Linear Algebra
Seminar 12: Special matrices
(make sure you can write on the board)

Problem 1. Which of the two matrices below is Hermitian?

$$(a)\quad \begin{pmatrix} 2 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 2 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 2 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 2 & \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ \frac{-i}{\sqrt{2}} & 0 & 2 \end{pmatrix}$$

For a Hermitian matrix, find its eigenvalues, eigenvectors, and perform its unitary diagonalization.

Problem 2. Which of the following matrices below is skew-Hermitian (i.e., satisfies $A^* = -A$)?

$$(a)\quad \begin{pmatrix} i & 1-i & 1-i \\ -1-i & 0 & i \\ -1-i & i & 0 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 1 & 1-i & 1-i \\ 1-i & 0 & i \\ 1-i & i & 0 \end{pmatrix}$$

For the skew-Hermitian matrix, find eigenvalues, eigenvectors, and perform its unitary diagonalization.

Problem 3. (a) Show that the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \\ i \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1+i \\ 0 \\ 1-i \end{pmatrix}$$

are orthogonal in $\mathbb{C}^3$ with the standard inner product.

(b) Normalize these vectors to get an orthonormal basis $(\mathbf{u}_1, \mathbf{u}_2)$ for the linear span of these vectors, $\operatorname{ls}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Extend this to an orthonormal basis of $\mathbb{C}^3$, by finding an appropriate vector $\mathbf{u}_3$ (try this two ways: using a Gram–Schmidt process or solving a system of linear equations; explain also why the cross-product does not help)

Problem 4. Under the notations of Problem 3

(a) Express the vector $\mathbf{a} = (1, 1, i)^\top$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

(b) Find the matrix E_1 of the orthogonal projection of $\mathbb{C}^3$ onto $\operatorname{ls}(\mathbf{u}_1)$ and show that E_1 is both Hermitian and idempotent. Find $E_1\mathbf{a}$.

(c) Find the matrices E_2 and E_3 of the orthogonal projections of $\mathbb{C}^3$ onto $\operatorname{ls}(\mathbf{u}_2)$ and $\operatorname{ls}(\mathbf{u}_3)$ respectively and verify that $I = E_1 + E_2 + E_3$.

Problem 5. Suppose that $\mathbf{u}, \mathbf{w} \in \mathbb{R}^2$ are the vectors

$$\mathbf{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

(a) Using the definition of a direct sum, show that $\mathbb{R}^2 = \operatorname{ls}\{\mathbf{u}\} \dotplus \operatorname{ls}\{\mathbf{w}\}$.

(b) Find the projection P of $\mathbb{R}^2$ onto $U = \operatorname{ls}\{\mathbf{u}\}$ parallel to $W = \operatorname{ls}\{\mathbf{w}\}$. Find the image $P(\mathbf{e}_1)$ of the vector $\mathbf{e}_1 = (1, 0)^\top$.

(c) Find the orthogonal projection T from $\mathbb{R}^2$ onto $\operatorname{ls}\{\mathbf{u}\}$. Then find the image of $\mathbf{e}_1 = (1, 0)^\top$ under this linear transformation

(d) Find a projection of $\mathbb{R}^3$ onto the subspace $U = \operatorname{ls}\{(1, 0, 1)^\top\}$ paralel to the subspace $W = \operatorname{ls}\{(1, 1, 0)^\top, (0, 1, 1)^\top\}$

(e) Find a projection of $\mathbb{R}^3$ onto the subspace $U = \operatorname{ls}\{(1, 0, 1)^\top\}$ paralel to the subspace $W = \operatorname{ls}\{(3, -2, 1)^\top, (0, 1, 1)^\top\}$

Problem 6. (a) Show that the matrix

$$P = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is idempotent.

(b) Conclude that P represents a projection in $\mathbb{R}^3$. Is that projection orthogonal?

(c) Find subspaces U and W such that P projects onto U parallel to W.

(d) Is it possible to fill in the missing entries in the matrix

$$A = \begin{pmatrix} 1 & * & 0 \\ 0 & \frac{1}{2} & * \\ * & * & * \end{pmatrix}$$

to get a matrix of an orthogonal projection in $\mathbb{R}^3$? If so, find the subspace U of $\mathbb{R}^3$ such that A is an orthogonal projection onto U.

