Linear Algebra
Seminar 1: Systems of Linear Equations

Problem 1. (2 pts)

(a) Transform the parametrized line equations $x_1 = 4 - 2t$, $x_2 = 3 + 6t$ into a non-parametrized form;

(b) Write the line equation $3x_2 + 2x_1 = 5$ in the parametrized form;

(c) Write non-parametric equation of the plane $x = 1 + s + t,\ y = 2 + 3s + 4t,\ z = s - t$.

Problem 2. (2 pts)

(a) Write a non-parametric equation of the plane through the points $(1, 3, 0)$, $(2, 0, 1)$, $(0, 0, 2)$.

(b) Write a parametric equation of the plane through the points $(1, 3, 0)$, $(2, 0, 1)$, $(0, 0, 2)$.

(c) Find a system of 3 equations in x, y, and z whose solution set is the line $x = s$, $y = 1 - 2s$, $z = 2 + s$ $(s \in \mathbb{R})$.

Problem 3. (4 pts) Determine all the values of c for which the matrix below is the augmented matrix of a consistent linear system.

$$(a)\ \left(\begin{array}{cc|c} 1 & c & 4 \\ 3 & 6 & 8 \end{array}\right) \qquad (b)\ \left(\begin{array}{cc|c} 1 & 0 & -5 \\ c & -4 & 6 \end{array}\right) \qquad (c)\ \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & c & -6 \end{array}\right) \qquad (d)\ \left(\begin{array}{cc|c} -4 & 12 & c \\ 2 & -6 & -3 \end{array}\right)$$

Problem 4. (4 pts) For what values of the parameters c and d does the following system have
(a) no solutions; **(b)** one solution; **(c)** infinitely many solutions?

$$\begin{aligned} 2x - y &= c, \\ dx + 2y &= 6. \end{aligned}$$

Problem 5. (4 pts) The following are coefficient matrices of linear systems. For each system, what can you say about the number of solutions to the corresponding system **(a)** in the homogeneous case (when $b_1 = \cdots = b_m = 0$) and **(b)** for a generic RHS?

$$(a) \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, \qquad (b) \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}, \qquad (c) \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 0 & 3 \end{pmatrix}, \qquad (d) \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Problem 6. (4 pts) Let

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right)$$

be the augmented matrix for a linear system. Find for what values of a and b the system has

(a) a unique solution;

(b) a one-parameter solution set;

(c) a two-parameter solution set;

(d) no solution.

Extra problems
(to be discussed if time permits)

Problem 7. Compute the rank of each of the following matrices:

$$(a)\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \qquad (b)\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \qquad (c)\begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix}, \qquad (d)\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}.$$

Problem 8. Write a system of linear equations consisting of m equations in n unknowns with

(a) no solutions;

(b) exactly one solution;

(c) infinitely many solutions

for (i) $m = n = 3$; (ii) $m = 3$ and $n = 2$; (iii) $m = 2$, $n = 3$.

Problem 9. Use Gauss–Jordan elimination in matrix form to solve the following linear system:

$$\begin{aligned} x + 3y - 2z + w &= 0 \\ 3x + 7y - 2z + 2w &= 9 \\ 5x + 13y - 9z + 3w &= 1 \\ x - z - 2w &= 0. \end{aligned}$$

Which variables are free and which basic (pivot)?

Problem 10. Transform the following matrices to row echelon form and to reduced row echelon form using the elementary row transformations:

$$(a)\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \qquad (b)\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix}, \qquad (c)\begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

1. a) $\begin{cases} x_1 = 4-2t & \Rightarrow 2t = 4-x_1 \\ x_2 = 3+6t & \Rightarrow x_2 = 3+12-3x_1 = 15-3x_1 \end{cases} \Rightarrow 3x_1 + x_2 = 15$

b) $3x_2 + 2x_1 = 5$

$\begin{cases} x_2 = \frac{2}{3}t \\ x_1 = \frac{5-2t}{2} = \frac{5}{2} - t \end{cases}$

c) $\begin{cases} x = 1+s+t \\ y = 2+3s+4t \\ z = s-t \end{cases} \quad \begin{cases} s = x-t-1 \\ y = 2+3s+4t \\ s = z+t \end{cases} \quad \begin{array}{l} x-t-1 = z+t \\ 2t = x-z-1 \\ t = \frac{x-z-1}{2} \end{array}$

