Linear Algebra

Lecture Notes

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Invertibility in a nutshell

A short summary of the previous lecture:

- not all matrices are invertible
- an $n \times n$ matrix A is invertible

  $\iff$ columns of A are linearly independent

  $\iff$ columns of A span $\mathbb{R}^n$
- the inverse matrix is unique (if exists)
- elementary matrices (*row multiplication, row replacement, row interchange*) are invertible
- a unique solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} := A^{-1}\mathbf{b}$
- $(AB)^{-1} = B^{-1}A^{-1}$ for invertible A and B

Lecture 3. Determinants,
or
Very useful formulae that are never used in practice

Outline

Outline

Solving the 2×2 system symbolically

$$\begin{aligned} ax + by &= e & &\text{(assume } a \neq 0\text{)} \\ cx + dy &= f & &\times a \\ ax + \quad by &= e & & \\ acx + ady &= af & &-c \times (1) \\ ax + \quad by &= e & & \\ (ad - bc)y &= af - ce & & \end{aligned}$$

Conclusions:

- The system (coeff. matrix) is non-singular $\iff ad - bc \neq 0$
- Solution:

$$y = \frac{af - ce}{ad - bc}; \qquad ax = e - by = \frac{ead - baf}{ad - bc}$$

Definition

The number $ad - bc$ is called the determinant of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and is denoted $\det A$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Solution in terms of determinants

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \quad \leftrightsquigarrow \quad \left(\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right)$$

- $\det A = 0 \iff$ the vectors $(a\ c)^\top$ and $(b\ d)^\top$ are collinear
  $\iff$ the matrix A is singular
  $\iff$ the system $A\mathbf{x} = \mathbf{b}$ is not always consistent
- $\det A \neq 0 \implies$ Cramer's rule for solutions:

$$x = \frac{de - bf}{ad - bc} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{af - ce}{ad - bc} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}},$$

Mnemonics (cross rule):

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\,d - b\,c$$

Example

$$\begin{aligned} x + y &= 1 \\ 2x + 3y &= 1 \end{aligned} \iff \begin{aligned} x + y &= 1 \\ y &= -1 \end{aligned} \iff \begin{aligned} x &= 2 \\ y &= -1 \end{aligned}$$

$$\widehat{A} = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 1 \end{array}\right)$$

$$x = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1 \cdot 3 - 1 \cdot 1}{1 \cdot 3 - 1 \cdot 2} = 2, \qquad y = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1 \cdot 1 - 1 \cdot 2}{1 \cdot 3 - 1 \cdot 2} = -1,$$

Properties of 2×2 determinants

1. If the columns (or rows) of A are exchanged to get A_1, then $\det(A) = -\det(A_1)$
2. $\det A$ is a *linear* function of any row/column:
$$\det\begin{pmatrix} \mathbf{a}_1 + \mathbf{a}_2 \\ \mathbf{a} \end{pmatrix} = \det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a} \end{pmatrix} + \det\begin{pmatrix} \mathbf{a}_2 \\ \mathbf{a} \end{pmatrix}$$
$$\det\begin{pmatrix} s\mathbf{a}_1 \\ \mathbf{a} \end{pmatrix} = s\det\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}$$
3. If columns (or rows) of A are collinear, then $\det A = 0$
4. $\det A$ does not change if we add to a row a number times another row
5. A invertible $\iff$ $\det(A) \neq 0$; then $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Remark

Item 2 does not say
$$\det(A+B) = \det A + \det B \quad \text{or} \quad \det(tA) = t\det(A)!$$

Example

- Row additivity:

$$\begin{vmatrix} \color{red}{1} & \color{blue}{1} \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} \color{red}{1} & 0 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 0 & \color{blue}{1} \\ 2 & 3 \end{vmatrix} = 3 + (-2) = 1$$

- Row linearity:

$$\begin{vmatrix} \color{red}{2} \cdot 1 & \color{red}{2} \cdot 1 \\ 2 & 3 \end{vmatrix} = \color{red}{2} \cdot 3 - \color{red}{2} \cdot 2 = 2 = \color{red}{2} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

- Non-additivity:

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \neq 0 + 0 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}$$

Outline

When is a 3×3 matrix non-singular?

