Linear Algebra

Lecture Notes

Rostyslav Hryniv

Ukrainian Catholic University
Business Analytics and Computer Science Programmes

4^{th} term
Spring 2019

Lecture 10. Eigenvalues and eigenvectors

Outline

Enlightening examples

The matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

seem pretty similar; however, they transform $\mathbb{R}^2$ in drastically different ways:

- A is the mirror symmetry w.r.t. the line $y = x$
- B rotates any vector by $\pi/2$ clockwise
- C is the orthogonal projection on the x-direction
- D is the shear transformation

The reason is A, B, C, and D have cardinally different eigenvalues and eigenvectors

Definition

Definition

Let A be a square matrix. An eigenvalue (EV) of A is a number λ for which $A - \lambda \cdot I$ is singular

Example

- $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}; \qquad A - 1 \cdot I = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$
- Diagonal matrix $A = \mathrm{diag}\{-1, 1\}$

$$A - (-1) \cdot I = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad A - 1 \cdot I = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

- Diagonal entries of a diagonal matrix D are its eigenvalues
- A square matrix A is singular $\iff$ 0 is an eigenvalue of A

Example

- $B = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ is singular. Another EV is 3.
- A matrix M whose entries are nonnegative and whose columns (or rows) each add to 1 is called a Markov matrix. *M.m.* play a major role in economic dynamics.

  Claim: $\lambda = 1$ is an EV of every Markov matrix M.
  Reason: Each column of $M - 1 \cdot I$ adds to 0, whence rows are linearly dependent, whence $M - 1 \cdot I$ is singular.

  $$M = \begin{pmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{pmatrix}; \qquad M - 1 \cdot I = \begin{pmatrix} -1/4 & 1/2 \\ 1/4 & -1/2 \end{pmatrix}$$

- $A - \lambda I$ is singular $\iff \det(A - \lambda I) = 0$;
- $p_A(\lambda) := \det(A - \lambda I)$ is a characteristic polynomial of λ of degree n (size of A); thus has n zeros counting multiplicities
- therefore, A possesses **at most n distinct eigenvalues**

Example

For $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ we find that

$$\det(A - \lambda \cdot I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = \lambda^2 - \underbrace{(a_{11} + a_{22})}_{\operatorname{tr} A}\lambda + \det A.$$

For $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\quad \det(A - \lambda I) = \lambda^2 + 1 \implies$ no real EV's!

- $A - \lambda I$ singular means the system of linear equations $(A - \lambda I)\mathbf{v} = 0$ has a nontrivial ($\mathbf{v} \neq \mathbf{0}$) solution
- For an EV λ any nonzero $\mathbf{v}$ satisfying $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is called an eigenvector (EVc) of A corresponding to EV λ
- $(A - \lambda I)\mathbf{v} = 0 \iff A\mathbf{v} = \lambda\mathbf{v}$.

For an $n \times n$ matrix and $\lambda \in \mathbb{C}$, TFAE:

(a) $A - \lambda I$ is singular;

(b) $\det(A - \lambda I) = 0$;

(c) $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero $\mathbf{v} \neq 0$.

Example (Computing eigenvalues and eigenvectors)

$$a)\ A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \implies \det(A - \lambda I) = \lambda^2 - \lambda - 6,$$

thus the EV's are $\lambda_1 = 3$ and $\lambda_2 = -2$

$$(A - 3I)\mathbf{v}_1 = 0 \iff \mathbf{v}_1 = (k, k)^\top;$$

$$(A + 2I)\mathbf{v}_2 = 0 \iff \mathbf{v}_2 = (2k, -3k)^\top$$

$$b)\ A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix} \implies \det(A - \lambda I) = (5 - \lambda)(\lambda^2 - 3\lambda - 4),$$

thus the EV's are $\lambda_1 = 5, \lambda_2 = 4, \lambda_3 = -1$

$$(A - 5I)\mathbf{v}_1 = 0 \iff \mathbf{v}_1 = (0, k, 0)^\top;$$

$$(A - 4I)\mathbf{v}_2 = 0 \iff \mathbf{v}_2 = (2k, 0, 3k)^\top;$$

$$(A + 1I)\mathbf{v}_3 = 0 \iff \mathbf{v}_3 = (k, 0, -k)^\top$$

EV's of upper- and lower-triangular matrices

Theorem

Diagonal entries of an upper-triangular (lower-triangular) matrix are its EV's

Reason:

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1kn} \\ 0 & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{pmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) \dots (a_{nn} - \lambda)$$

Eigenvector $\mathbf{v}_k$ for $\lambda = a_{kk}$:

$$\mathbf{v}_k = (v_1, \dots, \underbrace{v_k}_{\neq 0}, 0, \dots, 0)^\top$$

How to compute A^n?

