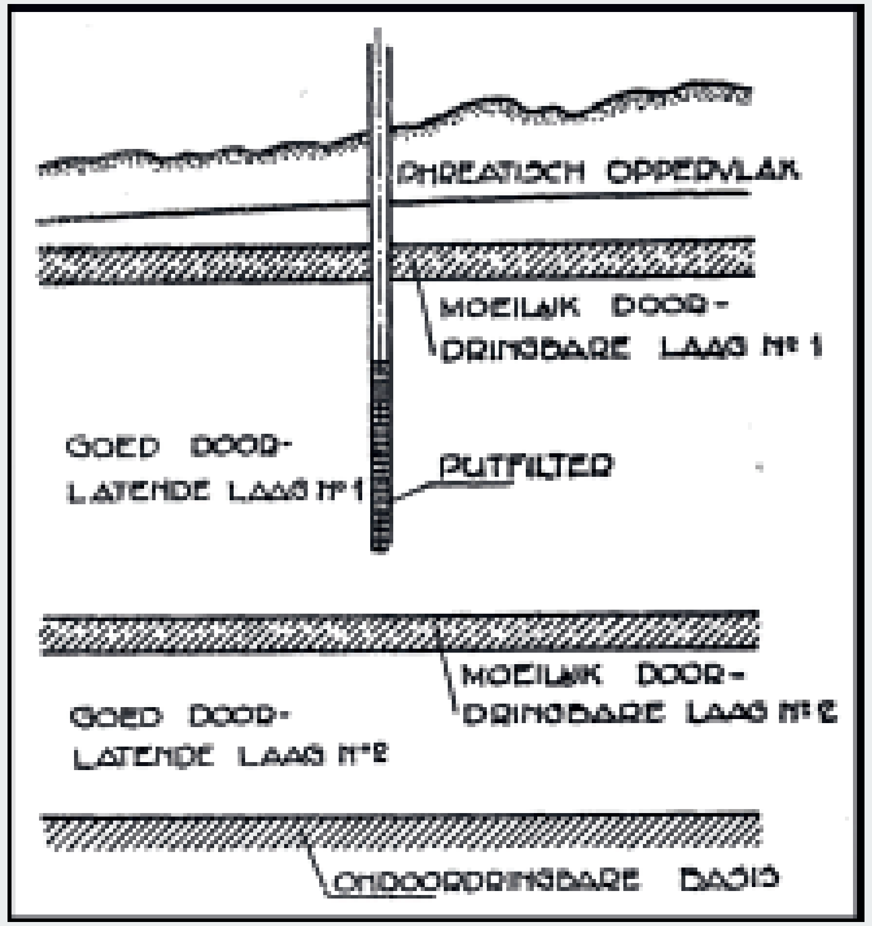
FACULTY OF SCIENCES

LABORATORY FOR APPLIED GEOLOGY AND HYDROGEOLOGY
DEPARTMENT OF GEOLOGY

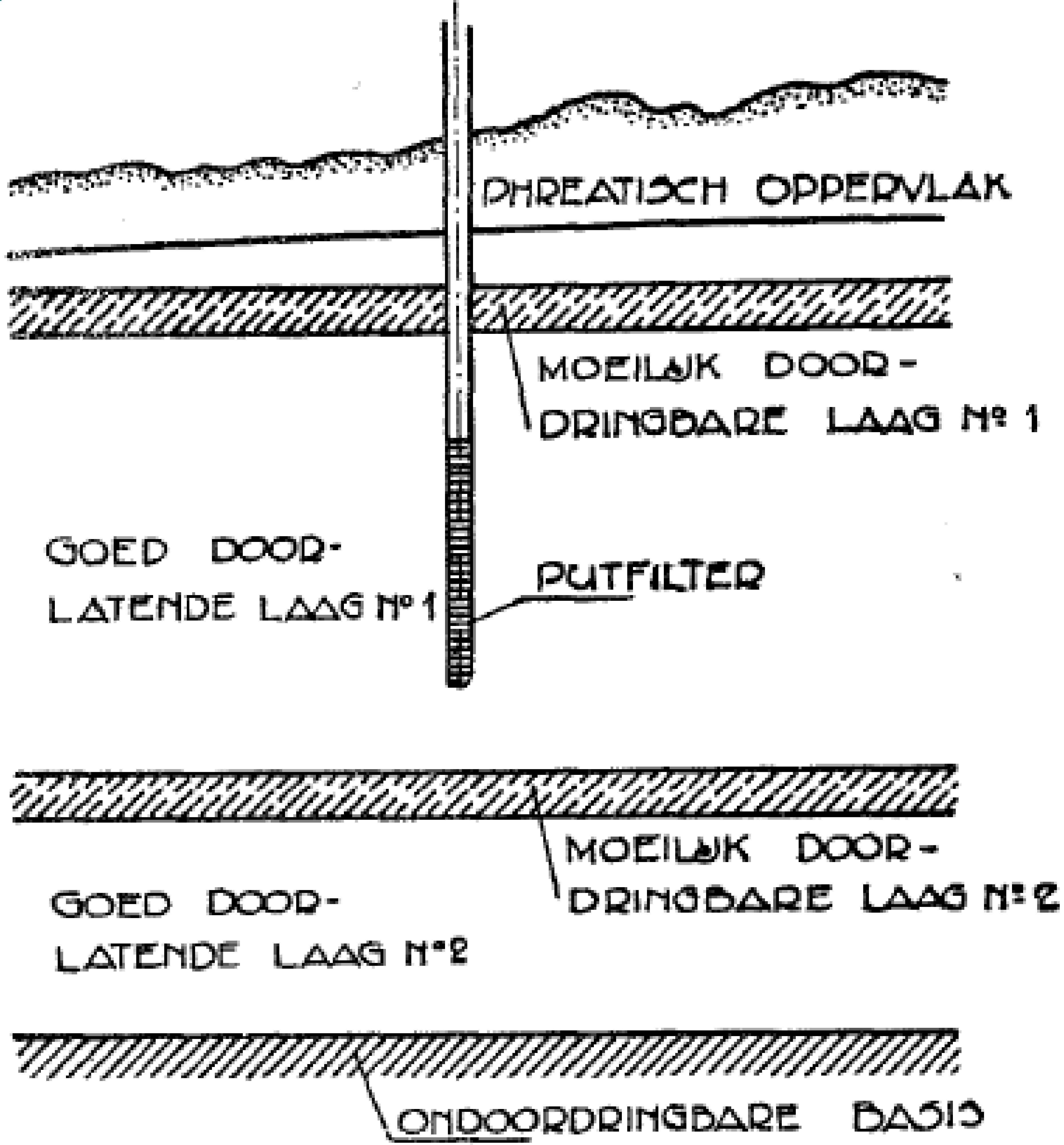
Axisymmetric Flow in Multilayer Aquifer Systems: Solutions and Theoretical Considerations



Andy Louwyck

September 2023

GHENT UNIVERSITY



OVER GRONDWATERSTROOMINGEN BIJ WATERONTTREKKING DOOR MIDDEL VAN PUTTEN.

PROEFSCHRIFT

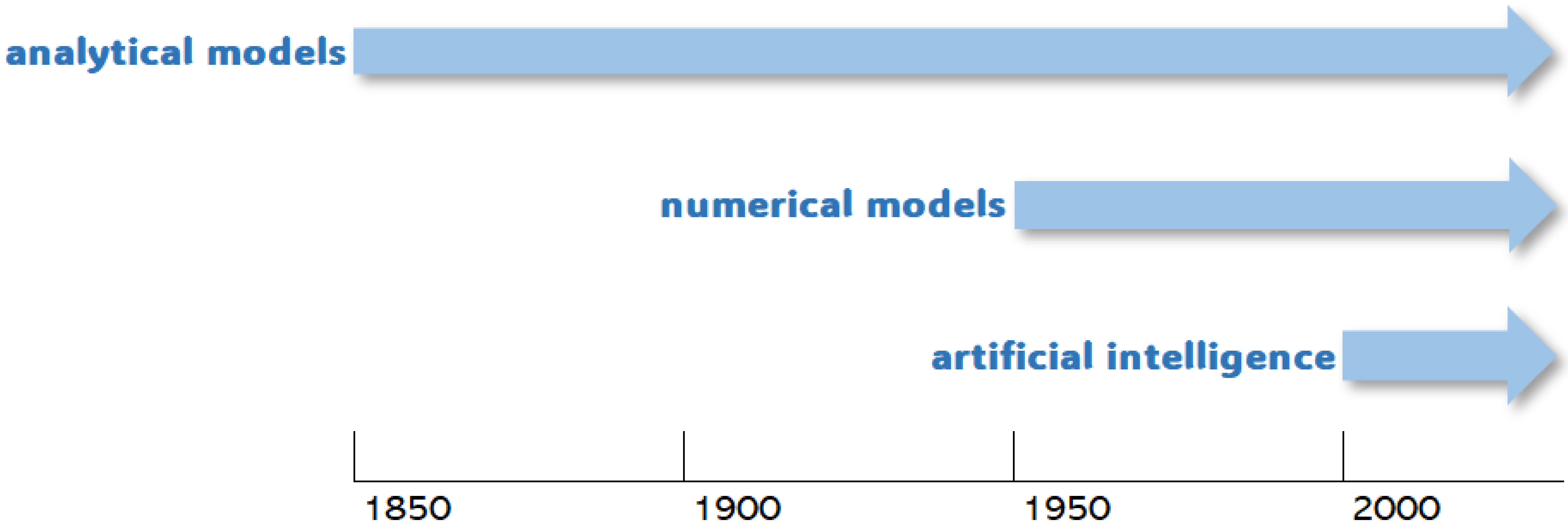
TER VERKRIJGING VAN DEN GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOOGESCHOOL TE DELFT, OP GEZAG VAN DEN RECTOR MAGNIFICUS IR. F. WESTENDORP, HOOGLEERAAR IN DE AFDEELING DER WERKTUIGBOUWKUNDE EN SCHEEPSBOUWKUNDE, VOOR EENE COMMISSIE UIT DEN SENAAT TE VERDEDIGEN OP WOENSDAG 2 APRIL 1930, DES NAMIDDAGS TE 3 UUR,

DOOR

GERRIT JAN DE GLEE,
CIVIEL-INGENIEUR,
GEBOREN TE ASSEN.

GEDRUKT BIJ DE TECHNISCHE BOEKHANDEL EN DRUKKERIJ
J. WALTMAN JR. DELFT. — 1930.

SOLUTION METHODS



RELEVANCE

Google Scholar — "analytical model" groundwater

Articles — About 1.950 results (0,31 sec)

- Any time
- Since 2024
- Since 2023
- Since 2020
- Custom range...

Comprehensive two-dimensional analytical modeling of groundwater levels in bi-directional sloping heterogeneous aquifers under variable recharge conditions

PC Hsieh, MC Wu - Journal of Hydrology, 2024 - Elsevier

… of **groundwater** … **analytical model** effectively captures varying recharge dynamics, both spatially and temporally, contributing to a better understanding and management of **groundwater** …

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Google Scholar — "artificial intelligence" groundwater

Articles — About 8.730 results (0,13 sec)

- Any time
- Since 2024
- Since 2023
- Since 2020
- Custom range...

[HTML] **Artificial Intelligence Advancements for Accurate Groundwater Level Modelling: An Updated Synthesis and Review**

S Pourmorad, M Kabolizade, LA Dimuccio - Applied Sciences, 2024 - mdpi.com

… , have proven to be important tools for accurate **groundwater** level (GWL) modelling. Through an … complex and nonlinear relationships in **groundwater** data, providing more accurate and …

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THE AI HYPE

Artificial intelligence curve matching method for pumping tests in leaky systems

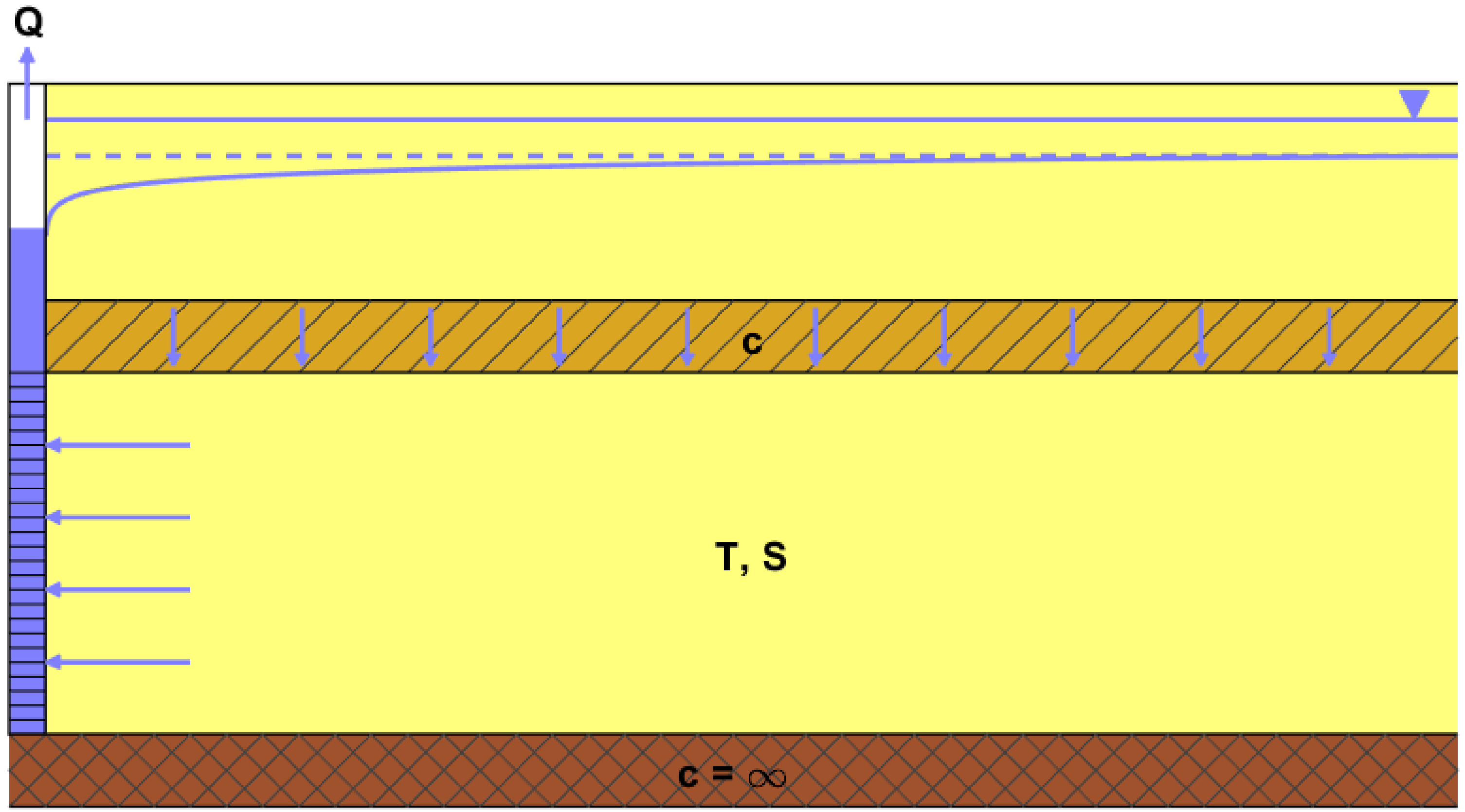
Yunyun Zhao* Yong Zhang,
Hebei University of Engineering, Handan 056038, Hebei, China
* Corresponding author
E-mail address: zhang_yong001@163.com

Earth Science Informatics
Springer Verlag

Abstract

For the unsteady-state flow pumping test, the standard curve fitting method is generally used to calculate the hydrogeological parameters. Because of differences from person to person, there will be visual errors in artificial curve fitting, especially when the well function is a family of curves, and there are countless standard curves in theory. It is impossible to draw all the standard curves in the standard curve template for curve fitting; thus, the manual curve fitting error will be very large. Therefore, the curve fitting method has been greatly limited in practical application. In this paper, the random weight particle swarm optimization algorithm (RandWPSO) is applied to the curve matching calculation for the unsteady-state flow pumping test in a leaky system, and intelligent optimization curve fitting of the curve family well function is performed. The calculation results show that the curve fitting parameters selected by artificial intelligence can be as accurate as r/B=0.3579764, and the curve fitting accuracy is much higher than that of manual selection, r/B=0.35. Artificial intelligence curve fitting avoids observation error due to manual curve fitting, solves the problem that it is difficult to apply the standard curve fitting method of the curve family in practice, and makes the curve fitting method more practical.

