Linear Algebra
Seminar 4: Linear vector spaces

Problem 1. (3pts)

(a) Determine whether the set of all pairs of real numbers of the form $(1, x)$ $(x \in \mathbb{R})$ with the operations

$$(1, x) + (1, x') = (1, x + x'), \qquad k \cdot (1, x) = (1, kx)$$

form a vector space.

(b) Determine which of the following are subspaces of the corresponding vector space V:

(1) the set of all vectors $(x, y, z)^\top$ in $V = \mathbb{R}^3$ with $y = x + z$;
(2) the set of all $n \times n$ symmetric matrices in $V = M_{n\times n}(\mathbb{R})$ (i.e. satisfying $A^\top = A$);
(3) the set of all $n \times n$ antisymmetric matrices A in $V = M_{n\times n}(\mathbb{R})$ (i.e., satisfying $A^\top = -A$);
(4) the set of all sequences $(x_1, x_2, \dots)$ in $V = \mathbb{R}^\infty$ with finitely many nonzero entries.

Problem 2. (3pts)

(a) Let V be the set of all ordered pairs of real numbers under the following addition and multiplication by scalar operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \qquad k \cdot \mathbf{u} = (ku_1, ku_2)$$

Determine whether V is a vector space.

(b) Determine which of the following are subspaces of the corresponding vector space V:

(1) the set of all $n \times n$ matrices A in $V = M_{n\times n}(\mathbb{R})$ such that $A^2 = A$;
(2) all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in $V = \mathcal{P}_3$ for which $a_0 + a_1 + a_2 + a_3 = 0$;
(3) all functions f in $V = \mathcal{F}(\mathbb{R})$ such that $f(-x) = f(x)$ for all $x \in \mathbb{R}$;
(4) the set of all $n \times n$ matrices A in $V = M_{n\times n}(\mathbb{R})$ such that $AB = BA$ for some fixed matrix B;

Problem 3. (4pts)

(a) Express the following vectors as linear combinations of $\mathbf{u} = (2, 1, 4)$, $\mathbf{v} = (1, -1, 3)$, and $\mathbf{w} = (3, 2, 5)$:

(1) $(-9, -7, -15)$ (2) $(6, 11, 6)$ (3) $(0, 0, 0)$ (4) $(7, 8, 9)$

(b) In each part, determine whether the given vectors span $\mathbb{R}^3$ and are linearly independent.

(1) $\mathbf{v}_1 = (1, 2, 6)$, $\mathbf{v}_2 = (3, 4, 1)$, $\mathbf{v}_3 = (4, 3, 1)$, $\mathbf{v}_4 = (3, 3, 1)$
(2) $\mathbf{v}_1 = (2, -1, 3)$, $\mathbf{v}_2 = (4, 1, 1)$, $\mathbf{v}_3 = (8, -1, 8)$

Problem 4. (4pts)

(a) In each part express the vector as a linear combination of $p_1(x) = 2 + x + 4x^2$, $p_2(x) = 1 - x + 3x^2$, and $p_3(x) = 3 + 2x + 5x^2$:

(1) $-9 - 7x - 15x^2$ (2) $6 + 11x + 6x^2$ (3) 0 (4) $7 + 8x + 9x^2$

(b) In each part, determine whether the given vectors span $\mathbb{R}^3$ and are linearly independent .

(1) $\mathbf{v}_1 = (3, 1, 4)$, $\mathbf{v}_2 = (2, -3, 5)$, $\mathbf{v}_3 = (5, -2, 9)$, $\mathbf{v}_4 = (1, 4, -1)$
(2) $\mathbf{v}_1 = (2, 2, 2)$, $\mathbf{v}_2 = (0, 1, 1)$, $\mathbf{v}_3 = (0, 0, 1)$

Problem 5. (4pts) Determine whether the solution space of the system $A\mathbf{x} = \mathbf{0}$ for the following coefficient matrix is a line through the origin, a plane through the origin, or the origin only. If it is a plane, find an equation for it. If it is a line, find parametric equations for it.