$y = 2+3\left(z+\frac{x-z-1}{2}\right) + 2(x-z-1)$

$y - 3z - \frac{3}{2}(x-z) - 2x + 2z = 2 - \frac{3}{2} - 2$

$-\frac{7}{2}x + y + \frac{z}{2} = -\frac{3}{2}$

2. a) $(1,3,0),\ (2,0,1),\ (0,0,2)$

$\begin{cases} a+3b = d \\ 2a+c = d \\ 2c = d \end{cases} \quad \begin{cases} 3b = d - \frac{d}{4} = \frac{3d}{4} \Rightarrow b = \frac{d}{4} \\ 2a = d - \frac{d}{2} = \frac{d}{2} \Rightarrow a = \frac{d}{4} \\ c = \frac{d}{2} \end{cases}$

$\frac{d}{4}x + \frac{d}{4}y + \frac{d}{2}z = d \quad | : d$

$\frac{x}{4} + \frac{y}{4} + \frac{z}{2} = 1$

b) $\begin{cases} x = 2s \\ y = 2t \\ z = 2\left(1 - \frac{x}{4} - \frac{y}{4}\right) = 2-s-t \end{cases}$

c) $\begin{cases} x = s \\ y = 1-2s \\ z = 2+s \end{cases} \quad \begin{cases} s = x \\ y = 1-2x \\ z = 2+x \Rightarrow x = z-2 \end{cases} \Rightarrow \begin{cases} x = z-2 \\ y = 1-2x \\ z = 2+x \end{cases}$

3. a) $\left(\begin{array}{cc|c} 1 & c & 4 \\ 3 & 6 & 8 \end{array}\right) -3r_1 \Rightarrow \left(\begin{array}{cc|c} 1 & c & 4 \\ 0 & 6-3c & -4 \end{array}\right)$ the system will be inconsistent for $6-3c=0 \Rightarrow c=2$, because we will have a row of the form: $(0\ 0\ 0\ |\ b \neq 0)$

The system is consistent for $c \in \mathbb{R} \setminus \{2\}$

b) $\left(\begin{array}{cc|c} 1 & 0 & -5 \\ c & -4 & 6 \end{array}\right) - c \cdot r_1 \Rightarrow \left(\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & -4 & 6+5c \end{array}\right)$ we have a pivot variable in every row and column, so we don't care about the right-hand side, as the system will always have a unique solution

The system is consistent for $c \in \mathbb{R}$

c) $\left(\begin{array}{cc|c} 1 & 4 & -2 \\ 3 & c & -6 \end{array}\right) -3r_1 \Rightarrow \left(\begin{array}{cc|c} 1 & 4 & -2 \\ 0 & c-12 & 0 \end{array}\right)$ if $c-12$ is pivot $(c \neq 12)$ $\Rightarrow$ pivot in every row and column $\Rightarrow$ an unique solution

else $(c=12) \Rightarrow$ we have a row $(0\ 0\ |\ b=0) \Rightarrow$ infinitely many solutions

The system is consistent for $c \in \mathbb{R}$

d) $\left(\begin{array}{cc|c} -4 & 12 & c \\ 2 & -6 & -3 \end{array}\right) +2r_2 \Rightarrow \left(\begin{array}{cc|c} 2 & -6 & -3 \\ 0 & 0 & c-6 \end{array}\right)$ if $c-6=0$ $(c=6)$ $\Rightarrow$ infinitely many solutions, else $(c \neq 6) \Rightarrow$ no solutions

The system is consistent for $c=6$

4. $\begin{cases} 2x - y = c \\ dx + 2y = 6 \end{cases}$

$\left(\begin{array}{cc|c} 2 & -1 & c \\ d & 2 & 6 \end{array}\right) \cdot 2 \Rightarrow \left(\begin{array}{cc|c} 2 & -1 & c \\ 2d & 4 & 12 \end{array}\right) - dr_1 \Rightarrow \left(\begin{array}{cc|c} 2 & -1 & c \\ 0 & 4+d & 12-dc \end{array}\right)$

a) 0 solutions: $\begin{cases} 4+d = 0 \\ 12-dc \neq 0 \end{cases} \quad \begin{cases} d = -4 \\ 12+4c \neq 0 \end{cases} \quad \begin{cases} d=-4 \\ c \neq -3 \end{cases}$

b) 1 solution: $4+d \neq 0 \quad d \neq -4$

c) infinitely many solutions: $\begin{cases} 4+d = 0 \\ 12-dc = 0 \end{cases} \quad \begin{cases} d = -4 \\ c = -3 \end{cases}$