Keywords: leaky aquifers; curve fitting; artificial intelligence optimization; pumping test; RandWPSO



Hantush-Jacob (1955) model:

$$s(r,t) = \frac{Q}{4\pi KD} W\left(\frac{r^2 S}{4tKD}, r\sqrt{\frac{1}{cKD}}\right)$$

Table 2 Comparative analysis of the calculation results of different curve fitting methods

Calculation parameters	Calculation results of different curve fitting methods			
$W\left(u, \frac{r}{B}\right)$ calculation method	Composite Simpson's rule; set m=1500 and h=1/3000	Composite Simpson's rule; set m=5000 and h=1/10000	Directly call built-in MATLAB numerical integral	Manual curve fitting
r/B	0.3579764	0.3578755	3.578325	0.35
T (m^2/d)	403. 9988961794492	404.080110307088	404.1146770792042	414.7
S	$1.39784369448563\times10^{-4}$	$1.39764226866348\times10^{-4}$	$1.39755658961577\times10^{-4}$	1.469×10^{-4}
K_1/M_1	0.001334001987695	0.001333518091093	0.001333311883692	0.0013
f_v	0.011562541890477	0.01156480806024	0.011565926829623	\
Computation time (min)	4.5	15.3	30.5	\

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DON’T BE DECEIVED

Artificial intelligence curve matching method for pumping tests in leaky systems

Yunyun Zhao* Yong Zhang,
Hebei University of Engineering, Handan 056038, Hebei, China
* Corresponding author.
E-mail address: zhang_yong001@163.com

Earth Science Informatics
Springer Verlag

REJECTED

Abstract

For the unsteady-state flow pumping test, the standard curve fitting method is generally used to calculate the hydrogeological parameters. Because of differences from person to person, there will be visual errors in artificial curve fitting, especially when the well function is a family of curves, and there are countless standard curves in theory. It is impossible to draw all the standard curves in the standard curve template for curve fitting; thus, the manual curve fitting error will be very large. Therefore, the curve fitting method has been greatly limited in practical application. In this paper, the random weight particle swarm optimization algorithm (RandWPSO) is applied to the curve matching calculation for the unsteady-state flow pumping test in a leaky system, and intelligent optimization curve fitting of the curve family well function is performed. The calculation results show that the curve fitting parameters selected by artificial intelligence can be as accurate as r/B=0.3579764, and the curve fitting accuracy is much higher than that of manual selection, r/B=0.35. Artificial intelligence curve fitting avoids observation error due to manual curve fitting, solves the problem that it is difficult to apply the standard curve fitting method of the curve family in practice, and makes the curve fitting method more practical.

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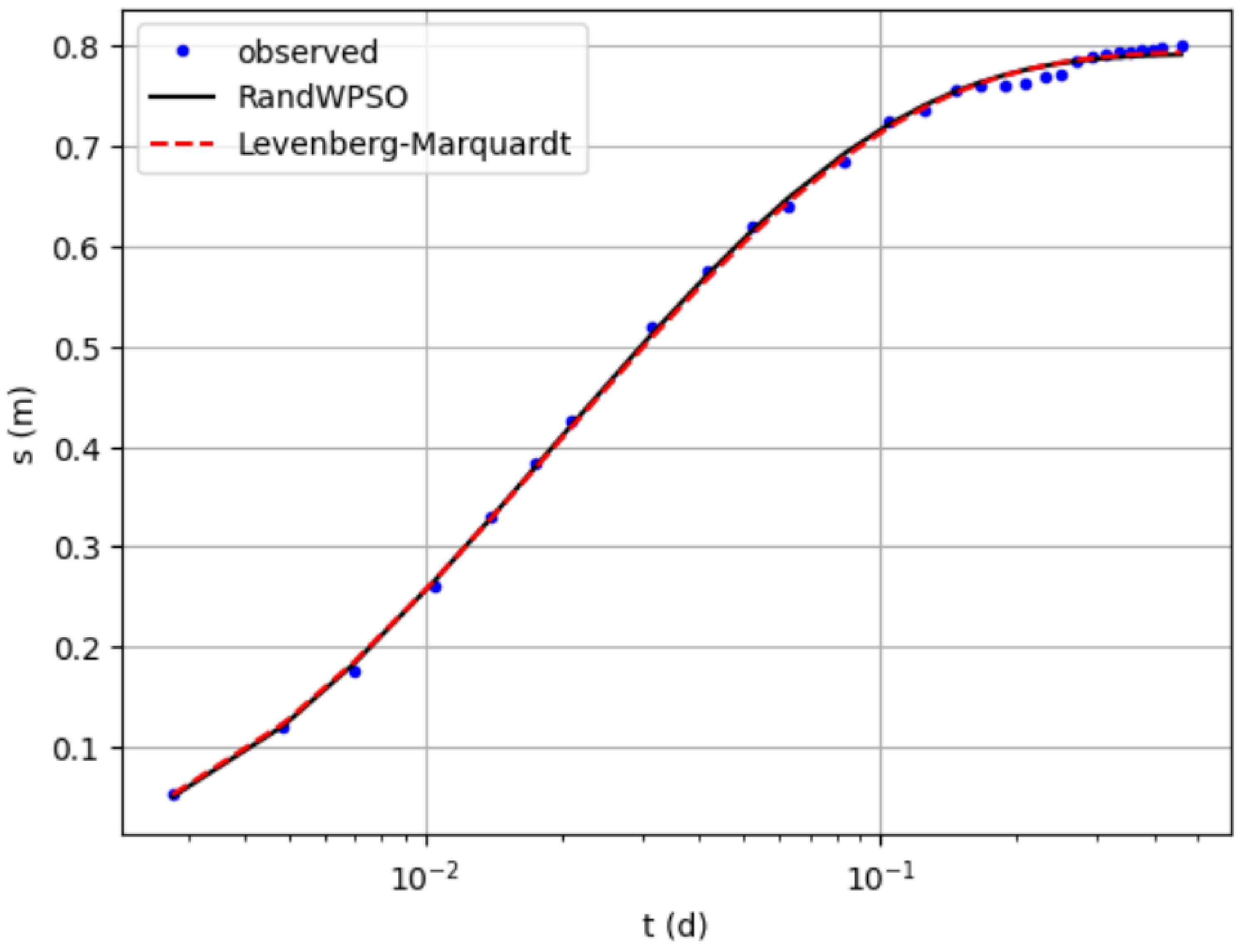


```
start = time()
result = least_squares(fun=residuals, x0=x0, method='lm',
                       ftol=1e-12, xtol=1e-12, gtol=1e-12)
print(f"elapsed time: {time() - start} sec")
```

```
elapsed time: 0.034934043884277344 sec
```

```
r/B = 0.33966118442493454
T = 418.38884431293474
S = 0.0001391052948927095
c = 804.0082714958618
K/M = 0.0012437682987259504
```

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MY RESUME

FLANDERS ENVIRONMENT AGENCY

Flanders State of the Art

Education:

2023: Doctor of Science: Geology (Ghent University)

2020: Micro Degree: AI & Data Science (KdG University Antwerp)

2015: Associate Degree: IT & Programming (CVO Brussels)

2001: Master of Science: Geology (Ghent University)

1999: Bachelor of Science: Geology (Catholic University of Leuven)

vives University of Applied Sciences

Professional Experience:

2023 - …	Voluntary Post-Doctoral Researcher in Hydrogeology (Ghent University)
2023 - …	AI Expert-Coordinator (Flanders Environment Agency)
2020 - …	Lecturer in AI (Vives University)
2020 - 2022	Research Associate in AI (Vives University)
2008 - 2020	Groundwater Modeler (Flanders Environment Agency)
2007 - 2008	Project Engineer Water Management (IMDC)
2006	Science Teacher (HH Secondary School Ninove)
2002 - 2005	PhD Fellow in Hydrogeology (Ghent University)

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WHERE DO I LIVE?

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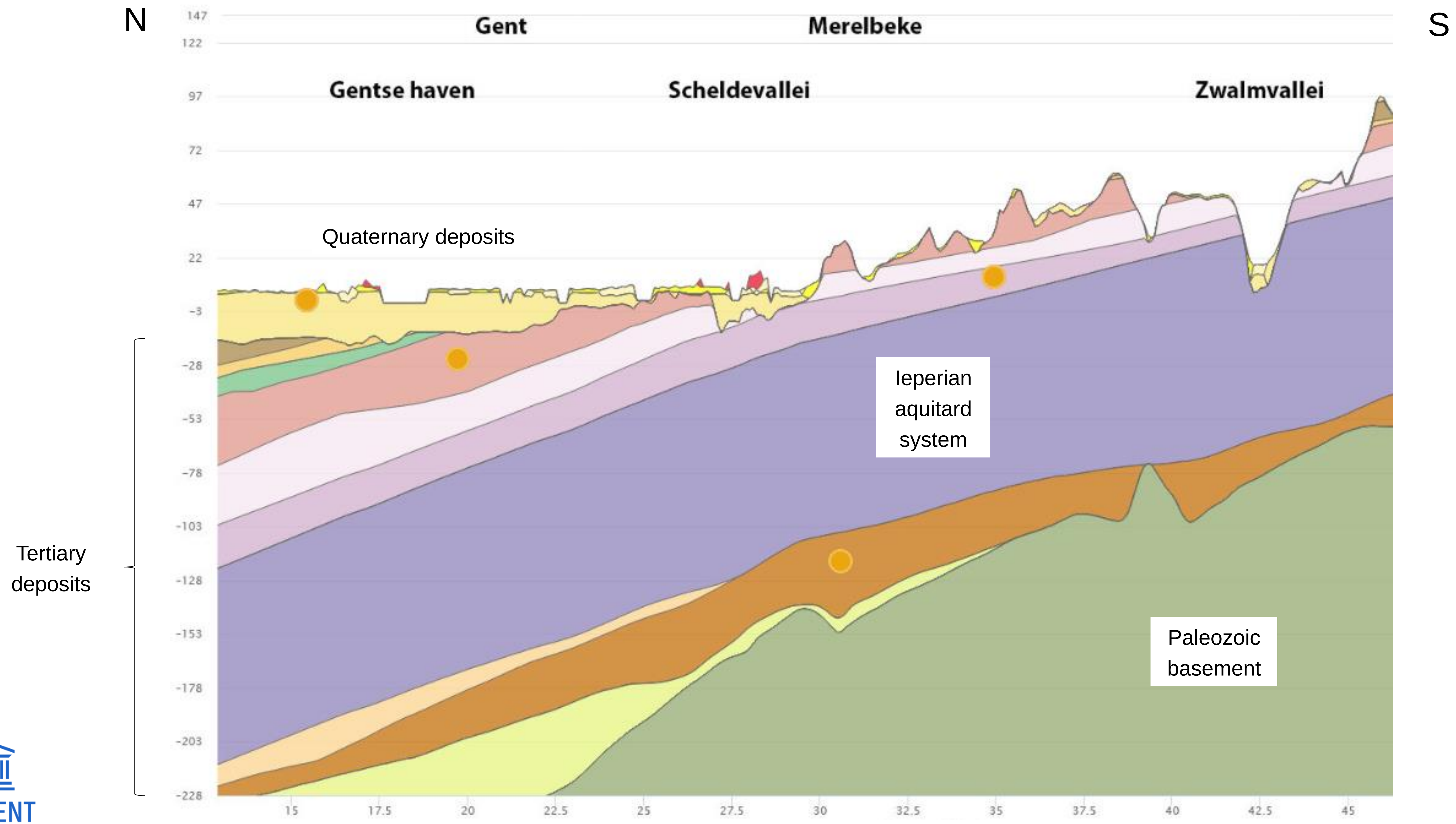
HYDROGEOLOGICAL STRATIGRAPHY



