Linear Algebra
Seminar 5: Bases
(make sure you can write on the board)

Problem 1. (2 pts.) Check whether the following set of vectors is a basis for $\mathbb{R}^3$? In case it is, find the coordinate vector of $v = (5, 4, 2)$ in this basis

(a) $(1, 1, 1)^\top$, $(1, -1, 2)^\top$, $(3, 4, -1)^\top$;

(b) $(1, 1, 0)^\top$, $(1, -1, 1)^\top$, $(-2, 2, -4)^\top$;

(c) $(1, 1, 1)^\top$, $(1, -1, 2)^\top$, $(-2, 2, -4)^\top$;

(d) $(1, 1, 1)^\top$, $(1, 1, 0)^\top$, $(0, 1, 2)^\top$, $(3, 0, -1)$;

(e) $(1, 1, 1)^\top$, $(1, 0, 1)^\top$.

Problem 2. (2 pts.) Check whether the following set of polynomials is basis for $\mathcal{P}_2$? In case it is, find the coordinate vector of $p = 5 + 4x + 2x^2$ in this basis

(a) $1 + x + x^2$, $1 - x + 2x^2$, $3 + 4x - x^2$;

(b) $1 - 3x + 2x^2$, $1 + x + 4x^2$, $1 - 7x$;

(c) $1 + x + x^2$, $1 + x$, $x + 2x^2$, $3 - x^2$;

(d) $1 + x + x^2$, $1 + x$.

Problem 3. (4 pts.)

(1) Find bases for the following subspaces of $\mathbb{R}^3$:

- (a) the plane $3x - 5y + 2z = 0$;
- (b) the plane $y - z = 0$;
- (c) the line $x = 3t$, $y = -2t$, $z = 5t$;
- (d) all vectors of the form (a, b, c), where $a - b = c$.

(2) Show that the set W of all polynomials p in $\mathcal{P}_3$ such that $p(2) = 0$ is a subspace of $\mathcal{P}_3$. Determine the dimension of W and suggest its basis.

Problem 4. (4 pts.)

(1) Find the dimensions for the following subspaces of $\mathbb{R}^4$:

- (a) all vectors of the form $(a, 0, b, c,)$;
- (b) all vectors of the form (a, b, c, d), where $d = a - b$ and $c = a - b$;
- (c) all vectors of the form (a, b, c, d), where $a = b = c = d$;
- (d) all vectors of the form (a, b, c, d), where $a + b + c + d = 0$.

(2) Show that the set W of all polynomials p in $\mathcal{P}_3$ such that $p(2) = p(3) = 0$ is a subspace of $\mathcal{P}_3$. Determine the dimension of W and suggest its basis.

Problem 5. (4 pts.)

(1) Find the rank and nullity of the matrix A:

(a) $\begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix}$

(2) verify that the values obtained satisfy the First Fundamental Theorem;

(3) find the number of leading (pivot) variables and the number of free variables in the solution of $A\mathbf{x} = \mathbf{0}$ without solving the system;

(4) find a basis for the null space of the above matrices A.

Problem 6. (4 pts.)

(1) In each part, use the information in the table to find the dimension of the row space of A, column space of A, null space of A, and null space of $A^\top$:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	9×5	4×4	6×2
$rank(A)$	3	2	1	2	2	0	2
$rank(A\|\mathbf{b})$	3	3	1	2	3	0	2

(2) In each part, determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. If so, state the number of parameters in its general solution.

Extra problems
(to be discussed if time permits)

Problem 7. (a) The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (2, 1, 1)$ are linearly independent. Find all $\mathbf{v}_3$ such that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$.

(b) Find standard basis vectors of $\mathbb{R}^4$ that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for $\mathbb{R}^4$, if $\mathbf{v}_1 = (1, -4, 2, -3)$ and $\mathbf{v}_2 = (-3, 8, -4, 6)$.

Problem 8. Let S be a basis for an n-dimensional vector space V. Show that if $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\mathbf{v}_1)_S$, …, $(\mathbf{v}_r)_S$ form a linearly independent set in $\mathbb{R}^n$, and conversely.

Problem 9. Find a basis for the space spanned by the given vectors $\mathbf{v}_1 = (1, 0, -2, 3)$, $\mathbf{v}_2 = (0, 1, 2, 3)$, $\mathbf{v}_3 = (2, -2, -8, 0)$, $\mathbf{v}_4 = (2, -1, 10, 3)$ and $\mathbf{v}_5 = (3, -1, -6, 9)$

Problem 10. Write the 3 by 3 identity matrix as a combination of the other five permutation matrices. Then show that those five matrices are linearly independent. The five permutations are a basis for the subspace of 3 by 3 matrices with raw and column sums all equal.