Extra problems
(to be discussed if time permits)

Problem 7. For the matrices

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

calculate e^{tA} and e^{tB} via the series expansion and by using the spectral decomposition.

Problem 8. Suppose that $U = \mathrm{ls}\{(1,0,-1,1)^\top, (0,0,1,-1)^\top\}$ and $W = \mathrm{ls}\{(0,1,1,0)^\top, (1,0,1,-1)^\top\}$. Is the sum $U + W$ direct? If so, why, and if not, why not? Find a basis for $U + W$.

Problem 9. (a) Show that any matrix P satisfying the relation $P^2 = P$ is a projector onto some subspace L parallel to M, and identify these L and M.

(b) Show that the projector P is an orthogonal projector if and only if the matrix P is symmetric.

(c) Assume that two transformations P_1 and P_2 of $\mathbb{R}^n$ satisfy the following conditions: $P_1 + P_2 = I_n$ and $P_1 P_2 = 0$. Prove that P_1 and P_2 are projectors and that $P_2 P_1 = 0$.

1 a) $A=\begin{pmatrix}2 & i/\sqrt2 & -i/\sqrt2\\ -i/\sqrt2 & 2 & 0\\ i/\sqrt2 & 0 & 2\end{pmatrix}$, $\overline{A}=\begin{pmatrix}2 & -i/\sqrt2 & i/\sqrt2\\ i/\sqrt2 & 2 & 0\\ -i/\sqrt2 & 0 & 2\end{pmatrix}$, $\overline{A}^T=\begin{pmatrix}2 & i/\sqrt2 & -i/\sqrt2\\ -i/\sqrt2 & 2 & 0\\ i/\sqrt2 & 0 & 2\end{pmatrix}=A \Rightarrow A$ is Hermitian matrix

$\begin{cases}\lambda_1=2,\\ \lambda_2+\lambda_3=4\\ \lambda_2\cdot\lambda_3=3\end{cases} \Rightarrow \begin{cases}\lambda_1=2\\ \lambda_2=1\\ \lambda_3=3\end{cases}$

$\det(A)=2\cdot4-\frac{i}{\sqrt2}\cdot\frac{-2i}{\sqrt2}-\frac{i}{\sqrt2}\cdot\frac{-2i}{\sqrt2}=8+2i^2=6$

$\lambda_1=1:\ \begin{pmatrix}1 & i/\sqrt2 & -i/\sqrt2\\ -i/\sqrt2 & 1 & 0\\ i/\sqrt2 & 0 & 1\end{pmatrix}\begin{matrix}-i/\sqrt2\, r_2\\ +r_3\\ \end{matrix} \sim \begin{pmatrix}1/2 & 0 & -\frac{i}{\sqrt2}\\ 0 & 1 & 1\\ i/\sqrt2 & 0 & 1\end{pmatrix}\begin{matrix}\cdot2\\ \\ -i\sqrt2 r_1\end{matrix} \sim \begin{pmatrix}1 & 0 & -i\sqrt2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{cases}x-i\sqrt2 z=0\\ y+z=0\\ z=t\end{cases} \Rightarrow \begin{cases}x=i\sqrt2 t\\ y=-t\\ z=t\end{cases} \Rightarrow \begin{bmatrix}i\sqrt2\\ -1\\ 1\end{bmatrix}=\vec{v_1}$

$\lambda_2=2:\ \begin{pmatrix}0 & i/\sqrt2 & -i/\sqrt2\\ -i/\sqrt2 & 0 & 0\\ i/\sqrt2 & 0 & 0\end{pmatrix}\begin{matrix}\cdot\sqrt2\\ \cdot\sqrt2\\ +r_2\end{matrix} \sim \begin{pmatrix}0 & i & -i\\ -i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{cases}-ix=0\\ iy-iz=0\\ z=t\end{cases} \Rightarrow \begin{cases}x=0\\ y=t\\ z=t\end{cases} \Rightarrow \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}=\vec{v_2}$