$$(a)\ \begin{pmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & -4 & -5 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & 2 & -3 \\ 2 & -5 & 3 \\ -1 & 0 & 8 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 1 & -1 & 1 \\ 2 & -1 & 4 \\ 3 & 1 & 11 \end{pmatrix} \qquad (d)\ \begin{pmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{pmatrix}$$

Problem 6. (4pts)

(a) Prove that the vectors $\mathbf{v}_1 = (1,6,4)$, $\mathbf{v}_2 = (2,4,-1)$, $\mathbf{v}_3 = (-1,2,5)$ and the vectors $\mathbf{w}_1 = (1,-2,-5)$, $\mathbf{w}_2 = (0,8,9)$ span the same subspace of $\mathbb{R}^3$.

(b) Show that the three vectors $\mathbf{v}_1 = (0,3,1,-1)$, $\mathbf{v}_2 = (6,0,5,1)$, and $\mathbf{v}_3 = (4,-7,1,3)$ form a linearly dependent set in $\mathbb{R}^4$.

(c) Express each vector in part (b) as a linear combination of the other two.

Extra problems

(to be discussed if time permits)

Problem 7. Which of the following are linear combinations of $\mathbf{u} = (0,-2,2)$ and $\mathbf{v} = (1,3,-1)$?
(a) $(2,2,2)$ (b) $(3,1,5)$ (c) $(0,4,5)$ (d) $(0,0,0)$

Problem 8. For what values of λ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$\mathbf{v}_1 = (\lambda,1,-1), \quad \mathbf{v}_2 = (1,\lambda,1), \quad \mathbf{v}_3 = (-1,1,\lambda).$$

Problem 9. Assume that $f_1, f_2, \dots, f_n$ are functions of a variable $x \in \mathbb{R}$. Prove that they are linearly independent if and only if there are real points $a_1, a_2, \dots, a_n$ such that $\det\bigl(f_i(a_j)\bigr)_{i,j=1}^n \neq 0$.
Hint: one part is straightforward; we need to prove that if that determinant is zero for all choices of the points $a_1, a_2, \dots, a_n$, then the functions are linearly dependent. Induction on n can be helpful here: for induction step, assume for $n = k$ the above determinant is not zero for some choice of $a_1, \dots, a_k$ and use the determinant of $(k+1)\times(k+1)$ matrix for functions $f_1, \dots, f_{k+1}$ and the points $a_1, \dots, a_k, x$.

Problem 10. (a) Assume that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of three vectors in a vector space V is linearly independent. Prove that the set $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is linearly independent as well.

(b) Assume that the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of n vectors in a vector space V is linearly independent and that

$$\mathbf{w}_j := \sum_k a_{jk}\mathbf{v}_k, \quad j = 1, \dots, m$$

Under what conditions on the $m \times n$ matrix $A = (a_{jk})$ is the set $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ linearly independent as well?

1. a) Addition: $(1,x)+(1,x')=(1,\ x+x')$: $x, x'\in\mathbb{R} \Rightarrow x+x'\in\mathbb{R} \Rightarrow$ it is closed under addition
Multiplication: $k\cdot(1,x)=(1,\ kx)$, $k,x\in\mathbb{R} \Rightarrow kx\in\mathbb{R} \Rightarrow$ it is closed under multiplication
Forms a vector space

b) ① $S=\{(x,y,z)^T\in\mathbb{R}^3 \mid y=x+z\}$

S is a subspace of $\mathbb{R}^3$ if: 1) $(x,y,z)^T+(x',y',z')^T=(x+x',\ y+y',\ z+z')^T$ is in S and
2) $\alpha\cdot(x,y,z)^T=(\alpha x,\ \alpha y,\ \alpha z)^T\in S$ too.

