Linear Algebra
Seminar 8: Orthogonal projections and least square solutions
(make sure you can write on the board)

Problem 1. For the vectors $\mathbf{a} = (1, 0, 1)$ and $\mathbf{b} = (-1, 2, 1)$

(a) find the matrix of the orthogonal projection P_V onto the line $V := \mathrm{ls}\{\mathbf{a}\}$;

(b) find the matrix of the orthogonal projection $P_{V^\perp}$ onto the plane $V^\perp$;

(c) find the components of the vector $\mathbf{b}$ with respect to the decomposition $\mathbb{R}^3 = V \oplus V^\perp$.

Problem 2. For the vectors $\mathbf{a}_1 = (1, 0, 1)$, $\mathbf{a}_2 = (0, 1, 2)$ and $\mathbf{b} = (-1, 2, 1)$

(a) find the matrix of the orthogonal projection P_W onto the plane $W := \mathrm{ls}\{\mathbf{a}_1, \mathbf{a}_2\}$;

(b) find the matrix of the orthogonal projection $P_{W^\perp}$ onto the plane $W^\perp$;

(c) find the components of the vector $\mathbf{b}$ with respect to the decomposition $\mathbb{R}^3 = W \oplus W^\perp$.

Problem 3. Let W be the plane given by the equation $5x - 3y + z = 0$.

(a) Find a basis for W

(b) Find the standard matrix for the orthogonal projection on W.

(c) Use the matrix obtained in part (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ on W.

(d) Find the distance between the point $P_0(1, -2, 4)$ and the plane W.

(e) Prove that the distance D between the point $P(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(f) Find the distance between two parallel planes $2x - y - z = 5$ and $-4x + 2y + 2z = 12$.

(g) Find the distance between two planes $2x - y - z = 5$ and $-4x + y + 2z = 12$.

Problem 4. Let W be the line with parametric equations $x = 2t, \quad y = -t, \quad z = 4t$

(a) Find a basis for W.

(b) Find the standard matrix for the orthogonal projection on W.

(c) Use the matrix obtained in part (b) to find the orthogonal projection of a point $P_0(x_0, y_0, z_0)$ on W.

(d) Find the distance between the point $P_0(2, 1, -3)$ and the line W.

(e) Find the formula for the distance between $P_0(x_0, y_0, z_0)$ and line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

(f) Find the distance between two parallel lines $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ and $\frac{x-2}{1} = \frac{y-1}{2} = \frac{z-1}{3}$

(g) Find the distance between two lines $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ and $\frac{x-2}{1} = \frac{y-1}{2} = \frac{z-1}{1}$

Problem 5. Given the system $A\mathbf{x} = \mathbf{b}$,

(a) write the normal equation $A^\top A\mathbf{x} = A^\top \mathbf{b}$;

(b) find the least squares solution of $A\mathbf{x} = \mathbf{b}$;

(c) find the smallest error $\mathbf{b} - A\mathbf{x}$ and its length

for

$$(i) \quad A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \qquad (ii) \quad A = \begin{pmatrix} 2 & -2 \\ 1 & 1 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix}$$

Problem 6. (a) Find the least squares straight line fit to the three points $(0, 1)$, $(1, 2)$, and $(2, 7)$.

(b) Find the quadratic polynomial that best fits the four points $(2, 0)$, $(3, -10)$, $(5, -48)$, and $(6, -76)$.

(c) Find the cubic polynomial that best fits the four points $(2, 0)$, $(3, -10)$, $(5, -48)$, and $(6, -76)$.

Extra problems
(to be discussed if time permits)

Problem 7. Assume that $\mathbf{a}_1 = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $\mathbf{a}_2 = (\frac{2}{3}, -\frac{1}{3}, \frac{2}{3})$, $\mathbf{a}_3 = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$, and $\mathbf{b} = (0, 1, 2)$.

(a) Explain why $\mathbf{a}_1^\top \mathbf{a}_1 + \mathbf{a}_2^\top \mathbf{a}_2 + \mathbf{a}_3^\top \mathbf{a}_3 = I_3$.

(b) Find the orthogonal projections of $\mathbf{b}$ onto the spaces $\mathrm{ls}\{\mathbf{a}_1\}$ and $\mathrm{ls}\{\mathbf{a}_2, \mathbf{a}_3\}$.