$\lambda_3=3:\ \begin{pmatrix}-1 & i/\sqrt2 & -i/\sqrt2\\ -i/\sqrt2 & -1 & 0\\ i/\sqrt2 & 0 & -1\end{pmatrix}\begin{matrix}+i/\sqrt2\, r_2\\ +r_3\\ \end{matrix} \sim \begin{pmatrix}-1/2 & 0 & -i/\sqrt2\\ 0 & -1 & -1\\ i/\sqrt2 & 0 & -1\end{pmatrix}\begin{matrix}\cdot(-2)\\ \cdot(-1)\\ +i\sqrt2 r_1\end{matrix} \sim \begin{pmatrix}1 & 0 & i\sqrt2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{cases}x+i\sqrt2 z=0\\ y+z=0\\ z=-t\end{cases} \Rightarrow \begin{cases}x=i\sqrt2 t\\ y=t\\ z=-t\end{cases} \Rightarrow \begin{pmatrix}i\sqrt2\\ 1\\ -1\end{pmatrix}=\vec{v_3}$

$\|\vec{v_1}\|=\sqrt{i\sqrt2(-i\sqrt2)+1+1}=\sqrt4=2 \Rightarrow \vec{u_1}=\frac12\begin{bmatrix}i\sqrt2\\ -1\\ 1\end{bmatrix}$

$\|\vec{v_2}\|=\sqrt2 \Rightarrow \vec{u_2}=\frac{1}{\sqrt2}\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}=\frac12\begin{bmatrix}0\\ \sqrt2\\ \sqrt2\end{bmatrix}$

$\|\vec{v_3}\|=\sqrt{i\sqrt2(-i\sqrt2)+1+1}=2 \Rightarrow \vec{u_3}=\frac12\begin{bmatrix}i\sqrt2\\ 1\\ -1\end{bmatrix}$

$U=\frac12\begin{pmatrix}i\sqrt2 & 0 & i\sqrt2\\ -1 & \sqrt2 & 1\\ 1 & \sqrt2 & -1\end{pmatrix}$, $D=U^{-1}AU=\begin{pmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix}$

b) $A=\begin{pmatrix}2 & i/\sqrt2 & -i/\sqrt2\\ i/\sqrt2 & 2 & 0\\ -i/\sqrt2 & 0 & 2\end{pmatrix}$, $\overline{A}=\begin{pmatrix}2 & -i/\sqrt2 & i/\sqrt2\\ -i/\sqrt2 & 2 & 0\\ i/\sqrt2 & 0 & 2\end{pmatrix}$, $\overline{A}^T=\begin{pmatrix}2 & -i/\sqrt2 & i/\sqrt2\\ -i/\sqrt2 & 2 & 0\\ i/\sqrt2 & 0 & 2\end{pmatrix}\neq A \Rightarrow A$ is not a Hermitian

2 a) $A=\begin{pmatrix}i & 1-i & 1-i\\ -1-i & 0 & i\\ -1-i & i & 0\end{pmatrix}$, $\overline{A}=\begin{pmatrix}-i & 1+i & 1+i\\ -1+i & 0 & -i\\ -1+i & -i & 0\end{pmatrix}$, $-A=\begin{pmatrix}-i & -1+i & -1+i\\ 1+i & 0 & -i\\ 1+i & -i & 0\end{pmatrix}=\overline{A}^T \Rightarrow A$ is a skew-Hermitian

$\begin{cases}\lambda_1=-i,\\ \lambda_2+\lambda_3=2i\\ \lambda_2\cdot\lambda_3=3\end{cases} \Rightarrow \begin{cases}\lambda_1=-i\\ \lambda_2=-i\\ \lambda_3=3i\end{cases}$

$\det(A)=i(-i^2)-(1-i)(i(1+i))+(1-i)(-i(1+i))=i-2(1-i)(i-1)=-3i$

$\lambda_1=\lambda_2=-i:\ \begin{pmatrix}2i & 1-i & 1-i\\ -1-i & i & i\\ -1-i & i & i\end{pmatrix}\begin{matrix}+r_2\\ \\ -r_2\end{matrix} \sim \begin{pmatrix}i-1 & 1 & 1\\ -1-i & i & i\\ 0 & 0 & 0\end{pmatrix}\begin{matrix}\\ -ir_1\\ \end{matrix} \sim \begin{pmatrix}i-1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{cases}(i-1)x+y+z=0\\ y=t\\ z=s\end{cases} \Rightarrow \begin{cases}x=\frac{t}{1-i}+\frac{s}{1-i}\\ y=t\\ z=s\end{cases} \Rightarrow \left\{\begin{bmatrix}\frac{1}{1-i}\\ 1\\ 0\end{bmatrix},\begin{bmatrix}\frac{1}{1-i}\\ 0\\ 1\end{bmatrix}\right\}=\left\{\underset{\vec{v_1}}{\begin{bmatrix}\frac{1+i}{2}\\ 1\\ 0\end{bmatrix}},\underset{\vec{v_2}}{\begin{bmatrix}\frac{1+i}{2}\\ 0\\ 1\end{bmatrix}}\right\}$