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OVERVIEW

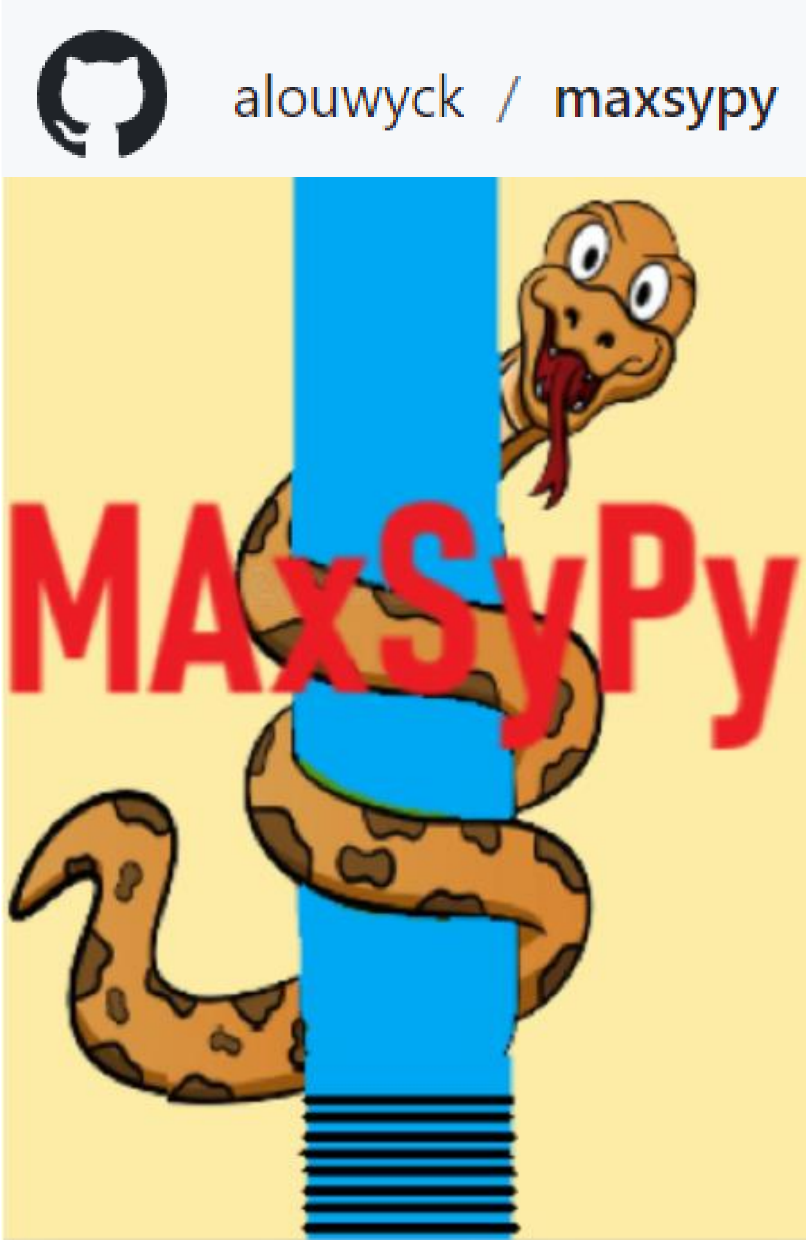
0. Axisymmetric flow
1. The very first axisymmetric models
2. Well-known 1D axisymmetric models
3. The radius of influence myth
4. The water budget myth and the superposition principle
5. More advanced axisymmetric models
6. Axisymmetric flow in multilayer aquifer systems
7. Aquifer tests
8. A theoretical case study
9. A practical case study

JUPYTER NOTEBOOKS WITH CODE EXAMPLES

https://github.com/alouwyck/maxsypy

- maxsypy
- seminaries
  - UGent_oct_2023
  - UM6P_jan_2025

Open in Colab



+ popular Python packages for groundwater modeling
  TimML, TTim and FloPy

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Seminary on Groundwater Modeling
January 6, 2025
Andy Louwyck

UM6P University Mohammed VI Polytechnic

AXISYMMETRIC FLOW

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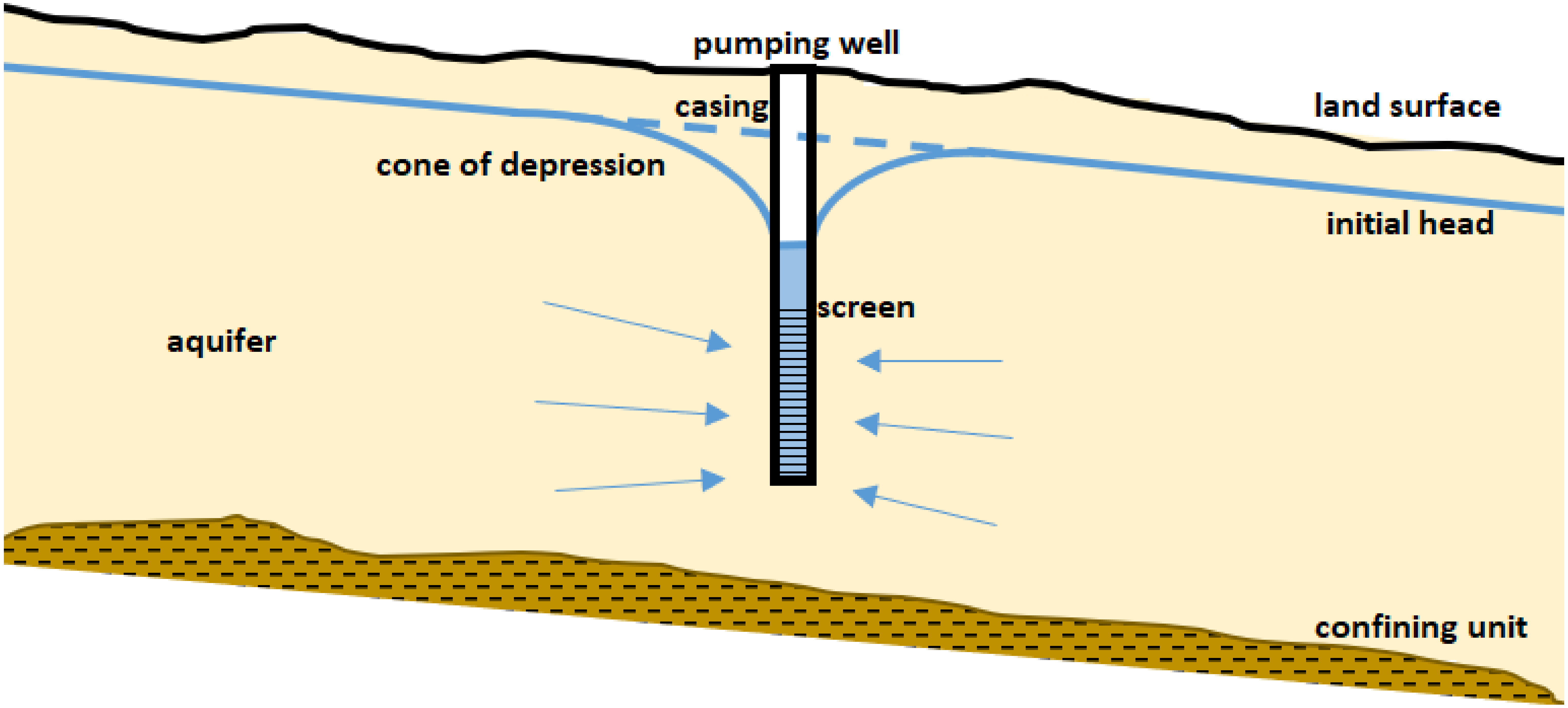
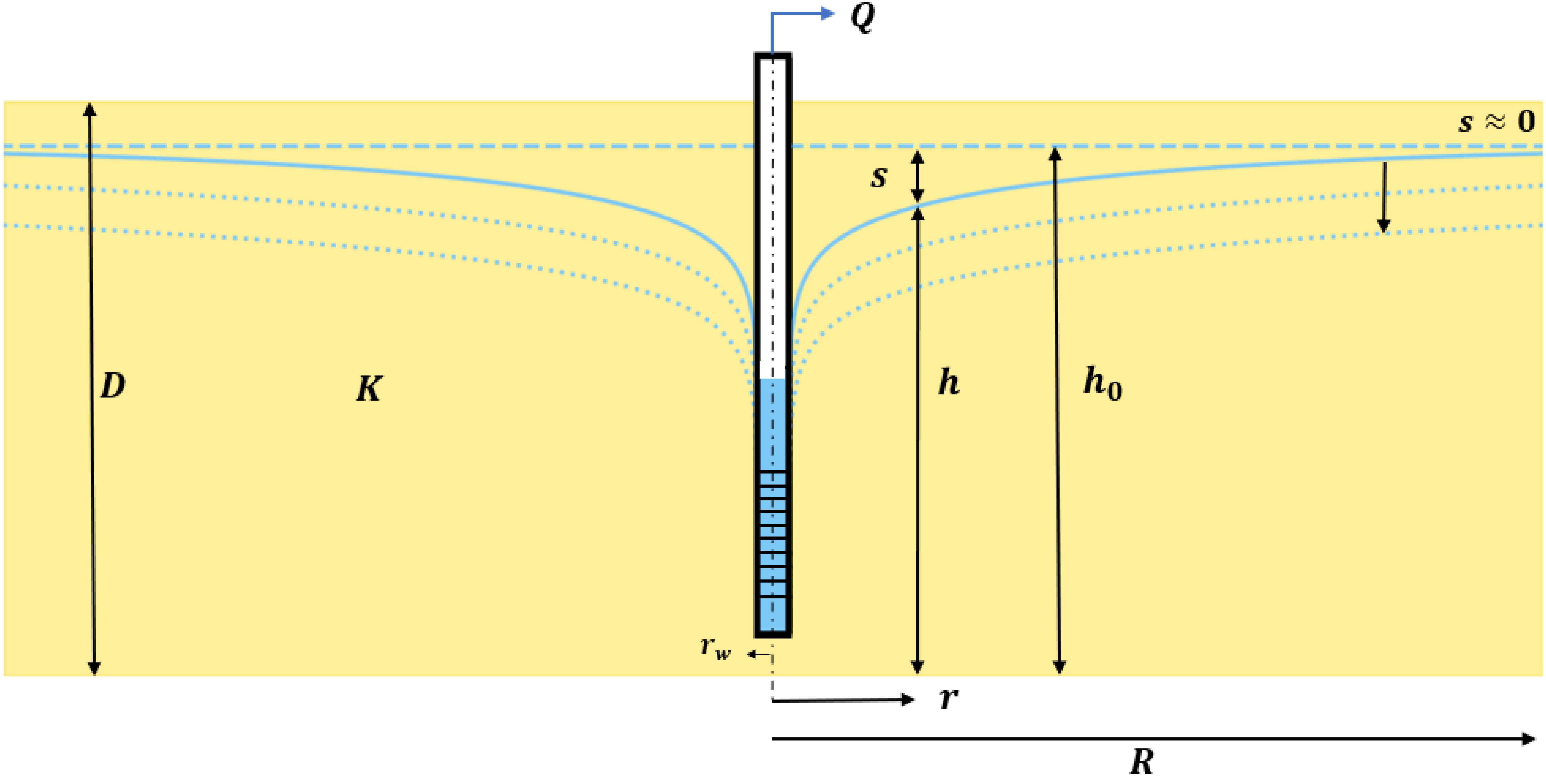
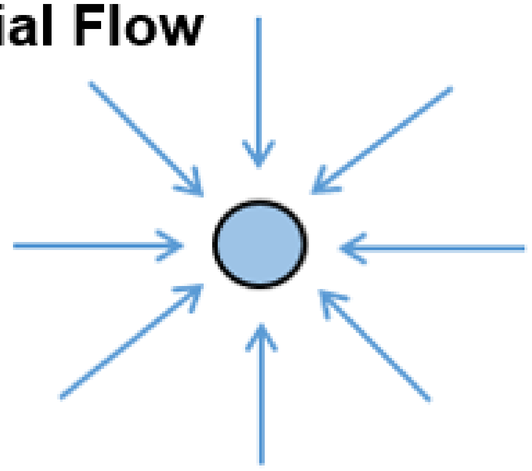
FLOW TO A PUMPING WELL



AXISYMMETRIC MODEL



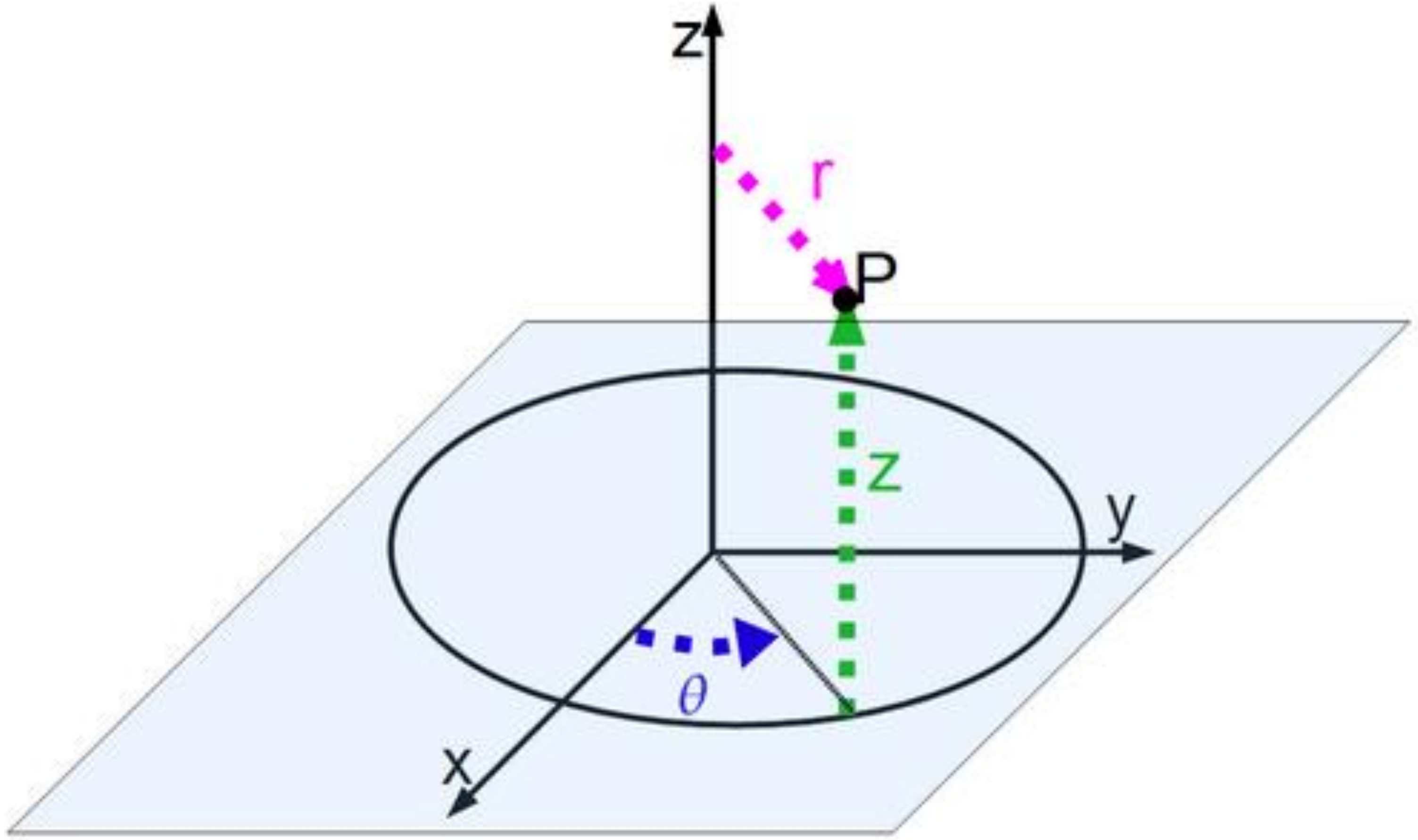
Radial Flow



pumping rate	Q
aquifer thickness	D
aquifer conductivity	K
aquifer transmissivity	$T = KD$
hydraulic head	h
initial head	h_0
drawdown	s
radial distance	r
well radius	r_w
radius of influence	R

CYLINDRICAL COORDINATES

- **Cartesian** coordinates: (x, y, z)
- **Cylindrical** coordinates: (r, θ, z)
  - Polar coordinates: (r, θ)
- **Axial symmetry**: (r, z)
  - No θ dimension!
  - 1D flow: only r