Problem 8. Let W be a subspace in $\mathbb{R}^2$ given by the equation $ax + by = 0$

(a) Find a basis for W and $W^\perp$ and the standard matrices of the orthogonal projections onto W and $W^\perp$

(b) Use the matrices obtained in part (a) to find the orthogonal projections of a vector $\mathbf{p} = (x_0, y_0)^\top$ on W and $W^\perp$

(c) Show that the distance between the point $P(x_0, y_0)$ and the line W is equal to

$$\frac{ax_0 + by_0}{\sqrt{a^2 + b^2}}$$

(d) comment on the changes to be made for the distance between P and the line $ax + by = c$

Problem 9. (a) In $\mathbb{R}^3$, consider the line l given by the equations

$$x = 0, \qquad y = t, \qquad z = 2t$$

and the line m given by the equations

$$x = s, \qquad y = 1 - s, \qquad z = 1.$$

Let P be a point on l, and let Q be a point on m. Find the values of t and s that minimize the distance between the lines by minimizing the squared distance $\|P - Q\|^2$, and find that minimal distance.

(b) Now denote by W the plane through the line l that is parallel to m. Find the distance from m to W and compare the result to that or part (a)

Problem 10. Is there any value of s for which $x_1 = 1$ and $x_2 = 2$ is the least squares solution of the following linear system?

$$\begin{aligned} x_1 - x_2 &= 1, \\ 2x_1 + 3x_2 &= 1, \\ 4x_1 + 5x_2 &= s. \end{aligned}$$

Explain your reasoning

1. $\vec{a} = (1,0,1)$, $\vec{b} = (-1,2,1)$

a) $P_V = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{\begin{bmatrix}1\\0\\1\end{bmatrix}[1\ 0\ 1]}{[1\ 0\ 1]\begin{bmatrix}1\\0\\1\end{bmatrix}} = \frac{\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}}{2} = \frac{1}{2}\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}$

b) $P_V + P_{V^\perp} = I_n = I_3$

$P_{V^\perp} = I_3 - P_V = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1/2&0&1/2\\0&0&0\\1/2&0&1/2\end{bmatrix} = \begin{bmatrix}1/2&0&-1/2\\0&1&0\\-1/2&0&1/2\end{bmatrix}$

c) $\vec{b} = \vec{v_1} + \vec{v_2}$, $\vec{v_1} \in V$, $\vec{v_2} \in V^\perp$

$\vec{v_1} = P_V\vec{b}$ – vector in V closest to $\vec{b}$ $= \frac{1}{2}\begin{bmatrix}1&0&1\\0&0&0\\1&0&1\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\1\end{bmatrix} = 0$

$\vec{v_2} = P_{V^\perp}\vec{b}$ – vector in $V^\perp$ closest to $\vec{b}$ $= \begin{bmatrix}1/2&0&-1/2\\0&1&0\\-1/2&0&1/2\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\1\end{bmatrix} = \begin{bmatrix}-1\\2\\1\end{bmatrix}$

2. $\vec{a_1} = (1,0,1)$, $\vec{a_2} = (0,1,2)$, $\vec{b} = (-1,2,1)$

a) $P_V = A(A^TA)^{-1}A^T = \begin{bmatrix}1&0\\0&1\\1&2\end{bmatrix}\cdot\begin{bmatrix}5/6&-1/3\\-1/3&1/3\end{bmatrix}\cdot\begin{bmatrix}1&0&1\\0&1&2\end{bmatrix} = \begin{bmatrix}5/6&-1/3\\-1/3&1/3\\1/6&1/3\end{bmatrix}\cdot\begin{bmatrix}1&0&1\\0&1&2\end{bmatrix} = \begin{bmatrix}5/6&-1/3&1/6\\-1/3&1/3&1/3\\1/6&1/3&5/6\end{bmatrix} = \frac{1}{6}\begin{bmatrix}5&-2&1\\-2&2&2\\1&2&5\end{bmatrix}$

$A = \begin{bmatrix}1&0\\0&1\\1&2\end{bmatrix}$, $A^T = \begin{bmatrix}1&0&1\\0&1&2\end{bmatrix}$