$\lambda_3=3i:\ \begin{pmatrix}-2i & 1-i & 1-i\\ -1-i & -3i & i\\ -1-i & i & -3i\end{pmatrix}\begin{matrix}-r_2\\ \\ -r_2\end{matrix} \sim \begin{pmatrix}1-i & 1+2i & 1-2i\\ -1-i & -3i & i\\ 0 & 4i & -4i\end{pmatrix}\begin{matrix}\\ +ir_1\\ \cdot(1/4)\end{matrix} \sim \begin{pmatrix}1-i & 1+2i & 1-2i\\ 0 & -2-2i & 2+2i\\ 0 & i & -i\end{pmatrix}\begin{matrix}-2r_3\\ \cdot\frac12+r_3\\ \cdot\frac1i\end{matrix} \sim \begin{pmatrix}1-i & 1 & 1\\ 0 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{matrix}+r_2\\ \\ +r_2\end{matrix} \sim \begin{pmatrix}1-i & 0 & 2\\ 0 & -1 & 1\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{cases}(1-i)x+2z=0\\ -y+z=0\\ z=t\end{cases} \Rightarrow \begin{cases}x=\frac{2t}{i-1}\\ y=t\\ z=t\end{cases} \Rightarrow \begin{bmatrix}\frac{2}{i-1}\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}-1-i\\ 1\\ 1\end{bmatrix}=\vec{v_3}$

$\vec{w_1}=\vec{v_1}=\begin{bmatrix}\frac{1+i}{2}\\ 1\\ 0\end{bmatrix}$, $\|\vec{w_1}\|=\sqrt{3/2}$

$\vec{w_2}=\vec{v_2}-\frac{\langle\vec{v_2},\vec{w_1}\rangle}{\|\vec{w_1}\|^2}\vec{w_1}=\begin{bmatrix}\frac{1+i}{2}\\ 0\\ 1\end{bmatrix}-\frac{1/2}{3/2}\begin{bmatrix}\frac{1+i}{2}\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}\frac{1+i}{2}-\frac13\cdot\frac{1+i}{2}\\ -\frac13\\ 1\end{bmatrix}=\begin{bmatrix}\frac{1+i}{3}\\ -\frac13\\ 1\end{bmatrix}=\frac13\begin{bmatrix}1+i\\ -1\\ 3\end{bmatrix}$, $\|\vec{w_2}\|=2/\sqrt3$

$\vec{w_3}=\vec{v_3}=\begin{bmatrix}-1-i\\ 1\\ 1\end{bmatrix}$, $\|\vec{w_3}\|=2$

$\vec{u_1}=\frac{1}{\sqrt{3/2}}\begin{bmatrix}\frac{1+i}{2}\\ 1\\ 0\end{bmatrix}=\sqrt2/\sqrt3\begin{bmatrix}\frac{1+i}{2}\\ 1\\ 0\end{bmatrix}=1/2\sqrt3\begin{bmatrix}\sqrt2+\sqrt2 i\\ \sqrt2/2\\ 0\end{bmatrix}$

$\vec{u_2}=\frac{1/3}{2/\sqrt3}\begin{bmatrix}1+i\\ -1\\ 3\end{bmatrix}=1/2\sqrt3\begin{bmatrix}1+i\\ -1\\ 3\end{bmatrix}$

$\vec{u_3}=1/2\begin{bmatrix}-1-i\\ 1\\ 1\end{bmatrix}=1/2\sqrt3\begin{bmatrix}-\sqrt3-\sqrt3 i\\ \sqrt3\\ \sqrt3\end{bmatrix}$

$U=\frac{1}{2\sqrt3}\begin{bmatrix}\sqrt2+\sqrt2 i & 1+i & -\sqrt3-\sqrt3 i\\ \sqrt2/2 & -1 & \sqrt3\\ 0 & 3 & \sqrt3\end{bmatrix}$, $D=U^{-1}AU=\begin{pmatrix}-i & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & 3i\end{pmatrix}$

b) $A=\begin{pmatrix}1 & 1-i & 1-i\\ 1-i & 0 & i\\ 1-i & i & 0\end{pmatrix}$ a_{11} is not purely imaginary $\Rightarrow$ is not skew-Hermitian