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r = \sqrt{x^2 + y^2}$$

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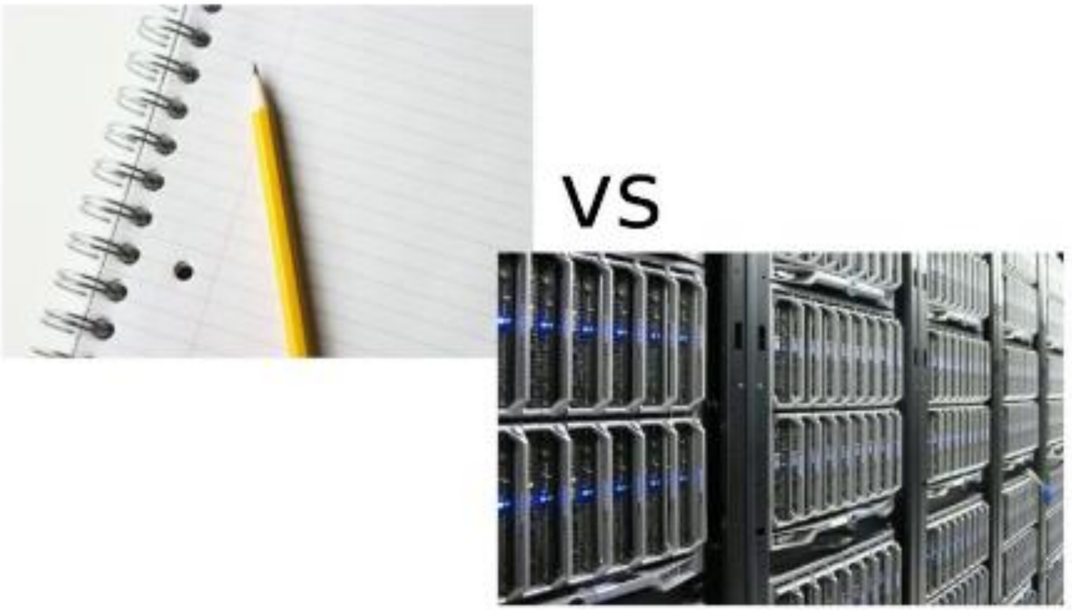
PARAMETERS AND UNITS

parameter	symbol	dimension
hydraulic head	h	L
initial head	h_0	L
drawdown	s	L
well drawdown	s_w	L
head change in well	H	L
initial head change in well	H_0	L
radial distance	r	L
time	t	T

parameter	symbol	dimension
pumping rate	Q	L^3/T
aquifer thickness	D	L
aquifer conductivity	K	L/T
aquifer transmissivity	$T = KD$	L^2/T
aquifer storativity	S	-
resistance	c	T
infiltration flux	N	L/T
radius of influence	R	L
well-screen radius	r_w	L
well-casing radius	r_c	L
well-skin radius	R_s	L

ANALYTICAL VS NUMERICAL MODELS

- Analytical solutions
  - exact
  - closed-form equations
  - methods from calculus
  - e.g. integral transforms



- Numerical solutions
  - approximate
  - discretization of the model domain
  - iterative methods
  - e.g. finite differences, finite elements, …

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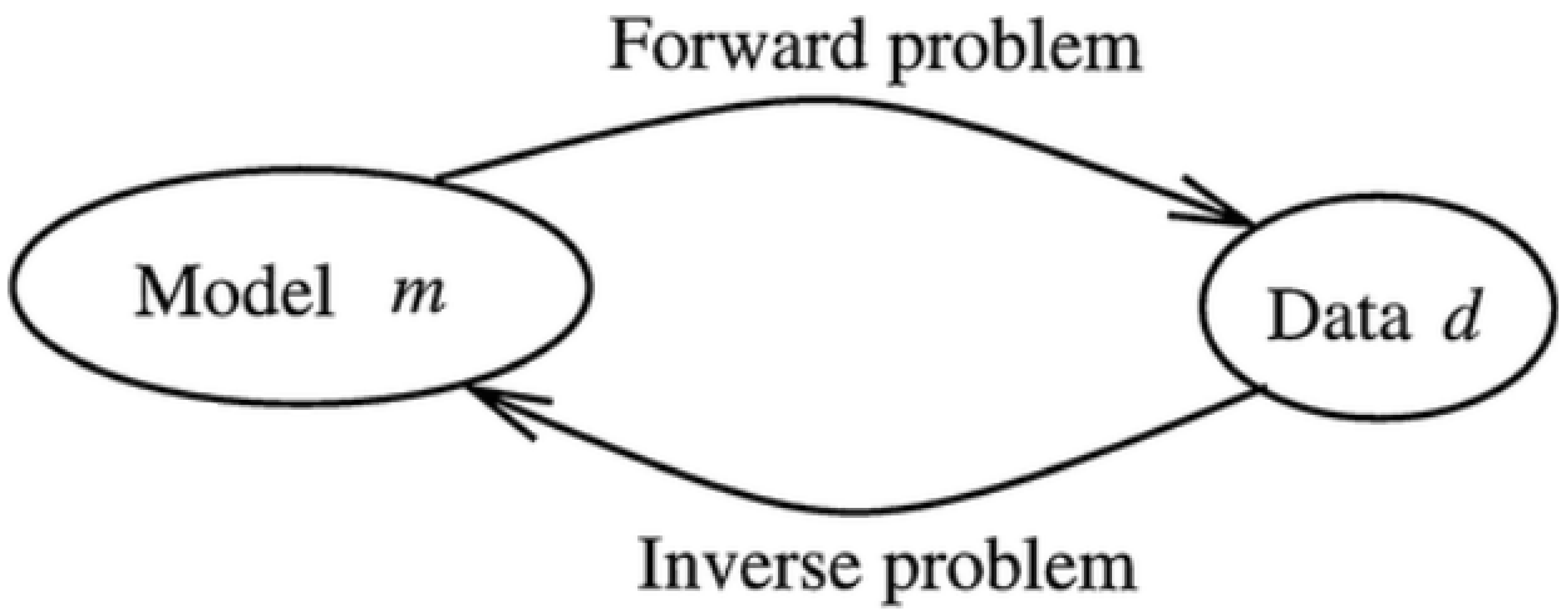
FORWARD AND INVERSE PROBLEMS

- **forward problem**
  - simulate head *h* or drawdown *s*
  - e.g. assessing the environmental impact of extractions

- **inverse problem type I**
  - derive transmissivity *T*
  - e.g. pumping test interpretation

- **inverse problem type II**
  - derive pumping rate *Q*
  - e.g. construction dewatering



THE VERY FIRST AXISYMMETRIC MODELS

THE THIEM-DUPUIT FORMULAS

– Steady confined flow (Thiem, 1870, 1906)

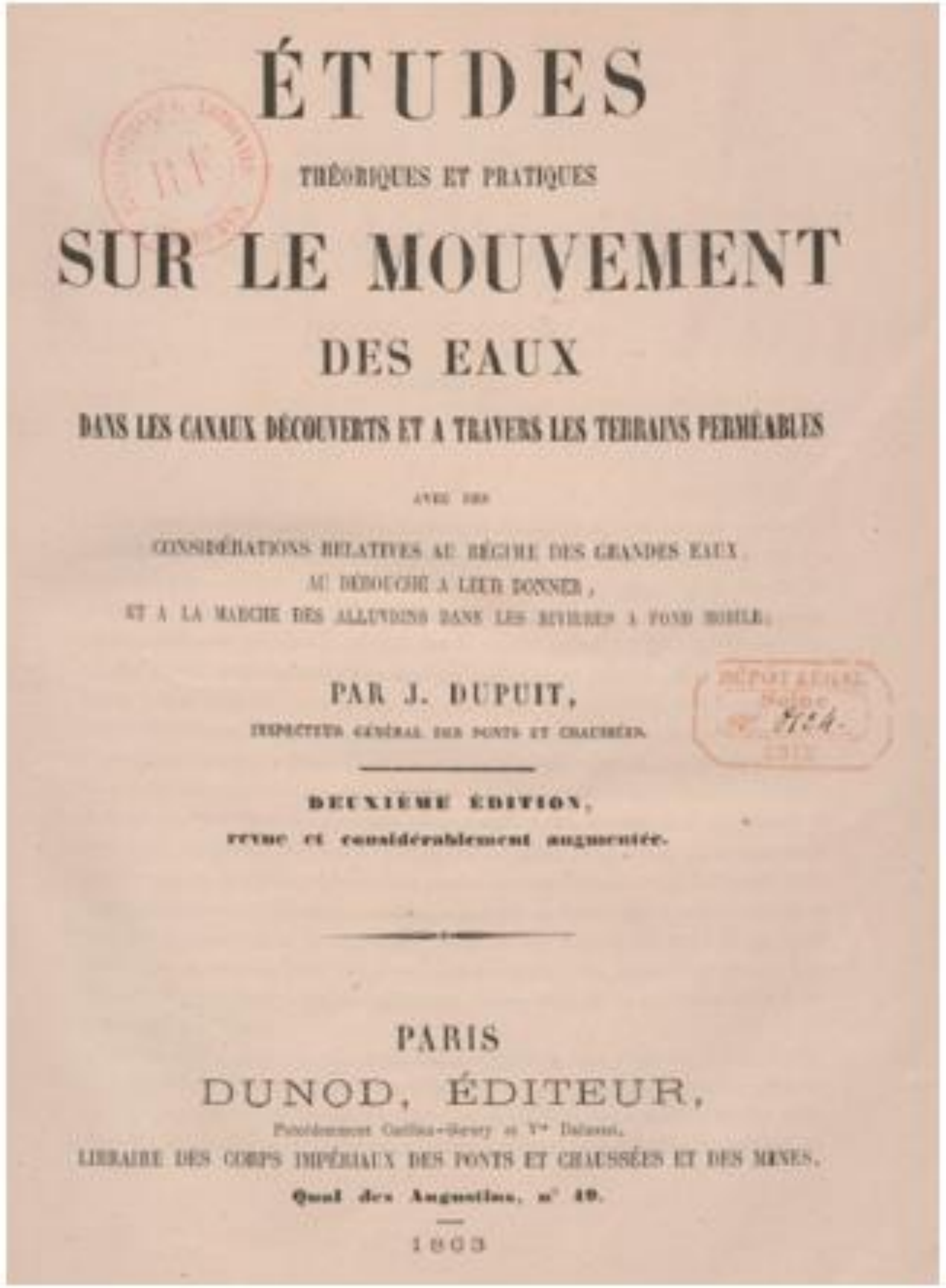
$$s(r) = \frac{Q}{2\pi KD} \ln\left(\frac{R}{r}\right)$$

Initial head h_0 is not required

– Steady unconfined flow (Dupuit, 1857, 1863)

$$s(r) = h_0 - \underbrace{\sqrt{h_0^2 - \frac{Q}{\pi K} \ln\left(\frac{R}{r}\right)}}_{h}$$

Initial head h_0 is required!


ÉTUDES
THÉORIQUES ET PRATIQUES
SUR LE MOUVEMENT
DES EAUX
DANS LES CANAUX DÉCOUVERTS ET A TRAVERS LES TERRAINS PERMÉABLES
AVEC DES
CONSIDÉRATIONS RELATIVES AU RÉGIME DES GRANDES EAUX,
AU DÉBOUCHÉ A LEUR DONNER,
ET A LA MARCHE DES ALLUVIONS DANS LES RIVIÈRES A FOND MOBILE.
PAR J. DUPUIT,
INSPECTEUR GÉNÉRAL DES PONTS ET CHAUSSÉES.
DEUXIÈME ÉDITION,
revue et considérablement augmentée.
PARIS
DUNOD, ÉDITEUR,
LIBRAIRE DES CORPS IMPÉRIAUX DES PONTS ET CHAUSSÉES ET DES MINES,
Quai des Augustins, n° 49.
1863


Jules Dupuit


Adolf Thiem


Günther Thiem

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CONFINED FLOW

- Constant saturated thickness ***D***
- If aquifer is homogeneous:
  - *K* is constant
  - *T* is constant
  - ***T = KD***
- **Linear** problem

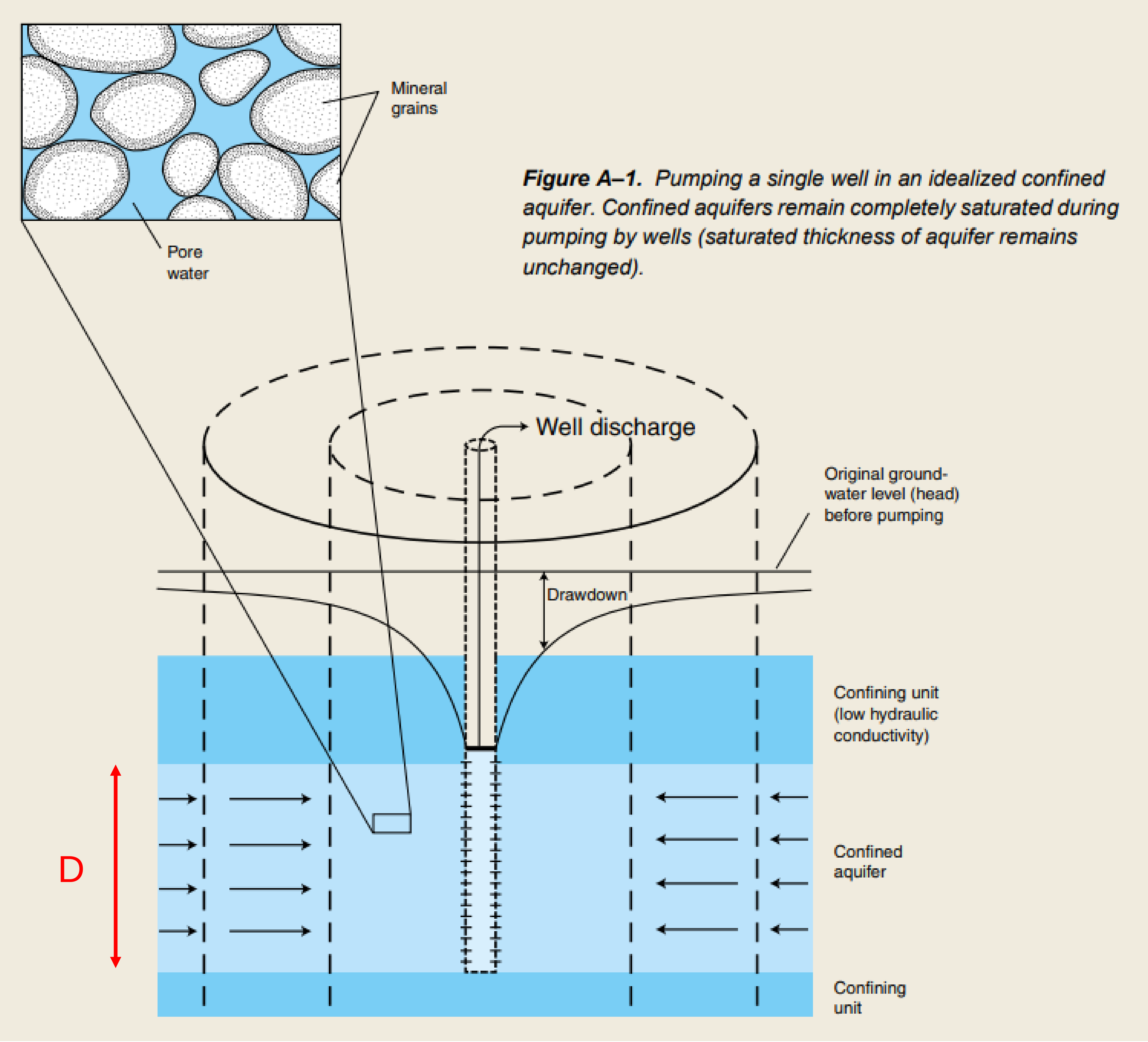

Figure A–1. *Pumping a single well in an idealized confined aquifer. Confined aquifers remain completely saturated during pumping by wells (saturated thickness of aquifer remains unchanged).*

https://pubs.usgs.gov/circ/circ1186/pdf/circ1186.pdf

UNCONFINED FLOW

- Saturated thickness = head ***h***

- If aquifer is homogeneous:
  - *K* is constant
  - *T* is head-dependent
  - ***T* = *Kh***

