Linear Algebra
Seminar 10: Eigenvalues and eigenvectors
(make sure you can write on the board)

Problem 1. (3 pts) Find all the eigenvalues of the following matrices by **inspection**:

$$\text{(a)}\ \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \qquad \text{(b)}\ \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \qquad \text{(c)}\ \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \qquad \text{(d)}\ \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \qquad \text{(e)}\ \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

Hint: Look for constant row/column sums, diagonal entries with zeros in the corresponding row or column otherwise, use eigenvalue sum/product rules, try subtracting λI for "tempting" candidates for λ etc

Problem 2. (3 pts) For the matrix A in each part below, find the eigenvalues and eigenvectors of A, A^2, A^{100}, A^{-1} and e^{tA}:

$$\text{(a)}\ \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \text{(b)}\ \begin{pmatrix} 1 & 0 & 0 \\ -2 & 5 & 3 \\ -4 & -6 & -4 \end{pmatrix}$$

Problem 3. (3 pts) For each of the following matrices A, find P so that $P^{-1}AP$ is in the Jordan form (ie, either diagonal or a Jordan block), and write this Jordan form:

$$\text{(a)}\ \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \qquad \text{(b)}\ \begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}, \qquad \text{(c)}\ \begin{pmatrix} -1 & -3 \\ 2 & 2 \end{pmatrix}, \qquad \text{(d)}\ \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}.$$

Hint: You do not have to calculate $P^{-1}AP$ explicitly!

Problem 4. (3 pts) Show that the matrix

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

has $\lambda = 3$ as an eigenvalue of multiplicity 3 with two corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Find the generalized eigenvector $\mathbf{v}_3$ such that $A\mathbf{v}_3 = 3\mathbf{v}_3 + \mathbf{v}_2$, then form the matrix $P = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3)$ and show that $P^{-1}AP$ has a Jordan form.

Problem 5. (4 pts)

(a) With the matrix A of Problem 4, solve the difference equation $\mathbf{x}_{n+1} = A\mathbf{x}$ subject to the initial condition $\mathbf{x}_0 = (1\ 1\ 1)^\top$;

(b) Assume that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ such that $\mathbf{u}^\top\mathbf{v} \neq 0$. Find the eigenvalues and the eigenvectors of the matrix $\mathbf{u}\mathbf{v}^\top$.

(c) Show that $\det(I + \mathbf{u}\mathbf{v}^\top) = 1 + \mathbf{u}^\top\mathbf{v}$

Problem 6. (4 pts)

(a) With the matrix A of Problem 4, solve the differential equation $\mathbf{x}'(t) = A\mathbf{x}$ subject to the initial condition $\mathbf{x}(0) = (1\ 1\ 1)^\top$

(b) Assume that $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ such that $\mathbf{u}^\top\mathbf{v} = 0$. Find the eigenvalues and the eigenvectors of the matrix $\mathbf{u}\mathbf{v}^\top$.

(c) Show that if A is an $n \times n$ matrix of rank 1, then there is a number k such that $A^2 = kA$.

Extra problems
(to be discussed if time permits)

Problem 7. (a) Prove that for any square matrices A and B of size n the eigenvalues of AB and BA are the same.

(b) Show that for any $m \times n$ matrix A and $n \times m$ matrix B the nonzero eigenvalues of AB and BA are the same.

Problem 8. (a) Prove that if λ^2 is an eigenvalue of a matrix A^2, then λ or/and $-\lambda$ is an eigenvalue of A.

(b) A matrix A is called the *involution* if $A^2 = I$. Find all eigenvalues of A and explain why A is diagonalizable.

