Linear Algebra

Lecture Notes

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Lecture 6. Linear transformations

Outline

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1. Coordinates and change of basis
   - Coordinate maps
   - Change of basis
2. Linear transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$
   - Linear transformations
   - Examples
3. Further properties
   - Matrix of a linear transformation
   - Change of basis

Coordinate map

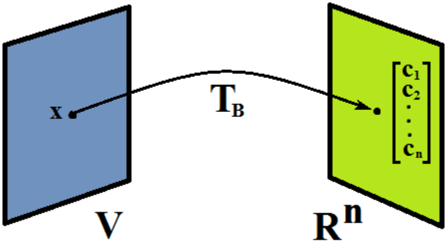
- Fix a basis $B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of a vector space V
- every $\mathbf{x} \in V$ gets its unique coordinates $c_1, c_2, \ldots, c_n$ in the basis B:

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Definition (Coordinate map $T_B : V \to \mathbb{R}^n$)

$$T_B \ : \ \mathbf{x} \mapsto (c_1, c_2, \ldots, c_n)^\top =: (\mathbf{x})_B$$

is called the coordinate map of V in the basis B



Example

- $S = \left(\mathbf{v}_1 = (1,2,0)^\top, \mathbf{v}_2 = (1,2,0)^\top, \mathbf{v}_3 = (1,1,1)^\top\right)$
- $T_S\mathbf{x} = \mathbf{c} \iff \mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \iff$

$$\underbrace{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{P_{S \to S'}} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

- x_1, x_2, x_3 are coordinates of $\mathbf{x}$ in the basis $S' = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$
  c_1, c_2, c_3 are coordinates of $\mathbf{x}$ in the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$
- $(1,2,0)^\top, (2,1,0)^\top, (1,1,1)^\top$ are coordinate vectors of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
- $\mathbf{c} \mapsto \mathbf{x}$ amounts to matrix multiplication by $P_{S \to S'}$
- $\mathbf{x} \mapsto \mathbf{c}$ amounts to matrix multiplication by

$$P_{S' \to S} = (P_{S \to S'})^{-1}$$

Coordinate map

Definition (Linear maps and isomorphisms)

Let V and W be linear vector spaces. A mapping $T : V \to W$ is

- linear if for all $\mathbf{x}, \mathbf{y} \in V$ and all $a, b \in \mathbb{R}$
$$\boxed{T(a\mathbf{x} + b\mathbf{y}) = aT(\mathbf{x}) + bT(\mathbf{y})}$$
- an isomorphism of V and W if it is linear, one-to-one, and onto

Definition (Isomorphic linear vector spaces)

Two linear vector spaces V and W are said to be isomorphic if there is an isomorphism $T : V \to W$

Isomorphism to $\mathbb{R}^n$

Lemma (Properties of T_S)

T_S is an isomorphism between V and $\mathbb{R}^n$

Proof.

T_S is one-to-one:

$$T_S(\mathbf{x}) = \mathbf{c} \qquad \Longleftrightarrow \qquad \mathbf{x} = \sum_j c_j \mathbf{v}_j$$

T_S is onto:

$$T_S^{-1}\mathbf{c} = \sum_j c_j \mathbf{v}_j \quad \text{is well defined}$$

T_S is linear:

$$\mathbf{x} = \sum_j c_j \mathbf{v}_j,\ \mathbf{y} = \sum_j d_j \mathbf{v}_j \implies a\mathbf{x} + b\mathbf{y} = \sum_j (ac_j + bd_j)\mathbf{v}_j$$

□

Corollary

Any two vector spaces of the same finite dimension are isomorphic

Corollary

Up to isomorphism, $\mathbb{R}^n$ is the only n-dimensional vector space

Outline

Change of basis

- Assume $\mathbf{x} \in V$ has coordinate vector $\mathbf{c} = T_S(\mathbf{x}) = (\mathbf{x})_S$ in basis $S = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$
- Take another basis $S' = (\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n)$; how can one calculate $\mathbf{c}' = T_{S'}(\mathbf{x}) = (\mathbf{x})_{S'}$?