3. a) $\langle v_1,v_2\rangle=v_1^T\cdot\overline{v_2}$, $\vec{v_1}=\begin{pmatrix}1\\ i\\ i\end{pmatrix}$, $\vec{v_2}=\begin{pmatrix}1+i\\ 0\\ 1-i\end{pmatrix}$

$\langle\vec{v_1},\vec{v_2}\rangle=1(1-i)+i\cdot0+i(1+i)=1-i+i+i^2=0$

b) $\|\vec{v_1}\|=\sqrt{1+i(-i)+i(-i)}=\sqrt{1+1+1}=\sqrt3$

$\|\vec{v_2}\|=\sqrt{(1+i)(1-i)+0+(1-i)(1+i)}=\sqrt{2(1-i^2)}=2$

$\vec{u_1}=1/\sqrt3\begin{pmatrix}1\\ i\\ i\end{pmatrix}$, $\vec{u_2}=1/2\begin{pmatrix}1+i\\ 0\\ 1-i\end{pmatrix}$

c) 1) $\begin{pmatrix}1/\sqrt3 & \frac{1+i}{2} & a_1\\ i/\sqrt3 & 0 & a_2\\ i/\sqrt3 & \frac{1-i}{2} & a_3\end{pmatrix}\begin{matrix}\\ -ir_1\\ -ir_1\end{matrix} \sim \begin{pmatrix}\frac{1}{\sqrt3} & \frac{1+i}{2} & a_1\\ 0 & \frac{1-i}{2} & a_2-ia_1\\ 0 & 0 & a_3-ia_1\end{pmatrix} \Rightarrow a_3-ia_1\neq0$

$\vec{v_3}=\begin{pmatrix}-1\\ i\\ i\end{pmatrix}$

$\vec{w_3}=\vec{v_3}-\frac{\langle\vec{v_3},\vec{w_1}\rangle}{\langle\vec{w_1},\vec{w_1}\rangle}\vec{w_1}-\frac{\langle\vec{v_3},\vec{w_2}\rangle}{\langle\vec{w_2},\vec{w_2}\rangle}\vec{w_2}=\begin{pmatrix}-1\\ i\\ i\end{pmatrix}-\frac13\begin{pmatrix}1\\ i\\ i\end{pmatrix}-\underset{-\frac{1-i}{2}}{\frac{-2+2i}{4}}\begin{pmatrix}1+i\\ 0\\ 1-i\end{pmatrix}=\begin{pmatrix}-1-\frac13+\frac{1-i}{2}\cdot(1+i)\\ i-\frac13 i\\ i-\frac13 i+\frac{1-i}{2}\cdot(1-i)\end{pmatrix}=\begin{pmatrix}-1/3\\ 2i/3\\ -i/3\end{pmatrix}$, $\|\vec{w_3}\|=\sqrt{2/3}$

$\vec{u_3}=\frac{\sqrt3}{\sqrt2}\cdot(-\frac13)\begin{pmatrix}1\\ -2i\\ i\end{pmatrix}=-\frac{1}{\sqrt6}\begin{pmatrix}1\\ -2i\\ i\end{pmatrix}$

OR

2) $\langle u_1,u_3\rangle=\langle u_2,u_3\rangle=0$, $u_3=(a_1,a_2,a_3)$

$\begin{cases}(a_1,a_2,a_3)\cdot\frac{1}{\sqrt3}(1,-i,-i)=0\\ (a_1,a_2,a_3)\cdot\frac12(1-i,0,1+i)=0\end{cases} \Rightarrow \begin{cases}a_1\cdot\frac{1}{\sqrt3}-\frac{i}{\sqrt3}a_2-\frac{i}{\sqrt3}a_3=0 & |\cdot\sqrt3\\ a_1\cdot\frac{1-i}{2}+\frac{1+i}{2}a_3=0 & |\cdot2\end{cases} \Rightarrow \begin{cases}a_1-ia_2-ia_3=0\\ a_1(1-i)+a_3(1+i)=0\end{cases} \Rightarrow \begin{cases}-a_3 i-ia_2-ia_3=0\\ a_1=\frac{-a_3(1+i)}{1-i}=-a_3 i\end{cases} \Rightarrow \begin{cases}2a_3 i+a_2 i=0\\ a_1=-a_3 i\end{cases} \Rightarrow \begin{cases}i(2a_3+a_2)=0\\ a_1=-a_3 i\\ a_2=t\end{cases} \begin{cases}a_3=\frac{-a_2}{2}\\ a_1=\frac{a_2}{2}i\\ a_2=t\end{cases}$