Problem 9. Let B be a square matrix of size n with 1's on the skew diagonal and zeros otherwise. Show that B is an involution (see the previous problem). Then explain why B is diagonazible and find a matrix P s.t. $P^{-1}BP$ is diagonal

Problem 10. Find the eigenvalues and eigenvectors of the following linear transformations:

(a) $X \mapsto X^\top$ in the space $M_{n\times n}(\mathbb{R})$ of square matrices of size n with real entries;

Hint: show that every X can be written as a sum of a symmetric X_{sym} and antisymmetric X_{asym} matrices

(b) $f \mapsto f'$ in the space $\mathcal{P}_n$ of polynomials of degree at most n;

(c) $f \mapsto xf'(x)$ in the space $\mathcal{P}_n$;

(d) $f \mapsto \frac{1}{x}\int_0^x f(t)\,dt$ in the space $\mathcal{P}_n$;

Hint: in (b)–(d), find the matrices of the transformations in the standard basis or:

the sum of each entry in each column $\left(\frac{1}{2}+\frac{1}{2}=\frac{3}{2}+\left(-\frac{1}{2}\right)=1\right)$
is the same, thus this sum is the eigenvalue

1. a) $A=\begin{pmatrix} 1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}$ $\lambda_1=-1$: $A-\lambda_1 I=\begin{pmatrix} 3/2 & 3/2 \\ 1/2 & 1/2 \end{pmatrix}$ – singular $\Rightarrow$ $\lambda_1+\lambda_2=\operatorname{trace}(A)=\frac{1}{2}+\left(-\frac{1}{2}\right)=0$, $\lambda_2=-\lambda_1=1$

b) $A=\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ $\lambda_1=5$ $A-\lambda_1 I=\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}$ – singular $\Rightarrow$ $\lambda_1+\lambda_2=\operatorname{trace}(A)=1+4=5$, $\lambda_2=5-\lambda_1=0$

c) $A=\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ $\lambda_1=-1$ $A-\lambda_1 I=\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$ – singular $\Rightarrow$ $\lambda_1+\lambda_2=\operatorname{trace}(A)=1+3=4$, $\lambda_2=4-\lambda_1=5$

d) $A=\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ $\lambda_1=-1$ $A-\lambda_1 I=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ – singular $\Rightarrow \begin{cases} \lambda_1+\lambda_2+\lambda_3=\operatorname{trace}(A) \\ \lambda_1\cdot\lambda_2\cdot\lambda_3=\det(A)= \\ \quad =-2\cdot(-1)+1\cdot 4=2+4=6 \end{cases} \Rightarrow \begin{cases} \lambda_2+\lambda_3=1 \\ \lambda_2\cdot\lambda_3=-6 \end{cases} \Rightarrow \begin{cases} \lambda_2=-2 \\ \lambda_3=3 \end{cases}$

e) $A=\begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ $\lambda_1=-1$ $A-\lambda_1 I=\begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$ – singular $\Rightarrow \begin{cases} \lambda_1+\lambda_2+\lambda_3=\operatorname{trace}(A) \\ \lambda_1\cdot\lambda_2\cdot\lambda_3=\det(A)= \\ \quad =2\cdot 4+2\cdot(-12)=8-24=-16 \end{cases} \Rightarrow \begin{cases} \lambda_2+\lambda_3=8 \\ \lambda_2\cdot\lambda_3=16 \end{cases} \Rightarrow \{\lambda_2=\lambda_3=4$ (of multiplicity 2)

2. a) $A=\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ column's entries sum: $2+2=3+1 \Rightarrow \lambda_1=4$, $\lambda_2=3-4=-1$

$\lambda_1=4$: $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}_{+r_1} \sim \begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} -2x+3y=0 \\ y=2/3k \end{cases} \Rightarrow \begin{cases} x=\frac{-3}{-2}\cdot\frac{2}{3}k=k \\ y=2/3k \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$e^{tA}=I+tA+\frac{1}{2}(tA)^2+\frac{1}{3!}(tA)^3+\dots$

$\lambda_2=-1$: $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}^{\cdot\frac{1}{3}}_{-\frac{2}{3}r_1} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ y=k \end{cases} \Rightarrow \begin{cases} x=-k \\ y=k \end{cases} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$e^{A}=PP^{-1}+PAP^{-1}+\frac{1}{2}PA^2P^{-1}+\dots=$
$=P\left(I+D+\frac{D^2}{2}+\dots\right)P^{-1}=P\begin{pmatrix} e^{d} & \\ & e^{d}\ddots \end{pmatrix}$
$=\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}^{-1}$

eigenvalues of A: $\{-1, 4\}$, A^2: $\{1, 16\}$, A^{100}: $\{1, 4^{100}\}$, A^{-1}: $\{1, 1/4\}$, e^{tA}: $\{e^{-t}, e^{4t}\}$