Lets check: 1) $y+y'=(x+x')+(z+z')=(x+z)+(x'+z')=y+y'$ ✓
2) $\alpha x+\alpha z=\alpha(x+z)=\alpha y$ ✓.
3) Zero vector also $\in\mathbb{R}$: $0=0+0$
It is a subspace

② $S=\{A_{n\times n}\in M_{n\times n}(\mathbb{R}) \mid A^T=A\}$

1) addition: $(A+B)^T=A^T+B^T=A+B$, because A,B are symmetric. $A+B$ is also symmetric $\Rightarrow A+B\in S$ ✓

2) multipl. $(\alpha A)^T=\alpha A^T=\alpha A$, because A is symmetric. αA is also symmetric $\Rightarrow \alpha A\in S$

3) zero vector $\vec{0}$ is also symmetric $\in S$

It is a subspace

③ $S=\{A_{n\times n}\in M_{n\times n}(\mathbb{R}) \mid A^T=-A\}$

1) $(A+B)^T=A^T+B^T=-A-B=-(A+B)$ is antisymmetric $\Rightarrow -(A+B)\in S$

2) $(\alpha A)^T=\alpha\cdot A^T=\alpha(-A)=-\alpha A$ is antisymmetric $\Rightarrow -\alpha A\in S$

3) $\vec{0}\in S$: $O^T=O=-O$, $O_{n\times n}$ – zero matrix

④ $S=\{(x_1,x_2,\dots)\in\mathbb{R}^\infty \mid (\exists n\in\mathbb{Z})\{x_1,\dots,x_{n-1}\neq 0 \wedge x_n, x_{n+1},\dots=0\}$

1) $(a_1+b_1,\ a_2+b_2,\ \dots,\ a_i+b_i,\ \dots)$

$(\exists n\in\mathbb{Z})\{a_1+b_1,\ \dots,\ a_{n-1}+b_{n-1}\neq 0 \wedge a_n+b_n,\ a_{n+1}+b_{n+1},\ \dots=0\}\in S$

2) $\alpha(a_1,a_2,\dots a_n,\dots)$, where $a_i, i<n\neq 0$ and $a_j, j\geq n=0$, then

3) $\vec{0}\in S \Rightarrow$ there are 0 nonzero values $\qquad \alpha\cdot a_j,\ j\geq n=0 \Rightarrow \in S$

It is a subspace

2. a) $u+v=(u_1+v_1+1,\ u_2+v_2+1)$, $k\cdot u=(ku_1,\ ku_2)$

let $\vec{v}=\vec{0}$: $\vec{u}+\vec{0}=(u_1+0+1,\ u_2+0+1)=(u_1+1,\ u_2+1)\neq\vec{u}$

$\vec{u}+\vec{0}\neq\vec{u}$ ← not satisfies the property that $\vec{u}+\vec{0}=\vec{u}$

It is **not** a vector space

b) ① $S=\{A_{n\times n}\in M_{n\times n}(\mathbb{R}) \mid A^2=A\}$

1) $(A+B)^2=A^2+AB+BA+B^2\neq A+B \Rightarrow$ it is not closed under addition

It is **not** a subspace

② $S=\{a_0+a_1x+a_2x^2+a_3x^3\in P_3 \mid a_0+a_1+a_2+a_3=0\}$

1) $a_0+b_0+a_1x+b_1x+a_2x^2+b_2x^2+a_3x^3+b_3x^3=(a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+(a_3+b_3)x^3$

$\sum_{i=0}^{3} a_i+b_i=0 \Rightarrow \in S$

2) $\alpha(a_0+a_1x+a_2x^2+a_3x^3)=\alpha a_0+\alpha a_1x+\alpha a_2x^2+\alpha a_3x^3$

$\sum_{i=0}^{3}\alpha a_i=0 \Rightarrow \in S$

3) $\vec{0}$: $a_0+a_1\cdot 0+a_2\cdot 0+a_3\cdot 0\in S$

It **is** a subspace

③ $S=\{f(x)\in\mathcal{F}(\mathbb{R}) \mid f(-x)=f(x)\}$ $\qquad f$ is an even function

1) $(f+g)(x)$ is even if g is even $\Rightarrow \in S$

2) $\alpha f(x)$ is even $\Rightarrow \in S$

3) $f(-0)=f(0) \Rightarrow \vec{0}\in S$

It **is** a subspace

④ $S=\{A_{n\times n}\in M_{n\times n}(\mathbb{R}) \mid AB=BA$ for some $B\}$

1) $(A+C)B=AB+CB=BA+BC=B(A+C)\in S$

2) $(\alpha A)B=\alpha(AB)=\alpha(BA)\in S$

3) $\vec{0}\cdot B=B\cdot\vec{0}\in S$

It **is** a subspace

3. a) $u=(2,1,4)$ $\quad V=(1,-1,3)$ $\quad W=(3,2,5)$

Find inverse matrix.

