Linear Algebra
Seminar 13: Singular value decomposition and applications
(make sure you can write on the board)

Problem 1. (4pts)

(a) Show that any $m \times n$ matrix of rank 1 can be written as $\mathbf{u}\mathbf{v}^\top$ for column vectors $\mathbf{u}$ and $\mathbf{v}$ of size m and n respectively.

(b) Assume that A is an $m \times n$ matrix of rank r. Without using the SVD, prove that A can be written as the sum of r matrices of rank 1. Is this representation unique?

(c) What are the ranks r for the matrices A and B below? Write A and B as the sum of r pieces $\mathbf{u}\mathbf{v}^\top$ of rank one. (Do not use the SVD yet!)

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}.$$

Problem 2. (4pts)

(a) Show that for any $m \times n$ matrix A, the nonzero singular values of A and $A^\top$ coincide.

(b) Find the singular values of the following matrices:

$$(a) \quad \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}; \qquad (b) \quad \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix}; \qquad (c) \quad \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}.$$

(c) If $A = U\Sigma V^\top$ is the SVD of A, what is the SVD of $A^\top$?

(d) Assume that $A = \mathbf{u}\mathbf{v}^\top$ is an $m \times n$ matrix of rank 1. Find the SVD of A.

Problem 3. (3 pts) Find the singular value decomposition of the matrix

$$A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}.$$

To this end, complete the following steps:

(a) How many singular values σ_j does A have? How many of them are nonzero? Find the nonzero singular values $\sigma_1, \dots, \sigma_r$.

(b) find the right singular vectors $A\mathbf{v}_j = \sigma_j \mathbf{u}_j$, $j = 1, \dots, r$;

(c) find the left singular vectors $A^\top \mathbf{u}_j = \sigma_j \mathbf{v}_j$, $j = 1, \dots, r$;

(d) form the unitary matrices U and V and write the singular value decomposition of A.

Problem 4. (3 pts) For the matrix A in Problem 3

(a) find the vector $\mathbf{x} \in \mathbb{R}^3$ for which $A\mathbf{x}$ has the maximal length α, and find that α;

(b) find the best rank one approximation for the matrix A of Problem 3 in the Frobenius norm.

Problem 5. Find the positive definite square root $S = V\Sigma V^\top$ of $A^\top A$ and its polar decomposition $A = QS$:

$$A = \frac{1}{\sqrt{10}} \begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix}$$

Problem 6. (3 pts)

(a) Assume that A has linearly independent columns. Show that the matrix $A^+ := (A^\top A)^{-1}A^\top$ is well defined and gives a left inverse of A. What is AA^+?

(b) Show that the above matrix A^+ is the Moore–Penrose pseudo-inverse of A (i.e., that $AA^+A = A$, $A^+AA^+ = A^+$, and the matrices AA^+ and A^+A are symmetric)

(c) Assume that $A = QR$ is a QR-factorization of A. Express the above A^+ in terms of Q and R.

Extra problems
(to be discussed if time permits)

Problem 7. Find the SVD and the Moore–Penrose pseudo-inverse $V\Sigma^+U^\top$ of the matrices

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 8. When columns of a matrix A are linearly dependent and a vector $\mathbf{b}$ is not in the column space of A, then the least square solution of $A\mathbf{x} = \mathbf{b}$ is not unique. Find the minimum-length least square solution $\mathbf{x}^+ = A^+\mathbf{b}$ of the equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

1. a) $(\forall \vec{v} \in \mathbb{R}^n)(\exists \vec{u} \in \mathbb{R}^m): A\vec{v} = k\vec{u}$ A maps every vector $\vec{v}$ in $\mathbb{R}^m$ to a scalar multiple of $\vec{u}$ in $\mathbb{R}^n$, so every column of A is a multiple of $\vec{u}$