eigenvectors for all A^n are the same because of the n distinct eigenvalues: $\left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

b) $A=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 5 & 3 \\ -4 & -6 & -4 \end{pmatrix}$ $\lambda_1=1$ $A-\lambda_1 I=\begin{pmatrix} 0 & 0 & 0 \\ -2 & 4 & 3 \\ -4 & -6 & -5 \end{pmatrix}$ singular $\Rightarrow \begin{cases} \lambda_2+\lambda_3=1+5-4-1=1 \\ \lambda_2\cdot\lambda_3=-20+18=-2 \end{cases} \Rightarrow \begin{cases} \lambda_2=-1 \\ \lambda_3=2 \end{cases}$

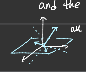
$\lambda_1=1$: $\begin{pmatrix} 0 & 0 & 0 \\ -2 & 4 & 3 \\ -4 & -6 & -5 \end{pmatrix}_{-2r_2} \sim \begin{pmatrix} 0 & 0 & 0 \\ -2 & 4 & 3 \\ 0 & -14 & -11 \end{pmatrix} \Rightarrow \begin{cases} -2x+4y+3z=0 \\ -14y-11z=0 \\ z=14/11k \end{cases} \Rightarrow \begin{cases} x=\frac{-3z-4y}{-2}=\frac{3}{2}z+2y=\frac{3}{2}\cdot\frac{14}{11}k-2k=\frac{-1}{11}k \\ y=\frac{11}{-14}z=-k \\ z=\frac{14}{11}k \end{cases} \Rightarrow \begin{pmatrix} -1/11 \\ -1 \\ 14/11 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 11 \\ -14 \end{pmatrix}$

$\lambda_2=-1$: $\begin{pmatrix} 2 & 0 & 0 \\ -2 & 6 & 3 \\ -4 & -6 & -3 \end{pmatrix}^{\cdot 1/2}_{+r_2} \sim \begin{pmatrix} 1 & 0 & 0 \\ -2 & 6 & 3 \\ -6 & 0 & 0 \end{pmatrix}^{+2r_1}_{+6r_1} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 6 & 3 \\ 0 & 0 & 0 \end{pmatrix}^{\cdot 1/3} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x=0 \\ 2y+z=0 \\ z=2k \end{cases} \Rightarrow \begin{cases} x=0 \\ y=-k \\ z=2k \end{cases} \Rightarrow \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

$\lambda_3=2$: $\begin{pmatrix} -1 & 0 & 0 \\ -2 & 3 & 3 \\ -4 & -6 & -6 \end{pmatrix}^{\cdot(-1)}_{-2r_2} \sim \begin{pmatrix} 1 & 0 & 0 \\ -2 & 3 & 3 \\ 0 & -12 & -12 \end{pmatrix}^{+2r_1}_{\cdot\left(-\frac{1}{4}\right)} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}^{\cdot\left(\frac{1}{3}\right)}_{-r_2} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x=0 \\ y+z=0 \\ z=k \end{cases} \Rightarrow \begin{cases} x=0 \\ y=-k \\ z=k \end{cases} \Rightarrow \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

eigenvalues of A: $\{-1, 1, 2\}$, A^2: $\{\underset{\text{mult.2}}{1}, 4\}$, A^{100}: $\{\underset{\text{mult.2}}{1}, 2^{100}\}$, A^{-1}: $\{-1, 1, \frac{1}{2}\}$, e^{tA}: $\{e^{-t}, e^{t}, e^{2t}\}$

eigenvectors of A = eigenvectors of A^n, when n is odd $=\left\{\begin{bmatrix} 1 \\ 11 \\ -14 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$
when n is even, the eigenvalue of A^n $\{1\}$ has multiplicity of 2,
and the dimension of a nullspace $=2$, thus, in $\mathbb{R}^3$ case:



all vectors in the plane, spanned by eigenvector of eigenvalue with mult.2,
will also be the eigenvectors of A^n, in addition to the given

3. a) $A=\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$ column's entries sum: $2+2=3+1 \Rightarrow \lambda_1=4$, $\lambda_2=3-4=-1$

