Linear Algebra

Lecture Notes

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4th term
Spring 2019

Lecture 9. Orthogonalization and QR

Outline

1. Orthonormal bases and orthogonal matrices
   - Orthonormal bases in $\mathbb{R}^n$
   - Orthogonal matrices
2. Gram–Schmidt orthogonalization and QR-decomposition
   - Gram–Schmidt orthogonalization
   - QR-decomposition
   - Applications of QR
3. Fourier transform
   - Hilbert spaces

Orthogonal systems

Definition

A set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of nonzero vectors in $\mathbb{R}^n$ is orthogonal if

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0 \text{ for } i \neq j.$$

An orthogonal set S s.t. $\|\mathbf{u}\| = 1$ for each $\mathbf{u} \in S$ is orthonormal.

Example

- The standard basis system of $\mathbb{R}^n$ is an orthonormal set
- The set $(1, 0, 1)$, $(1, 1, -1)$ and $(1, -2, -1)$ is orthogonal

Remark

Every *orthogonal* set S can be made *orthonormal* if we replace each $\mathbf{u}$ in S by $\mathbf{u}/\|\mathbf{u}\|$.

Orthogonal sets are linearly independent

Lemma

Every orthogonal set is linearly independent.

Proof.

Assume that a set $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is orthogonal and $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}$. Then

$$\begin{aligned} 0 = \langle \mathbf{u}_m, \mathbf{0} \rangle &= \langle \mathbf{u}_m, c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k \rangle \\ &= c_1\langle \mathbf{u}_m, \mathbf{u}_1 \rangle + \cdots + c_k\langle \mathbf{u}_m, \mathbf{u}_k \rangle = c_m\|\mathbf{u}_m\|^2 \end{aligned}$$

yielding $c_m = 0$. □

Every orthogonal set $S \subset \mathbb{R}^n$ is contained in an orthogonal basis of $\mathbb{R}^n$:

- If $\mathrm{ls}(S) \neq \mathbb{R}^n$, take any nonzero vector $\mathbf{u} \in \big(\mathrm{ls}(S)\big)^\perp$
- Denote $S' := S \cup \{\mathbf{u}\}$; then S' is orthogonal
- Continue (finitely many times) to get $\mathrm{ls}(S') = \mathbb{R}^n$

Coordinate representation in orthogonal bases

Theorem

Assume that $S = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ *is an orthonormal basis (ONB) of* $\mathbb{R}^n$*. Then, for every* $\mathbf{u} \in \mathbb{R}^n$*,*

$$\boxed{\mathbf{u} = \langle \mathbf{u}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}, \mathbf{u}_n \rangle \mathbf{u}_n}$$

Proof.

The vector $\mathbf{v} := \mathbf{u} - \langle \mathbf{u}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}, \mathbf{u}_n \rangle \mathbf{u}_n$ is orthogonal to every $\mathbf{u}_k$, thus to $\mathrm{ls}\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} = \mathbb{R}^n \implies \mathbf{v} = \mathbf{0}$. □

Corollary

Assume that $S = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ *is an orthogonal basis of* $\mathbb{R}^n$*. Then, for every* $\mathbf{u} \in \mathbb{R}^n$*,*

$$\boxed{\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 + \cdots + \frac{\langle \mathbf{u}, \mathbf{u}_n \rangle}{\|\mathbf{u}_n\|^2} \mathbf{u}_n}$$

$\langle \mathbf{u}, \mathbf{u}_k \rangle / \|\mathbf{u}_k\|^2$ *is the* k^{th} *coordinate of* $\mathbf{u}$ *in the basis* $\mathbf{u}_1, \ldots, \mathbf{u}_n$
$(\langle \mathbf{u}, \mathbf{u}_k \rangle / \|\mathbf{u}_k\|^2) \mathbf{u}_k$ *is the component* $=$ *projection of* $\mathbf{u}$ *onto* $\mathbf{u}_k$*.*

Orthogonal columns and least squares solutions

Properties of matrices with orthogonal columns:

- If an $m \times n$ matrix $A = Q$ has orthonormal columns ($\mathbf{q}_i^T \mathbf{q}_j = \delta_{ij}$), then
$$\boxed{Q^T Q = I_n}$$
- the least squares solution of $Q\mathbf{x} = \mathbf{b}$ is $\hat{\mathbf{x}} = (Q^T Q)^{-1} Q^T \mathbf{b} = Q^T \mathbf{b}$
- the projection $\mathbf{p}$, i.e., the best approximation in $\mathcal{C}(Q)$ is $QQ^T \mathbf{b}$
- the orthogonal projection operator onto $\mathcal{C}(Q) \subset \mathbb{R}^m$ is
$$\boxed{P = QQ^T}$$
- $\mathbf{p} = QQ^T \mathbf{b}$ is the basis decomposition of $\mathbf{p}$ in the ONB $\mathbf{q}_k$ of $\mathcal{C}(Q)$:
$$\mathbf{p} = \mathbf{q}_1(\mathbf{q}_1^T \mathbf{b}) + \mathbf{q}_2(\mathbf{q}_2^T \mathbf{b}) + \cdots + \mathbf{q}_n(\mathbf{q}_n^T \mathbf{b})$$
- When columns of Q are only orthogonal, then the above becomes
$$\mathbf{p} = \frac{\mathbf{q}_1^T \mathbf{b}}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 + \frac{\mathbf{q}_2^T \mathbf{b}}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 + \cdots + \frac{\mathbf{q}_n^T \mathbf{b}}{\|\mathbf{q}_n\|^2}\mathbf{q}_n$$

Orthogonal projectors

Explicit formula for the orthogonal projector P_W onto a subspace W

- Assume W is a subspace of $\mathbb{R}^m$ of dimension n
- Choose an orthogonal basis $\{\mathbf{q}_1, \dots, \mathbf{q}_n\}$ of W
- Denote by Q the $m \times n$ matrix with columns $\mathbf{q}_j$
- Then

$$\boxed{P_W = Q(Q^TQ)^{-1}Q^T = \frac{\mathbf{q}_1\mathbf{q}_1^T}{\|\mathbf{q}_1\|^2} + \frac{\mathbf{q}_2\mathbf{q}_2^T}{\|\mathbf{q}_2\|^2} + \cdots + \frac{\mathbf{q}_n\mathbf{q}_n^T}{\|\mathbf{q}_n\|^2}}$$

and the projection $P_W\mathbf{x}$ is equal to

$$\boxed{P_W\mathbf{x} = \frac{\mathbf{q}_1^T\mathbf{x}}{\|\mathbf{q}_1\|^2}\mathbf{q}_1 + \frac{\mathbf{q}_2^T\mathbf{x}}{\|\mathbf{q}_2\|^2}\mathbf{q}_2 + \cdots + \frac{\mathbf{q}_n^T\mathbf{x}}{\|\mathbf{q}_n\|^2}\mathbf{q}_n}$$

Example (Orthogonal projection)

Let $\mathbf{x} = (x_1, x_2, x_3)^\top$, $\mathbf{q}_1 = (1, 1, 0)^\top$, and $\mathbf{q}_2 = (1, -1, 0)^\top$. Find an orthogonal projection of the vector $\mathbf{x}$ onto the plane $W = \text{ls}\{\mathbf{q}_1, \mathbf{q}_2\}$.

Solution

Since $\mathbf{q}_1 \perp \mathbf{q}_2$ and $\|\mathbf{q}_1\| = \|\mathbf{q}_2\| = \sqrt{2}$, the orthogonal projector onto W is equal to

$$P_W = \frac{\mathbf{q}_1\mathbf{q}_1^\top}{2} + \frac{\mathbf{q}_2\mathbf{q}_2^\top}{2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{diag}\{1, 1, 0\}$$

i.e., P_W is the projector onto the xy-plane. Clearly, $P_W\mathbf{x} = (x_1, x_2, 0)^\top$. Alternatively, we find the orthogonal projection via

$$P_W\mathbf{x} = \frac{\mathbf{q}_1^\top\mathbf{x}}{2}\mathbf{q}_1 + \frac{\mathbf{q}_2^\top\mathbf{x}}{2}\mathbf{q}_2 = \frac{x_1 + x_2}{2}\mathbf{q}_1 + \frac{x_1 - x_2}{2}\mathbf{q}_2 = (x_1, x_2, 0)^\top$$

Orthogonal matrices

- When Q has orthonormal columns and is square, then
$$Q^{-1} = Q^T$$
- the least squares solution is then exact: $\hat{\mathbf{x}} = Q^T\mathbf{b} \implies Q\hat{\mathbf{x}} = \mathbf{b}$

Definition

An $n \times n$ matrix U is called orthogonal if $U^{-1} = U^T$, i.e., if
$$U^T U = UU^T = I_n$$

Criterion for orthogonality

$$\boxed{U \text{ is orthogonal} \iff \text{its columns form an ONB of } \mathbb{R}^n}$$

If $\mathbf{u}_j$ is the j^{th} column of U, then $U^\top U = I_n$ amounts to $\mathbf{u}_k^\top \mathbf{u}_j = \delta_{jk}$.