$$\left(\begin{array}{ccc|ccc} 2&1&3&1&0&0\\ 1&-1&2&0&1&0\\ 4&3&5&0&0&1 \end{array}\right)\begin{matrix}-2r_2\\ \\ -2r_1\end{matrix}\quad \left(\begin{array}{ccc|ccc} 0&3&-1&1&-2&0\\ 1&-1&2&0&1&0\\ 0&1&-1&-2&0&1 \end{array}\right)\begin{matrix}-3r_3\\ \\ \\ \end{matrix}\quad \left(\begin{array}{ccc|ccc} 0&0&2&7&-2&-3\\ 1&-1&2&0&1&0\\ 0&1&-1&-2&0&1 \end{array}\right)\quad \left(\begin{array}{ccc|ccc} 1&-1&2&0&1&0\\ 0&1&-1&-2&0&1\\ 0&0&2&7&-2&-3 \end{array}\right)\begin{matrix}-r_3\\ +\frac{1}{2}r_3\\ \cdot\frac{1}{2}\end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 1&-1&0&-7&3&3\\ 0&1&0&3/2&-1&-1/2\\ 0&0&1&7/2&-1&-3/2 \end{array}\right)\begin{matrix}+r_2\\ \\ \\ \end{matrix}\quad \left(\begin{array}{ccc|ccc} 1&0&0&-11/2&2&5/2\\ 0&1&0&3/2&-1&-1/2\\ 0&0&1&7/2&-1&-3/2 \end{array}\right)$$

$\vec{c}=A^{-1}\vec{b}$

① $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}-9\\-7\\-15\end{pmatrix}=\begin{pmatrix}-\frac{11}{2}(-9)+2(-7)+\frac{5}{2}(-15)\\ \frac{3}{2}(-9)+(-1)(-7)+(-\frac{1}{2})(-15)\\ \frac{7}{2}(-9)+(-1)(-7)+(-\frac{3}{2})(-15)\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$ $\qquad \vec{b}=-2\vec{u}+\vec{v}-2\vec{w}$

② $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}6\\11\\6\end{pmatrix}=\begin{pmatrix}-\frac{11}{2}\cdot 6+2\cdot 11+\frac{5}{2}\cdot 6\\ \frac{3}{2}\cdot 6+(-1)\cdot 11+(-\frac{1}{2})\cdot 6\\ \frac{7}{2}\cdot 6+(-1)\cdot 11+(-\frac{3}{2})\cdot 6\end{pmatrix}=\begin{pmatrix}4\\-5\\1\end{pmatrix}$ $\qquad \vec{b}=4\vec{u}-5\vec{v}+\vec{w}$

③ $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ $\qquad \vec{b}=0\vec{u}+0\vec{v}+0\vec{w}$

④ $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}-\frac{11}{2}\cdot 7+2\cdot 8+\frac{5}{2}\cdot 9\\ \frac{3}{2}\cdot 7-1\cdot 8-\frac{1}{2}\cdot 9\\ \frac{7}{2}\cdot 7-1\cdot 8-\frac{3}{2}\cdot 9\end{pmatrix}=\begin{pmatrix}0\\-2\\3\end{pmatrix}$ $\qquad \vec{b}=0\vec{u}-2\vec{v}+3\vec{w}$

b). Vectors span $\mathbb{R}^3 \Leftrightarrow$ matrix rank $=3$

$x=0$ is the only solution for $Ax=0 \Rightarrow$ vectors are linearly independent

① $\begin{pmatrix}1&2&6\\3&4&1\\4&3&1\\3&3&1\end{pmatrix}\begin{matrix}\\ \\ -r_1-r_2\\ -r_2\end{matrix}$ there is at least 1 free var, so the set of vectors is linearly dependent