$\lambda_1=4$: $\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}_{+r_1} \sim \begin{pmatrix} -2 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} -2x+3y=0 \\ y=2/3k \end{cases} \Rightarrow \begin{cases} x=\frac{-3}{-2}\cdot\frac{2}{3}k=k \\ y=2/3k \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 2/3 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda_2=-1$: $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}^{\cdot\frac{1}{3}}_{-\frac{2}{3}r_1} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ y=k \end{cases} \Rightarrow \begin{cases} x=-k \\ y=k \end{cases} \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$P=\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}$, $J=\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

b) $A=\begin{pmatrix} 2 & 1 \\ 5 & -2 \end{pmatrix}$ row sum: $\lambda_1=3$, $\lambda_2=0-3=-3$

$\lambda_1=3$: $\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}_{+5r_1} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} -x+y=0 \\ y=k \end{cases} \Rightarrow \begin{cases} x=k \\ y=k \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2=-3$: $\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix}_{-r_1} \sim \begin{pmatrix} 5 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} 5x+y=0 \\ y=5k \end{cases} \Rightarrow \begin{cases} x=-k \\ y=5k \end{cases} \Rightarrow \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

$P=\begin{pmatrix} 1 & -1 \\ 1 & 5 \end{pmatrix}$, $J=\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$

c) $A=\begin{pmatrix} -1 & -3 \\ 2 & 2 \end{pmatrix}$ $\begin{cases} \lambda_1+\lambda_2=-1+2=1 \\ \lambda_1\cdot\lambda_2=-2+6=4 \end{cases} \Rightarrow$ $p(\lambda)=\lambda^2-\lambda+4$, $D=1-16=-15$, $\sqrt{D}=i\sqrt{15}$, $\lambda_{1,2}=\frac{1\pm i\sqrt{15}}{2}$

$\lambda_1=\frac{1+i\sqrt{15}}{2}$: $\begin{pmatrix} -\frac{3+i\sqrt{15}}{2} & -3 \\ 2 & \frac{3-i\sqrt{15}}{2} \end{pmatrix}^{+\frac{r_2}{2}\cdot\frac{3+i\sqrt{15}}{2}} \sim \begin{pmatrix} 0 & 0 \\ 2 & \frac{3-i\sqrt{15}}{2} \end{pmatrix} \Rightarrow \begin{cases} 2x+\frac{3-i\sqrt{15}}{2}y=0 \\ y=\frac{-4}{3-i\sqrt{15}}k \end{cases} \Rightarrow \begin{cases} x=k \\ y=\frac{-4}{3-i\sqrt{15}}k \end{cases} \Rightarrow \begin{pmatrix} 1 \\ \frac{-4}{3-i\sqrt{15}} \end{pmatrix}$

$\lambda_2=\frac{1-i\sqrt{15}}{2}$: $\begin{pmatrix} -\frac{3-i\sqrt{15}}{2} & -3 \\ 2 & \frac{3+i\sqrt{15}}{2} \end{pmatrix}^{+\frac{r_2}{2}\cdot\frac{3-i\sqrt{15}}{2}} \sim \begin{pmatrix} 0 & 0 \\ 2 & \frac{3+i\sqrt{15}}{2} \end{pmatrix} \Rightarrow \begin{cases} 2x+\frac{3+i\sqrt{15}}{2}y=0 \\ y=\frac{-4}{3+i\sqrt{15}}k \end{cases} \Rightarrow \begin{cases} x=k \\ y=\frac{-4}{3+i\sqrt{15}}k \end{cases} \Rightarrow \begin{pmatrix} 1 \\ \frac{-4}{3+i\sqrt{15}} \end{pmatrix}$

$P=\begin{pmatrix} 1 & 1 \\ \frac{-4}{3-i\sqrt{15}} & \frac{-4}{3+i\sqrt{15}} \end{pmatrix}$, $J=\begin{pmatrix} \frac{1+i\sqrt{15}}{2} & 0 \\ 0 & \frac{1-i\sqrt{15}}{2} \end{pmatrix}$

d) $A=\begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$ row sum: $\lambda_1=3$, $\lambda_2=6-3=3$