$A = (v_1\vec{u} \;\; v_2\vec{u} \dots v_n\vec{u}) = \vec{u}(v_1, v_2, \dots, v_n)^T = \vec{u}\,\vec{v}^T$

b) we can express any non-zero i^{th} row of A as a Matrix of rank 1 as $\vec{u}\,\vec{w}^T$

c) $A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 4 & -6 & 8 \\ 3 & -6 & 9 & -12 \\ -4 & 8 & -12 & 16 \end{pmatrix}$, $\vec{u} = (1,-2,3,-4)^T$; $A = (\vec{u}, -2\vec{u}, 3\vec{u}, -4\vec{u}) = \vec{u}\,\vec{u}^T \Rightarrow \operatorname{rank} A = 1$

$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$, $\vec{u_1} = (1,2,3,4)^T$, $\vec{u_2} = (1,1,1,1)^T$; $B = (\vec{u_1}, \vec{u_1}+\vec{u_2}, \vec{u_1}+2\vec{u_2}, \vec{u_1}+3\vec{u_2}) = \vec{u_1}(1,1,1,1) + \vec{u_2}(0,1,2,3) \Rightarrow \operatorname{rank} B = 2$

2. a) $A = U\Sigma V^T$

$A^T = (U\Sigma V^T)^T = V(U\Sigma)^T = V\Sigma^T U^T$ – the singular values are preserved, yet not the singular vectors

b) a) $A = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}(0 \; 1 \; 2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$: $\lambda_1 = 0$, $\lambda_2 = 5 \Rightarrow \sigma_1 = \sqrt{5}$, $\sigma_2 = \sigma_3 = 0$

b) $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 5 \\ 5 & 13 & 13 \\ 5 & 13 & 13 \end{pmatrix}$: $\lambda_1 = 0$, $\lambda_2 + \lambda_3 = 28$, $\lambda_2 \cdot \lambda_3 = 0$ $\begin{pmatrix} 2-\lambda & 5 & 5 \\ 5 & 13-\lambda & 13 \\ 5 & 13 & 13-\lambda \end{pmatrix} \Rightarrow \det(A) = \lambda(-\lambda^2 + 28\lambda - 2) = 0 \Rightarrow \lambda_{1,2} = 14 \pm \sqrt{194} \Rightarrow$ $\sigma_{1,2} = \sqrt{14 \pm \sqrt{194}}$, $\sigma_3 = 0$

c) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$: $\begin{cases} \lambda_1 + \lambda_2 = 9 \\ \lambda_1 \cdot \lambda_2 = 16 \end{cases} \Rightarrow \lambda_{1,2} = \frac{\sqrt{17} \pm 9}{2} \Rightarrow \sigma_{1,2} = \sqrt{\frac{9 \pm \sqrt{17}}{2}}$

c) $A^T = (U\Sigma V^T)^T = V(U\Sigma)^T = V\Sigma^T U^T$

d) $A = \vec{u}\,\vec{v}^T$, $B = A^T A$

$\vec{v_1} = \frac{\vec{v}}{\|\vec{v}\|}$, $\vec{u_1} = \frac{\vec{u}}{\|\vec{u}\|}$, $\sigma_1 = \|A\vec{v_1}\| = \frac{\|A\vec{v}\|}{\|\vec{v}\|}$

$A = \sigma_1 \vec{u_1}\,\vec{v_1}^T$, $\Sigma = \underbrace{\begin{pmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}}_{m}$ n

$\left.\begin{array}{l} \vec{u_1} \\ \vec{u_2}, \dots, \vec{u_m} \text{ - ONB for } N(A^T) \end{array}\right\}$ ONB of $\mathbb{R}^m$

$\left.\begin{array}{l} \vec{v_1} \\ \vec{v_2}, \dots, \vec{v_n} \text{ - ONB for } N(A) \end{array}\right\}$ ONB of $\mathbb{R}^n$