- **Nonlinear** problem

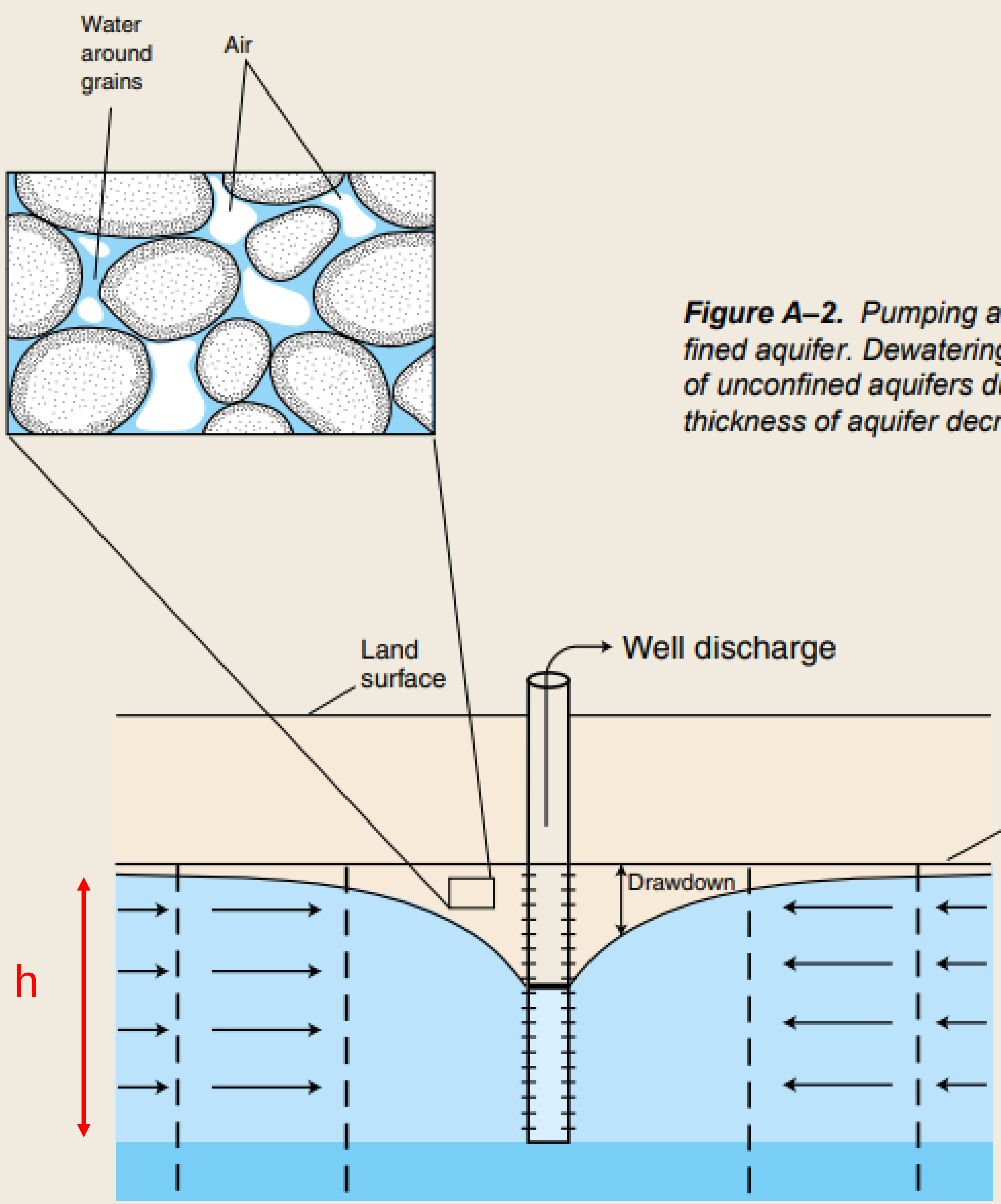

Figure A–2. *Pumping a single well in an idealized unconfined aquifer. Dewatering occurs in cone of depression of unconfined aquifers during pumping by wells (saturated thickness of aquifer decreases).*

https://pubs.usgs.gov/circ/circ1186/pdf/circ1186.pdf

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THIEM EQUATION: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Steady-state
  - Strictly horizontal
- Well:
  - Fully penetrating
  - Constant pumping rate
- Aquifer:
  - Homogeneous
  - **Constant saturated thickness**
  - Laterally bounded

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THIEM EQUATION: PROBLEM STATEMENT

Darcy's law:

$$Q = 2\pi r \boldsymbol{KD} \frac{dh}{dr}$$

(1)

Continuity of steady 1D axisymmetric flow:

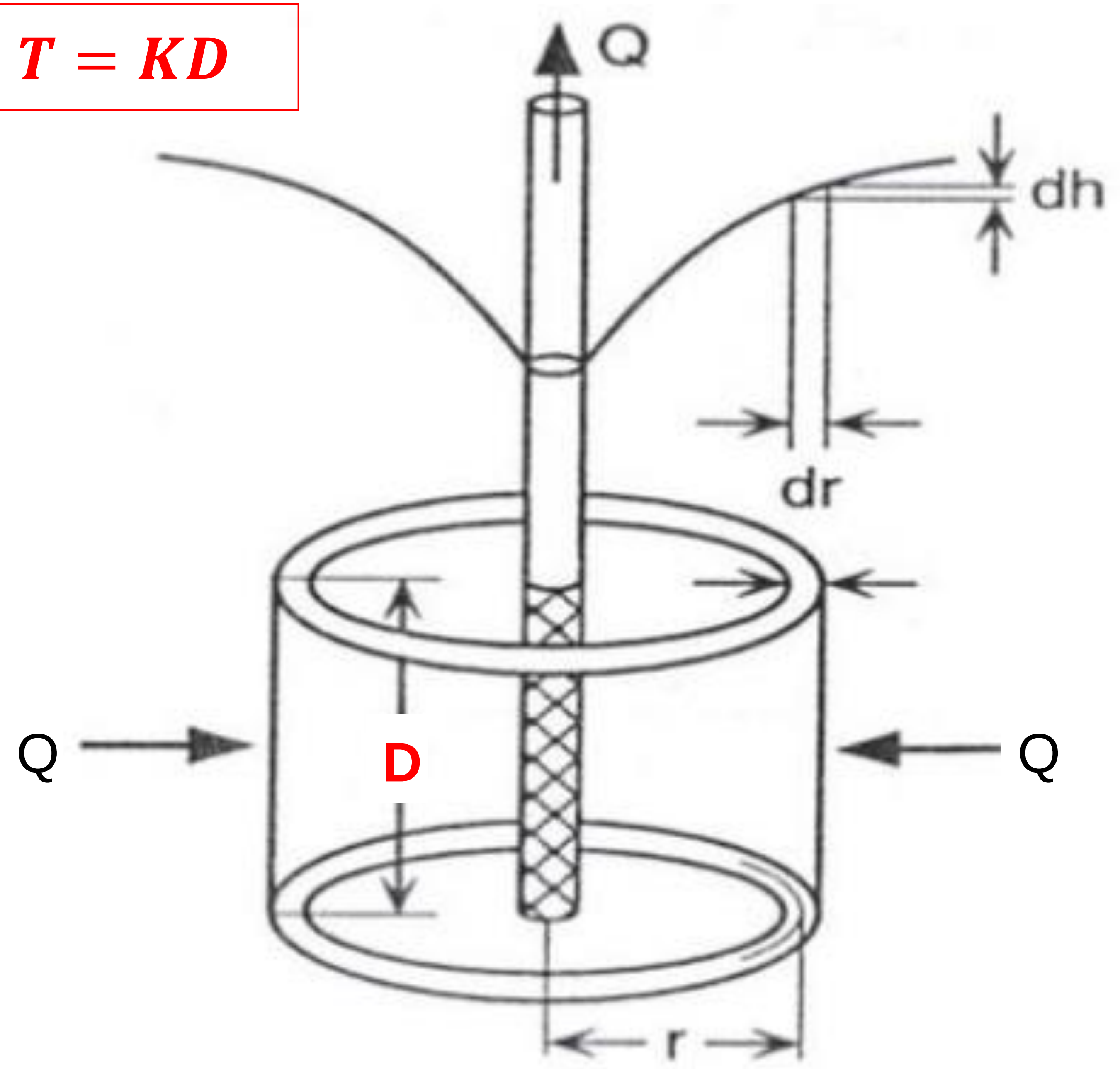
$$\frac{dQ}{dr} = 0 \quad \text{or} \quad \frac{d^2h}{dr^2} + \frac{1}{r}\frac{dh}{dr} = 0$$

Boundary condition: constant head h_0 at distance R:

$$h(R) = h_0$$

(2)

$\boldsymbol{T = KD}$



Source: Kresic, 1997

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THIEM EQUATION: DERIVATION

Rearranging (1):

$$dh = \frac{Q}{2\pi KD}\frac{dr}{r} \tag{3}$$

Integrating both sides of (3):

$$h(r) = \frac{Q}{2\pi KD}\ln r + C \tag{4}$$

Introducing (2) in (4):

$$h(R) = h_0 = \frac{Q}{2\pi KD}\ln R + C \tag{5}$$

Deriving integration constant C from (5):

$$C = h_0 - \frac{Q}{2\pi KD}\ln R \tag{6}$$

Introducing (6) in general solution (4):

$$h(r) = h_0 - \frac{Q}{2\pi KD}\ln\frac{R}{r} \tag{7}$$

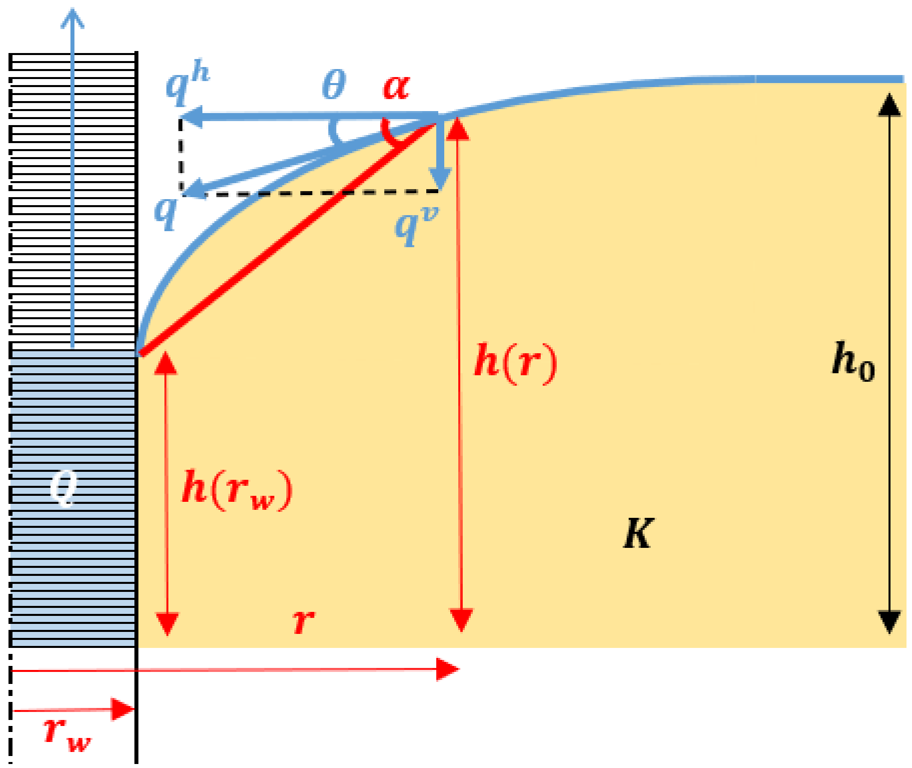
Applying definition of drawdown s to (7):

$$s(r) = h_0 - h(r) = \frac{Q}{2\pi KD}\ln\frac{R}{r} \tag{8}$$

DUPUIT EQUATION: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Steady-state
  - Strictly horizontal: $\theta < 30^\circ$

    = the Dupuit-Forchheimer approximation!