1. • The set of vector is a basis if:
  - it is linearly independent
  - it spans vector space

• Any linearly independent vectors $\vec{v_1}, \dots, \vec{v_n}$ in $\mathbb{R}^n$ form its basis

$\vec{v} = (5,4,2)$

a) $\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & -1 \end{pmatrix} \begin{matrix} \\ -r_1 \\ -r_1 \end{matrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -2 & 1 \\ 0 & 1 & -4 \end{pmatrix} +2r_3 \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & -7 \\ 0 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -7 \end{pmatrix}$ linearly independent.

Set S consists of 3 linearly independent vectors in $\mathbb{R}^3$ => S is a basis

$a\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + c\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$

$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 1 & -1 & 4 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right) \begin{matrix} \\ -r_1 \\ -r_1 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & -2 & 1 & -1 \\ 0 & 1 & -4 & -3 \end{array}\right) +2r_3 \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 0 & -7 & -7 \\ 0 & 1 & -4 & -3 \end{array}\right) \begin{matrix} -r_3 \\ \cdot(-\frac{1}{7}) \\ \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -4 & -3 \end{array}\right) \begin{matrix} -7r_2 \\ \\ +4r_2 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$

coordinates check: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$

b) $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 1 & -1 & 2 & 4 \\ 0 & 1 & -4 & 2 \end{array}\right) -r_1 \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -2 & 4 & -1 \\ 0 & 1 & -4 & 2 \end{array}\right) +2r_3 \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 0 & -4 & 3 \\ 0 & 1 & -4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & -4 & 3 \end{array}\right) \begin{matrix} -r_2 \\ -r_3 \\ \cdot(-\frac{1}{4}) \end{matrix}$ – lin. indep. => is a basis

$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3/4 \end{array}\right) -2r_3 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 9/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3/4 \end{array}\right)$

Check: $\frac{9}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{3}{4}\begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} - 1 + \frac{3}{2} \\ \frac{9}{2} + 1 - \frac{3}{2} \\ 0 - 1 + 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$

c) $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 1 & -1 & 2 & 4 \\ 1 & 2 & -4 & 2 \end{array}\right) \begin{matrix} \\ +r_1 \\ -r_1 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 2 & 0 & 0 & 9 \\ 0 & 1 & -2 & -3 \end{array}\right) \cdot(\frac{1}{2}) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 9/2 \\ 0 & 1 & -2 & -3 \\ 1 & 1 & -2 & 5 \end{array}\right) -r_1-r_2 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 9/2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 7/2 \end{array}\right)$ – is linearly dependent => is not a basis

d) The set contains four 3-d vectors, thus the set is linearly dependent => is not a basis

if rank(A) = 3 => it spans $\mathbb{R}^3$
but if rank(A) < 3 => it doesn't span $\mathbb{R}^3$

e) $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 0 & 4 \\ 1 & 1 & 2 \end{array}\right) \begin{matrix} -r_3 \\ \\ -r_2 \end{matrix} \sim \left(\begin{array}{cc|c} 0 & 0 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{array}\right)$ $3 \neq 0$ => the system is inconsistent => lin. dep. => not basis

$\exists x = (2,1,3) \neq c_1e_1 + c_2e_2$ – we can't get this vector with the given vectors, so it doesn't span $\mathbb{R}^3$

2. a) $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 1 & -1 & 4 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right) \begin{matrix} \\ -r_1 \\ -r_1 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & -2 & 1 & -1 \\ 0 & 1 & -4 & -3 \end{array}\right) +2r_3 \sim \left(\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 0 & -7 & -7 \\ 0 & 1 & -4 & -3 \end{array}\right) \begin{matrix} -r_3 \\ \cdot(-\frac{1}{7}) \\ \end{matrix}$ lin. indep. $\sim \left(\begin{array}{ccc|c} 1 & 0 & 7 & 8 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -4 & -3 \end{array}\right) \begin{matrix} -7r_2 \\ \\ +4r_2 \end{matrix} \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$