Motivating example: Leslie population dynamics model

Consider an organism that lives for two years. Denote

- x_n the number of individuals in their first year;
- y_n the number of individuals in their second year;
- b_1 the birth rate of first-year individuals;
- b_2 the birth rate of second-year individuals;
- d_1 the death rate of the first-year individuals.

Then

$$\begin{aligned} x_{n+1} &= b_1 x_n + b_2 y_n, \\ y_{n+1} &= (1 - d_1) x_n \end{aligned}$$

In matrix notation,

$$\mathbf{z}_{n+1} := \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A\mathbf{z}_n = \cdots = A^n \mathbf{z}_1 = A^{n+1} \mathbf{z}_0$$

Question: How does one compute A^n?

How to compute A^n?

Second example: Fibonacci numbers

Rabbit population evolution:

- f_n is the number of rabbits on year n
- initially, $f_1 = f_2 = 1$
- starting from the third year, $\boxed{f_{n+2} = f_{n+1} + f_n}$, $\qquad n \geq 1$
- f_n is the Fibonacci number: $1, 1, 2, 3, 5, 8, 13, 21, 44, 65, 109, \ldots$
- Clearly, $f_n \leq 2^{n-2}$, but how fast f_n grow?

For $\mathbf{z}_n := (f_n, f_{n+1})^\top$, we find

$$\mathbf{z}_{n+1} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n + f_{n+1} \end{pmatrix} =: A\mathbf{z}_n$$

In matrix notation,

$$\mathbf{z}_{n+1} = A\mathbf{z}_n = \cdots = A^n \mathbf{z}_1$$

Question: How does one compute A^n?

Easy for diagonal A:

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \implies A^k = \begin{pmatrix} a^k & 0 \\ 0 & d^k \end{pmatrix}$$

Idea: reduce to a diagonal form

if $\mathbf{z}_n = P\mathbf{Z}_n$, then

$$P\mathbf{Z}_{n+1} = \mathbf{z}_{n+1} = AP\mathbf{Z}_n \implies \mathbf{Z}_{n+1} = (P^{-1}AP)\mathbf{Z}_n$$

If $B := P^{-1}AP$ can be made diagonal, then

$$\mathbf{z}_n = P\mathbf{Z}_n = PB^n\mathbf{Z}_0 = PB^nP^{-1}\mathbf{z}_0$$

is easy to calculate!

- Assume $P^{-1}AP = D = \mathrm{diag}\{d_1, d_2\}$ and $P = (\mathbf{v}_1\ \mathbf{v}_2)$; then

$$AP = PD \iff A(\mathbf{v}_1\ \mathbf{v}_2) = (\mathbf{v}_1\ \mathbf{v}_2)\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

$$\iff (A\mathbf{v}_1\ A\mathbf{v}_2) = (d_1\mathbf{v}_1\ d_2\mathbf{v}_2)$$

- Thus $A\mathbf{v}_i = d_i\mathbf{v}_i$, i.e., d_i is an *EV* of A with an *EVc* $\mathbf{v}_i$.

Diagonalization:

(A) **Assume:**

- A is a $k \times k$ matrix,
- $r_1, \ldots, r_k$ are eigenvalues of A,
- $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are the corresponding eigenvectors,
- P is composed of columns $\mathbf{v}_i$.

- Then

$$\begin{aligned} AP &= A(\mathbf{v}_1 \cdots \mathbf{v}_k) = (A\mathbf{v}_1 \cdots A\mathbf{v}_k) = (r_1\mathbf{v}_1 \cdots r_k\mathbf{v}_k) \\ &= (\mathbf{v}_1 \cdots \mathbf{v}_k)\,\mathrm{diag}\{r_1, \ldots, r_k\} = P\,\mathrm{diag}\{r_1, \cdots, r_k\}. \end{aligned}$$

- If P is *invertible*, then

$$P^{-1}AP = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix} \tag{1}$$

Theorem

Assume (A) and let P be invertible; then (1) *holds.*

If P is s.t. (1) *holds, then r_i are EV's of A and the columns of P are the corresponding EVc's.*

Solving the k-dimensional systems

Theorem

Let $r_1, \dots, r_h$ be distinct EV's of A and $\mathbf{v}_1, \dots, \mathbf{v}_h$ be the corresponding EVc's. Then $\mathbf{v}_1, \dots, \mathbf{v}_h$ are linearly independent.

