Linear Algebra
Seminar 11: Symmetric matrices and quadratic forms
(make sure you can write on the board)

Problem 1. (3 pts) Find a matrix P that orthogonally diagonalizes A, and determine $P^{-1}AP$:

$$(a)\quad \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix} \qquad (d)\quad \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Problem 2. (3 pts) Decide for or against the positive definiteness of :

$$(a)\quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad (b)\quad \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix} \qquad (c)\quad \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^2$$

Problem 3. (3 pts) Let c be a constant and let

$$B = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix}$$

(a) For what value(s) of c will the matrix $B^\top B$ be invertible?

(b) For what value(s) of c will the matrix $B^\top B$ be positive definite?

Justify your answers.

Problem 4. (3 pts)

(a) Find the (generalized) Cholesky decomposition of the symmetric matrix

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

In other words, find a lower-triangular L with 1 on the diagonal and a diagonal matrix D such that $A = LDL^\top$.

Hint: $A = L(DL^\top)$ is an LU-factorization of A

(b) The *Gram* matrix $G = (g_{jk})_{j,k=1}^n$ of the set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ in $\mathbb{R}^m$ is defined via $g_{jk} = \mathbf{u}_j \cdot \mathbf{u}_k$. Show that G is always positive semi-definite; moreover, G is positive definite iff the set S is linearly independent. In the latter case, what is the Cholesky decomposition of G?

Hint: it is related to QR-decomposition of the $m \times n$ matrix U with columns $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$

Problem 5. (4 pts) Find an orthogonal change of variables that eliminates the cross product terms in the quadratic form Q, and express Q in terms of the new variables:

(a) $Q(x_1, x_2) = 2x_1^2 + 2x_2^2 - 2x_1x_2$;

(b) $Q(x_1, x_2, x_3) = 3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3$

(c) Sketch the curve $3x^2 + 4xy + 6y^2 = 14$ in the xy-plane.

Problem 6. (4 pts)

(a) Find all values of a and b for which there exists a 3×3 symmetric matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

(b) Reconsider the problem if $\lambda_3 = 3$.

(c) Find all symmetric matrices A satisfying the conditions of part (b).

Extra problems
(to be discussed if time permits)

Problem 7. Find all s and t for which the following matrices are (a) positive or (b) negative definite:

$$S = \begin{pmatrix} s & -4 & -4 \\ -4 & s & 4 \\ -4 & 4 & s \end{pmatrix}, \qquad T = \begin{pmatrix} t & -3 & 0 \\ -3 & t & 4 \\ 0 & 4 & t \end{pmatrix}$$

Problem 8. Find all values of k for which the quadratic form

$$5x_1^2 + x_2^2 + kx_3^2 + 4x_1x_2 - 2x_1x_3 - 2x_2x_3$$

is positive definite.

Problem 9. Consider the matrix A and the vector $\mathbf{v}_1$, where

$$A = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 6 & -2 \\ 1 & -2 & 3 \end{pmatrix}, \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

(a) Show that $\mathbf{v}_1$ is an eigenvector of A, and find its corresponding eigenvalue. Then find the other two eigenvalues and eigenvectors and orthogonally diagonalize A.

(b) Is the quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ positive definite, negative definite or indefinite? Find the principal axes of Q and the corresponding transition matrix.

1. a) $A = \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}$

1. $\lambda_1 = 2$, $\lambda_2 = 7$

2. $\lambda_1 = 2$: $A - \lambda_1 I = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \Rightarrow \begin{cases} -2x+y=0 \\ y=2t \end{cases} \Rightarrow \begin{cases} x=t \\ y=2t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{v_1}$

   $\lambda_2 = 7$: $A - \lambda_2 I = \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+2y=0 \\ y=t \end{cases} \Rightarrow \begin{cases} x=-2t \\ y=t \end{cases} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \vec{v_2}$

3. $\langle \vec{v_1}, \vec{v_2} \rangle = 0 \Rightarrow \{\vec{v_1}, \vec{v_2}\}$ is orthogonal

   $\vec{w_1} = \frac{\sqrt5}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w_2} = \frac{\sqrt5}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow P = \frac{\sqrt5}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

4. $D = P^{-1}AP = P^TAP = \frac{\sqrt5}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \cdot \frac{\sqrt5}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac15\begin{pmatrix} 2 & 4 \\ -14 & 7 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \frac15\begin{bmatrix} 10 & 0 \\ 0 & 35 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$

b) $\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix}$

1. $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 12 - 3 - 2 = 7$

2. $\lambda = 3$: $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x=0 \\ y=t \\ z=0 \end{cases} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{v_1}$