$A^TA = \begin{bmatrix}2&2\\2&5\end{bmatrix}$, $(A^TA)^{-1} = \begin{bmatrix}5/6&-1/3\\-1/3&1/3\end{bmatrix}$

b) $P_V + P_{V^\perp} = I_n = I_3$

$P_{V^\perp} = \frac{1}{6}\begin{bmatrix}6&0&0\\0&6&0\\0&0&6\end{bmatrix} - \frac{1}{6}\begin{bmatrix}5&-2&1\\-2&2&2\\1&2&5\end{bmatrix} = \frac{1}{6}\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-2&1\end{bmatrix}$

c) $\vec{b} = \vec{v_1} + \vec{v_2}$, $\vec{v_1} \in V$, $\vec{v_2} \in V^\perp$

$\vec{v_1} = P_V\vec{b}$ – vector in V closest to $\vec{b}$ $= \frac{1}{6}\begin{bmatrix}5&-2&1\\-2&2&2\\1&2&5\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}-8\\8\\8\end{bmatrix} = \begin{bmatrix}-4/3\\4/3\\4/3\end{bmatrix}$

$\vec{v_2} = P_{V^\perp}\vec{b}$ – vector in $V^\perp$ closest to $\vec{b}$ $= \frac{1}{6}\begin{bmatrix}1&2&-1\\2&4&-2\\-1&-2&1\end{bmatrix}\cdot\begin{bmatrix}-1\\2\\1\end{bmatrix} = \frac{1}{6}\begin{bmatrix}2\\4\\-2\end{bmatrix} = \begin{bmatrix}1/3\\2/3\\-1/3\end{bmatrix}$

3. $5x - 3y + z = 0$

a) $\begin{cases} y = 5s \\ z = 5t \\ 5x = 3y - z \Rightarrow x = \frac{3}{5}y - \frac{1}{5}z = 3s - t \end{cases}$

the basis: $s\begin{bmatrix}3\\5\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\5\end{bmatrix} \Rightarrow \left\{\begin{bmatrix}3\\5\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\5\end{bmatrix}\right\}$

b) $P = A(A^TA)^{-1}A^T = \begin{bmatrix}3&-1\\5&0\\0&5\end{bmatrix}\cdot\begin{bmatrix}26/875&3/875\\3/875&34/875\end{bmatrix}\cdot\begin{bmatrix}3&5&0\\-1&0&5\end{bmatrix} = \begin{bmatrix}3/35&-1/35\\26/175&3/175\\3/175&34/175\end{bmatrix}\begin{bmatrix}3&5&0\\-1&0&5\end{bmatrix} = \begin{bmatrix}2/7&3/7&-1/7\\3/7&26/35&3/35\\-1/7&3/35&34/35\end{bmatrix}$

$A = \begin{bmatrix}3&-1\\5&0\\0&5\end{bmatrix}$, $A^T = \begin{bmatrix}3&5&0\\-1&0&5\end{bmatrix}$,

$A^TA = \begin{bmatrix}34&-3\\-3&26\end{bmatrix} \Rightarrow \begin{bmatrix}26/875&3/875\\3/875&34/875\end{bmatrix}$

c) Let $\vec{OP_0} = (x_0, y_0, z_0)$ – vector from $O(0,0,0)$ origin to $P_0(x_0, y_0, z_0)$

$\vec{p} = P\cdot\vec{OP_0} = \begin{bmatrix}2/7&3/7&-1/7\\3/7&26/35&3/35\\-1/7&3/35&34/35\end{bmatrix}\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix} = \frac{1}{35}\begin{bmatrix}10x_0+15y_0-5z_0\\15x_0+26y_0+3z_0\\-5x_0+3y_0+34z_0\end{bmatrix} = \frac{1}{7}\begin{bmatrix}2\\3\\-1\end{bmatrix}x_0 + \frac{1}{35}\begin{bmatrix}15\\26\\3\end{bmatrix}y_0 + \frac{1}{35}\begin{bmatrix}-5\\3\\34\end{bmatrix}z_0$

d) $P_0(1,-2,4)$

$\vec{p} = P\cdot\vec{OP_0} = \frac{1}{35}\begin{bmatrix}10-30-20\\15-52+12\\-5-6+136\end{bmatrix} = \frac{1}{35}\begin{bmatrix}-40\\-25\\125\end{bmatrix} = \begin{pmatrix}-\frac{8}{7}\\-\frac{5}{7}\\\frac{25}{7}\end{pmatrix} \Rightarrow P_0'\left(-\frac{8}{7}, -\frac{5}{7}, \frac{25}{7}\right)$