$\vec{v_3}=(\frac{t}{2}i,\ t,\ -\frac{t}{2})$

let $t=-2i$: $\vec{v_3}=(1,-2i,i)$, $\|\vec{v_3}\|=\sqrt{1-4i^2-i^2}=\sqrt6$

$\vec{u_3}=\frac{1}{\sqrt6}(1,-2i,i)$

4 a) $\vec{a}=(1,1,i)^T$

$\left(\begin{matrix}1/\sqrt3 & \frac{1+i}{2} & -1/\sqrt6\\ i/\sqrt3 & 0 & 2i/\sqrt6\\ i/\sqrt3 & \frac{1-i}{2} & -i/\sqrt6\end{matrix}\middle|\begin{matrix}1\\ 1\\ i\end{matrix}\right)\begin{matrix}\cdot\sqrt3\\ \cdot\frac{\sqrt3}{i}\\ -r_2\end{matrix} \sim \left(\begin{matrix}1 & \frac{\sqrt3+\sqrt3 i}{2} & -\frac{1}{\sqrt2}\\ 1 & 0 & \frac{2}{\sqrt2}\\ 0 & \frac{1-i}{2} & -\frac{\sqrt3 i}{\sqrt2}\end{matrix}\middle|\begin{matrix}\sqrt3\\ \frac{\sqrt3}{i}\\ i-1\end{matrix}\right)\begin{matrix}-\sqrt3 r_3\\ \\ \cdot2\end{matrix} \sim \left(\begin{matrix}1 & 0 & \frac{-1+3i}{\sqrt2}\\ 1 & 0 & 2/\sqrt2\\ 0 & 1-i & -\sqrt6 i\end{matrix}\middle|\begin{matrix}2\sqrt3-\sqrt3 i\\ \sqrt3/i\\ 2i-2\end{matrix}\right)\begin{matrix}-r_2\\ \\ \\ \end{matrix} \sim \left(\begin{matrix}0 & 0 & \frac{-3+3i}{\sqrt2}\\ 1 & 0 & 2/\sqrt2\\ 0 & 1-i & -\sqrt6 i\end{matrix}\middle|\begin{matrix}2\sqrt3\\ -\sqrt3 i\\ 2i-2\end{matrix}\right)\begin{matrix}\cdot\frac{-3-3i}{9\sqrt2}\\ \\ +i\end{matrix} \sim$

$\left(\begin{matrix}0 & 0 & 1\\ 1 & 0 & 2/\sqrt2\\ 0 & 1 & (1-\sqrt6)i\end{matrix}\middle|\begin{matrix}-\frac{\sqrt2(1+i)}{\sqrt3}\\ -\sqrt3 i\\ 3i-2\end{matrix}\right)\begin{matrix}\\ -\frac{2}{\sqrt2}r_1\\ -(1-\sqrt6)i\,r_1\end{matrix} \sim \left(\begin{matrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{matrix}\middle|\begin{matrix}-\frac{\sqrt2}{\sqrt3}(1+i)\\ \frac{2+5i}{\sqrt3}\\ \frac{(\sqrt3+\sqrt2)i-\sqrt2}{\sqrt3}\end{matrix}\right) \Rightarrow$

$\vec{a}=\frac{2+5i}{\sqrt3}\vec{u_1}+\frac{(\sqrt3+\sqrt2)i-\sqrt2}{\sqrt3}\vec{u_2}-\frac{\sqrt2(1+i)}{\sqrt3}\vec{u_3}$

b) $E_1=\sum_{i=1}\vec{u_i}\cdot(\overline{\vec{u_i}})^T=\frac{1}{\sqrt3}\begin{pmatrix}1\\ i\\ i\end{pmatrix}\frac{1}{\sqrt3}(1,-i,-i)=\frac13\begin{pmatrix}1 & -i & -i\\ i & -i^2 & -i^2\\ i & -i^2 & -i^2\end{pmatrix}=\frac13\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}$