   $\lambda_2 = 2$: $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+2z=0 \\ y=0 \\ z=t \end{cases} \Rightarrow \begin{cases} x=-2t \\ y=0 \\ z=t \end{cases} \Rightarrow \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \vec{v_2}$

   $\lambda_3 = 7$: $\begin{pmatrix} -4 & 0 & 2 \\ 0 & -4 & 0 \\ 2 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} 2x-z=0 \\ y=0 \\ z=2t \end{cases} \Rightarrow \begin{cases} x=t \\ y=0 \\ z=2t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \vec{v_3}$

3. $\langle \vec{v_1}, \vec{v_2} \rangle = \langle \vec{v_1}, \vec{v_3} \rangle = \langle \vec{v_2}, \vec{v_3} \rangle = 0 \Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is orthogonal

   $\vec{w_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w_2} = \frac{\sqrt5}{5}\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\vec{w_3} = \frac{\sqrt5}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \Rightarrow P = \frac{\sqrt5}{5}\begin{pmatrix} 0 & -2 & 1 \\ 5/\sqrt5 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

4. $D = P^{-1}AP = P^TAP = \frac{\sqrt5}{5}\begin{pmatrix} 0 & 5/\sqrt5 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 6 \end{pmatrix} \cdot \frac{\sqrt5}{5}\begin{pmatrix} 0 & -2 & 1 \\ 5/\sqrt5 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \frac15\begin{pmatrix} 0 & 3\sqrt5 & 0 \\ -4 & 0 & 2 \\ 7 & 0 & 14 \end{pmatrix}\begin{pmatrix} 0 & -2 & 1 \\ 5/\sqrt5 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} = \frac15\begin{pmatrix} 15 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 35 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

c) $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$

1. $\lambda_1 = 4$ $\begin{cases} \lambda_1+\lambda_2+\lambda_3 = \operatorname{trace}(A) \\ \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(A) \end{cases} \Rightarrow \begin{cases} \lambda_2 + \lambda_3 = 8 \\ \lambda_2 \cdot \lambda_3 = \frac{52}{4} = 13 \end{cases} \Rightarrow p(\lambda) = \lambda^2 - 8\lambda + 13 \Rightarrow \lambda_{2,3} = \frac{8 \pm 2\sqrt3}{2} = 4 \pm \sqrt3$

   $\det(A) = 3(20-1) + 1\cdot(-5) = 52$, $D = 64 - 52 = 12$

2. $\lambda_1 = 4$: $\begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ y+z=0 \\ z=t \end{cases} \Rightarrow \begin{cases} x=t \\ y=-t \\ z=t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \vec{v_1}$

   $\lambda_2 = 4-\sqrt3$: $\begin{pmatrix} -1+\sqrt3 & -1 & 0 \\ -1 & \sqrt3 & 1 \\ 0 & 1 & 1+\sqrt3 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{\sqrt3+1}{2} & 0 \\ -1 & \sqrt3 & 1 \\ 0 & 1 & \sqrt3+1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{\sqrt3+1}{2} & 0 \\ 0 & \frac{\sqrt3-1}{2} & 1 \\ 0 & 1 & \sqrt3+1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{\sqrt3+1}{2} & 0 \\ 0 & 1 & \sqrt3+1 \\ 0 & 1 & \sqrt3+1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{\sqrt3+1}{2} & 0 \\ 0 & 1 & \sqrt3+1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x - \frac{\sqrt3+1}{2}y = 0 \\ y + (\sqrt3+1)z = 0 \\ z = -t \end{cases} \Rightarrow \begin{cases} x = \frac{(\sqrt3+1)^2}{2}t = (\sqrt3+2)t \\ y = (\sqrt3+1)t \\ z = -t \end{cases} \Rightarrow$

   $\Rightarrow \begin{bmatrix} \sqrt3+2 \\ \sqrt3+1 \\ -1 \end{bmatrix} = \vec{v_2}$

   $\lambda_3 = 4+\sqrt3$: $\begin{pmatrix} -1-\sqrt3 & -1 & 0 \\ -1 & -\sqrt3 & 1 \\ 0 & 1 & 1-\sqrt3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{\sqrt3-1}{2} & 0 \\ -1 & -\sqrt3 & 1 \\ 0 & 1 & 1-\sqrt3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{\sqrt3-1}{2} & 0 \\ 0 & -\frac{\sqrt3+1}{2} & 1 \\ 0 & 1 & 1-\sqrt3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{\sqrt3-1}{2} & 0 \\ 0 & 1 & 1-\sqrt3 \\ 0 & 1 & 1-\sqrt3 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{\sqrt3-1}{2} & 0 \\ 0 & 1 & 1-\sqrt3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x + \frac{\sqrt3-1}{2}y = 0 \\ y + (1-\sqrt3)z = 0 \\ z = t \end{cases} \Rightarrow \begin{cases} x = \frac{(\sqrt3-1)^2}{2}t = (\sqrt3-2)t \\ y = (\sqrt3-1)t \\ z = t \end{cases} \Rightarrow$