- Aquifer:
  - Homogeneous
  - **Head-dependent saturated thickness**
  - Laterally bounded

- Well:
  - Fully penetrating
  - Constant pumping rate
  - No seepage face

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DUPUIT EQUATION: PROBLEM STATEMENT

Darcy's law:

$$Q = 2\pi r \boldsymbol{Kh} \frac{dh}{dr} \qquad (1)$$

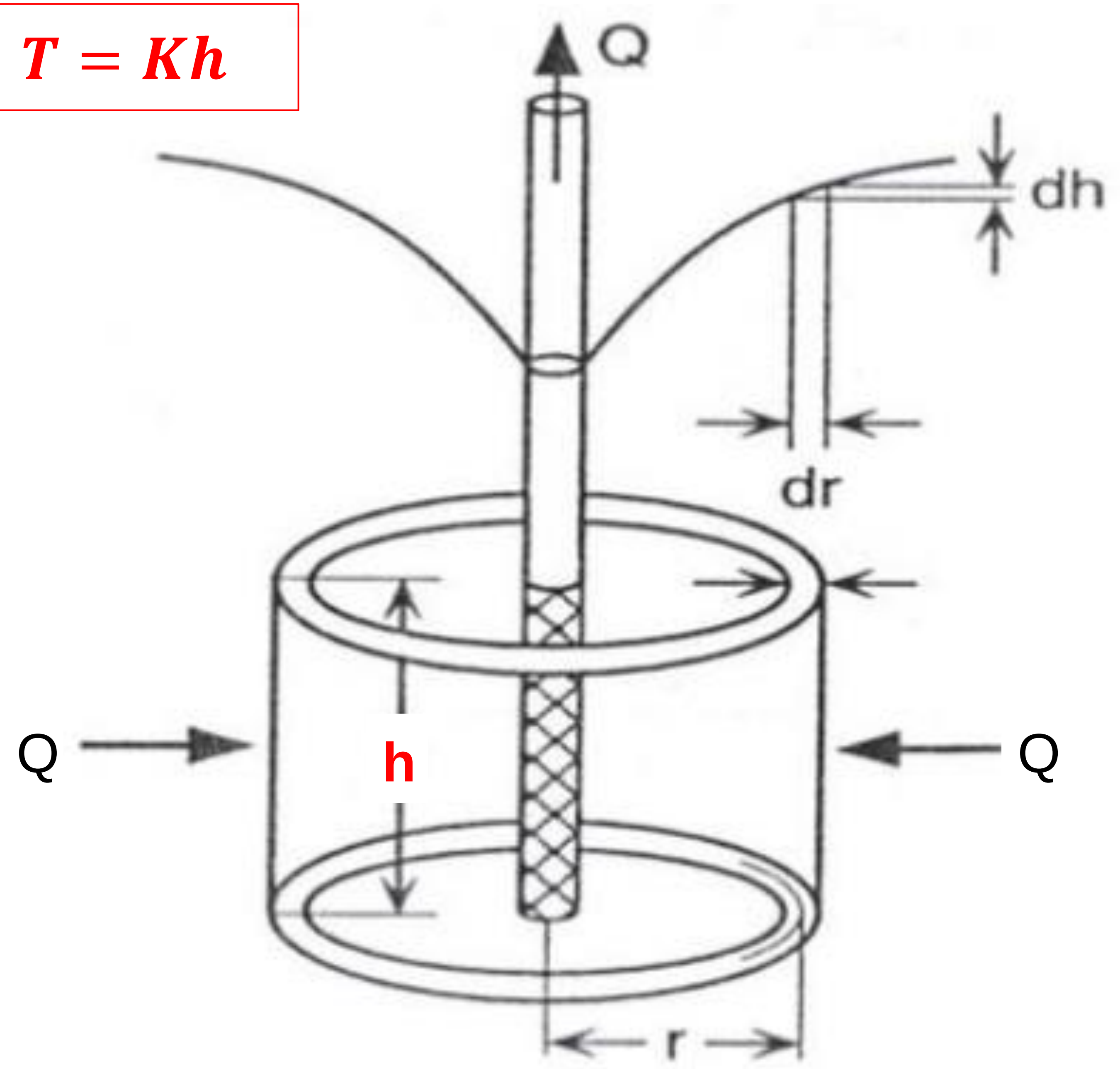
$$\boldsymbol{T = Kh}$$

Continuity of steady 1D axisymmetric flow:

$$\frac{dQ}{dr} = 0 \quad \text{or} \quad \frac{d}{dr}\left(rh\frac{dh}{dr}\right) = 0$$

Boundary condition: constant head h_0 at distance R:

$$h(R) = h_0 \qquad (2)$$



Source: Kresic, 1997

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DUPUIT EQUATION: DERIVATION

Rearranging (1):

$$hdh = \frac{Q}{2\pi K}\frac{dr}{r} \tag{3}$$

Integrating both sides of (3):

$$h^2(r) = \frac{Q}{\pi K}\ln r + C \tag{4}$$

Introducing (2) in (4):

$$h^2(R) = h_0^2 = \frac{Q}{\pi K}\ln R + C \tag{5}$$

Deriving integration constant C from (5):

$$C = h_0^2 - \frac{Q}{\pi K}\ln R \tag{6}$$

Introducing (6) in general solution (4):

$$h(r) = \sqrt{h_0^2 - \frac{Q}{\pi K}\ln\frac{R}{r}} \tag{7}$$

Applying definition of drawdown s to (7):

$$s(r) = h_0 - \sqrt{h_0^2 - \frac{Q}{\pi K}\ln\frac{R}{r}} \tag{8}$$

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DUPUIT VS THIEM

Rearranging (8):

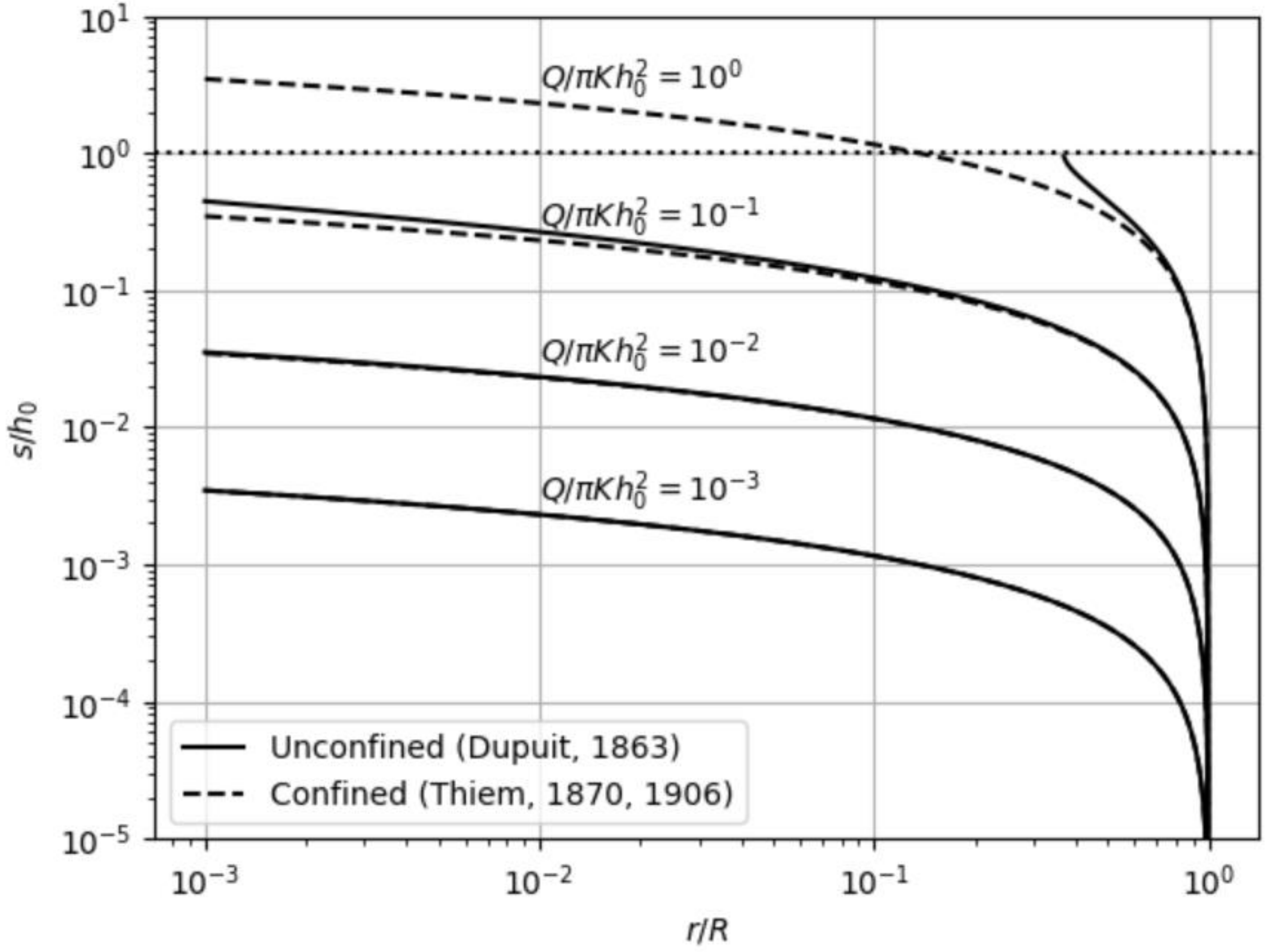
$$s(r) = h_0\left(1 - \boxed{\sqrt{1 - \frac{Q}{\pi K h_0^2}\ln\frac{R}{r}}}\right) \tag{9}$$

Series expansion:

$$\boxed{\sqrt{1-x}} \approx 1 - \frac{x}{2} \qquad (x \to 0) \tag{10}$$

Applying (10) to (9):

$$s(r) \approx \frac{Q}{2\pi K \boldsymbol{h_0}}\ln\frac{R}{r} \qquad (s < 0.1 h_0)$$



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WELL-KNOWN 1D AXISYMMETRIC MODELS

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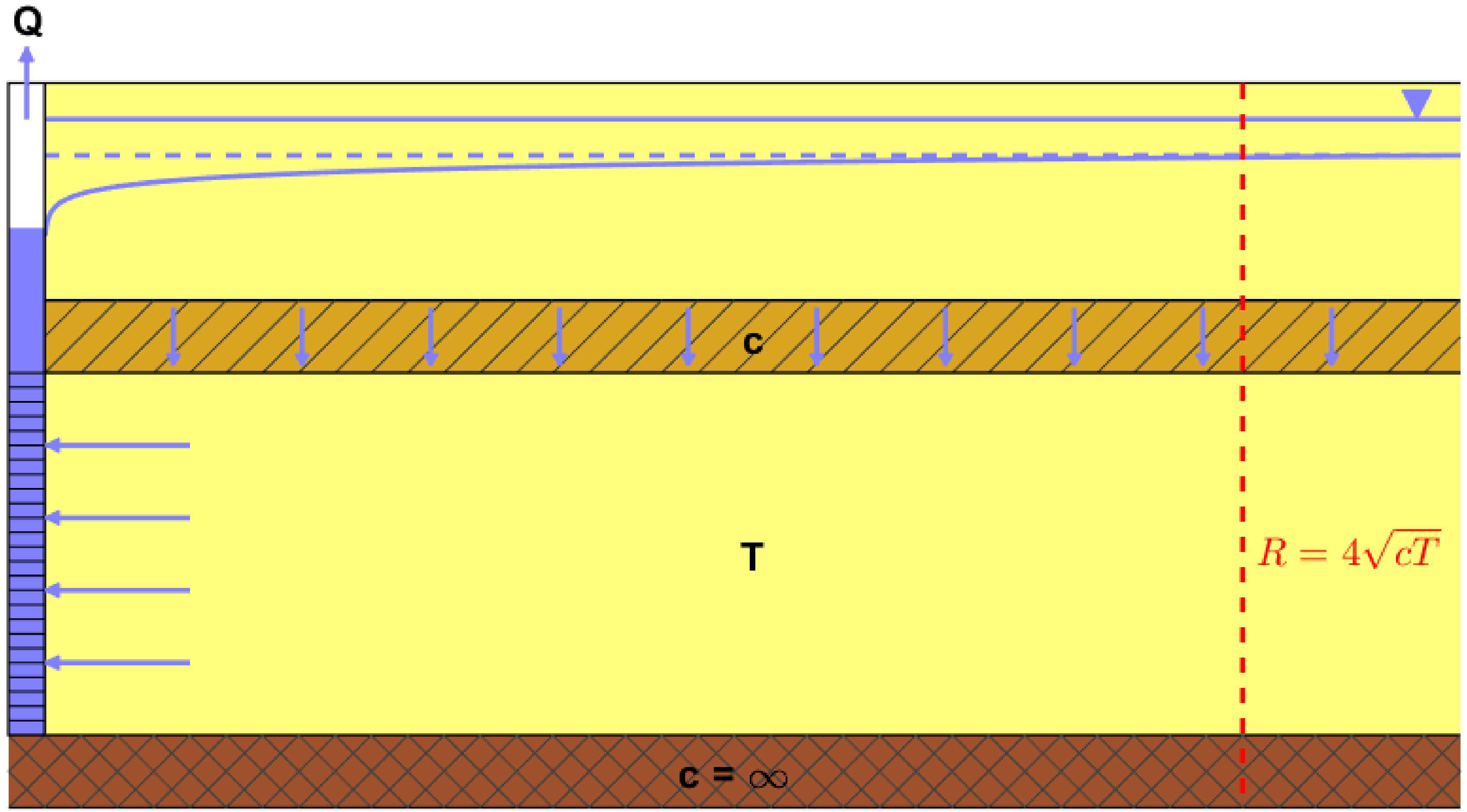
OVERVIEW

- **de Glee (1930)**: steady well-flow in a leaky aquifer
- **Theis (1935)**: transient well-flow in a confined aquifer
- **Hantush-Jacob (1955)**: transient well-flow in a leaky aquifer
- **Ernst (1971)**: steady well-flow in a phreatic aquifer subject to uniform infiltration and drainage

THE DE GLEE FORMULA



leakage

OVER GRONDWATERSTROOMINGEN BIJ WATERONTTREKKING DOOR MIDDEL VAN PUTTEN.

PROEFSCHRIFT

TER VERKRIJGING VAN DEN GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOOGESCHOOL TE DELFT, OP GEZAG VAN DEN RECTOR MAGNIFICUS IR. F. WESTENDORP, HOOGLEERAAR IN DE AFDEELING DER WERKTUIGBOUWKUNDE EN SCHEEPSBOUWKUNDE, VOOR EENE COMMISSIE UIT DEN SENAAT TE VERDEDIGEN OP WOENSDAG 2 APRIL 1930, DES NAMIDDAGS TE 3 UUR,

DOOR

GERRIT JAN DE GLEE,

CIVIEL-INGENIEUR,

GEBOREN TE ASSEN.

GEDRUKT BIJ DE TECHNISCHE BOEKHANDEL EN DRUKKERIJ J. WALTMAN JR. DELFT. — 1930.

