Linear Algebra
Seminar 2: Matrix Algebra
(make sure you can write on the board)

Problem 1. (4 pts.) For the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$$

(a) Find the LU factorization $A = LU$

(b) Solve $A\mathbf{x} = \mathbf{1}$ by solving first $L\mathbf{y} = \mathbf{1}$ and then $U\mathbf{x} = \mathbf{y}$

(c) Solve $A\mathbf{x} = \mathbf{1}$ using Gauss-Jordan elimination

Problem 2. (4 pts.) Invert the coefficient matrix to solve the following system of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + \ \ x_2 \qquad\quad &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 + 3x_2 + 9x_3 &= 3; \end{aligned} \qquad c)\ \begin{aligned} 2x_1 + 4x_2 \qquad\quad &= 2 \\ 4x_1 + 6x_2 + 3x_3 &= 1 \\ -6x_1 - 10x_2 \qquad\quad &= -6. \end{aligned}$$

Problem 3. (2 pts.) For what numbers c is this matrix not invertible, and why not?

$$A = \begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix}$$

Problem 4. (2 pts.) Find all numbers c such that the system of vectors is linearly independent:

$$\text{(a)} \quad \begin{pmatrix} 3 \\ -6 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 6 \\ c \\ 2 \end{pmatrix} \qquad \text{(b)} \quad \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ -6 \end{pmatrix}, \begin{pmatrix} c \\ 3 \\ -3 \end{pmatrix}$$

Problem 5. (4 pts.) Let $\mathbf{0}$ denote the zero matrix of size n (say, n=2021).

1. Is there an $n \times n$ matrix A such that and $A \neq \mathbf{0}$ and $A^2 = \mathbf{0}$? Justify your answer.
2. Is there an $n \times n$ matrix A such that and $A \neq \mathbf{0}$ and $A^2 = A$? Justify your answer.
3. Is there an $n \times n$ matrix A such that and $A \neq I_n$ and $A^2 = I_n$? Justify your answer.
4. Are there $n \times n$ matrices A and B such that $A \neq \mathbf{0}$, $B \neq \mathbf{0}$ but $AB = \mathbf{0}$?

Problem 6. (4 pts.) Choose a 3×3 matrix B so that for every 3×3 matrix A

(a) $BA = 4A$

(b) $BA = 4B$

(c) BA has rows 1 and 3 of A reversed and row 2 unchanged

(d) All rows of BA are the same as row 1 of A

Extra problems
(to be discussed if time permits)

Problem 7. The solution set of the equation $A\mathbf{x} = \mathbf{0}$ is spanned by the vector $\mathbf{u} = (2, -1, 1)^\top$, and the vectors $\mathbf{v}_1 = (1, 2, 1, 0)^\top$ and $\mathbf{v}_2 = (0, 1, 2, 1)^\top$ belong to the column space of A.

(a) Determine the size of A and the number of pivot and free columns. What is the rank of A?

(b) Find the reduced row echelon form R of A.

(c) Find an invertible matrix B such that the first two columns of BR are $\mathbf{v}_1$ and $\mathbf{v}_2$. Explain why this BR can be taken to be A.

(d) Find conditions on b_1, b_2, b_3, b_4 for $A\mathbf{x} = (b_1, b_2, b_3, b_4)^\top$ to have a solution.

(e) If those conditions are satisfied by b_1, b_2, b_3, b_4, what are all the solutions $\mathbf{x}$ (the solution set of $A\mathbf{x} = \mathbf{b}$)?

Problem 8. Show that if a square matrix A satisfies $A^2 - 3A + I = \mathbf{0}$, then $A^{-1} = 3I - A$.

Problem 9. Explain why the matrix

$$A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

rotates all vectors on angle θ counterclockwise. Show that $A(\theta_1)A(\theta_2) = A(\theta_1 + \theta_2)$. What is the inverse of $A(\theta)$?