P^2 and $\mathbb{R}^3$ are isomorphic (working with polynomials is the same as with vectors)

check: $1 + x + x^2 + 1 - x + 2x^2 + 3 + 4x - x^2 = 5 + 4x + 2x^2$

b) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ -3 & 1 & -7 & 4 \\ 2 & 4 & 0 & 2 \end{array}\right) +7r_1 \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 4 & 8 & 0 & 39 \\ 2 & 4 & 0 & 2 \end{array}\right)$ – $r_2 = 2r_3$ => lin. dependent => is not a basis

c) $\begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$ The set contains four 3-d vectors => lin. depen. => not a basis

d) $\left(\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 1 & 4 \\ 1 & 0 & 2 \end{array}\right) \begin{matrix} -r_2 \\ -r_3 \\ \end{matrix} \sim \left(\begin{array}{cc|c} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 2 \end{array}\right)$ $1 \neq 0$ => the system is inconsistent => not a basis.

3. 1) a) $3x - 5y + 2z = 0$

put parametric description for the solutions: $\begin{cases} z = s \\ y = t \\ x = \frac{5t - 2s}{3} = \frac{5}{3}t - \frac{2}{3}s \end{cases} \Rightarrow \begin{cases} z = 0t + 1s \\ y = 1t + 0s \\ x = \frac{5}{3}t - \frac{2}{3}s \end{cases} \Rightarrow$

the vectors $(\frac{5}{3}, 1, 0), (-\frac{2}{3}, 0, 1)$ span the solution space. They are also independent => they form a basis. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5/3 \\ 1 \\ 0 \end{pmatrix} y + \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix} z$

b) $y - z = 0$

The points on the plane are of the form (x, y, y), which we can transform to $x(1,0,0) + y(0,1,1)$. The set of two vectors $\{(1,0,0), (0,1,1)\}$ spans the solution space and is linearly independent => is a basis for the plane.

c) $\begin{cases} x = 3t \\ y = -2t \\ z = 5t \end{cases}$

$(3, -2, 5)$ spans the line

d) $a - b = c$

$(a, b, a-b) \Rightarrow a(1,0,1) + b(0,1,-1)$ -> vectors span the space.

2) $p(2) = 0$ is subspace of P_3; $\alpha_1 p_1(2) + \alpha_2 p_2(2) = \alpha_1 p_1(2) + \alpha_2 p_2(2) = 0$

$p(x) = ax^3 + bx^2 + cx + d$

$p(2) = 8a + 4b + 2c + d = 0$ => express d as a linear combination of a, b, c

$(\forall p \in W)$: $p(x) = ax^3 + bx^2 + cx + (-8a - 4b - 2c) = a(x^3 - 8) + b(x^2 - 4) + c(x - 2)$

the basis: $\{(x^3 - 8), (x^2 - 4), (x - 2)\}$ – $\{(-8,0,0,1), (-4,0,1,0), (-2,1,0,0)\}$ dim = 3

4. a) $(a, 0, b, c) = a(1,0,0,0) + b(0,0,1,0) + c(0,0,0,1)$ => the set forms the basis => dim = 3

b) (a, b, c, d), $d = a - b$, $c = a - b$

$(a, b, a-b, a-b) = a(1,0,1,1) + b(0,1,-1,-1)$ => dim = 2

c) (a, b, c, d), $a = b = c = d$

$(a, a, a, a) = a(1,1,1,1)$ => dim = 1

d) (a, b, c, d), $a + b + c + d = 0$

$\begin{cases} d = s \\ c = t \\ b = p \\ a = -s - t - p \end{cases} \quad \begin{cases} d = 1s + 0t + 0p \\ c = 0s + 1t + 0p \\ b = 0s + 0t + 1p \\ a = -1s - 1t - 1p \end{cases}$

$\{(-1,0,0,1), (-1,0,1,0), (-1,1,0,0)\}$, dim = 3

2) $p(2) = p(3) = 0$; P_3

$p(x) = ax^3 + bx^2 + cx + d$

$p(2) = p(3) = 0$: $8a + 4b + 2c + d = 27a + 9b + 3c + d = 0$

$19a + 5b + c = 0$ $\quad d = -8a - 4b - 2c = -8a - 4b + 38a + 10b =$

$c = -(19a + 5b)$ $\quad = 30a + 6b$

$(\forall p \in W)$: $p(x) = ax^3 + bx^2 - (19a + 5b)x + 30a + 6b = a(x^3 - 19x + 30) + b(x^2 - 5x + 6)$

the basis $S = \{(30, 19, 0, 1), (6, -5, 1, 0)\}$, dim W = 2

$\{p : p(2) = p(3) = 0\} = \{p : (x-2)(x-3)(a_0 + a_1 x)\}$

$e_1 = (x-2)(x-3)$

$e_2 = (x-2)(x-3)x$

5. rank = Number of pivots

nullity = $\dim N(A) = n - \operatorname{rank}(A)$ — Fundamental theorem,

a) $\begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix} \begin{matrix} \\ -5r_1 \\ -r_2 - 2r_1 \end{matrix} \sim \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{pmatrix}$