Therefore, if A has k distinct EV's, then P is *invertible*!

- Then $\mathbf{z}_{n+1} = A\mathbf{z}_n$ can be transformed to $\mathbf{Z}_{n+1} = D\mathbf{Z}_n$ with $D = \operatorname{diag}\{r_1, \dots, r_k\} = P^{-1}AP$
- the solution is $\mathbf{Z}_n = D^n\mathbf{Z}_0$, or $(Z_i)_n = c_i r_i^n, \quad i = 1, \dots, k$;
- finally,

$$\mathbf{z}_n = P\mathbf{Z}_n = (\mathbf{v}_1, \dots, \mathbf{v}_k)\begin{pmatrix} c_1 r_1^n \\ \vdots \\ c_k r_k^n \end{pmatrix} = c_1 r_1^n \mathbf{v}_1 + \cdots + c_k r_k^n \mathbf{v}_k$$

for suitable $c_1, \dots, c_k$. To find c_i, take $n = 0$: $\mathbf{z}_0 = P\mathbf{c}$

The powers of a matrix

Assume that
$$P^{-1}AP = D = \operatorname{diag}\{r_1, \ldots, r_k\}; \tag{2}$$
then $A = PDP^{-1}$,
$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD^2P^{-1}$$
and, in general,
$$A^n = PD^nP^{-1} = P \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix} P^{-1} \tag{3}$$

Theorem

Assume that there is a nonsingular P satisfying (2). Then A^n is given by (3) and the solution of the difference equation $\mathbf{z}_{n+1} = A\mathbf{z}_n$ with initial vector $\mathbf{z}_0$ is
$$\mathbf{z}_n = P \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix} P^{-1}\mathbf{z}_0 = P \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix} \mathbf{c}$$

Examples

The eigenvalues of

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

are 3, -2 with eigenvectors $\binom{1}{1}$, $\binom{2}{-3}$. Therefore

$$C = PDP^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}^{-1}$$

and

$$C^n = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 0,6 & 0,4 \\ 0,2 & -0,2 \end{pmatrix}$$

This can be used to calculate other functions of C, e.g.

$$C^{1/3} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 3^{1/3} & 0 \\ 0 & (-2)^{1/3} \end{pmatrix} \begin{pmatrix} 0,6 & 0,4 \\ 0,2 & -0,2 \end{pmatrix},$$

$$e^C = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} e^3 & 0 \\ 0 & e^{-2} \end{pmatrix} \begin{pmatrix} 0,6 & 0,4 \\ 0,2 & -0,2 \end{pmatrix}$$

Leslie model reconsidered:

Take $b_1 = 1$, $b_2 = 3$, $d_1 = 1/3$ in the Leslie model:

$$\begin{aligned} x_{n+1} &= 1x_n + 3y_n, \\ y_{n+1} &= \tfrac{2}{3}x_n \end{aligned} \qquad \Longleftrightarrow \qquad \mathbf{z}_{n+1} = \begin{pmatrix} 1 & 3 \\ \frac{2}{3} & 0 \end{pmatrix} \mathbf{z}_n$$

- EV's of A: $\lambda_1 = 2$, $\lambda_2 = -1$ ($\operatorname{tr} A = 1$, $\det A = -2$)
- eigenvectors: $\mathbf{v}_1 = (3, 1)^\top$ and $\mathbf{v}_2 = (3, -2)^\top$
- now

$$P = \begin{pmatrix} 3 & 3 \\ 1 & -2 \end{pmatrix}, \qquad P^{-1} = \frac{1}{9}\begin{pmatrix} 2 & 3 \\ 1 & -3 \end{pmatrix},$$

- A generic solution is ($\mathbf{c}_0 := P^{-1}\mathbf{z}_0$)

$$\mathbf{z}_n = PD^nP^{-1}\mathbf{z}_0 = PD^n\mathbf{c}_0 = c_1 2^n \mathbf{v}_1 + c_2(-1)^n \mathbf{v}_2$$

is asymptotically $c_1 2^n \mathbf{v}_1$ and thus unstable

Fibonacci numbers

For the Fibonacci numbers,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

- EV's of A: $\lambda_1 = \frac{1+\sqrt{5}}{2}$, $\lambda_2 = \frac{1-\sqrt{5}}{2}$ (tr $A = 1$, det $A = -1$)
- eigenvectors: $\mathbf{v}_1 = (1, \lambda_1)^\top$ and $\mathbf{v}_2 = (1, \lambda_2)^\top$
- now