Another derivation of $\mathbf{u} = \langle \mathbf{u}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{u}, \mathbf{u}_n \rangle \mathbf{u}_n$

- $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n$ is just the matrix equality $\mathbf{u} = U\mathbf{c}$
- if $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ forms an ONB of $\mathbb{R}^n$, then $\mathbf{c} = U^{-1}\mathbf{u} = U^T\mathbf{u}$
- then $\mathbf{u} = U\mathbf{c} = UU^T\mathbf{u}$ reads componentwise
$$\mathbf{u} = \mathbf{u}_1(\mathbf{u}_1^T\mathbf{u}) + \mathbf{u}_2(\mathbf{u}_2^T\mathbf{u}) + \cdots + \mathbf{u}_n(\mathbf{u}_n^T\mathbf{u})$$

Properties of orthogonal matrices

An orthogonal matrix U does not change scalar products

Reason: $\langle U\mathbf{u}, U\mathbf{v}\rangle = \langle U^T U\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$

Orthogonality criterion for square matrices

$$\boxed{\text{A matrix } U \text{ is orthogonal} \iff U \text{ does not change length of vectors}}$$

$\Longrightarrow$ from the above

$\Longleftarrow$ if $\forall \mathbf{x} \quad \|U\mathbf{x}\| = \|\mathbf{x}\|$, then $\forall \mathbf{u}, \mathbf{v}$: $\langle U^T U\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle \implies U^T U = I$

Corollary: An orthogonal matrix preserves the angle between vectors

Reason: $\langle \mathbf{u}, \mathbf{v}\rangle = \|\mathbf{u}\|\|\mathbf{v}\| \cos\phi$

Orthogonal matrices and bases

$$\boxed{\text{A matrix } U \text{ is orthogonal} \iff U \text{ sends every ONB into ONB}}$$

$\Longleftarrow$ because then U is square and does not change length of vectors

Gram–Schmidt orthogonalization

Remark

Having a matrix with orthogonal columns or an orthogonal set is very useful

Problem: Construct an orthogonal system $\mathbf{w}_1, \mathbf{w}_2, \dots$ given a system of linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$

- set $\mathbf{w}_1 = \mathbf{v}_1$; $\mathbf{v}_2$ need not be orthogonal to $\mathbf{w}_1$
- subtract from $\mathbf{v}_2$ its projection onto $\mathbf{w}_1$:

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{w}_1^\top \mathbf{v}_2}{\mathbf{w}_1^\top \mathbf{w}_1}\mathbf{w}_1 \implies \mathbf{w}_2^T \perp \mathbf{w}_1 = 0$$

- subtract from $\mathbf{v}_3$ its projection onto the plane $\mathrm{ls}\{\mathbf{w}_1, \mathbf{w}_2\}$:

$$\mathbf{w}_3 := \mathbf{v}_3 - \frac{\mathbf{v}_3^T \mathbf{w}_1}{\mathbf{w}_1^T \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3^T \mathbf{w}_2}{\mathbf{w}_2^T \mathbf{w}_2}\mathbf{w}_2$$

- and so on for $\mathbf{w}_4, \mathbf{w}_5, \dots$

Gram–Schmidt orthogonalization

Example (Gram–Schmidt orthogonalization)

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \; \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \; \mathbf{v}_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$$

Solution:

$$\mathbf{w}_1 = \mathbf{v}_1; \qquad \mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2^\top \mathbf{w}_1}{\mathbf{w}_1^\top \mathbf{w}_1}\mathbf{w}_1 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3^T \mathbf{w}_1}{\mathbf{w}_1^T \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3^T \mathbf{w}_2}{\mathbf{w}_2^T \mathbf{w}_2}\mathbf{w}_2 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

QR factorization = matrix form of Gram–Schmidt

- Assume A has linearly independent columns $\mathbf{a}_1, \dots, \mathbf{a}_n$
- perform Gram–Schmidt orthogonalization to get an orthogonal set $\mathbf{w}_1, \dots, \mathbf{w}_n$
- then normalize to get $\mathbf{q}_1, \dots, \mathbf{q}_n$
- at each step, $\mathbf{q}_k$ is a linear combination of $\mathbf{a}_k$, $\mathbf{q}_1, \dots, \mathbf{q}_{k-1}$
- thus $\mathbf{a}_k$ is in the span of $\mathbf{q}_1, \dots, \mathbf{q}_k$
- $\mathbf{a}_k = P_1\mathbf{a}_k + \cdots + P_k\mathbf{a}_k = \mathbf{q}_1\mathbf{q}_1^\top\mathbf{a}_k + \cdots + \mathbf{q}_k\mathbf{q}_k^\top\mathbf{a}_k$
- in matrix form, this becomes a *QR* factorization:

$$\underbrace{\begin{pmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n\end{pmatrix}}_{A} = \underbrace{\begin{pmatrix}\mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n\end{pmatrix}}_{Q} \underbrace{\begin{pmatrix}\mathbf{q}_1^\top\mathbf{a}_1 & \mathbf{q}_1^\top\mathbf{a}_2 & \dots & \mathbf{q}_1^\top\mathbf{a}_n \\ & \mathbf{q}_2^\top\mathbf{a}_2 & \dots & \mathbf{q}_2^\top\mathbf{a}_n \\ & & \dots & \dots \\ & & & \mathbf{q}_n^\top\mathbf{a}_n\end{pmatrix}}_{R}$$

- then $R = Q^\top A$: indeed, $Q^\top A = Q^\top(QR) = (Q^\top Q)R = R$

QR factorization, or QR decomposition

QR factorization of an $m \times n$ matrix $A \equiv$ its representation as $A = QR$, where

- an $m \times n$ matrix Q has orthonormal columns;
- an $n \times n$ matrix R is upper-triangular.

A has linearly independent columns $\iff$ R is non-singular

$$A^\top A = (QR)^\top QR = R^\top (Q^\top Q) R = R^\top I_n R = R^\top R$$

Another reason: $\operatorname{rank} A = n \iff \operatorname{rank}(QR) = n \overset{?}{\iff} \operatorname{rank} R = n$

Uniqueness of QR-factorization of A with linearly independent columns:

- $Q_1 R_1 = Q_2 R_2 \iff Q_2^\top Q_1 = R_2 R_1^{-1}$ upper triangular
- however, $Q_2^\top Q_1$ is lower triangular since linear spans of the first k columns of Q_1 and Q_2 are the same for $k = 1, \ldots, n$
- thus $Q_2^\top Q_1$ is a diagonal matrix $D \implies R_2 = DR_1$ and $Q_1 = Q_2 D$
- columns of Q_j are of length 1 $\implies D = \operatorname{diag}\{\pm 1, \pm 1, \ldots, \pm 1\}$

Lifehack: normalize at the very end!

- In the above algorithm, $\mathbf{q}_k = \mathbf{w}_k/\|\mathbf{w}_k\|$, where $\mathbf{w}_1, \dots, \mathbf{w}_n$ are obtained by GS algorithm from columns of A ($\mathbf{w}_k$ not normalized!)
- denote by W the matrix with columns $\mathbf{w}_1, \dots, \mathbf{w}_n$ and by D the diagonal matrix $\operatorname{diag}\{\|\mathbf{w}_1\|, \dots, \|\mathbf{w}_n\|\}$
- then $Q = WD^{-1}$ and $R = Q^\top A = D^{-1}W^\top A$
- therefore, must divide by $\|\mathbf{w}_k\|$ the k^{th} column of W and the k^{th} row of $W^\top A$ to get Q and R respectively

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}, \quad D = \operatorname{diag}\{\sqrt{2}, \sqrt{6}, \sqrt{3}\}$$

$$Q \leftarrow \quad \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{3}}$$

$$DR = W^T A = \begin{pmatrix} 2 & 2 & 6 \\ 0 & 6 & -6 \\ 0 & 0 & 3 \end{pmatrix} \begin{array}{l} :\sqrt{2} \\ :\sqrt{6} \\ :\sqrt{3} \end{array} \quad \mapsto \quad R$$

Full QR factorization and applications

- In the above form, Q is $m \times n$ and R is $n \times n$
- often called the reduced QR factorization
- Q is not orthogonal (as it is not square)
- add $m - n$ columns to get an orthogonal $\tilde{Q}$ and add $m - n$ zero rows to R; then $A = \tilde{Q}\tilde{R}$ is the full QR factorization

Application of QR to least squares:

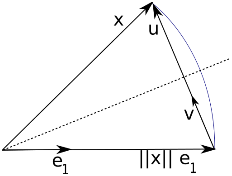
- $A^\top A = (QR)^\top (QR) = R^\top R$;
- since $A^\top A$ is invertible, such is also $R^\top R$ and thus R
- therefore, $R^\top R\,\hat{\mathbf{x}} = R^\top Q^T \mathbf{b} \implies R\hat{\mathbf{x}} = Q^\top \mathbf{b}$
- as R is upper-triangular, this is very fast!

