Linear Algebra
Seminar 7: Orthogonal vectors and subspaces
(make sure you can write on the board)

Problem 1. (3pt) Let $\mathbf{x}$ and $\mathbf{y}$ be vectors of $\mathbb{R}^n$.

(a) Prove that $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^n x_i y_i$ is an inner product on $\mathbb{R}^n$.

(b) Prove that $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ is a norm on $\mathbb{R}^n$.

(c) Prove that $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ is a distance on $\mathbb{R}^n$.

(d) For $\mathbf{x} = (2, 3)$, $\mathbf{y} = (3, 2)$ calculate the dot product $\langle \mathbf{x}, \mathbf{y} \rangle$.

(e) For $\mathbf{x} = (2, 3)$, calculate the norm of $\|\mathbf{x}\|$.

(f) For $\mathbf{x} = (2, 3)$, $\mathbf{y} = (3, 2)$ calculate the distance $\rho(\mathbf{x}, \mathbf{y})$ between $\mathbf{x}$ and $\mathbf{y}$.

Problem 2. (3pt) For any polynomials p and q in $\mathcal{P}_2$, set

$$\langle p, q \rangle := p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

(a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{P}_2$.

(b) For $p = 1 + x - x^2$ and $q = x + x^2$ calculate the dot product $\langle p, q \rangle$.

(c) For $p = 1 + x - x^2$, calculate the norm $\|p\| := \langle p, p \rangle$ of p.

(d) For $p = 1 + x - x^2$ and $q = x + x^2$ calculate the distance $\rho(p, q) := \|p - q\|$ between p and q.

Problem 3. (2pt)

(a) Find two unit vectors that are orthogonal to both $\mathbf{u} = (1, 0, 1)^\top$ and $\mathbf{v} = (0, 1, 1)^\top$.

(b) Find two unit vectors that are orthogonal to all three of the vectors $\mathbf{u} = (2, 1, -4, 0)^\top$, $\mathbf{v} = (-1, -1, 2, 2)^\top$, and $\mathbf{w} = (3, 2, 5, 4)^\top$

Hint: you can try to generalize the notion of cross-product of two vectors in $\mathbb{R}^3$ to three vectors in $\mathbb{R}^4$.

Problem 4. (2pt) Find a basis for the orthogonal complement of the subspace of $\mathbb{R}^n$ spanned by the vectors:

(a) $\mathbf{v}_1 = (1, -1, 3)^\top$, $\mathbf{v}_2 = (5, -4, -4)^\top$, $\mathbf{v}_3 = (7, -6, 2)^\top$;

(b) $\mathbf{v}_1 = (2, 0, -1)^\top$, $\mathbf{v}_2 = (4, 0, -2)^\top$;

(c) $\mathbf{v}_1 = (1, 4, 5, 2)^\top$, $\mathbf{v}_2 = (2, 1, 3, 0)^\top$, $\mathbf{v}_3 = (-1, 3, 2, 2)^\top$.

Problem 5. (2pt)

(a) Let W be the plane in $\mathbb{R}^3$ with equation $x - 2y - 3z = 0$. Find the parametric equations for $W^\perp$.

(b) Let W be the line in $\mathbb{R}^3$ with parametric equations $x = 2t$, $y = -5t$, $z = 4t$. Find an equation for $W^\perp$.

(c) Let W be the intersection of the two planes $x + y + z = 0$ and $x - y + z = 0$ in $\mathbb{R}^3$. Find an equation for $W^\perp$.

Problem 6. (2pt)

(a) Find the distance between the point $P = (1, 1)$ and the line $x - 2y = 3$.

(b) Find the distance between the point $P = (1, 1, 0)$ and the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+1}{2}$.

(c) Find the distance between the point $P = (1, 1, 0)$ and the plane $2x - y + z = 0$.

Problem 7. (3pt) In the linear vector space $\mathcal{P}_2$ of polynomials of degree at most 2 with inner product

$$\langle p, q\rangle := p(-1)q(-1) + p(0)q(0) + p(1)q(1).$$

(a) Find the orthogonal complement $L^\perp$ of the subspace L spanned by the constant polynomial $p_0(t) \equiv 1$ and suggest a basis of $L^\perp$.

(b) Find the distance from p_0 to the subspace spanned by the polynomials t and t^2.

Problem 8. (3pt) For $n \times n$ matrices A and B with real entries, set

$$\langle A, B\rangle := \operatorname{tr}(A^\top B).$$

(a) Show that $\langle \cdot, \cdot\rangle$ is an inner product in the linear vectors space $V = M_n(\mathbb{R})$ of $n \times n$ matrices with real entries.