Problem 10. The $n \times 1$ column vectors $\mathbf{u}$ and $\mathbf{v}$ are such that $\mathbf{v}^\top\mathbf{u} \neq 1$. Prove that the matrix $I_n - \mathbf{v}\mathbf{u}^\top$ is invertible and find its inverse.

1. $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix}$

a) $A = LU$

$$U = REF \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{pmatrix} \begin{matrix} \\ -2r_1 \\ -3r_1 - 2r_2 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \qquad E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$E | I \rightarrow I | E^{-1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ -3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ +2r_1 \\ +3r_1 \end{matrix} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right) \Rightarrow E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

$$L = E_1^{-1} \cdot E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

b) $Ax = \underline{1}$, $Ly = 1$, $Ux = y$

$Ly = 1$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right) \begin{matrix} \\ -2r_1 \\ -3r_1 \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -2 \end{array}\right) -2r_2 \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{matrix} y_1 = 1 \\ y_2 = -1 \\ y_3 = 0 \end{matrix}$$

$Ux = y$:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{array}\right) \begin{matrix} \\ \cdot\frac{1}{2} \\ \cdot\frac{1}{2} \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)^{-r_3} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{matrix} x_1 = 1 \\ x_2 = -\frac{1}{2} \\ x_3 = 0 \end{matrix}$$

c) $Ax = \underline{1}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 2 & 2 & 1 \\ 3 & 4 & 5 & 1 \end{array}\right) \begin{matrix} \\ -2r_1 \\ -3r_1 \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 4 & 2 & -2 \end{array}\right) \begin{matrix} \\ \cdot\frac{1}{2} \\ -2r_2 \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 2 & 0 \end{array}\right) \cdot\frac{1}{2} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)^{-r_3} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{matrix} x_1 = 1 \\ x_2 = -\frac{1}{2} \\ x_3 = 0 \end{matrix}$$

2. $Ax = b$

$$\underbrace{A^{-1}A}_{I}x = A^{-1}b$$

$$x = A^{-1}b$$

a)
$$\begin{matrix} 2x_1 + x_2 = 5 \\ x_1 + x_2 = 3 \end{matrix}$$

$$A|I = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)^{-r_2} \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \end{array}\right) - r_1 \Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

$$x: \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 2 & 3 \end{array}\right) \cdot\frac{1}{2} \Rightarrow \left(\begin{array}{cc|c} 1 & -1 & 5 \\ 0 & 1 & \frac{3}{2} \end{array}\right) + r_2 \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & 13/2 \\ 0 & 1 & 3/2 \end{array}\right)$$

b)
$$\begin{matrix} 2x_1 + x_2 = 4 \\ 6x_1 + 2x_2 + 6x_3 = 20 \\ -4x_1 + 3x_2 + 9x_3 = 3 \end{matrix}$$

$$A|I = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 6 & 2 & 6 & 0 & 1 & 0 \\ -4 & 3 & 9 & 0 & 0 & 1 \end{array}\right)$$

$$\overset{A \qquad\quad b}{\left(\begin{array}{ccc|c} 2 & 1 & 0 & 4 \\ 6 & 2 & 6 & 20 \\ -4 & 3 & 9 & 3 \end{array}\right)} \begin{matrix} \cdot\frac{1}{2} \\ -3r_1 \\ +2r_1 \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2 \\ 0 & -1 & 6 & 8 \\ 0 & 5 & 9 & 11 \end{array}\right) \begin{matrix} \\ \cdot(-1) \\ +5r_2 \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & 39 & 51 \end{array}\right) \cdot\frac{1}{39} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & 1 & \frac{17}{13} \end{array}\right) + 6r_3 \Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 1 & 0 & -\frac{2}{13} \\ 0 & 0 & 1 & \frac{17}{13} \end{array}\right)^{-\frac{1}{2}r_2} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{27}{13} \\ 0 & 1 & 0 & \frac{2}{13} \\ 0 & 0 & 1 & \frac{4}{13} \end{array}\right)$$