$d = \rho(P_0, P_0') = \sqrt{\left(1+\frac{8}{7}\right)^2 + \left(-2+\frac{5}{7}\right)^2 + \left(4-\frac{25}{7}\right)^2} = \sqrt{\frac{225+81+9}{49}} = \frac{1}{7}\sqrt{315} = \frac{3}{7}\sqrt{35}$

e) Let A be the point on the plane closest to P. The distance from P to the plane = distance from P to A.

(X,Y,Z) equation of the plane $= ax + by + cz + d = 0 \Rightarrow \vec{n}$ normal vector $= (a, b, c)$

$\vec{n_1}$ – normal unit vector $= \frac{\vec{n}}{\|\vec{n}\|} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$

Let $\vec{v}$ – vector from A to $P = (x_0 - X, y_0 - Y, z_0 - Z)$. The distance from P to plane is the projection of $\vec{v}$ onto the unit normal vector $\vec{n_1}$. Since its length = 1 $\Rightarrow$

$D = \operatorname{abs}(\langle\vec{v}, \vec{n_1}\rangle)$; $\langle\vec{QP}, \vec{n}\rangle = \|n\|\cdot|\operatorname{Proj}_{\vec{n}}\vec{QP}|$ ← distance

$D = |\langle\vec{v}, \vec{n_1}\rangle| = |(x_0 - X, y_0 - Y, z_0 - Z)\cdot\vec{n_1}| = \frac{|a(x_0-X) + b(y_0-Y) + c(z_0-Z)|}{\sqrt{a^2+b^2+c^2}} =$

$= \frac{|ax_0 - aX + by_0 - bY + cz_0 - cZ|}{\sqrt{a^2+b^2+c^2}}$ $\Rightarrow$ substitute

from plane equation $\Rightarrow d = -ax - by - cz \Rightarrow d = -aX - bY - cZ$

$\Rightarrow D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$

f) $2x - y - z = 5$; $: l_1$

$-4x + 2y + 2z = 12$ $: l_2$

Take $P_0(-3,0,0) \in l_2 \Rightarrow d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}} = \frac{|-3\cdot2-5|}{\sqrt{4+1+1}} = \frac{|-11|}{\sqrt{6}} = \frac{11}{\sqrt{6}}$

g) $\frac{2}{-4} \neq \frac{-1}{1} \Rightarrow$ the planes are not parallel $\Rightarrow d = 0$

4. $x = 2t$
$y = -t$
$z = 4t$

a) $\left\{\begin{bmatrix}2\\-1\\4\end{bmatrix}\right\}$

b) $P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{\begin{bmatrix}2\\-1\\4\end{bmatrix}[2\ -1\ 4]}{[2\ -1\ 4]\begin{bmatrix}2\\-1\\4\end{bmatrix}} = \frac{\begin{bmatrix}4&-2&8\\-2&1&-4\\8&-4&16\end{bmatrix}}{21} = \frac{1}{21}\begin{bmatrix}4&-2&8\\-2&1&-4\\8&-4&16\end{bmatrix}$

c) $P_0(x_0, y_0, z_0) \Rightarrow \vec{OP}(x_0, y_0, z_0)$

$\vec{p} = P\cdot\vec{OP} = \frac{1}{21}\begin{bmatrix}4&-2&8\\-2&1&-4\\8&-4&16\end{bmatrix}\cdot\begin{bmatrix}x_0\\y_0\\z_0\end{bmatrix} = \frac{1}{21}\begin{bmatrix}4x_0-2y_0+8z_0\\-2x_0+y_0-4z_0\\8x_0-4y_0+16z_0\end{bmatrix} = \frac{2}{21}\begin{bmatrix}2\\-1\\4\end{bmatrix}x_0 + \frac{1}{21}\begin{bmatrix}-2\\1\\-4\end{bmatrix}y_0 + \frac{4}{21}\begin{bmatrix}2\\-1\\4\end{bmatrix}z_0$

d) $P_0(2,1,-3)$

$\vec{p} = \frac{1}{21}\begin{bmatrix}8-2-24\\-4+1+12\\16-4-48\end{bmatrix} = \frac{1}{21}\begin{bmatrix}-18\\9\\-36\end{bmatrix} \Rightarrow P_0'\left(-\frac{6}{7}, \frac{3}{7}, \frac{-12}{7}\right)$