$A = (a_{ij})_{3\times 3}$; assume $a_{11} \neq 0$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{pmatrix}$$

$$\sim \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{23} - a_{13}a_{21} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{13}a_{31} \end{pmatrix}$$

$\Longrightarrow$ The 2×2 blue submatrix must be nonsingular!

$$\Delta := (a_{11}a_{22} - a_{12}a_{21})(a_{11}a_{33} - a_{13}a_{31}) - (a_{11}a_{23} - a_{13}a_{21})(a_{11}a_{32} - a_{12}a_{31}) \neq 0$$

$$\frac{\Delta}{a_{11}} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) =: \det(A)$$

Mnemonic rules for 3×3 matrices

- Row expansion:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- arrow rule

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

- triangle rule:

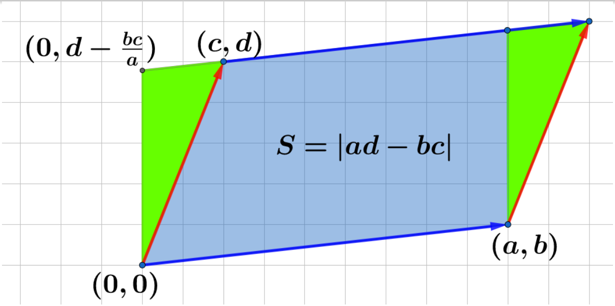
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} - \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

Outline

Geometrical meaning of determinants

In dimension 2

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is the (signed) area or the parallelogram formed by vectors $(a\ b)$ and $(c\ d)$ on the xy-plane



In dimension 3

$\det(A)$ is the (signed) volume of the parallelepiped formed by the rows of A in the xyz-space

Geometrical meaning of determinant

Example (Change of variables in integrals)

If $\mathbf{x} = h(\mathbf{y})$ is a 1-1 change of variables, then

$$\int_K f(\mathbf{x})\,d\mathbf{x} = \int_{h^{-1}(K)} f(h(\mathbf{y}))J_h(\mathbf{y})\,d\mathbf{y};$$

J_h is the Jacobian $\equiv$ absolute value of the determinant of $\left(\frac{\partial h_j}{\partial y_k}\right)$
J_h tells us how much a small volume has got expanded under h

Example (Area of an ellipsis)

The area of the ellipsis E of semi-axes a and b is πab
Idea: under the transformation $\mathbf{x} = (x_1, x_2) = (ay_1, by_2) = h(\mathbf{y})$, the ellipsis E,

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$$

becomes a unit circle $y_1^2 + y_2^2 \leq 1$

Outline

Definition (Abstract definition of determinant)

The determinant is a function $\det$ on square matrices possessing the following three properties:

1. The determinant of the identity matrix is 1 (Norming)

$$\det(I_2) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1; \qquad \det(I_3) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

2. The determinant changes its sign when two rows are exchanged (Antisymmetry)

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc) = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3. The determinant depends linearly on the first row (Row multilinearity)

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}, \qquad \begin{vmatrix} sa & sb \\ c & d \end{vmatrix} = s\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Outline

Further properties of determinants

4. If two rows of A are equal, then $\det(A) = 0$ (Equal rows)

Interchange those rows and apply Rule 2

5. Subtracting a multiple of one row from another row leaves the determinant unchanged (Row operation)

Apply Rule 3 and Rule 4

6. If A has a row of zeros, then $\det(A) = 0$ (Zero row)

Add any row to the zero row and use Rules 5 and 4

7. If A is triangular, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$ (Triangularity)

Reduce A to a diagonal D by row operations to get

$$\det(A) = \det(D) = a_{11}a_{22}\cdots a_{nn}\det(I_n)$$

8. A nonsingular $\implies \det(A) \neq 0$ (Singularity)

Reduce A to U and apply Rule 7

For nonsingular A, have the Product of pivots rule:

$$\det(A) = \pm\det(U) = \pm d_1 d_2 \cdots d_n$$

Further properties of determinants

9. $\det(AB) = \det(A)\det(B)$ **(Product rule)**

Assume A and B nonsingular as otherwise nothing to prove

Check $d(A) := \det(AB)/\det(B)$ has properties 1–3

thus $d(A) = \det(A)$

Corollary: $\det(A^{-1}) = 1/\det(A)$

10. $\det(A^\top) = \det(A)$ **(Transpose rule)**

For nonsingular A, write as $PA = LDU$, with row exchanges P, diagonal D, lower/upper-triangular L / D with 1 on the diagonal