   $\Rightarrow \begin{bmatrix} \sqrt3-2 \\ \sqrt3-1 \\ 1 \end{bmatrix} = \vec{v_3}$

3. $\langle \vec{v_1}, \vec{v_2} \rangle = \langle \vec{v_2}, \vec{v_3} \rangle = \langle \vec{v_1}, \vec{v_3} \rangle = 0 \Rightarrow \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ is orthogonal

   $\vec{w_1} = \frac{1}{\sqrt3}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{w_2} = \frac{1}{\sqrt{6\sqrt3+12}}\begin{bmatrix} \sqrt3+2 \\ \sqrt3+1 \\ -1 \end{bmatrix}$, $\vec{w_3} = \frac{1}{\sqrt{-6\sqrt3+12}}\begin{bmatrix} \sqrt3-2 \\ \sqrt3-1 \\ 1 \end{bmatrix}$

   $P = \begin{pmatrix} 1/\sqrt3 & \frac{\sqrt3+2}{\sqrt{6\sqrt3+12}} & \frac{\sqrt3-2}{\sqrt{-6\sqrt3+12}} \\ -1/\sqrt3 & \frac{\sqrt3+1}{\sqrt{6\sqrt3+12}} & \frac{\sqrt3-1}{\sqrt{-6\sqrt3+12}} \\ 1/\sqrt3 & \frac{-1}{\sqrt{6\sqrt3+12}} & \frac{1}{\sqrt{-6\sqrt3+12}} \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4-\sqrt3 & 0 \\ 0 & 0 & 4+\sqrt3 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$

1. $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 4$

2. $\lambda_1 = 1$: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 3 & -3 \\ 0 & 3 & -3 \\ 1 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow \begin{cases} x+y-2z=0 \\ y-z=0 \\ z=t \end{cases} \Rightarrow \begin{cases} x=t \\ y=t \\ z=t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \vec{v_1}$

   $\lambda_2 = \lambda_3 = 4$: $\begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+y+z=0 \\ y=-t \\ z=-s \end{cases} \Rightarrow \begin{cases} x=t+s \\ y=-t \\ z=-s \end{cases} \Rightarrow \left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\} = \{\vec{v_2}, \vec{v_3}\}$

3. $\vec{w_1} = \frac{1}{\sqrt3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{w_2} = \frac{1}{\sqrt2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v_3' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{\sqrt2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{0}{\sqrt3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-1/\sqrt2 \\ 1/\sqrt2 \\ -1 \end{pmatrix} \Rightarrow w_3 = \frac{1}{\sqrt{3-\sqrt2}}\begin{pmatrix} 1-1/\sqrt2 \\ 1/\sqrt2 \\ -1 \end{pmatrix}$

   $P = \begin{pmatrix} 1/\sqrt3 & 1/\sqrt2 & (1-1/\sqrt2)/\sqrt{3-\sqrt2} \\ 1/\sqrt3 & -1/\sqrt2 & 1/\sqrt2\sqrt{3-\sqrt2} \\ 1/\sqrt3 & 0 & -1/\sqrt{3-\sqrt2} \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

2. a) $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$ $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = 0$ $\Rightarrow$ positive semidefinite

b) $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}$ $\lambda_1 = 4$, $\lambda_2 = 4-\sqrt3$, $\lambda_3 = 4+\sqrt3$ $\Rightarrow$ positive

c) $A^2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^2$ eigenvalues of A: $\lambda_1 = -2$, $\lambda_{2,3} = 1 \pm \sqrt3$ $\Rightarrow A^2$ is positive

3. $B = \begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix}$

a) $B^TB = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 3 & c \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -1 & 3 \\ 2 & c \end{pmatrix} = \begin{pmatrix} 6 & -6+2c \\ -6+2c & 18+c^2 \end{pmatrix}$ — is symmetric

$\det(B^TB) = 6(18+c^2) - (2c-6)^2 = 108 + 6c^2 - 4c^2 + 24c - 36 = 2c^2 + 24c + 72 = 0$

$c^2 + 12c + 36 = 0$

$(c+6)^2 = 0$

$c = -6$

B^TB is invertible for $c \in \mathbb{R} \setminus \{-6\}$

b) Sylvester criterion: A symmetric A is positive definite $\Leftrightarrow$ all principals minors are positive