5. a) $\left(\begin{array}{cc|c} 1 & 4 & b_1 \\ 2 & 1 & b_2 \end{array}\right) -2r_1 \Rightarrow \left(\begin{array}{cc|c} 1 & 4 & b_1 \\ 0 & -7 & b_2-2b_1 \end{array}\right) \Rightarrow$ we have a pivot variable in every row and column, so there is no impact of the right-hand side values.
a) & b) an unique solution

b) $\left(\begin{array}{ccc|c} 1 & 4 & 3 & b_1 \\ 2 & 1 & 0 & b_2 \end{array}\right) -2r_1 \Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 3 & b_1 \\ 0 & -7 & -6 & b_2-2b_1 \end{array}\right) \Rightarrow$ there is a free variable:
a) & b) infinitely many solutions

c) $\left(\begin{array}{cc|c} 2 & 1 & b_1 \\ 1 & 4 & b_2 \\ 0 & 3 & b_3 \end{array}\right) - \frac{4}{3}r_3 \Rightarrow \left(\begin{array}{cc|c} 2 & 1 & b_1 \\ 1 & 0 & b_2 - \frac{4}{3}b_3 \\ 0 & 3 & b_3 \end{array}\right) -2r_2 - \frac{1}{3}r_3 \Rightarrow \left(\begin{array}{cc|c} 0 & 0 & b_1 - 2(b_2 - \frac{4}{3}b_3) - \frac{1}{3}b_3 \\ 1 & 0 & b_2 - \frac{4}{3}b_3 \\ 0 & 3 & b_3 \end{array}\right) \Rightarrow$

$\Rightarrow \left(\begin{array}{cc|c} 1 & 0 & b_2 - \frac{4}{3}b_3 \\ 0 & 3 & b_3 \\ 0 & 0 & b_1 - 2b_2 - 3b_3 \end{array}\right) \Rightarrow$

a) if $(b_1 - 2b_2 - 3b_3) = 0$, then we have 2×2 matrix with pivot in every row and column $\Rightarrow$ there is an unique solution

b) if $(b_1 - 2b_2 - 3b_3) \neq 0$, then there is no solutions

d) $\left(\begin{array}{ccc|c} 1 & 4 & 3 & b_1 \\ 2 & 1 & 0 & b_2 \\ 1 & 1 & 1 & b_3 \end{array}\right) \begin{array}{l} -r_3 \\ -2r_3 \\ \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & 3 & 2 & b_1 - b_3 \\ 0 & -1 & -2 & b_2 - 2b_3 \\ 1 & 1 & 1 & b_3 \end{array}\right) +3r_2 \Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & -4 & b_1 - b_3 + 3(b_2 - 2b_3) \\ 0 & -1 & -2 & b_2 - 2b_3 \\ 1 & 1 & 1 & b_3 \end{array}\right) \Rightarrow$

$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & b_3 \\ 0 & -1 & -2 & b_2 - 2b_3 \\ 0 & 0 & -4 & b_1 + 3b_2 - 7b_3 \end{array}\right)$ there is a pivot in every row and column
a) & b) an unique solution

6. $\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right)$

$a = 0$

$\left(\begin{array}{ccc|c} 0 & 0 & b & 2 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & b \end{array}\right)$

d) if $b \neq 2$: no solutions

c) else if $b = 2$: infinitely many solutions with 2 parameters:

$\begin{cases} x = s \\ y = t \\ z = 1 \end{cases}$

$a \neq 0$

$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) -r_2 \Rightarrow \left(\begin{array}{ccc|c} 0 & -a & b-4 & -2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) +r_3 \Rightarrow$

$\Rightarrow \left(\begin{array}{ccc|c} 0 & 0 & b-2 & b-2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} a & a & 4 & 4 \\ 0 & a & 2 & b \\ 0 & 0 & b-2 & b-2 \end{array}\right) \Rightarrow$

a) if $b \neq 2$: there is a pivot variable in every row and column $\Rightarrow$ an unique solution

b) else if $b = 2$: there is one free variable $\Rightarrow$ infinitely many solutions with 1 parameter:

$\begin{cases} ax + ay + 4z = 4 \\ ay + 2z = b \\ z = t \end{cases} \quad \begin{cases} ay + 2z = 4 - 2z - ax \\ ay + 2z = b \\ z = t \end{cases}$