$\begin{pmatrix}1&2&6\\3&4&1\\0&-3&-6\\0&-1&0\end{pmatrix}\begin{matrix}\\ -3r_1\\ -3r_3\\ \\ \end{matrix}\quad \begin{pmatrix}1&2&6\\0&-2&-17\\0&0&-6\\0&-1&0\end{pmatrix}\begin{matrix}\\ -2r_4\\ \\ \\ \end{matrix}\quad \begin{pmatrix}1&2&6\\0&0&-17\\0&0&-6\\0&-1&0\end{pmatrix}\begin{matrix}\\ -\frac{17}{6}r_3\\ \\ \\ \end{matrix}\quad \begin{pmatrix}1&2&6\\0&0&0\\0&0&-6\\0&-1&0\end{pmatrix}\quad \begin{pmatrix}1&2&6\\0&-1&0\\0&0&-6\\0&0&0\end{pmatrix}$ rank $=3$ vectors span $\mathbb{R}^3$

② $\begin{pmatrix}2&-1&3\\4&1&1\\8&-1&8\end{pmatrix}\begin{matrix}\\ -2r_1\\ -4r_1\end{matrix}\quad \begin{pmatrix}2&-1&3\\0&3&-5\\0&3&-4\end{pmatrix}\begin{matrix}\\ \\ -r_2\end{matrix}\quad \begin{pmatrix}2&-1&3\\0&3&-5\\0&0&1\end{pmatrix}$ rank $=3 \Rightarrow$ Vectors span $\mathbb{R}^3$, vectors are linearly independent

4. a) $p_1(x)=2+x+4x^2$,
$p_2(x)=1-x+3x^2$
$p_3(x)=3+2x+5x^2$

$P=\begin{pmatrix}2&1&4\\1&-1&3\\3&2&5\end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 2&1&4&1&0&0\\ 1&-1&3&0&1&0\\ 3&2&5&0&0&1 \end{array}\right)\begin{matrix}-2r_2\\ \\ -3r_2\end{matrix}\quad \left(\begin{array}{ccc|ccc} 0&3&-2&1&-2&0\\ 1&-1&3&0&1&0\\ 0&5&-4&0&-3&1 \end{array}\right)\begin{matrix}\\ \\ -\frac{5}{3}r_1\end{matrix}\quad \left(\begin{array}{ccc|ccc} 0&3&-2&1&-2&0\\ 1&-1&3&0&1&0\\ 0&0&-\frac{2}{3}&-\frac{5}{3}&\frac{1}{3}&1 \end{array}\right)\begin{matrix}\cdot\frac{1}{3}\\ \\ \\ \end{matrix}\quad \left(\begin{array}{ccc|ccc} 0&1&-\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}&0\\ 1&-1&3&0&1&0\\ 0&0&-\frac{2}{3}&-\frac{5}{3}&\frac{1}{3}&1 \end{array}\right)\begin{matrix}-r_3\\ \\ \cdot(-\frac{3}{2})\end{matrix}$$

$$\left(\begin{array}{ccc|ccc} 0&1&0&2&-1&-1\\ 1&-1&3&0&1&0\\ 0&0&1&5/2&-1/2&-3/2 \end{array}\right)\begin{matrix}\\ +r_1-3r_3\\ \\ \end{matrix}\quad \left(\begin{array}{ccc|ccc} 0&1&0&2&-1&-1\\ 1&0&0&-11/2&3/2&7/2\\ 0&0&1&5/2&-1/2&-3/2 \end{array}\right)\quad \left(\begin{array}{ccc|ccc} 1&0&0&-11/2&3/2&7/2\\ 0&1&0&2&-1&-1\\ 0&0&1&5/2&-1/2&-3/2 \end{array}\right)$$

$\vec{c}=(A^{-1})^T\vec{b}$

① $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}-9\\-7\\-15\end{pmatrix}=\begin{pmatrix}-\frac{11}{2}(-9)+2(-7)+\frac{5}{2}(-15)\\ \frac{3}{2}(-9)+(-1)(-7)+(-\frac{1}{2})(-15)\\ \frac{7}{2}(-9)+(-1)(-7)+(-\frac{3}{2})(-15)\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$