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}, \qquad P^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\lambda_2 & 1 \\ \lambda_1 & -1 \end{pmatrix}$$

- initial condition: $f_0 = 0$, $f_1 = 1$; thus $\mathbf{c}_0 := P^{-1}\mathbf{z}_0 = (1, -1)^\top / \sqrt{5}$

$$\mathbf{z}_n = PD^n P^{-1}\mathbf{z}_0 = PD^n \mathbf{c}_0 = \frac{\lambda_1^n \mathbf{v}_1 - \lambda_2^n \mathbf{v}_2}{\sqrt{5}}$$

so that $f_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} \asymp \frac{\lambda_1^n}{\sqrt{5}}$

Example (Employment model)

- x_n and y_n the fractions of employed/unemployed;
- $a = 0,9$ be the probability of keeping a job
- $d = 0,6$ the probability of remaining unemployed.

Then we get a difference equation

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0,9 & 0,4 \\ 0,1 & 0,6 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

M is a Markov matrix with eigenvalues $r = 1$ and $r = 0,5$;
a general solution is

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 1^n \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 (\tfrac{1}{2})^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0,8 \\ 0,2 \end{pmatrix},$$

and the eventual (stable state) distribution of employed vs unemployed is $4 : 1$

Markov processes

A Markov matrix M is called regular if $\exists$ r s.t. M^r has only positive entries. If $r = 1$, M is called positive

Theorem (23.15)

Assume that M is a regular Markov matrix. Then

(a) 1 *is an eigenvalue of multiplicity* 1

(b) *every other EV r of M satisfies $|r| < 1$*

(c) *eigenvector $\mathbf{w}_1$ corresponding to $r = 1$ has all positive components*

(d) *normalize $\mathbf{w}_1$ so that the entries sum up to 1; then each solution of $\mathbf{x}_{n+1} = M\mathbf{x}_n$ tends to $\mathbf{w}_1$ as $n \to \infty$.*

Remark

Also holds for general positive or even regular matrices with nonnegative entries.

Examples

- Families can be classified as *urban*, *suburban*, and *rural*
- each year
  - 20% of urban fam. move to suburbs and 5% to rural;
  - 2% of suburban fam. move to urban areas and 8% to rural
  - 10% of rural fam. move to cities and 20% to suburbs
- U_n, S_n, and R_n are fractions of families in the corresponding areas in n years from now

Then

$$\begin{pmatrix} U_{n+1} \\ S_{n+1} \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} 0,75 & 0,02 & 0,1 \\ 0,2 & 0,9 & 0,2 \\ 0,05 & 0,08 & 0,7 \end{pmatrix} \begin{pmatrix} U_n \\ S_n \\ R_n \end{pmatrix}$$

The EV's are $r = 1; 0,7$, and $0,65$; the solution is

$$\begin{pmatrix} U_n \\ S_n \\ R_n \end{pmatrix} = \underbrace{\begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}}_{\text{steady state}} + c_2(0,7)^n \begin{pmatrix} 8 \\ -5 \\ -3 \end{pmatrix} + c_3(0,65)^n \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Problem

A flea hops randomly on vertices of a triangle. It is twice as likely to jump clockwise as counterclockwise.

(a) Find the stationary distribution of the Markov chain

(b) Find the probability p_n that after n hops the flea is back where it started.

Solution

- The stationary probability distribution π solves $\pi = \pi P$ with
$$P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix} = \frac{2}{3}J + \frac{1}{3}J^2, \qquad J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
- As $J^3 = I$, its EV's satisfy $\lambda^3 = 1$ and thus are $1, e^{2\pi i/3}, e^{-2\pi i/3}$ with EVc's $\mathbf{v} = (\lambda^2, \lambda, 1)$;
- therefore, EV's of P are $\frac{2}{3}\lambda + \frac{1}{3}\lambda^2$ with the same EVc's
- p_n is the first entry of $(1\ 0\ 0)P^n = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2 + c_3\lambda_3^n\mathbf{v}_3$

Can one always diagonalize?

Question

Is there a nonsingular P s.t. $P^{-1}AP = D$?