$$\det(P)\det(A) = \det(L)\det(D)\det(U) = \det(D)$$

$$\det(A^\top)\det(P^\top) = \det(U^\top)\det(D)\det(L^\top) = \det(D)$$

$$\implies \boxed{\det(A^\top) = \det(A)} \qquad \text{as} \qquad \det(P) = \det(P^\top)$$

Corollary:

All row rules become column rules!

Outline

Example: calculating the determinant

$$A = \begin{pmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{pmatrix}$$

$$\color{red}{2\times} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 5 & 5 \end{pmatrix} \sim \color{red}{2\times} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 10 \end{pmatrix}$$

$$\implies \det(A) = 20$$

Vandermonde determinant

$$B = \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \sim \begin{matrix} \\ \color{red}{(b-a)\times} \\ \color{red}{(c-a)\times} \end{matrix} \begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{pmatrix}$$

$$\implies \det(B) = (a-b)(b-c)(c-a)$$

Example: Equation of a plane in $\mathbb{R}^3$

Given: three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$

Need: equation of the plane π through P_1, P_2, and P_3

Solution:

- let $P(x, y, z)$ be a generic point of the plane
- set $\mathbf{u} := \overrightarrow{P_1P_2}$, $\mathbf{v} := \overrightarrow{P_1P_3}$, and $\mathbf{w} := \overrightarrow{P_1P}$
- then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$
- Conclusion: $P \in \pi \iff$ the matrix

$$A(x, y, z) := \begin{pmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{pmatrix}$$

  is non-invertible

- $$\det A(x, y, z) = 0 \iff ax + by + cz = d, \quad \text{for some} \quad a, b, c, d$$

Outline

Row cofactor expansion

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{(1,1)-minor}} - a_{12} \underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\text{(1,2)-minor}} + a_{13} \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{(1,3)-minor}}$$

$$= a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{(1,1)-cofactor}} + a_{12} \underbrace{(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\text{(1,2)-cofactor}} + a_{13} \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{(1,3)-cofactor}}$$

Row cofactor expansion

Definition (Minors and cofactors)

Assume A is an $n \times n$ matrix.

- The (i,j)-minor M_{ij} of A is the determinant of the $(n-1) \times (n-1)$ submatrix obtained after removing i^{th} row and j^{th} column from A.
- $C_{ij} := (-1)^{i+j} M_{ij}$ is the (i,j)-cofactor of A.

Theorem (Row Cofactor Expansion)

$$\boxed{\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}}$$

Taking the transposes, one gets

Theorem (Column Cofactor Expansion)

$$\boxed{\det(A) = a_{1i}C_{1i} + a_{2i}C_{2i} + \cdots + a_{ni}C_{ni}}$$

Example

$$A_4 := \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Use the cofactor expansion along the first row:

$$C_{11} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = \det(A_3)$$

$$C_{12} = {\color{red}-} \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = {\color{red}+} \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = \det(A_2)$$

$$\begin{aligned} \det(A_4) &= 2C_{11} + (-1)C_{12} \\ &= 2\det(A_3) - \det(A_2) = 2(4) - 3 = 5 \\ \det(A_n) &= 2\det(A_{n-1}) - \det(A_{n-2}) = 2(n) - (n-1) = n+1 \end{aligned}$$

Cross-product of vectors

- $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are non-collinear vectors in $\mathbb{R}^3$
- denote by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ basis vectors in $\mathbb{R}^3$ and formally use row cofactor expansion in

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = C_{11}\mathbf{e}_1 + C_{12}\mathbf{e}_2 + C_{13}\mathbf{e}_3 = (C_{11}, C_{12}, C_{13})$$

Definition and properties

- the vector (C_{11}, C_{12}, C_{13}) is called the cross-product, or vector product of $\mathbf{u}$ and $\mathbf{v}$ and is denoted $\mathbf{u} \times \mathbf{v}$
- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$
- the length of $\mathbf{u} \times \mathbf{v}$ is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$

