \dots \\ \cos(x), \cos(2x), \dots \end{matrix} \right\}$

To show this is orthogonal set: $\langle f,g\rangle = 0$:

1) $f = 1(x),\ g = \sin(kx)$: $\langle f,g\rangle = \int_0^{2\pi}\sin(kx)\,dx = \frac{-\cos(kx)}{k}\Big|_0^{2\pi} = \frac{-\overset{1 \text{ for } k\in\mathbb{N}}{\cos(2\pi k)} + 1}{k} = 0$

2) $f = 1(x),\ g = \cos(kx)$: $\langle f,g\rangle = \int_0^{2\pi}\cos(kx)\,dx = \frac{\sin(kx)}{k}\Big|_0^{2\pi} = \frac{\overset{0 \text{ for } k\in\mathbb{N}}{\sin(2\pi k)}}{k} = 0$

3) $f = \sin(kx),\ g = \cos(nx)$: $\langle f,g\rangle = \int_0^{2\pi}\sin(kx)\cdot\cos(nx)\,dx = \left[\sin x\cdot\cos y = \frac{\sin(x+y)}{2} - \frac{\sin(x-y)}{2}\right] = \frac{1}{2}\int_0^{2\pi}\sin(kx+nx)\,dx - \frac{1}{2}\int_0^{2\pi}\cos(kx-nx)\,dx =$

$= \left(\frac{-1}{2(k+n)}\cos(kx+nx) - \frac{-1}{2(k-n)}\cos(kx-nx)\right)\Big|_0^{2\pi} = \left(\frac{\cos(kx-nx)}{2(k-n)} - \frac{\cos(kx+nx)}{2(k+n)}\right)\Big|_0^{2\pi} =$

$= \frac{\cos(2\pi k - 2\pi n)}{2(k-n)} - \frac{\cos(2\pi k + 2\pi n)}{2(k+n)} - \frac{\cos(0)}{2(k-n)} + \frac{\cos(0)}{2(k+n)} = \frac{(k+n)\overset{1}{\cos(2\pi(k-n))} - (k-n)\overset{1}{\cos(2\pi(k+n))} - (k+n) + (k-n)}{2k^2 - 2n^2} =$

$= \frac{\cancel{k}+\cancel{n}-\cancel{k}+\cancel{n}-\cancel{k}-\cancel{n}+\cancel{k}-\cancel{n}}{2k^2 - 2n^2} = 0.$

Thus, B is actually orthogonal.

New, make it orthonormal:

1) $\|1(x)\| = \sqrt{\int_0^{2\pi} dx} = \sqrt{2\pi}$

2) $\|\sin(kx)\| = \sqrt{\langle\sin(kx), \sin(kx)\rangle} = \sqrt{\int_0^{2\pi}\sin^2(kx)\,dx} = \sqrt{\frac{1}{2k}\int(1-\cos(2kx))\,d(kx)} = \sqrt{\left(\frac{x}{2} - \frac{\sin(2kx)}{4k}\right)\Big|_0^{2\pi}} = \sqrt{\pi}$

3) $\|\cos(kx)\| = \sqrt{\langle\cos(kx), \cos(kx)\rangle} = \sqrt{\int_0^{2\pi}\cos^2(kx)\,dx} = \sqrt{\frac{1}{2k}\int(1+\cos(2kx))\,d(kx)} = \sqrt{\left(\frac{x}{2} + \frac{\sin(2kx)}{4k}\right)\Big|_0^{2\pi}} = \sqrt{\pi}$

$B = \left\{ \begin{matrix} \frac{1}{\sqrt{2\pi}}\,1(x) \\ \frac{1}{\sqrt{\pi}}(\sin x, \sin(2x), \dots \\ \frac{1}{\sqrt{\pi}}(\cos x, \cos(2x), \dots \end{matrix} \right\}$ — orthonormal set

3) $F(x) = \sum_{b\in B} \alpha_b\cdot b$ — we can represent an arbitrary vector as a linear combination of a basis vectors

$\alpha_b = \langle F(x), b\rangle = \int_0^{2\pi} F(x)\cdot b(x)\,dx$.

Let a be a constant for $1(x)$, b_k for $\sin(kx)$, c_k for $\cos(kx)$ $\Rightarrow F(x) = \frac{a}{\sqrt{2\pi}}\cdot 1(x) + \sum_{k=1}^{\infty}\frac{b_k}{\sqrt{\pi}}\sin(kx) + \sum_{k=1}^{\infty}\frac{c_k}{\sqrt{\pi}}\cos(kx)$

Let $F(x) = f(x) = \begin{cases} 0, & 0<x<\pi \\ 1, & \pi<x<2\pi \end{cases}$

$a = \int_0^{2\pi} F(x)\,dx = \left[\ \right] = \pi$

$b = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} F(x)\sin(kx)\,dx = \frac{1}{\sqrt{\pi}}\int_\pi^{2\pi}\sin(kx)\,dx = \frac{1}{\sqrt{\pi}}\cdot\frac{-\cos(kx)}{k}\Big|_\pi^{2\pi} = \frac{-\overset{1}{\cos(2\pi k)} + \overset{-1 \text{ or } 1}{\cos(\pi k)}}{\sqrt{\pi}k} = \begin{cases} 0, & k \text{ even} \\ -\frac{2}{\sqrt{\pi}k}, & k \text{ odd} \end{cases}$

$c = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} F(x)\cos(kx)\,dx = \frac{1}{\sqrt{\pi}}\int_\pi^{2\pi}\cos(kx)\,dx = \frac{1}{\sqrt{\pi}}\frac{\sin(kx)}{k}\Big|_\pi^{2\pi} = 0$

$F(x) = \frac{\pi}{\sqrt{2\pi}} + \sum_{k=1,3,5,\dots}^{\infty}\frac{-2\sqrt{\pi}}{\pi k\cdot\sqrt{\pi}}\sin(kx) = \frac{\pi}{\sqrt{2\pi}} - \frac{2}{\pi}\sum_{k=1,3,5\dots}^{\infty}\frac{\sin(kx)}{k}$

b) $e^{ix} = \cos\theta + i\sin\theta$

$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}$

$\sin(nx) = \frac{e^{inx} - e^{-inx}}{2}$

$F(x) = \pi - \frac{2}{\pi}\sum_{k=1,3,5\dots}^{\infty}\frac{\sin(kx)}{k} = \pi - \frac{2}{\pi}\sum_{k=1,3,5\dots}^{\infty}\frac{e^{ikx} - e^{-ikx}}{2k} = \pi - \frac{1}{\pi}\sum_{k=1,3,5\dots}^{\infty}\frac{e^{ikx} - e^{-ikx}}{k}$