- $\overline{E_1}=\frac13\begin{pmatrix}1 & i & i\\ -i & 1 & 1\\ -i & 1 & 1\end{pmatrix}$, $\overline{E_1}^T=\frac13\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}=E_1 \Rightarrow$ is Hermitian
- $E_1\cdot E_1=\frac19\begin{pmatrix}3 & -3i & -3i\\ 3i & 3 & 3\\ 3i & 3 & 3\end{pmatrix}=\frac13\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}=E_1 \Rightarrow$ is idempotent

$E_1\vec{a}=\frac13\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}\begin{pmatrix}1\\ 1\\ i\end{pmatrix}=\frac13\begin{pmatrix}2-i\\ 1+2i\\ 1+2i\end{pmatrix}$

c) $E_2=u_2\cdot\overline{u_2}^T=\frac12\begin{pmatrix}1+i\\ 0\\ 1-i\end{pmatrix}\cdot\frac12(1-i,0,1+i)=\frac14\begin{pmatrix}2 & 0 & 2i\\ 0 & 0 & 0\\ -2i & 0 & 2\end{pmatrix}=\frac12\begin{pmatrix}1 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 1\end{pmatrix}$

$E_3=u_3\cdot\overline{u_3}^T=-\frac{1}{\sqrt6}\begin{pmatrix}1\\ -2i\\ i\end{pmatrix}\cdot-\frac{1}{\sqrt6}(1,2i,-i)=\frac16\begin{pmatrix}1 & 2i & -i\\ -2i & 4 & -2\\ i & -2 & 1\end{pmatrix}$

$E_1+E_2+E_3=\frac26\begin{pmatrix}1 & -i & -i\\ i & 1 & 1\\ i & 1 & 1\end{pmatrix}+\frac36\begin{pmatrix}1 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 1\end{pmatrix}+\frac16\begin{pmatrix}1 & 2i & -i\\ -2i & 4 & -2\\ i & -2 & 1\end{pmatrix}=\frac16\begin{pmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=I$

5. $\vec{u}=\begin{pmatrix}1\\ 3\end{pmatrix}$, $\vec{w}=\begin{pmatrix}-2\\ 5\end{pmatrix}$

a) $\begin{vmatrix}1 & -2\\ 3 & 5\end{vmatrix}=11\neq0 \Rightarrow$ linearly independent $\Rightarrow ls(\vec{u})\dotplus ls(\vec{w})=ls(\vec{u},\vec{w})$

b) $ls\{\vec{u}\}=col(P)$, $ls\{\vec{w}\}=Null(P)$

$P=(a\cdot\vec{u},\ b\cdot\vec{u})$, $P\cdot\vec{w}=0$ $\quad P\vec{u}=\vec{u}$

$(a\vec{u},b\vec{u})\vec{w}=0$ $\quad\begin{pmatrix}a & b\\ \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\ 3\end{pmatrix}=\begin{pmatrix}1\\ \vdots\end{pmatrix}$

$a\vec{u}w_1+b\vec{u}w_2=0$

$aw_1+bw_2=0$

$\begin{cases}-2a+5b=0\\ a+3b=1\end{cases} \Rightarrow \begin{cases}-2+6b+5b=0\\ a=1-3b\end{cases} \Rightarrow \begin{cases}11b=2\\ a=1-3b\end{cases} \Rightarrow \begin{cases}b=2/11\\ a=\frac{11-6}{11}=\frac{5}{11}\end{cases}$

$P=\frac{1}{11}\begin{pmatrix}5 & 2\\ 15 & 6\end{pmatrix}$

c) $\vec{v}=\frac{\vec{u}}{\|\vec{u}\|}=\frac{1}{\sqrt{10}}\begin{pmatrix}1\\ 3\end{pmatrix}$

$T=\vec{v}\cdot\vec{v}^T=\frac{1}{\sqrt{10}}\begin{pmatrix}1\\ 3\end{pmatrix}\cdot\frac{1}{\sqrt{10}}(1,3)=\frac{1}{10}\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}$

$T\vec{e_1}=\frac{1}{10}\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\frac{1}{10}\begin{pmatrix}1\\ 3\end{pmatrix}$

d) $U\in C(P)$, $W\in N(P)$

$P=(a\vec{u},\ b\vec{u},\ c\vec{u})$, $P\vec{w_1}=P\vec{w_2}=0$, $P\vec{u}=\vec{u}$