Theorem (Change of basis)

$$\boxed{(\mathbf{x})_{S'} = P_{S \to S'}\,(\mathbf{x})_S}$$

where the transition matrix $P_{S \to S'}$ is a square matrix with columns

$$T_{S'}(\mathbf{v}_1),\ T_{S'}(\mathbf{v}_2),\ \ldots,\ T_{S'}(\mathbf{v}_n)$$

Proof.

$$(\mathbf{x})_{S'} = T_{S'}(\mathbf{x}) = T_{S'}\left(\sum_k c_k \mathbf{v}_k\right) = \sum_k c_k\, T_{S'}(\mathbf{v}_k)$$
$$= P_{S \to S'}\mathbf{c} = P_{S \to S'}\,(\mathbf{x})_S$$

□

Trigonometric polynomials

Recall the Euler formulae ($i := \sqrt{-1}$)

$$\boxed{e^{ix} = \cos x + i \sin x} \quad \Longleftrightarrow \quad \boxed{\cos x = \frac{e^{ix}+e^{-ix}}{2}, \quad \sin x = \frac{e^{ix}-e^{-ix}}{2i}}$$

In the linear space V of trigonometric polynomials spanned by $B := (1, \cos x, \sin x)$ another basis is $B' := (1, e^{ix}, e^{-ix})$

Then every trigonometric polynomial $a_0 + a_1 \cos x + a_2 \sin x$ can also be written as $b_1 + b_1 e^{ix} + b_2 e^{-ix}$, where

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2i} \\ 0 & \frac{1}{2} & \frac{-1}{2i} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & i & -i \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}$$

Computing the transition matrices in $\mathbb{R}^n$

- We have an "old" basis $S = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$
  and a "new" basis $S' = (\mathbf{v}'_1, \mathbf{v}'_2, \dots, \mathbf{v}'_n)$
- matrix B has columns $\mathbf{v}_k$ (in the standard basis $S_0 = (\mathbf{e}_1, \dots, \mathbf{e}_n)$)
- matrix B' has columns $\mathbf{v}'_k$ (in the standard basis $S_0 = (\mathbf{e}_1, \dots, \mathbf{e}_n)$)
- use elementary row transformations to get

$$(B' \mid B) \sim (I_n \mid P_{S \to S'})$$

- mnemonic rule:

$$\boxed{(\text{“new basis”} \mid \text{“old basis”}) \sim (I_n \mid P_{S \to S'})}$$

- reason: $B' = P_{S' \to S_0}$, so that

$$(B')^{-1} B = (P_{S' \to S_0})^{-1} P_{S \to S_0} = P_{S_0 \to S'} P_{S \to S_0} = P_{S \to S'}$$

- yet another reason:

$$P_{S' \to S_0} (\mathbf{x})_{S'} = \mathbf{x} = P_{S \to S_0} (\mathbf{x})_S \iff (\mathbf{x})_{S'} = (P_{S' \to S_0})^{-1} P_{S \to S_0} (\mathbf{x})_S$$

Example in $\mathbb{R}^2$

- Old basis S: $\mathbf{v}_1 = (1,1)^\top$, $\mathbf{v}_2 = (1,-1)^\top$
- new basis S': $\mathbf{v}_1' = (1,2)^\top$, $\mathbf{v}_2' = (2,-1)^\top$
- find the transition matrix $P_{S\to S'}$:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} \end{array}\right)$$

- enough to check for $\binom{1}{0}$ and $\binom{0}{1}$ that are coefficient vectors in S of $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively:

$$T_S(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \qquad T_{S'}(\mathbf{v}_1) = P_{S\to S'}\, T_S(\mathbf{v}_1) = \begin{pmatrix} \frac{3}{5} \\ \frac{1}{5} \end{pmatrix}$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tfrac{3}{5}\mathbf{v}_1' + \tfrac{1}{5}\mathbf{v}_2' = \tfrac{3}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{1}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix};$$

similarly for $\mathbf{v}_2$

Outline

Linear transformations = matrix multiplications

- Recall that a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is called linear if $\forall \alpha_1, \alpha_2 \in \mathbb{R}, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$

$$T(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) = \alpha_1 T(\mathbf{x}_1) + \alpha_2 T(\mathbf{x}_2)$$

- In particular, $T(\mathbf{0})$ is a zero vector of $\mathbb{R}^m$
- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis of $\mathbb{R}^n$ and $\mathbf{u}_j := T(\mathbf{e}_j)$
- Denote by A the $m \times n$ matrix with columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$