Johan Kooper

Charles E. Jacob

Steady leaky flow (Kooper, 1914; de Glee, 1930; Jacob, 1946)

$$s(r) = \frac{Q}{2\pi KD} \mathrm{K}_0\left(r\sqrt{\frac{1}{cKD}}\right) \approx \frac{Q}{2\pi KD} \ln\left(\frac{2e^{-\gamma}\sqrt{KDc}}{r}\right)$$

DE GLEE EQUATION: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Steady-state
  - Strictly horizontal
- Well:
  - Fully penetrating
  - Constant pumping rate
  - Infinitesimal radius
- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded
  - Leaky top

DE GLEE EQUATION: PROBLEM STATEMENT

Continuity of steady 1D leaky flow:

$$T\left(\frac{d^2h}{dr^2}+\frac{1}{r}\frac{dh}{dr}\right)=\boxed{\frac{h-h_0}{c}}$$

leakage

(1)

Inner boundary condition at zero:

$$Q=\lim_{r\to 0}\left(2\pi rT\frac{dh}{dr}\right)$$

(2)

Outer boundary condition at infinity:

$$h(\infty)=h_0$$

(3)

Modified Bessel differential equation:

$$\frac{d^2h}{dr^2}+\frac{1}{r}\frac{dh}{dr}=ah-b$$

General solution:

$$h=\alpha \mathrm{I}_0(r\sqrt{a})+\beta \mathrm{K}_0(r\sqrt{a})+\frac{b}{a}$$

with:

- I_0, K_0: the zero order modified Bessel functions of the first and second kind, resp.
- α, β: integration constants

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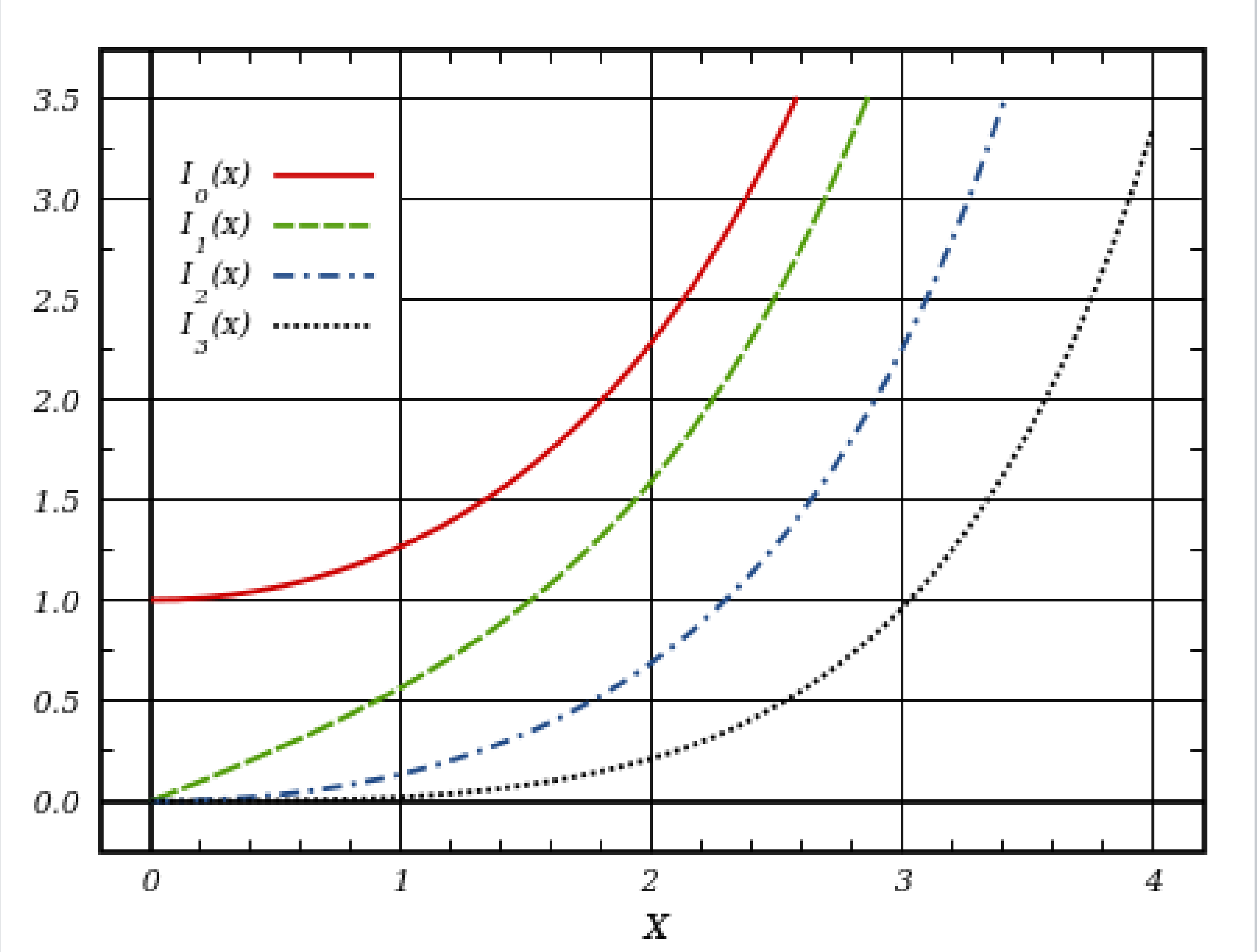
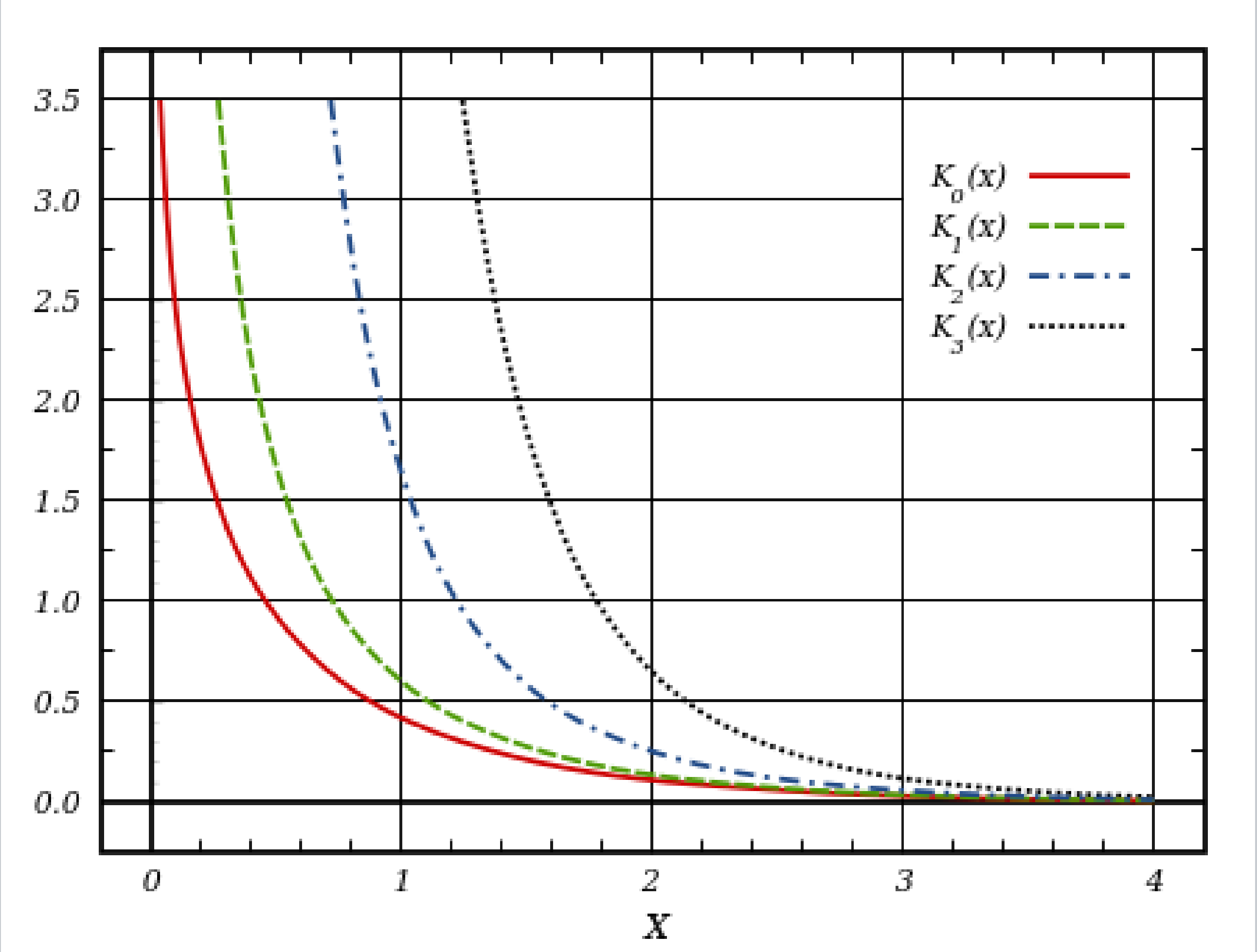
MODIFIED BESSEL FUNCTIONS



Modified Bessel functions of the first kind, $I_\alpha(x)$, for $\alpha = 0, 1, 2, 3$



Modified Bessel functions of the second kind, $K_\alpha(x)$, for $\alpha = 0, 1, 2, 3$

$$\frac{d\mathrm{I}_0(ax)}{dx} = a\mathrm{I}_1(ax)$$

$$\frac{d\mathrm{K}_0(ax)}{dx} = -a\mathrm{K}_1(ax)$$

	$\mathrm{I}_0(x)$	$\mathrm{K}_0(x)$	$x\mathrm{I}_1(x)$	$x\mathrm{K}_1(x)$
$x \to 0$	1	∞	0	1
$x \to \infty$	∞	0	∞	0

DE GLEE EQUATION: DERIVATION

General solution of (1):

$$h = \alpha \mathrm{I}_0\left(\frac{r}{\lambda}\right) + \beta \mathrm{K}_0\left(\frac{r}{\lambda}\right) + h_0$$

with $\lambda = \sqrt{Tc}$ the leakage factor

(4)

Applying BC (3): $\mathrm{I}_0(x) \to \infty \text{ if } x \to \infty$

$$\alpha = 0$$

(5)

Introducing (5) in (4):

$$h(r) = h_0 + \beta \mathrm{K}_0\left(\frac{r}{\lambda}\right)$$

(6)

Applying BC (2): $x\mathrm{K}_1(x) \to 1 \text{ if } x \to 0$

$$\beta = \frac{-Q}{2\pi T}$$

(7)

Introducing (7) in (6):

$$h(r) = h_0 - \frac{Q}{2\pi T}\mathrm{K}_0\left(\frac{r}{\lambda}\right)$$

(8)

Applying definition of drawdown s to (8):

$$s(r) = h_0 - h(r) = \frac{Q}{2\pi T}\mathrm{K}_0\left(\frac{r}{\lambda}\right)$$

(9)

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DE GLEE VS THIEM

Series expansion:

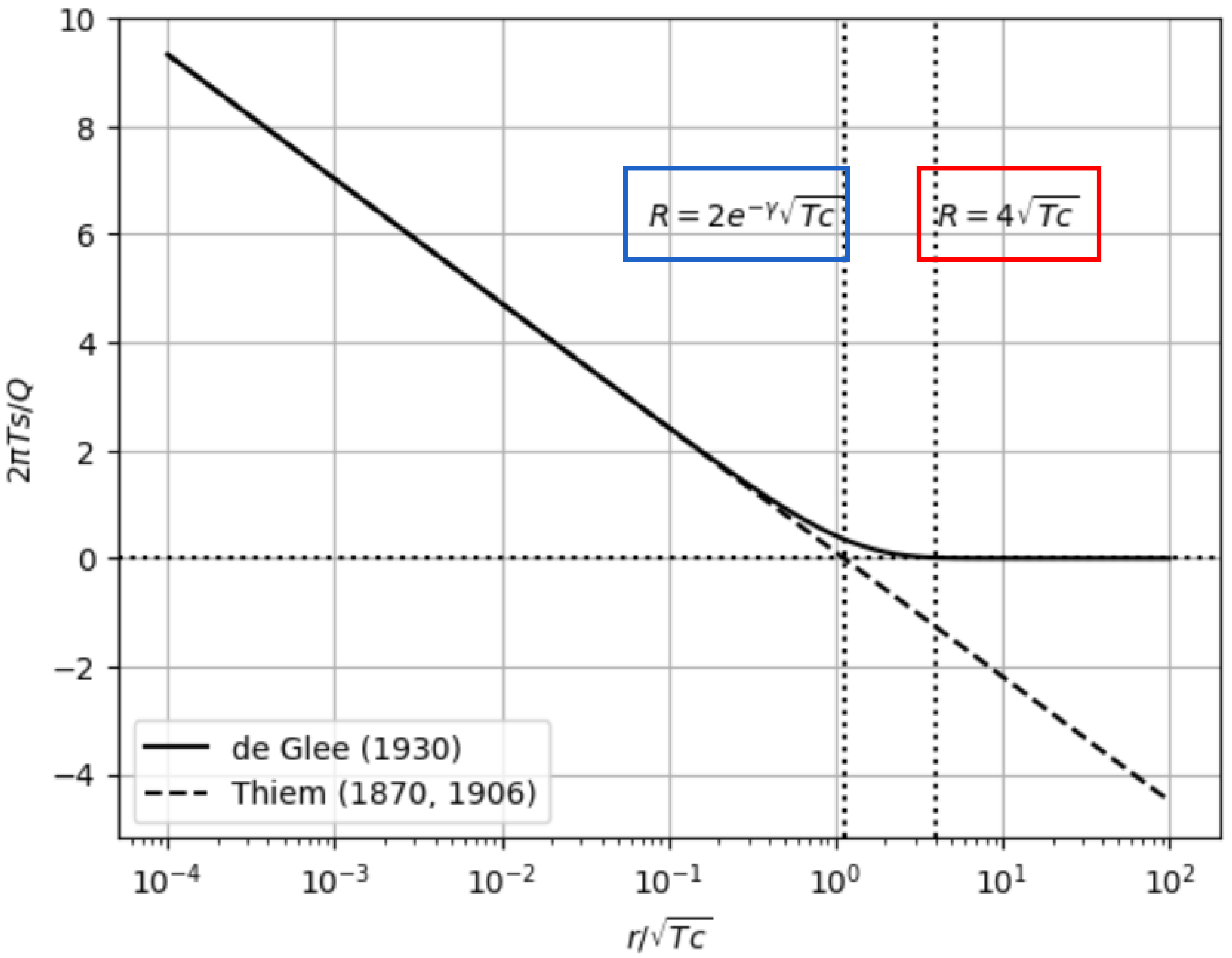
$$\mathrm{K}_0(x) \approx -\gamma - \ln\frac{x}{2} \quad (x \to 0) \tag{10}$$

Applying (10) to (9):

$$s(r) \approx \frac{Q}{2\pi T}\ln\left(\frac{\boxed{2e^{-\gamma}\lambda}}{r}\right) \quad (r < 0.1\lambda) \tag{11}$$