1) rank = 2;

general element of $N(A)$: $\begin{cases} z = t \\ x - y + 3z = 0 \\ y - 19z = 0 \end{cases} \quad \begin{cases} z = t \\ y = 19z = 19t \\ x = y - 3z = 19t - 3t = 16t \end{cases}$

$(x, y, z) = t(16, 19, 1)$ is the basis for $N(A)$ => the nullity = 1.

2) $3 - 2 = 1 = 1$ ✓

3) the 3rd row is the linear combination of the first and the second, while they (1^{st}, 2^{nd}) are linearly independent => rank = 2

4) $(16, 19, 1)$

b) $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} -2r_1 \sim \begin{pmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1) rank = 1

$2x - z = 0$; solution set: $\begin{cases} 2x - z = 0 \\ y = s \\ z = 2t \end{cases} \quad \begin{cases} x = \frac{z}{2} = t \\ y = s \\ z = 2t \end{cases} \quad \begin{cases} x = 1 \cdot t + 0 \cdot s \\ y = 0 \cdot t + 1 \cdot s \\ z = 2 \cdot t + 0 \cdot s \end{cases}$

$(x, y, z) = t(1, 0, 2) + s(0, 1, 0)$ => nullity = 2

2) $3 - 1 = 2 = 2$ ✓

3) the second row is the transformation of the first one, the third is the zero row => rank = 1

4) $\{(1,0,2), (0,1,0)\}$

c) $\begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{pmatrix} \begin{matrix} \\ -2r_1 \\ +r_1 \end{matrix} \sim \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{pmatrix} +r_2 \sim \begin{pmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

1) rank = 2

$\begin{cases} x_4 = 7/4\, t \\ x_3 = s \\ x_1 + 4x_2 + 5x_3 + 2x_4 = 0 \\ -7x_2 - 7x_3 - 4x_4 = 0 \end{cases} \quad \begin{cases} x_4 = 0s + \frac{7}{4}t \\ x_3 = 1s + 0t \\ x_1 = -4x_2 - 5x_3 - 2x_4 = 4s + 4t - 5s - \frac{7}{2}t = -1s + \frac{1}{2}t \\ x_2 = \frac{7x_3 + 4x_4}{-7} = -s - t \end{cases}$

$(x_1, x_2, x_3, x_4) = s(-1, -1, 1, 0) + t(\frac{1}{2}, -1, 0, \frac{7}{4})$ => nullity = 2

2) $4 - 2 = 2 = 2$ ✓

3) the 3^{rd} row is the linear combination of the first two rows, which by themselves two are linearly independent => rank = 2

4) $\{(-1, -1, 1, 0), (\frac{1}{2}, -1, 0, \frac{7}{4})\}$

6.

	a	b	c	d	e	f	g
Size (A)	3 x 3	3 x 3	3 x 3	5 x 9	9 x 5	4 x 4	6 x 2
rank(A)	3	2	1	2	2	0	2
rank(A|B)	3	3	1	2	3	0	2
dim R(A) = = rank(A)	3	2	1	2	2	0	2
dim C(A) = = rank(A)	3	2	1	2	2	0	2
dim N(A) = = n_cols − rank(A)	0	1	2	9−2= 7	5−2= 3	4	2−2= 0
(left nullspace) dim N(A^T) = = n_rows − rank(A)	0	1	2	5−2= 3	9−2= 7	4	6−2= 4
is Ax=b consistent; number of parameters in the general solution	the vectors are linearly independent => there is one unique solution. 0 parameters	inconsistent, because it has the last row of the form: (0 0 0 | a), where a≠0	$\left(\begin{array}{ccc\|c} a & b & c & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\begin{cases} a + b + c = k \\ b = s \\ c = t \end{cases}$ 2 parameters	7 parameters	there is zero row in A, while the relative coefficient in vector b is non-zero => inconsistent	4 parameters	$\left(\begin{array}{cc\|c} a & b & k_1 \\ 0 & c & k_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ consistent, no parameters