② $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}6\\11\\6\end{pmatrix}=\begin{pmatrix}-\frac{11}{2}\cdot 6+2\cdot 11+\frac{5}{2}\cdot 6\\ \frac{3}{2}\cdot 6+(-1)\cdot 11+(-\frac{1}{2})\cdot 6\\ \frac{7}{2}\cdot 6+(-1)\cdot 11+(-\frac{3}{2})\cdot 6\end{pmatrix}=\begin{pmatrix}4\\-5\\1\end{pmatrix}$

③ $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

④ $\begin{pmatrix}-11/2&2&5/2\\ 3/2&-1&-1/2\\ 7/2&-1&-3/2\end{pmatrix}\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}-\frac{11}{2}\cdot 7+2\cdot 8+\frac{5}{2}\cdot 9\\ \frac{3}{2}\cdot 7-1\cdot 8-\frac{1}{2}\cdot 9\\ \frac{7}{2}\cdot 7-1\cdot 8-\frac{3}{2}\cdot 9\end{pmatrix}=\begin{pmatrix}0\\-2\\3\end{pmatrix}$

b) ① $\begin{pmatrix}3&1&4\\2&-3&5\\5&-2&9\\1&4&-1\end{pmatrix}\begin{matrix}-3r_4\\ -2r_4\\ -r_1-r_2\\ \\ \end{matrix}$ is linear dependent – there is at least one free variable (+third row is a combination of r_1, r_2)

$\begin{pmatrix}0&-11&7\\0&-11&7\\0&0&0\\1&4&-1\end{pmatrix}\begin{matrix}\\ +r_1\\ \\ \\ \end{matrix}\quad \begin{pmatrix}0&-11&7\\0&0&0\\0&0&0\\1&4&-1\end{pmatrix}\quad \begin{pmatrix}1&4&1\\0&-11&7\\0&0&0\\0&0&0\end{pmatrix}$ doesn't span $\mathbb{R}^3$ because rank $=2\neq 3$

② $\begin{pmatrix}2&2&2\\0&1&1\\0&0&1\end{pmatrix}$ span $\mathbb{R}^3$ (rank=3)
there is a unique solution for homogenious case,
thus vectors are linear independent

5. a) $\begin{pmatrix}-1&1&1\\3&-1&0\\2&-4&-5\end{pmatrix}\begin{matrix}\\ +3r_1\\ +2r_1\end{matrix}\quad \begin{pmatrix}-1&1&1\\0&2&3\\0&-2&-3\end{pmatrix}\begin{matrix}\\ +r_3\\ \\ \end{matrix}\quad \begin{pmatrix}-1&1&1\\0&2&3\\0&0&0\end{pmatrix}\begin{matrix}\\ \cdot\frac{1}{2}\\ \\ \end{matrix}\quad \begin{pmatrix}-1&1&1\\0&1&3/2\\0&0&0\end{pmatrix}\begin{matrix}-r_2\\ \\ \\ \end{matrix}\quad \begin{pmatrix}-1&0&-\frac{1}{2}\\0&1&\frac{3}{2}\\0&0&0\end{pmatrix}$

$\begin{cases} z=s\\ -x-\frac{1}{2}z=0\\ y+3/2z=0\end{cases}\qquad \begin{cases} z=s\\ x=-\frac{1}{2}z\\ y=-\frac{3}{2}z\end{cases}$ or $\begin{cases} z=2s\\ x=-s\\ y=-3s\end{cases}$ a line

b) $\begin{pmatrix}1&2&-3\\2&-5&3\\-1&0&8\end{pmatrix}\begin{matrix}\\ -2r_1\\ +r_1\end{matrix}\quad \begin{pmatrix}1&2&-3\\0&-9&9\\0&0&5\end{pmatrix}$ ← one unique solution – a point $(0,0,0)$