Theorem

For every $\mathbf{x} \in \mathbb{R}^n$,

$$\boxed{T(\mathbf{x}) = A\mathbf{x}}$$

Proof.

$$\begin{aligned} T(\mathbf{x}) &= T(x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n) = x_1 \mathbf{u}_1 + \cdots + x_n \mathbf{u}_n = A\mathbf{x} \end{aligned}$$

□

Corollary

$$\boxed{T = T_A \leftrightarrow A}$$

Outline

Linear transformations of the plane

Example (Mirror symmetry w.r.t. $x = y$)

$$T(\mathbf{e}_1) = \mathbf{e}_2,\ T(\mathbf{e}_2) = \mathbf{e}_1 \implies A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Example (Rotation on angle θ)

$$T(\mathbf{e}_1) = (\cos\theta, \sin\theta)^\top,\ T(\mathbf{e}_2) = (-\sin\theta, \cos\theta)^\top \implies$$

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example (Shear translation along $\mathbf{e}_1$)

$$T(\mathbf{e}_1) = \mathbf{e}_1,\ T(\mathbf{e}_2) = s\mathbf{e}_1 + \mathbf{e}_2 \implies A = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Linear transformations between vector spaces

- A linear mapping $T : V \to W \iff \forall \alpha_1, \alpha_2 \in \mathbb{R}, \mathbf{u}_1, \mathbf{u}_2 \in V$
$$T(\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2) = \alpha_1 T(\mathbf{u}_1) + \alpha_2 T(\mathbf{u}_2) \in W$$
- In particular, $T(\mathbf{0}_V)$ is a zero vector $\mathbf{0}_W$ of W
- Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of V and $\mathbf{w}_1, \ldots, \mathbf{w}_m$ a basis of W
- Set $\mathbf{u}_j := \big(T(\mathbf{v}_j)\big)_W$ and enote by A the $m \times n$ matrix with columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$

Theorem

For every $\mathbf{v} \in V$, $\boxed{\big(T(\mathbf{v})\big)_W = A(\mathbf{v})_V}$

Proof.

With $\mathbf{c} := (\mathbf{v})_V$,
$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1 T(\mathbf{v}_1) + \cdots + c_n T(\mathbf{v}_n);$$
$$\big(T(\mathbf{v})\big)_W = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = A\mathbf{c} = A(\mathbf{v})_V$$
$\square$

Linear transformations between vector spaces

Problem

Assume T is a linear mapping from a 2-dim vector space V to a 3-dim vector space W whose matrix in bases $(\mathbf{v}_1, \mathbf{v}_2)$ of V and $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ of W is

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(a) Express the vector $T\mathbf{v}_1$ in terms of the basis vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. If $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ and $T\mathbf{v} = d_1\mathbf{w}_1 + d_2\mathbf{w}_2 + d_3\mathbf{w}_3$, find the relation between c's and d's

(b) Identify the nullspace $\mathrm{Nul}(T) = \{\mathbf{v} \in V \mid T\mathbf{v} = \mathbf{0}\}$ of T and find its dimension

(c) Describe the range $\mathrm{Ran}(T) = \{\mathbf{w} = T\mathbf{v} \mid \mathbf{v} \in V\}$

(d) Find a basis of $\mathrm{Ran}(T)$

(e) Is the mapping T an isomorphism between the spaces V and W?

Linear mappings in $\mathcal{P}_n$

Example (Shift)

The mapping $T_a : \mathcal{P}_n \to \mathcal{P}_n$ given by $(T_a p)(x) := p(x+a)$ is linear. Since

$$(x+a)^k = \sum_{j=0}^{k} \binom{k}{j} a^{k-j} x^j,$$

in the standard basis $B_0 = \{1, x, \dots, x^n\}$ the matrix M_a of T_a is

$$M_a = \begin{pmatrix} 1 & a & \dots & a^k & \dots & a^n \\ 0 & 1 & \dots & ka^{k-1} & \dots & na^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \dots & \binom{n}{k-1} a^{n-k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

T_a is a homeomorphism; its inverse is given by $(T_a)^{-1} = T_{-a}$, so that $M_{-a} M_a = I_{n+1}$

Outline

Composition of linear transformations

Definition

If $T_1 : \mathbb{R}^m \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^n$ are linear transformations, then their composition $T_2 \circ T_1 : \mathbb{R}^m \to \mathbb{R}^n$ is given by

$$(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

Theorem (Matrix of composition mapping)

If T_1 has matrix $A \in M_{k\times m}(\mathbb{R})$ and T_2 has matrix $B \in M_{n\times k}(\mathbb{R})$, then the matrix of $T_2 \circ T_1$ is BA

Proof.