c)
$$\begin{matrix} 2x_1 + 4x_2 = 2 \\ 4x_1 + 6x_2 + 3x_3 = 1 \\ -6x_1 - 10x_2 = -6 \end{matrix}$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 4 & 6 & 3 & 1 \\ -6 & -10 & 0 & -6 \end{array}\right) \begin{matrix} \\ -2r_1 \\ +r_1 + r_2 \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 0 & -2 & 3 & -3 \\ 0 & 0 & 3 & -3 \end{array}\right) \begin{matrix} \\ -r_3 \\ \cdot\frac{1}{3} \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 2 & 4 & 0 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)^{+2r_2} \Rightarrow \left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \begin{matrix} \cdot\frac{1}{2} \\ \cdot(-\frac{1}{2}) \\ \end{matrix} \Rightarrow$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{matrix} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{matrix}$$

3. $A = \begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix}$

$$\det(A) = 2(c^2 - 7c) - c(c^2 - 8c) + c(7c - 8c) =$$

$$= 2c^2 - 14c - c^3 + \cancel{8c^2} + 7c^2 - \cancel{8c^2} = -c^3 + 9c^2 - 14c$$

if $\det(A) = 0 \Rightarrow$ NOT invertible

$$-c^3 + 9c^2 - 14c = 0$$

$$-c(c^2 - 9c + 14) = 0$$

$c_1 = 0 \quad c_2 = 2 \quad c_3 = 7$

$c \neq 0$
$$\begin{pmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{pmatrix} \quad c \neq 2$$

$$\begin{pmatrix} 2 & c & 8 \\ c & c & 7 \\ c & c & c \end{pmatrix} \quad c \neq 7$$

4. a)
$$\left(\begin{array}{ccc|c} 3 & 6 & 6 & 0 \\ -6 & 4 & c & 0 \\ 1 & 3 & 2 & 0 \end{array}\right) \begin{matrix} -3r_3 \\ +2r_1 \\ \end{matrix} \Rightarrow \left(\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 16 & c+12 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right) + \frac{16}{3}r_1 \Rightarrow \left(\begin{array}{ccc|c} 0 & -3 & 0 & 0 \\ 0 & 0 & c+12 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & c+12 & 0 \end{array}\right)$$

$c \neq -12$

b)
$$\begin{pmatrix} 1 & 2 & c \\ 5 & 4 & 3 \\ -3 & -6 & -3 \end{pmatrix} \begin{matrix} \\ -5r_1 \\ +3r_1 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 2 & c \\ 0 & -6 & 3-5c \\ 0 & 0 & -3+3c \end{pmatrix} \qquad \begin{matrix} -3+3c \neq 0 \\ c \neq 1 \end{matrix}$$

5. 1) $A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

2) $A = I_n$

3) no

4) each cell on the main diagonal of $(A+B)$ matrix should have non empty value

6. a) $BA = 4A$

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

b) $BA = 4B$ | reversed io is impossible

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

c) BA

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

d) $B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Extra problems:

7. a) Ax, $x \in \mathbb{R}^3$ ($u \in \mathbb{R}^3$) $\Rightarrow$ matrix has length $m \times 3$

$u \in \mathbb{R}^3$, $v_1 \in \mathbb{R}^4 \Rightarrow m = 4 \quad \Rightarrow \quad A$ is 4×3

- spanned by $u^n \Rightarrow$ one free variable

3 columns and 1 free variable $\Rightarrow$ 2 pivot columns $\Rightarrow$ rank $= 2$

b) there are 3 possible forms:

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

possible — $\neq 0$ not possible — $\neq 0$, $\neq 0$ not possible

u spans solution, $u = (2, -1, 1)^T$:

$2 \cdot 1 + 0(-1) + 1a = 0 \Rightarrow a = -2$

$2 \cdot 0 + 1 \cdot (-1) + b \cdot 1 = 0 \Rightarrow b = 1$ $\Rightarrow$ RREF: $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$