c) $\begin{pmatrix}1&-1&1\\2&-1&4\\3&1&11\end{pmatrix}\begin{matrix}\\ -2r_1\\ -3r_1\end{matrix}\quad \begin{pmatrix}1&-1&1\\0&1&2\\0&4&8\end{pmatrix}\begin{matrix}\\ \\ -4r_2\end{matrix}\quad \begin{pmatrix}1&-1&1\\0&1&2\\0&0&0\end{pmatrix}\begin{matrix}+r_2\\ \\ \\ \end{matrix}\quad \begin{pmatrix}1&0&3\\0&1&2\\0&0&0\end{pmatrix}$ a line
$z=s$ $\begin{cases} x+3z=0\\ y+2z=0\end{cases}\qquad \begin{cases} z=s\\ x=-3s\\ y=-2s\end{cases}$

d) $\begin{pmatrix}1&-3&1\\2&-6&2\\3&-9&3\end{pmatrix}\begin{matrix}\\ -2r_1\\ -r_1-r_2\end{matrix}\quad \begin{pmatrix}1&-3&1\\0&0&0\\0&0&0\end{pmatrix}$ a plane: $x-3y+z=0$

6. a) Two sets of vectors S_1, S_2 in the same vector space span the same subspace if and only if:
- each vector in S_1 can be written as a linear combination of the vectors in S_2
- each vector in S_2 can be written as a linear combination of the vectors in S_1

$V=\begin{pmatrix}1&6&4\\2&4&-1\\-1&2&5\end{pmatrix}\qquad W=\begin{pmatrix}1&-2&-5\\0&8&9\end{pmatrix}$

$V=\begin{pmatrix}1&6&4\\2&4&-1\\-1&2&5\end{pmatrix}\begin{matrix}-r_2-r_3\\ +2r_3\\ \\ \end{matrix}\sim\begin{pmatrix}0&0&0\\0&8&9\\-1&2&5\end{pmatrix}\sim\begin{pmatrix}-1&2&5\\0&8&9\\0&0&0\end{pmatrix}\begin{matrix}\cdot(-1)\\ \\ \\ \end{matrix}\sim\begin{pmatrix}\boxed{\begin{matrix}1&-2&-5\\0&8&9\end{matrix}}\\ \begin{matrix}0&0&0\end{matrix}\end{pmatrix}=W \Rightarrow$ they span the same subspace

$\mathrm{span}(V)=\mathrm{span}(W)$

b) $\begin{pmatrix}0&3&1&-1\\6&0&5&1\\4&-7&1&3\end{pmatrix}$ ~~there is at least one free variable $\Rightarrow$ the set is linear dependent~~

c) $\begin{pmatrix}0&3&1&-1\\6&0&5&1\\4&-7&1&3\end{pmatrix}\begin{matrix}\\ \\ -\frac{4}{6}r_2\end{matrix}\quad \begin{pmatrix}0&3&1&-1\\6&0&5&1\\0&-7&-7/3&7/3\end{pmatrix}\begin{matrix}\\ \\ +\frac{7}{3}r_1\end{matrix}\quad \begin{pmatrix}0&3&1&-1\\6&0&5&1\\0&0&0&0\end{pmatrix}\begin{matrix}u\\ v\\ w\end{matrix}$ $\Rightarrow w=-\frac{7}{3}u+\frac{2}{3}v$,
$u=\frac{2}{7}v-\frac{3}{7}w$
$v=\frac{7}{2}u+\frac{3}{2}w$

Extra problems

7. $\vec{u}=(0,-2,2)$, $\quad\vec{v}=(1,3,-1)$

a) $\left(\begin{array}{cc|c}0&1&2\\-2&3&2\\2&-1&2\end{array}\right)\begin{matrix}\\ +r_3\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&2\\0&2&4\\2&-1&2\end{array}\right)\begin{matrix}\\ -2r_1\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&2\\0&0&4\\2&-1&2\end{array}\right)\quad \left(\begin{array}{cc|c}2&-1&2\\0&1&2\\0&0&4\end{array}\right)$ there are no solutions, and we don't know whether or not the columns are independent. But we know, that $\vec{b}$ is not a linear combination of the columns

b) $\left(\begin{array}{cc|c}0&1&3\\-2&3&1\\2&-1&5\end{array}\right)\begin{matrix}\\ +r_3\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&3\\0&2&6\\2&-1&5\end{array}\right)\begin{matrix}\\ -2r_1\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&3\\0&0&0\\2&-1&5\end{array}\right)\quad \left(\begin{array}{cc|c}2&-1&5\\0&1&3\\0&0&0\end{array}\right)$ It has a unique solution, so the columns are lin. independent, and $\vec{b}$ is a linear combination