$d = \rho(P_0, P_0') = \sqrt{\left(-\frac{6}{7}-2\right)^2 + \left(\frac{3}{7}-1\right)^2 + \left(\frac{-12}{7}+3\right)^2} = \sqrt{\left(-\frac{20}{7}\right)^2 + \left(-\frac{4}{7}\right)^2 + \left(\frac{9}{7}\right)^2} = \frac{1}{7}\sqrt{491}$

e) $l: \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \Rightarrow \begin{cases}x = x_1 + at\\y = y_1 + bt\\z = z_1 + ct\end{cases}$, $P_0(x_0, y_0, z_0)$

Pick a point $P_1(x_1, y_1, z_1) \in l$, normal unit vector $\vec{n_1} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$

$\rho(P_0, l) = \|(\vec{P_1P_0})\times\vec{n_1}\| = \frac{\|(x_0-x_1, y_0-y_1, z_0-z_1)\times(a,b,c)\|}{\sqrt{a^2+b^2+c^2}}$

f) $l_1 = \frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3} \Rightarrow \begin{cases}x = 1+t\\y = 1+2t\\z = 1+3t\end{cases}$, $P_1 = (1,1,1) \in l_1$

$l_2 = \frac{x-2}{1} = \frac{y-1}{2} = \frac{z-1}{3} \Rightarrow \begin{cases}x = 2+t\\y = 1+2t\\z = 1+3t\end{cases}$, $P_2 = (2,1,1) \in l_2$

$\vec{P_1P_2} = (2,1,1) - (1,1,1) = (1,0,0)$ $\quad \vec{n} = (1,2,3)$

$\rho(l_1, l_2) = \frac{\|\vec{P_1P_2}\times\vec{n}\|}{\|\vec{n}\|} = \frac{\|(1,0,0)^T\times(1,2,3)^T\|}{\sqrt{1+4+9}} = \frac{\left\|\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&0&0\\1&2&3\end{vmatrix}\right\|}{\sqrt{14}} = \frac{\|-\vec{j}\cdot3+\vec{k}\cdot2\|}{\sqrt{14}} = \frac{\sqrt{9+4}}{\sqrt{14}} = \sqrt{\frac{13}{14}}$

g) $l_1 = \frac{x-1}{1} = \frac{y-1}{2} = \frac{x-1}{3}$; $l_2 = \frac{x-2}{1} = \frac{y-1}{2} = \frac{z-1}{1}$

$P_1(1,1,1)$ $\quad \vec{P_1P_2} = (1,0,0)$ the projection of the line of any two points on l_1 and l_2 on the normal $\vec{n}$

$P_2(2,1,1)$

the normal of the skewed planes is $\perp$ to both l_1, l_2 $\Rightarrow$

$\vec{v_1} = (1,2,3)^T$, $\vec{v_2} = (1,2,1)^T$

$\vec{n} = \vec{v_1}\times\vec{v_2} = \begin{vmatrix}i&j&k\\1&2&3\\1&2&1\end{vmatrix} = (2-6, -1+3, 2-2)^T = (-4,2,0)^T$

$D = \frac{|\vec{P_1P_2}\cdot\vec{n}|}{\|n\|} = \frac{|-4|}{\sqrt{16+4}} = \frac{4}{\sqrt{20}} = \frac{2}{\sqrt{5}}$

5. (i) $A = \begin{bmatrix}1&-1\\2&3\\4&5\end{bmatrix}$, $b = \begin{bmatrix}2\\-1\\5\end{bmatrix}$

a) $A^TA = \begin{bmatrix}1&2&4\\-1&3&5\end{bmatrix}\begin{bmatrix}1&-1\\2&3\\4&5\end{bmatrix} = \begin{bmatrix}21&25\\25&35\end{bmatrix}$