$\begin{cases} 4 - 2t - ax = b \\ y = \frac{b-2t}{a} \\ z = t \end{cases} \quad \begin{cases} x = \frac{4-2t-b}{a} \\ y = \frac{b-2t}{a} \\ z = t \end{cases}$

Extra problems:

7. a) $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} +2r_2 \Rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$ rank = 1

b) $\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix} +2r_2 \Rightarrow \begin{pmatrix} 0 & 0 & 4 \\ -1 & 2 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ rank = 2

c) $\begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix} -r_3 \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 6 & 2 & 0 \\ 1 & 3 & -8 & 4 \end{pmatrix} \begin{array}{l} \\ -r_1 \\ -r_1 \end{array} \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & 0 & 9 & -3 \\ 0 & -3 & -1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & 9 & -3 \end{pmatrix}$ rank = 3

d) $\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix} -r_3 \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 6 & 2 & 0 & -3 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix} \begin{array}{l} \\ -r_1 \\ -r_1 \end{array} \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & 0 & 9 & -3 & -4 \\ 0 & -3 & -1 & 1 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & -3 & -1 & 1 & 4 \\ 0 & 0 & 9 & -3 & -4 \end{pmatrix}$ rank = 3

8. $m \times n$

I. $m = n = 3$

a) no solutions: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 4 & 5 & 3 \\ 0 & 4 & 5 & 9 \end{array}\right)$

b) 1 solution: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right)$

c) infinitely many solutions: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 4 & 5 & 3 \\ 0 & 4 & 5 & 3 \end{array}\right)$

II. $m = 3 \quad n = 2$

a) no solutions: $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right)$

b) 1 solution: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{array}\right)$

c) infinitely many solutions: $\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right)$

III. $m = 2,\ n = 3$

a) no solutions: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{array}\right)$

b) 1 solution: no example – there will always be a free variable $\Rightarrow$ none or infinitely many solutions

c) infinitely many solutions: $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \end{array}\right)$

9. $\begin{cases} x + 3y - 2z + w = 0 \\ 3x + 7y - 2z + 2w = 9 \\ 5x + 13y - 9z + 3w = 1 \\ x - z - 2w = 0 \end{cases}$

$\left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 3 & 7 & -2 & 2 & 9 \\ 5 & 13 & -9 & 3 & 1 \\ 1 & 0 & -1 & -2 & 0 \end{array}\right) \begin{array}{l} \\ -3r_4 \\ -r_2 \\ -r_1 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 0 & 7 & 1 & 8 & 9 \\ 2 & 5 & -7 & 1 & -8 \\ 0 & -3 & 1 & -3 & 0 \end{array}\right) \begin{array}{l} +r_4 \\ \\ -2r_1 \\ \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 7 & 1 & 8 & 9 \\ 0 & -1 & -3 & -1 & -8 \\ 0 & -3 & 1 & -3 & 0 \end{array}\right) \begin{array}{l} \\ +2r_4+r_3 \\ \cdot(-1) \\ -3r_3 \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 & 8 \\ 0 & 0 & 10 & 0 & 24 \end{array}\right) \Rightarrow$

$\Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 3 & 1 & 8 \\ 0 & 0 & 10 & 0 & 24 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} +\frac{1}{10}r_3 \\ -\frac{3}{10}r_3 \\ \cdot(\frac{1}{10}) \\ \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 2{,}4 \\ 0 & 1 & 0 & 1 & 5{,}6 \\ 0 & 0 & 1 & 0 & 2{,}4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{l} +2r_4 \\ -r_4 \\ \\ \end{array} \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4{,}4 \\ 0 & 1 & 0 & 0 & 4{,}6 \\ 0 & 0 & 1 & 0 & 2{,}4 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$

$\begin{cases} x_1 = 4{,}4 \\ x_2 = 4{,}6 \\ x_3 = 2{,}4 \\ x_4 = 1 \end{cases}$

10. a) $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} +2r_1 \Rightarrow \underset{\text{REF}}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} -r_2 \Rightarrow \underset{\text{RREF}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

b) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix} -2r_1 \Rightarrow \underset{\text{REF}}{\begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -1 \end{pmatrix}} \begin{array}{l} +3r_2 \\ \cdot(-1) \end{array} \Rightarrow \underset{\text{RREF}}{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}}$

c) $\begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{array}{l} +r_3 \\ +r_1-r_3 \\ \end{array} \Rightarrow \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \underset{\text{REF}}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}} \cdot(-1) \Rightarrow \underset{\text{RREF}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$