c) $\left(\begin{array}{cc|c}0&1&0\\-2&3&4\\2&-1&5\end{array}\right)\begin{matrix}\\ +r_3\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&0\\0&2&9\\2&-1&5\end{array}\right)\begin{matrix}\\ -2r_1\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&0\\0&0&9\\2&-1&5\end{array}\right)$ ← $\vec{b}$ is not a linear combination of $\vec{u}$ and $\vec{v}$

d) $\left(\begin{array}{cc|c}0&1&0\\-2&3&0\\2&-1&0\end{array}\right)\begin{matrix}\\ +r_3\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&0\\0&2&0\\2&-1&0\end{array}\right)\begin{matrix}\\ -2r_1\\ \\ \end{matrix}\quad \left(\begin{array}{cc|c}0&1&0\\0&0&0\\2&-1&0\end{array}\right)\quad \left(\begin{array}{cc|c}2&-1&0\\0&1&0\\0&0&0\end{array}\right)$ $\vec{b}$ is a linear combination of $\vec{u}$ and $\vec{v}$

8.

$$A=\begin{pmatrix} \lambda & 1 & -1 \\ 1 & \lambda & 1 \\ -1 & 1 & \lambda \end{pmatrix}$$

$$\det(A) = \lambda\cdot(\lambda^2-1) - 1(\alpha+1) - 1(1+\lambda) = \lambda^3 - \lambda - \lambda - 1 - 1 - \lambda = \lambda^3 - 3\lambda - 2 = 0$$

guess $\lambda_1 = 2$: $8-6-2=0$

$$(\lambda^3 - 3\lambda - 2) : (\lambda - 2) = \lambda^2 + 2\lambda + 1$$

$$\underline{\lambda^3 - 2\lambda^2}$$

$$-\ 2\lambda^2 - 3\lambda$$

$$\underline{2\lambda^2 - 4\lambda}$$

$$-\lambda - 2$$

$$\underline{\lambda - 2}$$

$$0$$

$\lambda^2 + 2\lambda + 1 = 0$

$(\lambda+1)^2 = 0 \Rightarrow \lambda_2 = -1$

Thus, for $\lambda=2$ or $\lambda=-1$ the column vectors are linearly dependent.

OR:

$$\begin{pmatrix} \lambda & 1 & -1 \\ 1 & \lambda & 1 \\ -1 & 1 & \lambda \end{pmatrix} \begin{matrix} -\lambda r_2 \\ \\ +r_2 \end{matrix} \quad \begin{pmatrix} 0 & 1-\lambda^2 & -1-\lambda \\ 1 & \lambda & 1 \\ 0 & 1+\lambda & 1+\lambda \end{pmatrix} \sim \begin{pmatrix} 1 & \lambda & 1 \\ 0 & (1-\lambda)(1+\lambda) & -(1+\lambda) \\ 0 & 1+\lambda & 1+\lambda \end{pmatrix} \begin{matrix} \\ -(1-\lambda)r_3 \\ \end{matrix}$$

if $1+\lambda=0$, $\lambda=-1$ $\Rightarrow$ inf. solutions $\Rightarrow$ linearly dependent vectors

$$\begin{pmatrix} 1 & \lambda & 1 \\ 0 & 0 & -(1+\lambda)-(1-\lambda)(1+\lambda) \\ 0 & 1+\lambda & 1+\lambda \end{pmatrix} \sim \begin{pmatrix} 1 & \lambda & 1 \\ 0 & 0 & (1+\lambda)(\lambda-2) \\ 0 & 1+\lambda & 1+\lambda \end{pmatrix} \sim \begin{pmatrix} 1 & \lambda & 1 \\ 0 & 1+\lambda & 1+\lambda \\ 0 & 0 & (1+\lambda)(\lambda-2) \end{pmatrix}$$

for $\lambda=-1$ and $\lambda=2$, there is inf. solutions thus the vectors are linearly dependent