- The matrix of $T := T_2 \circ T_1$ has columns $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_m)$;
- $$T(\mathbf{e}_j) = T_2\big(T_1(\mathbf{e}_j)\big) = BT_1(\mathbf{e}_j)$$
- j^{th} column of the matrix of T is equal to B times j^{th} column of A
- thus the matrix of T is equal to BA

$\square$

General linear transformation

- Assume that $\mathbf{u}_1, \ldots, \mathbf{u}_n$ are linearly independent vectors in $\mathbb{R}^n$ (thus a basis S)
- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are arbitrary vectors in $\mathbb{R}^m$

Theorem

There exists a unique linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $T(\mathbf{u}_j) = \mathbf{v}_j$ for all $j = 1, \ldots, n$.

Proof.

- Any $\mathbf{x} \in \mathbb{R}^n$ has the form $\mathbf{x} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ for a unique coefficient vector $(c_1, \ldots, c_n)^\top =: T_S(\mathbf{x})$
- Existence: Denote by A the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$; Then $T_A \circ T_S$ is the required transformation:
$$(T_A \circ T_S)(\mathbf{u}_k) = T_A(T_S(\mathbf{u}_k)) = T_A(\mathbf{e}_k) = A\mathbf{e}_k = \mathbf{v}_k$$
- Uniqueness: such a T must satisfy
$$T(\mathbf{x}) = T(c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n) = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = A\mathbf{c}$$
- Its standard matrix is $AP_{S_0 \to S}$

Example

- Take $\mathbf{u}_1 = (1,1)^\top$ and $\mathbf{u}_2 = (2,-1)^\top$ in $\mathbb{R}^2$;
- $\mathbf{v}_1 = (1,0,1)^\top$ and $\mathbf{v}_2 = (0,1,1)^\top$ in $\mathbb{R}^3$;
- our task is to construct a linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T\mathbf{u}_1 = \mathbf{v}_1$ and $T\mathbf{u}_2 = \mathbf{v}_2$
- The standard matrix B of T satisfies

$$B \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

so that

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}$$

- it can be checked that $B\mathbf{u}_1 = \mathbf{v}_1$ and $B\mathbf{u}_2 = \mathbf{v}_2$

Outline

Transformation in different bases

- Assume that T is a linear transformation in $\mathbb{R}^n$ that has matrix A_0 in the standard basis $S_0 = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$
- In particular, j^{th} column of A_0 is just $T(\mathbf{e}_j)$
- What is the matrix A of T in a different basis $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$?
- Clearly, j^{th} column of A is just the coordinate vector of $T(\mathbf{v}_j)$ relative to basis S
- Thus

$$\boxed{A = P_{S_0 \to S}\, A_0\, P_{S \to S_0}}$$

- Another reason: as $T\mathbf{x} = A_0\mathbf{x}$ and $(T\mathbf{x})_S = A(\mathbf{x})_S$, we see that

$$A\,(\mathbf{x})_S = (T\mathbf{x})_S = P_{S_0 \to S}(T\mathbf{x}) = P_{S_0 \to S}(A_0\mathbf{x}) = P_{S_0 \to S} A_0 P_{S \to S_0}(\mathbf{x})_S$$

Example

- In $\mathbb{R}^2$, set $T(\mathbf{e}_1) = \mathbf{e}_2$ and $T(\mathbf{e}_2) = \mathbf{e}_1$
- its transformation matrix in the basis $\mathbf{e}_1, \mathbf{e}_2$ is $A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Take the basis $\mathbf{v}_1 = (1,1)^\top$, $\mathbf{v}_2 = (1,-1)^\top$; then

$$P_{S\to S_0} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P_{S_0\to S} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$