(b) For the identity matrix I_n and zero matrix $\mathbf{0}$ calculate the dot product $\langle I_n, \mathbf{0}\rangle$

(c) For I_n, calculate the norm of $\|I_n\|$

(d) For $2I_n$, $\mathbf{0}$ calculate $\rho(2I_n, \mathbf{0})$ the distance between $2I_n$ and $\mathbf{0}$

Extra problems
(to be discussed if time permits)

Problem 9. In the linear vectors space $V = M_n(\mathbb{R})$ of $n \times n$ matrices with real entries with inner product

$$\langle A, B\rangle := \operatorname{tr}(A^\top B).$$

(a) Find the orthogonal complement $L^\perp$ when

- (i) L is the subspace in V of all diagonal matrices;
- (ii) L is the subspace in V of all matrices A with $\operatorname{tr} A = 0$.

(b) Find the distance from the identity matrix to the subspace of matrices of zero trace.

Problem 10. Assume that A is an $m \times n$ matrix with linearly independent columns.

(a) Prove that the matrix $A^\top A$ is non-singular.

(b) Show that

$$\langle \mathbf{x}, \mathbf{y}\rangle := \mathbf{x}^\top (A^\top A)\mathbf{y}$$

is an inner product on $V := \mathbb{R}^n$.

(c) Find the orthogonal complement $L^\perp$ of the subspace L spanned by $\mathbf{e}_1 = (1, 0, \dots, 0)^\top$.

Problem 11. The last two columns of a 3×3 matrix A are $\mathbf{u}_2 = (1, 1, 0)^\top$ and $\mathbf{u}_3 = (0, 1, 2)^\top$ respectively, while its RREF is

$$R = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

What is the first column of A?

1. a) Prove $\langle \vec{x}, \vec{y} \rangle := \vec{x}^T \vec{y} = \sum_{i=1}^{n} x_i y_i$ is the inner product on $\mathbb{R}^n$

Show that all properties hold:

I. Positivity $\langle \vec{u}, \vec{u} \rangle \geq 0 \quad \langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$

II. Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

III. Linearity: $\langle c_1 \vec{u_1} + c_2 \vec{u_2}, \vec{v} \rangle = c_1 \langle \vec{u_1}, \vec{v} \rangle + c_2 \langle \vec{u_2}, \vec{v} \rangle \quad \forall c_1, c_2 \in \mathbb{R}$

I. 1.) $\langle \vec{x}, \vec{x} \rangle = \vec{x}^T \vec{x} = x_1^2 + x_2^2 + \ldots + x_n^2 \geq 0$

2.) If $\langle \vec{x}, \vec{x} \rangle = x_1^2 + x_2^2 + \ldots + x_n^2 = 0 \Rightarrow x_i = 0, \ i = \overline{1,n} \Rightarrow \vec{x} = \vec{0}$

if $\vec{x} = \vec{0} \Rightarrow 0 + 0 + 0 = 0$

II. $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$
$\langle \vec{y}, \vec{x} \rangle = \vec{y}^T \vec{x} = y_1 x_1 + y_2 x_2 + \ldots + y_n x_n$ $\Rightarrow \langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

III. 1.) $\langle c_1 \vec{x}, \vec{y} \rangle = c_1 \langle \vec{x}, \vec{y} \rangle$

$\langle c_1 \vec{x}, \vec{y} \rangle = c_1 \vec{x}^T \vec{y} = c_1 x_1 y_1 + c_1 x_2 y_2 + \ldots + c_1 x_n y_n = c_1 (x_1 y_1 + x_2 y_2 + \ldots + x_n y_n) =$
$= c_1 \langle \vec{x}, \vec{y} \rangle$

$\langle \vec{x} + \vec{z}, \vec{y} \rangle = (\vec{x} + \vec{z})^T \vec{y} = (\vec{x}^T + \vec{z}^T) \vec{y} = \vec{x}^T \vec{y} + \vec{z}^T \vec{y} = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$

b) Prove $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on $\mathbb{R}^n$

$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$

$\sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\vec{x}^T x} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \|\vec{x}\|$

c) Prove $\rho(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ is a distance on $\mathbb{R}^n$

$\rho(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$

$\|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)} = \rho(\vec{x}, \vec{y})$

d) $\vec{x} = (2, 3), \ \vec{y} = (3, 2), \ \langle \vec{x}, \vec{y} \rangle - ?$

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = 2 \cdot 3 + 3 \cdot 2 = 12$

e) $\|\vec{x}\| = \sqrt{4 + 9} = \sqrt{13}$

f) $\rho(\vec{x}, \vec{y}) = \sqrt{(2-3)^2 + (3-2)^2} = \sqrt{2}$

2. $(\forall p, q \in \mathcal{P}_2)$: $\langle p, q \rangle := p(-1) \cdot q(-1) + p(0) \cdot q(0) + p(1) \cdot q(1)$

a) Show that all properties hold:

I. Positivity $\langle \vec{u}, \vec{u} \rangle \geq 0 \quad \langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = \vec{0}$

II. Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

III. Linearity: $\langle c_1 \vec{u_1} + c_2 \vec{u_2}, \vec{v} \rangle = c_1 \langle \vec{u_1}, \vec{v} \rangle + c_2 \langle \vec{u_2}, \vec{v} \rangle \quad \forall c_1, c_2 \in \mathbb{R}$

I. 1) $\langle p, p \rangle = p^2(-1) + p^2(0) + p^2(1) \geq 0$

2) if $\langle p, p \rangle = 0 \Rightarrow p^2(-1) + p^2(0) + p^2(1) = 0 \Rightarrow p$ is zero polynomial

if p is zero polynomial $\Rightarrow p^2(-1) + p^2(0) + p^2(1) = 0 \Rightarrow \langle p, p \rangle = 0$

II. $\langle p, q \rangle = p(-1) q(-1) + p(0) q(0) + p(1) q(1)$
$\langle q, p \rangle = q(-1) p(-1) + q(0) p(0) + q(1) p(1)$ $\Rightarrow \langle p, q \rangle = \langle q, p \rangle$

III. $\langle c_1 p, q \rangle = c_1 p(-1) q(-1) + c_1 p(0) q(0) + c_1 p(1) q(1) = c_1 \big(p(-1) q(-1) + p(0) q(0) + p(1) q(1) \big) =$
$= c_1 \langle p, q \rangle$

$\langle p_1 + p_2, q \rangle = (p_1 + p_2)(-1) \cdot q(-1) + (p_1 + p_2)(0) \cdot q(0) + (p_1 + p_2)(1) \cdot q(1) =$
$= (p_1(-1) + p_2(-1)) q(-1) + (p_1(0) + p_2(0)) q(0) + (p_1(1) + p_2(1)) q(1) =$
$= p_1(-1) q(-1) + p_2(-1) q(-1) + p_1(0) q(0) + p_2(0) q(0) + p_1(1) q(1) + p_2(1) q(1)$
$= \langle p_1, q \rangle + \langle p_2, q \rangle$

b) $p = 1 + x - x^2, \ q = x + x^2$

$\langle p, q \rangle = (1 - 1 - 1)(-1 + 1) + 1 \cdot 0 + (1 + 1 - 1)(1 + 1) = 2$

c) $p = 1 + x - x^2$

$\|p\| := \langle p, p \rangle = (-1)^2 + 1^2 + 1^2 = 3$

d) $\rho(p, q) := \|p - q\| = \langle p - q, p - q \rangle = \langle p, p - q \rangle - \langle q, p - q \rangle = \langle p, p \rangle - \langle p, q \rangle - \langle q, p \rangle + \langle q, q \rangle =$
$= \langle p, p \rangle - 2 \langle p, q \rangle + \langle q, q \rangle = 3 - 2 \cdot 2 + 4 = 3$

$\langle q, q \rangle = (-1 + 1)^2 + 0 + (1 + 1)^2 = 4$

3. a) $\vec{w} = (1, 0, 1)^T, \ \vec{v} = (0, 1, 1)^T$

The cross product $\vec{u} \times \vec{v}$ gives the vector, orthogonal to both $\begin{pmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$

$\vec{u} \times \vec{v} = (u_1, u_2, u_3) \times (v_1, v_2, v_3) = \left(\vec{e_1} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \vec{e_2} \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \vec{e_3} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) =$

1) $= (1, 0, 1) \times (0, 1, 1) = \left(\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (-1, -1, 1) = \vec{a}$

2) $= (0, 1, 1) \times (1, 0, 1) = \left(\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) = (1, 1, -1) = -\vec{a} = \vec{b}$

$\|\vec{a}\| = \sqrt{1 + 1 + 1} = \sqrt{3}$

$\frac{\vec{a}}{\sqrt{3}} = \left(-\frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}\right) \quad ; \quad \frac{\vec{b}}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}}\right)$

b) $\vec{u} = (2, 1, -4, 0)^T, \ \vec{v} = (-1, -1, 2, 2)^T, \ \vec{w} = (3, 2, 5, 4)^T$

$$\vec{w} \times \vec{v} \times \vec{w} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} & \vec{e_4} \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_3 & w_3 & w_4 \end{vmatrix} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} & \vec{e_4} \\ 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{vmatrix} =$$

$$= \left\{ \vec{e_1} \begin{vmatrix} 1 & -4 & 0 \\ -1 & 2 & 2 \\ 2 & 5 & 4 \end{vmatrix}, -\vec{e_2} \begin{vmatrix} 2 & -4 & 0 \\ -1 & 2 & 2 \\ 3 & 5 & 4 \end{vmatrix}, \vec{e_3} \begin{vmatrix} 2 & 1 & 0 \\ -1 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}, -\vec{e_4} \begin{vmatrix} 2 & 1 & -4 \\ -1 & -1 & 2 \\ 3 & 2 & 5 \end{vmatrix} \right\} =$$