$A^Tb = \begin{bmatrix}1&2&4\\-1&3&5\end{bmatrix}\begin{bmatrix}2\\-1\\5\end{bmatrix} = \begin{bmatrix}20\\20\end{bmatrix}$

$(A^TA)x = A^Tb$:

$\begin{bmatrix}21&25\\25&35\end{bmatrix}\hat{x} = \begin{bmatrix}20\\20\end{bmatrix}$

b) $\left[\begin{array}{cc|c}21&25&20\\25&35&20\end{array}\right] - \frac{25}{21}r_1 \sim \left[\begin{array}{cc|c}21&25&20\\0&\frac{110}{21}&-\frac{80}{21}\end{array}\right] - \frac{21}{110}\cdot25r_2 \sim \left[\begin{array}{cc|c}21&0&\frac{420}{11}\\0&\frac{110}{21}&-\frac{80}{21}\end{array}\right]\begin{matrix}\cdot\frac{1}{21}\\\cdot\frac{21}{110}\end{matrix} \sim \left[\begin{array}{cc|c}1&0&\frac{20}{11}\\0&1&-\frac{8}{11}\end{array}\right] \Rightarrow \begin{bmatrix}\hat{x}_1 = \frac{20}{11}\\\hat{x}_2 = -\frac{8}{11}\end{bmatrix}$

c) $A\hat{x} = \begin{bmatrix}1&-1\\2&3\\4&5\end{bmatrix}\begin{bmatrix}20/11\\-8/11\end{bmatrix} = \begin{bmatrix}28/11\\16/11\\40/11\end{bmatrix}$

$b - A\hat{x} = \begin{bmatrix}2\\-1\\5\end{bmatrix} - \begin{bmatrix}28/11\\16/11\\40/11\end{bmatrix} = \begin{bmatrix}-6/11\\-27/11\\15/11\end{bmatrix}$

$\|b - A\hat{x}\| = \sqrt{\frac{990}{121}} = 3\sqrt{\frac{110}{121}}$

(ii) $A = \begin{pmatrix}2&-2\\1&1\\1&3\end{pmatrix}$, $\vec{b} = \begin{pmatrix}7\\0\\-7\end{pmatrix}$

a) $A^TA = \begin{bmatrix}2&1&1\\-2&1&3\end{bmatrix}\begin{bmatrix}2&-2\\1&1\\1&3\end{bmatrix} = \begin{bmatrix}6&0\\0&14\end{bmatrix}$

$A^Tb = \begin{bmatrix}2&1&1\\-2&1&3\end{bmatrix}\begin{bmatrix}7\\0\\-7\end{bmatrix} = \begin{bmatrix}7\\-35\end{bmatrix}$

$\begin{bmatrix}6&0\\0&14\end{bmatrix}x = \begin{bmatrix}7\\-35\end{bmatrix}$

b) $\left[\begin{array}{cc|c}6&0&7\\0&14&-35\end{array}\right]\begin{matrix}\cdot\frac{1}{6}\\\cdot\frac{1}{14}\end{matrix} \sim \left[\begin{array}{cc|c}1&0&7/6\\0&1&-5/2\end{array}\right] \Rightarrow \hat{x} = \begin{bmatrix}7/6\\-5/2\end{bmatrix}$

c) $A\hat{x} = \begin{bmatrix}2&-2\\1&1\\1&3\end{bmatrix}\begin{bmatrix}7/6\\-5/2\end{bmatrix} = \begin{bmatrix}22/3\\-4/3\\-19/3\end{bmatrix}$

$b - A\hat{x} = \begin{bmatrix}7\\0\\-7\end{bmatrix} - \begin{bmatrix}22/3\\-4/3\\-19/3\end{bmatrix} = \begin{bmatrix}-1/3\\4/3\\-2/3\end{bmatrix}$

$\|b - A\hat{x}\| = \sqrt{\frac{1+16+4}{9}} = \frac{1}{3}\sqrt{21}$

6. a) $(0,1), (1,2), (2,7)$, $p(t) = \alpha + \beta t$ – ?