$\begin{pmatrix}au & bu & cu\\ \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}=\begin{pmatrix}1\\ \vdots\end{pmatrix}$

$\begin{cases}a\cdot1+b\cdot1+c\cdot0=0\\ a\cdot0+b\cdot1+c\cdot1=0\\ a\cdot1\cdot1+b\cdot0\cdot0+c\cdot1\cdot1=1\end{cases} \Rightarrow \begin{cases}a+b=0\\ b+c=0\\ a+c=1\end{cases} \begin{cases}1-c-c=0\\ b=-c\\ a=1-c\end{cases} \begin{cases}c=1/2\\ b=-1/2\\ a=1/2\end{cases}$

$P=\frac12\begin{pmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 1 & -1 & 1\end{pmatrix}$

e) $U\in C(P)$, $W\in N(P)$

$P=(a\vec{u},\ b\vec{u},\ c\vec{u})$, $P\vec{w_1}=P\vec{w_2}=0$, $P\vec{u}=\vec{u}$

$\begin{pmatrix}au & bu & cu\\ \vdots & \vdots & \vdots\end{pmatrix}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}=\begin{pmatrix}1\\ \vdots\end{pmatrix}$

$\begin{cases}3a-2b+c=0\\ b+c=0\\ a+c=1\end{cases} \begin{cases}3-3c+2c+c=0\\ b=-c\\ a=1-c\end{cases} \begin{cases}3=0\\ \\ \end{cases}$ there is no such projection

$3\cdot(1,0,1)=(3,-2,1)+2(0,1,1)$

6. a) $P=\frac12\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & 1\\ 0 & 0 & 2\end{pmatrix}$

$P^2=\frac14\begin{pmatrix}2 & -2 & 2\\ -2 & 2 & 2\\ 0 & 0 & 4\end{pmatrix}=\frac12\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & 1\\ 0 & 0 & 2\end{pmatrix}=P$

b) Every idempotent A is a matrix of projection of $\mathbb{R}^n$ onto the $Col(A)$ parallel to the $Null(A)$

P is not symmetric $\Rightarrow$ not orthogonal

c) $P=\frac12\Big($ $\quad=\frac12\begin{pmatrix}1 & -1 & 1\\ -1 & 1 & 1\\ 0 & 0 & 2\end{pmatrix}$

$U=Col(P)=ls\left\{\begin{bmatrix}1\\ -1\\ 0\end{bmatrix},\begin{bmatrix}1\\ 1\\ 2\end{bmatrix}\right\}$

$W=Nul(P)$: $P\vec{x}=\vec{0}$

$\left(\begin{matrix}1 & -1 & 1\\ -1 & 1 & 1\\ 0 & 0 & 2\end{matrix}\middle|\begin{matrix}0\\ 0\\ 0\end{matrix}\right)\begin{matrix}\\ +r_1\\ \end{matrix} \sim \begin{pmatrix}1 & -1 & 1\\ 0 & 0 & 2\\ 0 & 0 & 2\end{pmatrix}\begin{matrix}-\frac12 r_3\\ \cdot\frac12\\ -r_2\end{matrix} \sim \begin{pmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \begin{cases}x-y=0\\ z=0\\ y=t\end{cases} \Rightarrow \begin{cases}x=t\\ y=t\\ z=0\end{cases} \Rightarrow \begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$

$W=ls\left\{\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}\right\}$

d) $A=\begin{pmatrix}1 & * & 0\\ 0 & 1/2 & *\\ * & * & *\end{pmatrix}$

A is orthogonal projection $\Leftrightarrow A$ is symmetric idempotent

$A=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac12 & *\\ 0 & * & *\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac12 & b\\ 0 & a & c\end{pmatrix}$

$A^2=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{4a^2+1}{4} & \frac{a+2ab}{2}\\ 0 & \frac{a+2ab}{2} & a^2+b^2\end{pmatrix}=A=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac12 & a\\ 0 & a & b\end{pmatrix}$

$\begin{cases}\frac{4a^2+1}{4}=\frac12\\ \frac{a+2ab}{2}=a\\ a^2+b^2=b\end{cases} \quad \begin{cases}a^2=\frac14\\ al\\ \end{cases}$