$M_2 = \det(B^TB) = (c+6)^2 \Rightarrow c \neq -6$

$M_1 = 6 \Rightarrow c \in \mathbb{R}$

B^TB is positive definite for $c \in \mathbb{R} \setminus \{-6\}$

4. a) $A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$

$DL^T = U$

apply elementary row operations: $\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 2 \\ x & 1 & 0 & -4 & 1 & -2 \\ x & x & 1 & 2 & -2 & -2 \end{array}\right) \begin{matrix} \\ +4r_1 \\ -2r_1 \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 2 \\ -4 & 1 & 0 & 0 & -15 & 6 \\ 2 & x & 1 & 0 & 6 & -6 \end{array}\right) + \frac{6}{15}r_2 \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 2 \\ -4 & 1 & 0 & 0 & -15 & 6 \\ 2 & -\frac{6}{15} & 1 & 0 & 0 & -\frac{18}{5} \end{array}\right)$ (L | U)

now apply the same row operations as column operations on U: $\left(\begin{array}{ccc|ccc} 1 & -4 & 2 & 1 & x & x \\ 0 & -15 & 6 & 0 & 1 & x \\ 0 & 0 & -\frac{18}{5} & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 2 \\ 0 & -15 & 6 & 0 & 1 & x \\ 0 & 0 & -\frac{18}{5} & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & 2 \\ 0 & -15 & 0 & 0 & 1 & -\frac{6}{15} \\ 0 & 0 & -\frac{18}{5} & 0 & 0 & 1 \end{array}\right)$ (D | L^T) or $D = \begin{pmatrix} d_1 = M_1 & 0 & \dots \\ 0 & d_2 = \frac{M_2}{d_1} & 0 \dots \\ 0 & 0 & d_3 = \frac{d_1 \cdot M_3}{d_2} \dots \end{pmatrix}$

($+4c_1, -2c_1$; $+\frac{6}{15}c_2$)

Check: $A = LDL^T = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & -\frac{6}{15} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -15 & 0 \\ 0 & 0 & -18/5 \end{pmatrix}\begin{pmatrix} 1 & -4 & 2 \\ 0 & 1 & -6/15 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix} = A$

b) $G_{jk} = u_j \cdot u_k$, $u_k \in \mathbb{R}^m$, $g_{jk} = \vec{u_j} \cdot \vec{u_k}$

1) positive semidefiniteness:

$\vec{x}^T G \vec{x} = \sum_{i,j} x_i \langle \vec{u_i}, \vec{u_j} \rangle x_j = \sum_{i,j} x_i x_j \langle \vec{u_i}, \vec{u_j} \rangle = \sum_{i,j} \langle x_i\vec{u_i}, x_j\vec{u_j} \rangle = \langle \sum_i x_i\vec{u_i}, \sum_j x_j\vec{u_j} \rangle = \| \sum_i x_i \vec{u_i} \|^2 \geq 0$

2) positive definiteness:

$\forall x \neq 0$: $\sum_i x_i \vec{u_i} \neq 0$

$\sum_i x_i \vec{u_i} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}\begin{pmatrix} \vec{u_1} & \vec{u_2} & \dots & \vec{u_n} \end{pmatrix}$ = linear combination of $\{\vec{u_1}, \dots, \vec{u_n}\} \Rightarrow \sum_i x_i\vec{u_i} = 0$ has only the trivial solution $\Leftrightarrow S$ is lin. indep.

3) Cholesky decomposition

$G = U^TU$ – by definition

$U = QR \Rightarrow G = (QR)^TQR = R^TQ^TQR = \left[\begin{matrix} Q \text{ is orthogonal} \\ Q^T = Q^{-1} \end{matrix}\right] = R^TR$

Q^T is left-triangular $\Rightarrow R = DL^T \Rightarrow G = R^TR = (DL^T)^T DL^T = LD^TDL^T = LD^2L^T = LD^*L^T$

5. a) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\lambda_1 = 3$, $\lambda_2 = 1$

$\lambda_1 = 3$: $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ y=t \end{cases} \Rightarrow \begin{cases} x=-t \\ y=t \end{cases} \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{v_1}$

$\lambda_2 = 1$: $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + r_1 \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x-y=0 \\ y=t \end{cases} \Rightarrow \begin{cases} x=t \\ y=t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v_2}$

$\vec{w_1} = \frac{\sqrt2}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{w_2} = \frac{\sqrt2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow P = \frac{\sqrt2}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = P\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow$ principal axis theorem $\Rightarrow Q(P\vec{y}) = \vec{y}^T(P^TAP)\vec{y} = r_1y_1^2 + r_2y_2^2 + \dots + r_ny_n^2 \Rightarrow Q(P(y_1, y_2)) = 3y_1^2 + y_2^2$

b) $A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 4 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