$= \{ 1(8 - 10) + 4(-4 - 4); -[2(8 - 10) + 4(-4 - 6)], \ 2(-4 - 4) - 1(-4 - 6), -[2(-5 - 4) - 1(-5 - 6) - 4(-2 + 3)] \} =$

$= \{ -2 - 32; -[-4 - 40] \ -16 + 10; -[-18 + 11 - 4] \} =$

$= \{ -34, \ 44, \ -6, \ 11 \} = \vec{a}$

$\|\vec{a}\| = \sqrt{34^2 + 44^2 + 6^2 + 11^2} = \sqrt{3249} = 57$

$\vec{a_1} = \left\{ \frac{-34}{57}, \frac{44}{57}, \frac{-6}{57}, \frac{11}{57} \right\} = \left\{ \frac{-34}{57}, \frac{44}{57}, \frac{-2}{57}, \frac{11}{57} \right\}$

$\vec{a_2} = \left\{ \frac{34}{57}, \frac{-44}{57}, \frac{6}{57}, -\frac{11}{57} \right\} = \left\{ \frac{34}{57}, \frac{-44}{57}, \frac{2}{57}, \frac{-11}{57} \right\}$

4. a) $\vec{V_1} = (1, -1, 3)^T, \quad \vec{V_2} = (5, -4, -4)^T, \quad \vec{V_3} = (7, -6, 2)^T$

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \begin{matrix} \\ -5r_1 \\ -7r_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \begin{matrix} +r_1 \\ \\ -r_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -16 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x - 16z = 0 \\ y - 19z = 0 \\ z = t \end{cases} \Rightarrow \begin{cases} x = 16t \\ y = 19t \\ z = t \end{cases} \Rightarrow \{(16, 19, 1)\}$$

b) $\vec{V_1} = (2, 0, -1)^T, \ \vec{V_2} = (4, 0, -2)^T$

$$\left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 4 & 0 & -2 & 0 \end{array}\right] \begin{matrix} \\ -2r_1 \end{matrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} 2x - z = 0 \\ y = t, \ z = s \end{cases} \Rightarrow \begin{cases} x = \frac{s}{2} \\ y = t \\ z = s \end{cases} \Rightarrow \left\{ \left(\tfrac{1}{2}, 0, 1\right), (0, 1, 0) \right\}$$

c) $\vec{v_1} = (1, 4, 5, 2)^T, \quad \vec{v_2} = (2, 1, 3, 0)^T, \ \vec{v_3} = (-1, 3, 2, 2)^T$

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \begin{matrix} \\ -2r_1 \\ +r_1 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \begin{matrix} \\ \\ +r_2 \end{matrix} \sim \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 4x_2 + 5x_3 + 2x_4 = 0 \\ -7x_2 - 7x_3 - 4x_4 = 0 \\ x_3 = t \\ x_4 = \frac{7}{4}s \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = -4x_2 - 5x_3 - 2x_4 = -4(-t - s) - 5t + 2 \cdot \frac{7}{4}s = 4t + 4s - 5t - \frac{7}{2}s = -t + \frac{1}{2}s \\ x_2 = \frac{7x_3 + 4x_4}{-7} = \frac{7t + 4 \cdot \frac{7}{4}s}{-7} = -t - s \\ x_3 = t \\ x_4 = \frac{7}{4}s \end{cases} \Rightarrow \left\{ (-1, -1, 1, 0), \left(\tfrac{1}{2}, -1, 0, \tfrac{7}{4}\right) \right\}$$

5. a) The orthogonal complement of the plane is the line that contains the vector $(1, -2, -3)^T$, so $t(2, 1, 3)$ is the orthogonal complement of the plane

b) $\begin{cases} x = 2t, \\ y = -5t, \\ z = 4t \end{cases} \Rightarrow (2, -5, 4) \Rightarrow 2x - 5y + 4z = 0$

c) $\begin{cases} x + y + z = 0 \\ x - y + z = 0 \end{cases} \Rightarrow y = 0 \Rightarrow x + z = 0$

6. a) $P = (1, 1), \quad x - 2y = 3 \ = \ell$

$d = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$, where $ax + by + cz + d = 2x - y + z$, $(2, 1, 1, 0)$
$(x_1, y_1, z_1) = P = (1, 1, 0)$

$= \dfrac{|2 - 1 + 0 + 0|}{\sqrt{4 + 1 + 1}} = \dfrac{1}{\sqrt{6}}$