  (the last two statements is a problem from HW1)

Plane in $\mathbb{R}^3$

Given: three points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$

Need: equation of the plane π through P_1, P_2, and P_3

Solution: $P(x, y, z)$ be a generic point of the plane; then

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

$$- \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

$$= C_{11}x + C_{12}y + C_{13}z - d = 0$$

- Therefore, in $ax + by + cz = d$ the normal vector $\mathbf{n} := (a, b, c)$ can be taken to be the cross-product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u} = \overrightarrow{P_1P_2}$ and $\mathbf{v} = \overrightarrow{P_1P_3}$

Outline

Row linearity of determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix}$$

$$+ \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Formula for the determinant:

Permutation of rows:

$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ one-to-one

$$\sigma \rightsquigarrow \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix}$$

P_σ is the matrix performing the row permutations σ

Theorem (Formula for the determinant)

$$\boxed{\det(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \det(P_\sigma)}$$

Remark

- *There are $n!$ permutations of $\{1, 2, \ldots, n\}$*
- *All permutations form a group denoted S_n*
- *"group" here is an algebraic notion (like a field)*
- *the formula is never used for numerical computations!*

Example: characteristic polynomial

Definition

Assume A is an $n \times n$ matrix. Then $p_A(\lambda) := \det(A - \lambda I)$ is called the characteristic polynomial of A.

Properties

- p_A is a polynomial of degree n:
$$p_A(\lambda) = (-\lambda)^n + \mathrm{tr}(A)(-\lambda)^{n-1} + \cdots + c_1\lambda + \det(A)$$
- indeed,
$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = \dots$$
- therefore, p_A has at most n zeros

Question

Assume A and B are non-singular $n \times n$ matrices. Are there real r such that both $A - r \cdot B$ and $B - r \cdot A$ are non-singular?

Outline

Inverse matrix via determinants

Theorem (Inverse of A)

Assume A is an $n \times n$ matrix with $\det(A) \neq 0$. Then A is invertible and $(A^{-1})_{ij} = C_{ji} / \det(A)$

Proof: The row cofactor expansion gives $AC^{\top} = \det(A) I_n$:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix} = \det(A) I_n$$

Definition

$C^{\top}$ is called the adjugate (or classical adjoint) $\operatorname{adj}(A)$ of A

Example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Cramer's rule

Theorem (Cramer's rule)

Assume that A is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$. Then the j^{th} entry of $\mathbf{x} := A^{-1}\mathbf{b}$ is equal to

$$x_j = \frac{\det(B_j)}{\det(A)},$$

where B_j is A with the j^{th} column replaced by $\mathbf{b}$.

Proof.

j^{th} column expansion of $\det(B_j)$ gives

$$\begin{aligned}\det(B_j) &= b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj} \\ &= (C^\top)_{j1} b_1 + (C^\top)_{j2} b_2 + \cdots + (C^\top)_{jn} b_n,\end{aligned}$$

which is the j^{th} entry of $C^\top \mathbf{b}$ □

Example solving systems via Cramer's rule:

$$\begin{aligned} x + \ \ y &= 1 \\ 2x + 3y &= 1 \end{aligned} \iff \begin{aligned} x + y &= \ \ 1 \\ y &= -1 \end{aligned} \iff \begin{aligned} x \quad &= \ \ 2 \\ y &= -1 \end{aligned}$$

$$\widehat{A} = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 3 & 1 \end{array}\right)$$

$$x = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1 \cdot 3 - 1 \cdot 1}{1 \cdot 3 - 1 \cdot 2} = 2, \qquad y = \frac{\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1 \cdot 1 - 1 \cdot 2}{1 \cdot 3 - 1 \cdot 2} = -1,$$

Determinants in a nutshell

A short summary of determinants

- Determinant is an antisymmetric multilinear function on square matrices
- $\det A = 0 \quad \iff \quad A$ is singular (non-ivertible)
- geometrically, $\det A$ is a (signed) volume of a parallelepiped formed by columns (or rows) of A
- can be expanded via cofactors
- explicit formulae for A^{-1} and $A^{-1}\mathbf{b}$
- never used directly in numerical computations