3. $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$

a) $B = A^T A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix}$

$\lambda_1 = 9$, $\lambda_2 = 25$, $\lambda_3 = 0$ $\Rightarrow$ $\sigma_1 = 3$, $\sigma_2 = 5$, $\sigma_3 = 0$ $\Rightarrow \Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}$

b) $\vec{w_1} = 4\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $\vec{w_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w_3} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$

$\vec{v_1} = \frac{1}{4 \cdot 3\sqrt{2}} \cdot 4\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $\vec{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v_3} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \Rightarrow V = \frac{1}{3\sqrt{2}}\begin{pmatrix} 1 & 3 & -2\sqrt{2} \\ -1 & 3 & 2\sqrt{2} \\ 4 & 0 & \sqrt{2} \end{pmatrix}$

c) $\vec{u_1} = \frac{A \cdot \vec{v_1}}{\sigma_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{u_2} = \frac{A \cdot \vec{v_2}}{\sigma_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\Rightarrow U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

d) $A = U\Sigma V^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} \cdot \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 & -1 & 4 \\ 3 & 3 & 0 \\ -2\sqrt{2} & 2\sqrt{2} & \sqrt{2} \end{bmatrix}$

4. a) $\|A\vec{x}\|^2 = (A\vec{x})^T(A\vec{x}) = \vec{x}^T A^T A \vec{x} = \sigma_x^2 \cdot \|\vec{x}\|^2 = \sigma_2^2 \cdot \|\vec{v_2}\| = 25$

$\|A\vec{x}\| = \sqrt{25} = 5$

b) $\|A - A_1\|_F \Rightarrow \sigma_2 \vec{u_2}\,\vec{v_2}^T = 5 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}^T = \frac{5}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

5. $A = \frac{1}{\sqrt{10}}\begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix}$

$M = A^T A = \frac{1}{10}\begin{pmatrix} 100 & 60 \\ 60 & 100 \end{pmatrix} = \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$

$S = V\Sigma V^T$:

$\lambda_1 = 4$: $\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{w_1} \Rightarrow \vec{v_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\sigma_1 = 2$

$\lambda_2 = 16$: $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{w_2} \Rightarrow \vec{v_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\sigma_2 = 4$

$V = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$

$S = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \cdot 2\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$A = QS = Q \cdot V\Sigma V^T$, $Q = UV^T$

$\vec{u_1} = \frac{A\vec{v_1}}{\sigma_1} = \frac{2\sqrt{5}}{5} \cdot \frac{1}{2}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\vec{u_2} = \frac{A\vec{v_2}}{\sigma_2} = \frac{4\sqrt{5}}{5} \cdot \frac{1}{4}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\Rightarrow U = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$

$Q = \frac{1}{\sqrt{5}} \cdot \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$

$A = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{1}{\sqrt{10}}\begin{pmatrix} 10 & 6 \\ 0 & 8 \end{pmatrix}$

6. a) $A = QR$

if A has indep. columns $\Rightarrow A^T A$ is invertible

$A \cdot A^+ = A(A^T A)^{-1} A^T = A \cdot A^{-1} \cdot (A^T)^{-1} A^T = I \cdot I = I_n$

b) $AA^+A = A(A^T A)^{-1} A^T A = I_n \cdot A = A$

$A^+ A A^+ = (A^T A)^{-1} A^T \cdot A \cdot (A^T A)^{-1} A^T = (A^T A)^{-1} A^T \cdot I_n = A^+$

$AA^+ = I_n \Rightarrow$ is symmetric

$A^+ A = (A^T A)^{-1} A^T A = C^{-1} C = I_n \Rightarrow$ is symmetric

c) $A = QR$

$A^+ = (A^+ A)^{-1} A^T = ((QR)^T (QR))^{-1} (QR)^T = (R^T Q^T Q R)^{-1} R^T Q^T = \left[Q^T Q = I \right] = R^{-1} (R^T)^{-1} R^T Q^T = R^{-1} Q^T$