Answer:

Only if there are k linearly independent eigenvectors

Example

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

has EV λ of multiplicity 2 but just one EVc $\mathbf{v} = (1,0)^\top$
However, $\mathbf{v}_2 := (0,1)^\top$ is s.t. $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$

Definition

$\mathbf{v}_2, \mathbf{v}_3, \ldots$ are generalized eigenvectors of A for EV λ if

$$(A-\lambda)\mathbf{v}_k = \mathbf{v}_{k-1} \text{ (with } \mathbf{v}_{-1} := \mathbf{0}\text{).}$$

The above A is the simplest Jordan block of size 2

Jordan blocks

Definition

Matrices A and B are called similar if there is a nonsingular P s.t. $P^{-1}AP = B$ (written: $A \sim B$)

Not every matrix is similar to a diagonal one!

Assume $P = (\mathbf{v}_1 \mathbf{v}_2 \dots \mathbf{v}_k)$ is s.t.

$$P^{-1}AP = J_k(\lambda) := \begin{pmatrix} \lambda & 1 & \dots & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & \dots & \lambda \end{pmatrix}$$

Then:

- $\mathbf{v}_1$ is the EVc and $\mathbf{v}_2, \dots, \mathbf{v}_k$ are generalized EVc's
- $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ span a root subspace of A for EV λ
- root subspaces are formed by vectors satisfying $(A - \lambda)^k \mathbf{v} = 0$ and are invariant w.r.t. A!

Jordan normal form

Theorem

$A \sim \oplus J_{k_j}(\lambda_j)$

- Jordan blocks are simple enough to work with!
- The corresponding root subspaces are invariant under $A \implies$ can work with Jordan blocks separately!

What are all matrices satisfying $A^2 = \mathbf{0}$ or $A^2 = A$?

- Look what Jordan blocks satisfy the same relation
- In the first case, the only eigenvalue is $\lambda = 0$ and Jordan blocks are of size at most 2
- In the second case, A is diagonalizable, with eigenvalues 0 or 1 $\implies$ similar to an orthogonal projection

Linear differential equations and stability:

The equation $\dot{\mathbf{x}} = A\mathbf{x}$ subject to $\mathbf{x}(0) = \mathbf{x}_0$ has a solution

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0$$

If $A = PDP^{-1}$, then

$$\mathbf{x}(t) = Pe^{tD}P^{-1}\mathbf{x}_0 = c_1 e^{t\lambda_1}\mathbf{v}_1 + \cdots + c_k e^{t\lambda_k}\mathbf{v}_k$$

Theorem

The steady state $\mathbf{x} \equiv \mathbf{0}$ of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ is

(a) *asymptotically stable if all EV's have negative real part;*

(b) *unstable if there is at least one EV with positive real part;*

(c) *unstable if there are pure imaginary or zero EV's without complete set of eigenvectors;*

(d) *neutrally stable if all EV's have nonpositive real parts and there are enough eigenvectors for pure imaginary and zero EV's*

Example

Example

The system $\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1$ describes a uniform circular motion $(x_1(t), x_2(t)) = \big(r\cos(t+\varphi_0), r\sin(t+\varphi_0)\big)$ on the plane

Reason:

- From the system, $\ddot{x}_1 = -x_1$ and thus $x_1(t) = r\cos(t+\varphi_0)$
- alternatively, the solution to $\dot{\mathbf{x}} = A\mathbf{x}$ is $\mathbf{x}(t) = e^{tA}\mathbf{x}(0)$
- since $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we see that $A^2 = -I$ and thus

$$e^{tA} = I + tA + \tfrac{t^2}{2}A^2 + \cdots = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

- alternatively: the EV's are $\pm i$, thus $e^{tA} = P\,\mathrm{diag}\{e^{it}, e^{-it}\}P^{-1}$
- with $\mathbf{x}(0) = (r\cos\varphi_0, r\sin\varphi_0)$ get the same answer

Invariants of similar matrices

Theorem

If $A \sim B$, then

- $\det A = \det B$;
- $\operatorname{tr} A = \operatorname{tr} B$;
- *A invertible iff B invertible;*
- $\operatorname{rank} A = \operatorname{rank} B$;
- $\dim \ker A = \dim \ker B$;
- $p_A(\lambda) = p_B(\lambda)$;
- *the eigenvalues of A and B are the same*
- *dimensions of root subspaces are the same*

Corollary (Determinant and trace rule)

For a matrix A with eigenvalues $\lambda_1, \dots, \lambda_k$

- $\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_k$;
- $\operatorname{tr} A = \lambda_1 + \lambda_2 + \cdots + \lambda_k$.