$\lambda_1=\lambda_2=3$: $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}_{-r_1} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x-y=0 \\ y=k \end{cases} \Rightarrow \begin{cases} x=k \\ y=k \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

generalized eigenvector: $\vec{V}_{k-1}=(A-\lambda)\vec{V}_k$

$\vec{V}_1=(A-3)\vec{V}_2 \Rightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\vec{V}_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{V}_2=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$AV_2=3V_2+V_1$

$P=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $J=\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

4. $A=\begin{pmatrix} 3 & -2 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ $\det(A-\lambda I)=\det\begin{pmatrix} 3-\lambda & -2 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{pmatrix}=(3-\lambda)^3$

$\lambda=3$: $\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{+2r_3} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} y=0 \\ x=s \\ z=t \end{cases} \Rightarrow \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} \Rightarrow \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}\right\}$

$(A-3I)\vec{V}_3=\vec{V}_2 \Rightarrow \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\vec{V}_3=\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{V}_3=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$P=\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $P^{-1}=\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $J=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

5. a) $\vec{x}_0=(1,1,1)^T$

$\vec{X}_{n+1}=A\vec{x}_n=A(A\vec{x}_{n-1})=A^{n+1}\vec{x}_0=PJ^{n+1}P^{-1}\vec{x}_0=\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3^{n+1} & 0 & 0 \\ 0 & 3^{n+1} & \\ 0 & 0 & 3^{n+1} \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=$

$=\begin{pmatrix} 3^{n+1} & -2\cdot 3^{n+1} & -2(n+1)3^n \\ 0 & 0 & 3^{n+1} \\ 0 & 3^{n+1} & (n+1)3^n \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 3^{n+1} & -2(n+1)3^n & 0 \\ 0 & 3^{n+1} & 0 \\ 0 & (n+1)3^n & 3^{n+1} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 3^n(1-2n) \\ 3^{n+1} \\ 3^n(n+4) \end{pmatrix}$

b) $u^Tv\neq 0$ – trace of a matrix $vu^T=A$

$u^Tv=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(v_1, v_2)=\begin{pmatrix} u_1v_1 & u_1v_2 \\ u_2v_1 & u_2v_2 \end{pmatrix}$: $r_2=\frac{u_2}{u_1}r_1$ – it is singular $\Rightarrow$ $\lambda_1=0$ of multiplicity at least $n-1$

$\operatorname{trace}(A)\neq 0 \Rightarrow$ there is other nonzero eigenvalue

$\lambda_n=\operatorname{trace}(A)=\sum_{i=1}^{n}u_iv_i=u^Tv$

$\lambda_1=0$ $\underbrace{uv^T\cdot x=0}_{\substack{v^Tx=0 \\ x \text{ is orthogonal to } v}}$ $\Rightarrow x\in\{v\}^{\perp}$
$u\neq 0$

$\lambda_n=u^Tv$ $uv^Tx=u^Tvx$
$x=u$

c) $\det(I+uv^T)=1+u^Tv$

$I+uv^T$: $\lambda_1=\dots=\lambda_{n-1}=1$
$\operatorname{trace}(I+uv^T)=n+u^Tv$ $\lambda_n=n+u^Tv-(n-1)\cdot 1=1+u^Tv$

$\det(I+uv^T)=\lambda_1\cdot\lambda_2\cdot\ldots\cdot\lambda_n=1+u^Tv$

6. c) $A=\begin{pmatrix} \text{row}_1 \\ a\cdot\text{row}_1 \\ \vdots \end{pmatrix}=u\cdot v^T$
$A^2=u\cdot\underbrace{v^T\cdot u}_{k}\cdot v^T=k\cdot u\cdot v^T$

7. a) prove: $ABx=\lambda x$: $\exists y$: $BAy=\lambda y$

$BA\underbrace{Bx}_{\lambda x}=B\lambda x=\lambda Bx \Rightarrow |y=Bx|$
$Bx\neq 0 \Rightarrow \lambda$ is eigenvalue
① $\lambda\neq 0$ $Bx\neq 0$
② $\lambda=0$ $ABx=0$
$BAy=0$ $Ay=x$ $Ay\neq x$ $\cancel{Ay=0}$ $BAy=0$