$\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = 7$

$\lambda_1 = 1$: $\begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & -2 \\ 0 & -2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ y-2z=0 \\ z=t \end{cases} \Rightarrow \begin{cases} x=-2t \\ y=2t \\ z=t \end{cases} \Rightarrow \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \vec{v_1}$

$\lambda_2 = 4$: $\begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 0 \\ 0 & 4 & -2 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 1 \end{pmatrix} \Rightarrow \begin{cases} -x+2y=0 \\ -2y+z=0 \\ z=2t \end{cases} \Rightarrow \begin{cases} x=2t \\ y=t \\ z=2t \end{cases} \Rightarrow \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \vec{v_2}$

$\lambda_3 = 7$: $\begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \begin{cases} -2x+y=0 \\ y+z=0 \\ z=2t \end{cases} \Rightarrow \begin{cases} x=t \\ y=2t \\ z=-2t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \vec{v_3}$

$Q(y_1, y_2, y_3) = y_1 + 4y_2 + 7y_3$

c) $3x^2 + 4xy + 6y^2 = 14$

$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ $\lambda_1 = 2$, $\lambda_2 = 7$

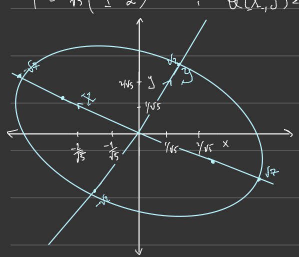
$\lambda_1 = 2$: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} - 2r_1 \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} x+2y=0 \\ y=t \end{cases} \Rightarrow \begin{cases} x=-2t \\ y=t \end{cases} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \vec{v_1}$

$\lambda_2 = 7$: $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} + 2r_2 \sim \begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} \Rightarrow \begin{cases} 2x-y=0 \\ y=2t \end{cases} \Rightarrow \begin{cases} x=t \\ y=2t \end{cases} \Rightarrow \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{v_2}$

$\vec{w_1} = \frac{1}{\sqrt5}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\vec{w_2} = \frac{1}{\sqrt5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P = \frac{1}{\sqrt5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$, $Q(X, Y) = 2X^2 + 7Y^2 = 14$ $| :14$

$\frac{X^2}{7} + \frac{Y^2}{2} = 1$ – ellipse



6. a) $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = 7$

$\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

symmetric matrices have perpendicular eigenvectors:

$\langle \vec{v_1}, \vec{v_2} \rangle = 0$

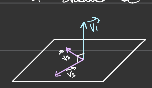
$\langle \vec{v_1}, \vec{v_3} \rangle = b \Rightarrow b = 0$

$\langle \vec{v_2}, \vec{v_3} \rangle = a + b \Rightarrow a = 0 \Rightarrow$ there are no such a and b for a symmetric A

b) $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 3$

$\vec{v_1} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v_3} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$

$\vec{v_1}$ should be orthogonal to the plane spanned by $\{\vec{v_2}, \vec{v_3}\}$



so: $\langle \vec{v_1}, \vec{v_2} \rangle = \langle \vec{v_1}, \vec{v_3} \rangle = 0$, but $\langle \vec{v_2}, \vec{v_3} \rangle$ = any

$\langle \vec{v_1}, \vec{v_3} \rangle = b = 0$, $\langle \vec{v_2}, \vec{v_3} \rangle = a \in \mathbb{R} \setminus \{0\} \Rightarrow \vec{v_3} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

c) $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$, $A = PDP^{-1}$, $P^{-1} = \frac12\begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & -1 \end{pmatrix}$

$A = PDP^{-1} = [illegible] = \frac12\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}$

7. S: $\lambda_1 = s+8$, $\lambda_2 = s-4 = \lambda_3$ — a) positive: $s+8>0$, $s-4>0 \Rightarrow s>4$ — b) negative: $s+8<0$, $s-4<0 \Rightarrow s<-8$

T: $\lambda_1 = t$, $\lambda_2 = t-5$, $\lambda_3 = t+5$ — $t>0$, $t-5>0$, $t+5>0 \Rightarrow t>5$ — $t<0$, $t-5<0$, $t+5<0 \Rightarrow t<-5$

8. $\begin{pmatrix} 5 & 2 & -1 \\ 2 & 1 & -1 \\ -1 & -1 & k \end{pmatrix}$ $M_1 = 5$

$M_2 = 5 - 4 = 1$

$M_3 = k - 2 > 0$

$k > 2$