Advantages of QR-decomposition:

Orthogonal columns of Q make algorithm stable
(norms do not increase or decrease)

Householder's reflection algorithm of QR

- another methods to find Q and R involve Householder's reflections $Q = I - 2\mathbf{v}\mathbf{v}^T$ with $\|v\| = 1$
- Householder's reflection Q can be chosen so that $Q\mathbf{x} = \|\mathbf{x}\|\mathbf{e}_1$: set $\mathbf{u} = \mathbf{x} - \|\mathbf{x}\|\mathbf{e}_1$ and $\mathbf{v} = \mathbf{u}/\|\mathbf{u}\|$



- the impact on A: take $\mathbf{x}$ to be the first column of A; then

$$QA = \begin{pmatrix} \|\mathbf{x}\| & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & * & \dots & * \end{pmatrix}$$

- Then consider $(n-1) \times (n-1)$ submatrix and continue

Givens rotations and QR

The idea in dimension 2

Rotate a vector $(x, y)^\top$ of length r to make it collinear to $\mathbf{e}_1 = (1, 0)^\top$:

$$G(\theta)\mathbf{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mathbf{x} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

We easily find that $\cos\theta = x/r = x/\sqrt{x^2+y^2}$ and $\sin\theta = -y/\sqrt{x^2+y^2}$

Algrithm:

Apply Givens rotations in coordinates 1 and n, then 1 and $n-1$ etc to make all entries below the $(1, 1)$-entry zero

Pros:

- easily parallelized
- fast for sparse matrices

Applications

- *QR* eigenvalue algorithm
- Fourier transform and Fast Fourier transform
- ...

Hilbert space $L_2(0, 2\pi)$

- Consider the space of functions over $[0, 2\pi]$ that are square integrable:
$$\|f\|^2 := \int_0^{2\pi} |f(t)|^2 \, dt$$
- this is the inner product space called the Hilbert space $L_2(0, 2\pi)$
- scalar product:
$$\langle f, g \rangle := \int_0^{2\pi} f(t)\overline{g(t)} \, dt$$
- the set of functions 1, $\cos nx$, $\sin nx$, $n = 1, 2 \ldots,$ forms an orthogonal set in $L_2(0, \pi)$:

$$\sin nx \cos mx = \tfrac{1}{2}\sin(n+m)x + \tfrac{1}{2}\sin(n-m)x \implies$$
$$\langle \sin nx, \cos mx \rangle = 0;$$
$$\langle 1, \sin nx \rangle = \langle 1, \cos mx \rangle = 0 \qquad \text{etc}$$

Fourier series

- $\{1/\sqrt{2\pi}, \sin nx/\sqrt{\pi}, \cos nx/\sqrt{\pi}\}$ forms an orthonormal basis of $L_2(0, 2\pi)$
- every function $f \in L_2(0, 2\pi)$ is equal (in some sense) to its Fourier series

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{k>0}(a_k \cos kx + b_k \sin kx)$$

with

$$a_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\, dt, \qquad b_k := \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\, dt,$$

being the Fourier coefficients of f

Complex trigonometric Fourier series

- This time we consider complex-valued functions over $(0, 2\pi)$
- can associate any such function with a function on the unit circle
- Another basis for $L_2(0, 2\pi)$ is $\{e^{inx}\}$, $n \in \mathbb{Z}$, with $i = \sqrt{-1}$
- This is an orthogonal basis:

$$\langle e^{inx}, e^{imx} \rangle = \int_0^{2\pi} e^{inx}\overline{e^{imx}}\,dx = \int_0^{2\pi} e^{i(n-m)x}\,dx = 2\pi\delta_{n,m}$$

- Fourier series of a function f:

$$f \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx},$$

  where

$$c_k := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt$$

  are the Fourier coefficients

The Fourier transform

Definition

The mapping $\mathscr{F} : L_2(0,2\pi) \to \ell_2(\mathbb{Z})$ given by

$$\mathscr{F}(f) = (c_k)$$

is the Fourier transform

- ℓ_2 is a vector space of all complex-valued sequences $\mathbf{c} = (c_k)_{k=-\infty}^{\infty}$ with norm $\|\mathbf{c}\| = \sqrt{\sum |c_k|^2}$
- The Pythagorean thm = Parseval thm:

$$\|f\|^2 = \sum |c_k|^2$$

- Plancherel thm: $f \sim \sum c_k e^{ikx}$, $g \sim \sum d_k e^{ikx} \implies$

$$\langle f, g \rangle = \sum c_k \overline{d_k}$$

Example (Fourier transform of $f = \cos 2x$)

$\cos 2x = \frac{1}{2} e^{2ix} + \frac{1}{2} e^{-2ix}$ is the Fourier series of f;

$\mathscr{F}(f) = (\ldots, 0, 0, \ldots, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \ldots, 0, 0, \ldots)$