$A = \begin{pmatrix}1&0\\1&1\\1&2\end{pmatrix}$, $\hat{x} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$, $b = \begin{pmatrix}1\\2\\7\end{pmatrix}$

$\hat{x}$ minimizes $\|Ax - b\|^2$ is:

$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix}5/6&-1/2\\-1/2&1/2\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix} = \begin{bmatrix}5/6&1/3&-1/6\\-1/2&0&1/2\end{bmatrix}\begin{bmatrix}1\\2\\7\end{bmatrix} = \begin{bmatrix}1/3\\3\end{bmatrix}$

$A^TA = \begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}\begin{bmatrix}1&0\\1&1\\1&2\end{bmatrix} = \begin{bmatrix}3&3\\3&5\end{bmatrix} \Rightarrow (A^TA)^{-1} = \begin{bmatrix}5/6&-1/2\\-1/2&1/2\end{bmatrix}$

$p(t) = \frac{1}{3} + 3t$

b) $(2,0), (3,-10), (5,-48), (6,-76)$, $p(t) = \alpha + \beta t + \gamma t^2$ – ?

$A = \begin{pmatrix}1&2&4\\1&3&9\\1&5&25\\1&6&36\end{pmatrix}$, $\hat{x} = \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$, $\vec{b} = \begin{pmatrix}0\\-10\\-48\\-76\end{pmatrix}$

$x^0\ \ x\ \ x^2$

$\hat{x} = (A^TA)^{-1}A^Tb = \begin{bmatrix}33/10&-8/5&-12/5&17/10\\-23/15&37/30&43/30&-17/15\\1/6&-1/6&-1/6&1/6\end{bmatrix}\begin{bmatrix}0\\-10\\-48\\-76\end{bmatrix} = \begin{bmatrix}2\\5\\-3\end{bmatrix}$

$A^TA = \begin{bmatrix}1&1&1&1\\2&3&5&6\\4&9&25&36\end{bmatrix}\begin{bmatrix}1&2&4\\1&3&9\\1&5&25\\1&6&36\end{bmatrix} = \begin{bmatrix}4&16&74\\16&74&376\\74&376&2018\end{bmatrix}$

$(A^TA)^{-1} = \begin{bmatrix}221/10&-62/5&3/2\\-62/5&649/90&-8/9\\3/2&-8/9&1/9\end{bmatrix}$

$p(t) = 2 + 5t - 3t^2$

c) $(2,0), (3,-10), (5,-48), (6,-76)$, $p(t) = \alpha + \beta t + \gamma t^2 + \varphi t^3$ – ?

$A = \begin{bmatrix}1&2&4&8\\1&3&9&27\\1&5&25&125\\1&6&36&216\end{bmatrix}$, $\vec{b} = \begin{pmatrix}0\\-10\\-48\\-76\end{pmatrix}$

$\hat{x} = (A^TA)^{-1}A^T\vec{b} = \begin{bmatrix}15/2&-10&6&-5/2\\-21/4&26/3&-8&31/12\\7/6&-13/6&11/6&-5/6\\-1/12&1/6&-1/6&1/12\end{bmatrix}\begin{bmatrix}0\\-10\\-48\\-76\end{bmatrix} = \begin{bmatrix}2\\5\\-3\\0\end{bmatrix}$

$A^TA = \begin{bmatrix}1&1&1&1\\2&3&5&6\\4&9&25&36\\8&27&125&216\end{bmatrix}\begin{bmatrix}1&2&4&8\\1&3&9&27\\1&5&25&125\\1&6&36&216\end{bmatrix} = \begin{bmatrix}4&16&74&376\\16&74&376&2018\\74&376&2018&11176\\376&2018&11176&63074\end{bmatrix}$

$(A^TA)^{-1} = \begin{bmatrix}397/2&-337/2&87/2&-7/2\\-337/2&10465/72&-685/18&223/72\\87/2&-685/18&91/9&-5/6\\-7/2&223/72&-5/6&5/72\end{bmatrix}$

$p(t) = 2 + 5t - 3t^2$ – is in fact quadratic

if we substitute each point we see that the system is consistent so it is not only the least square solution, but the regular solution itself