Comparing (11) to the Thiem equation:

$$\boxed{R = 2e^{-\gamma}\lambda}$$



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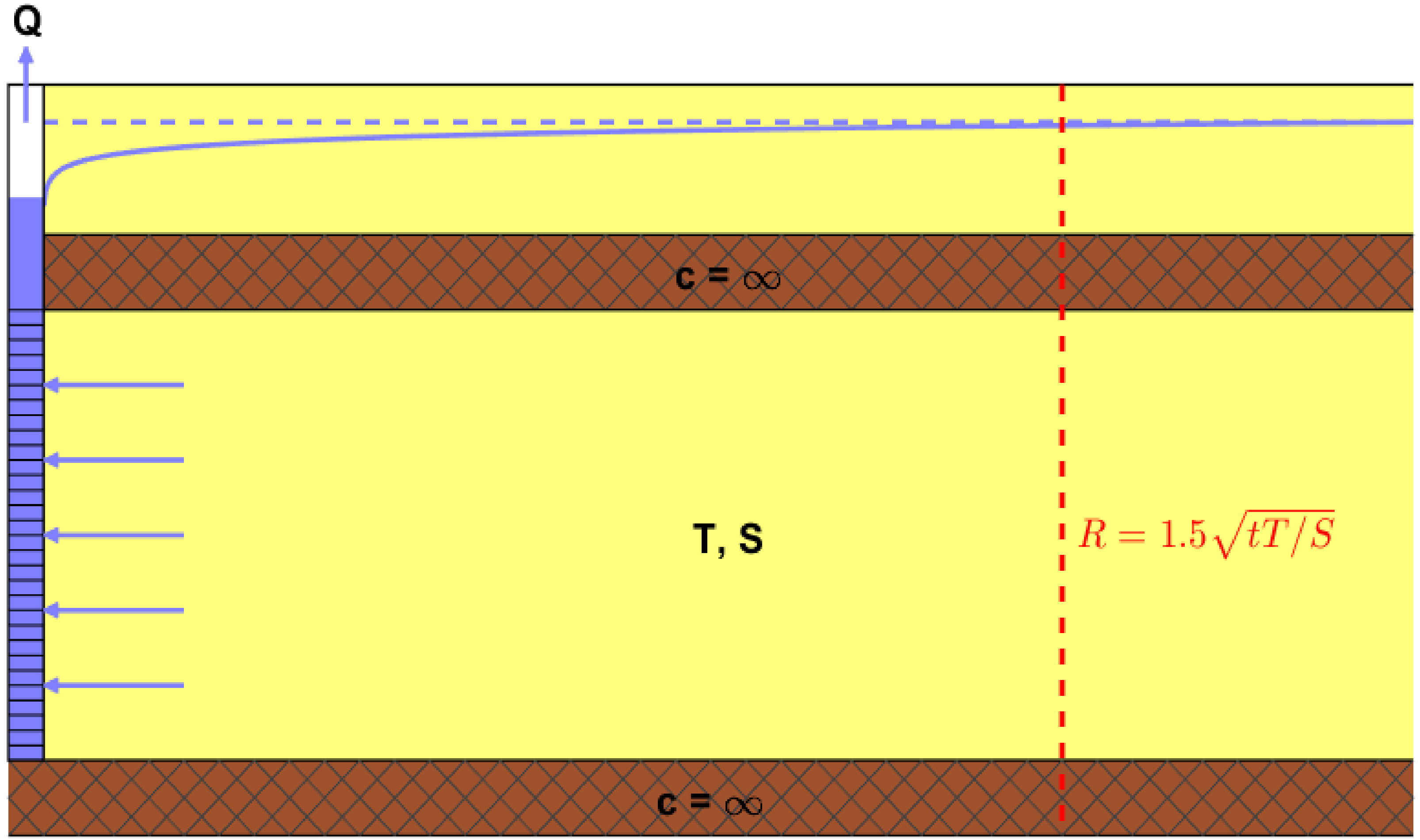
THE THEIS EQUATION



UNITED STATES
DEPARTMENT OF THE INTERIOR
GEOLOGICAL SURVEY
WATER RESOURCES DIVISION
GROUND WATER BRANCH
Washington 25, D. C.

GROUND WATER NOTES
HYDRAULICS

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THE RELATION BETWEEN THE LOWERING OF THE PIEZOMETRIC SURFACE AND THE RATE AND DURATION OF DISCHARGE OF A WELL USING GROUND WATER STORAGE

By
Charles V. Theis

This paper develops some of the basic concepts on which much of our present-day theory of ground-water hydraulics is founded. Although published in the Transactions of the American Geophysical Union, part 2 (pp. 519-524), August 1935, the supply of reprints has long since been exhausted and the paper is now generally to be found only in the better-stocked reference libraries.

It is reproduced as part of the new series of Ground Water Notes to permit distribution to all ground-water field personnel for their ready reference and use. Minor changes have been made in notation only, to be consistent with current Branch usage.

When a well is pumped or otherwise discharged, water levels in its neighborhood are lowered. Unless this lowering occurs instantaneously it represents a loss of storage, either by the unwatering of a portion of the previously saturated sediments if the aquifer is nonartesian or by release of stored water by the compaction of the aquifer due to lowered pressure if the aquifer is artesian. The mathematical theory of ground-water hydraulics has been based, apparently entirely, on a postulate that equilibrium has been attained and therefore that water-levels are no longer falling. In a great number of hydrologic problems, involving a well or pumping district near or in which water-levels are falling, the

Open file

Charles V. Theis

Hilton H. Cooper, Jr.

Transient confined flow (Theis, 1935; Cooper & Jacob, 1946)

$$s(r,t) = \frac{Q}{4\pi KD} \mathrm{W}\left(\frac{r^2 S}{4tKD}\right) \approx \frac{Q}{2\pi KD} \ln\left(\frac{1}{r}\sqrt{\frac{4tKD}{e^{\gamma} S}}\right)$$

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THEIS EQUATION: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Transient-state
  - Strictly horizontal
- Well:
  - Fully penetrating
  - Constant pumping rate
  - Infinitesimal radius
- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded

THEIS EQUATION: PROBLEM STATEMENT

Continuity of transient 1D confined flow:

$$T\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\right) = \boxed{S\frac{\partial h}{\partial t}} \qquad (1)$$

storage change

Partial differential equation (PDE):

- head *h* is function of *r* and *t*
- apply Laplace transform w.r.t. *t*

Inner boundary condition at zero:

$$Q = \lim_{r \to 0}\left(2\pi r T \frac{\partial h}{\partial r}\right) \qquad (2)$$

Outer boundary condition at infinity:

$$h(\infty, t) = h_0 \qquad (3)$$

Initial condition at *t* = 0:

$$h(r, 0) = h_0 \qquad (4)$$

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THE LAPLACE TRANSFORM

= convert PDE in ODE by eliminating derivative w.r.t. time

Definition:

$$\bar{h}(r,p) = \mathcal{L}\{h(r,t)\}(p) = \int_0^{\infty} h(r,t)e^{-pt}dt \tag{5}$$

Derivative w.r.t. time *t*:

$$\mathcal{L}\left\{\frac{\partial h}{\partial t}\right\}(p) = p\bar{h}(r,p) - \boxed{h(r,0)} = p\bar{h}(r,p) - \boxed{h_0} \tag{6}$$

Initial condition (4)

Laplace transform of a constant:

$$\mathcal{L}\{Q\}(p) = \frac{Q}{p} \tag{7}$$

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THEIS EQUATION: LAPLACE TRANSFORM

Laplace transform of PDE (1):

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r}\frac{d\bar{h}}{dr} = \frac{S}{T}\left(p\bar{h} - h_0\right) \tag{8}$$

Laplace transform of inner BC (2):

$$\frac{Q}{p} = \lim_{r\to 0}\left(2\pi r T\frac{d\bar{h}}{dr}\right) \tag{9}$$

Laplace transform of outer BC (3):

$$\bar{h}(\infty, p) = \frac{h_0}{p} \tag{10}$$

Modified Bessel differential equation:

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r}\frac{d\bar{h}}{dr} = a\bar{h} - b$$

General solution:

$$\bar{h} = \alpha \mathrm{I}_0(r\sqrt{a}) + \beta \mathrm{K}_0(r\sqrt{a}) + \frac{b}{a}$$

with:

- I_0, K_0: the zero order modified Bessel functions of the first and second kind, resp.
- α, β: integration constants

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THEIS EQUATION: LAPLACE SOLUTION

General solution of (8):

$$\bar{h} = \alpha \mathrm{I}_0(r\omega) + \beta \mathrm{K}_0(r\omega) + \frac{h_0}{p}$$

with $\omega = \sqrt{Sp/T}$

(11)

Applying BC (9): $\mathrm{I}_0(x) \to \infty \text{ if } x \to \infty$

$$\alpha = 0$$

(12)

Introducing (12) in (11):

$$\bar{h}(r,p) = \frac{h_0}{p} + \beta \mathrm{K}_0(r\omega)$$

(13)

Applying BC (10): $x\mathrm{K}_1(x) \to 1 \text{ if } x \to 0$

$$\beta = \frac{-Q}{2\pi T p}$$

(14)

Introducing (14) in (13):

$$\bar{h}(r,p) = \frac{h_0}{p} - \frac{Q}{2\pi T p}\mathrm{K}_0(r\omega)$$

(15)

Applying definition of drawdown to (15):

$$\bar{s}(r,p) = \frac{h_0}{p} - \bar{h} = \frac{Q}{2\pi T p}\mathrm{K}_0(r\omega)$$

(16)

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THEIS EQUATION: LAPLACE INVERSION

Inverting (16) analytically:

$$s(r,t) = \frac{Q}{4\pi T}\mathrm{W}(u)$$

with $u = \frac{r^2 S}{4tT}$ (17)

Theis' well function W:

$$\mathrm{W}(u) = \int_u^{\infty} \frac{e^{-x}}{x} dx \qquad (18)$$

	$\bar{f}(p) = L\{\bar{f}(t)\}$	$f(t) = L^{-1}\{\bar{f}(p)\}$
1.	$1/p$	1
2.	$1/p^n$	$t^{n-1}/(n-1)\,!$
3.	$1/p(p+a)$	$(1/a)[1-\exp(-at)]$
4.	$1/\sqrt{p}$	$1/\sqrt{\pi t}$
5.	$1/p^k$	$t^{k-1}/\Gamma(k)$
6.	$(1/\sqrt{p})\exp(-k\sqrt{p})$	$(1/\sqrt{\pi t})\exp(-k^2/4t)$
7.	$(1/p)\exp(-k\sqrt{p})$	$\operatorname{erfc}(k/\sqrt{4t})$
8.	$(1/p^{1+n/2})\exp(-k\sqrt{p})$	$(4t)^{n/2} i^n \operatorname{erfc}(k/\sqrt{4t})$
9.	$K_0(k\sqrt{p})$	$(1/2t)\exp(-k^2/4t)$
10.	$(1/p)K_0(k\sqrt{p})$	$(1/2)W(k^2/4t)$
11.	$K_0(k\sqrt{p+a})$	$(1/2t)\exp(-at-k^2/4t)$
12.	$(1/p)K_0(k\sqrt{p+a})$	$(1/2)W(k^2/4t,\, k\sqrt{a})$
13.	$(1/p)K_0(k\sqrt{p+a\sqrt{p}})$	$(1/2)H(k^2/4t,\, ka/4)$
14.	$K_0(k\sqrt{p})/pK_0(k_1\sqrt{p})$	$A(t/k_1^2, k/k_1)$
15.	$K_0(k\sqrt{p})/p(k_1\sqrt{p})K_1(k_1\sqrt{p})$	$(1/2)S(t/k_1^2,\, k/k_1)$
16.	$K_0(k\sqrt{p+a})/pK_0(k_1\sqrt{p+a})$	$Z(t/k_1^2,\, k/k_1,\, k_1\sqrt{a})$

p.303 of "Hydraulics of Wells" (Hantush, 1964)

THE COOPER-JACOB APPROXIMATION

Series expansion of (18):

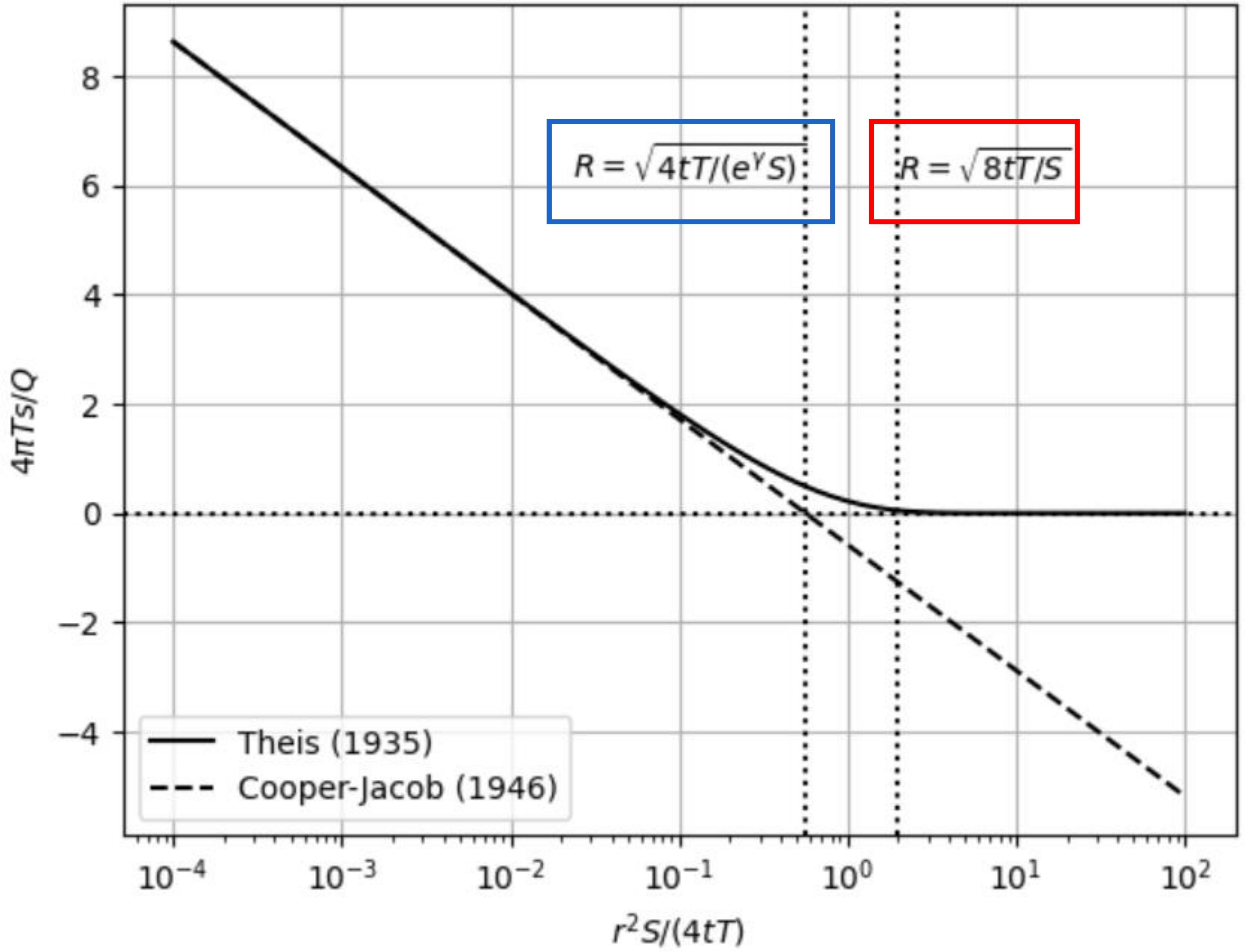
$$W(u) = \boxed{-\gamma - \ln(u)} - \sum_{n=1}^{\infty} \frac{(-u)^n}{n.n!} \qquad (19)$$

Truncating (19) and applying to (17):

$$s(r) \approx \frac{Q}{2\pi T} \ln\left(\frac{1}{r} \boxed{\sqrt{\frac{4tT}{e^\gamma S}}}\right) \quad (u < 0.1) \qquad (20)$$

Comparing (20) to the Thiem equation:

$$\boxed{R = \sqrt{\frac{4tT}{e^\gamma S}}}$$



$= u$

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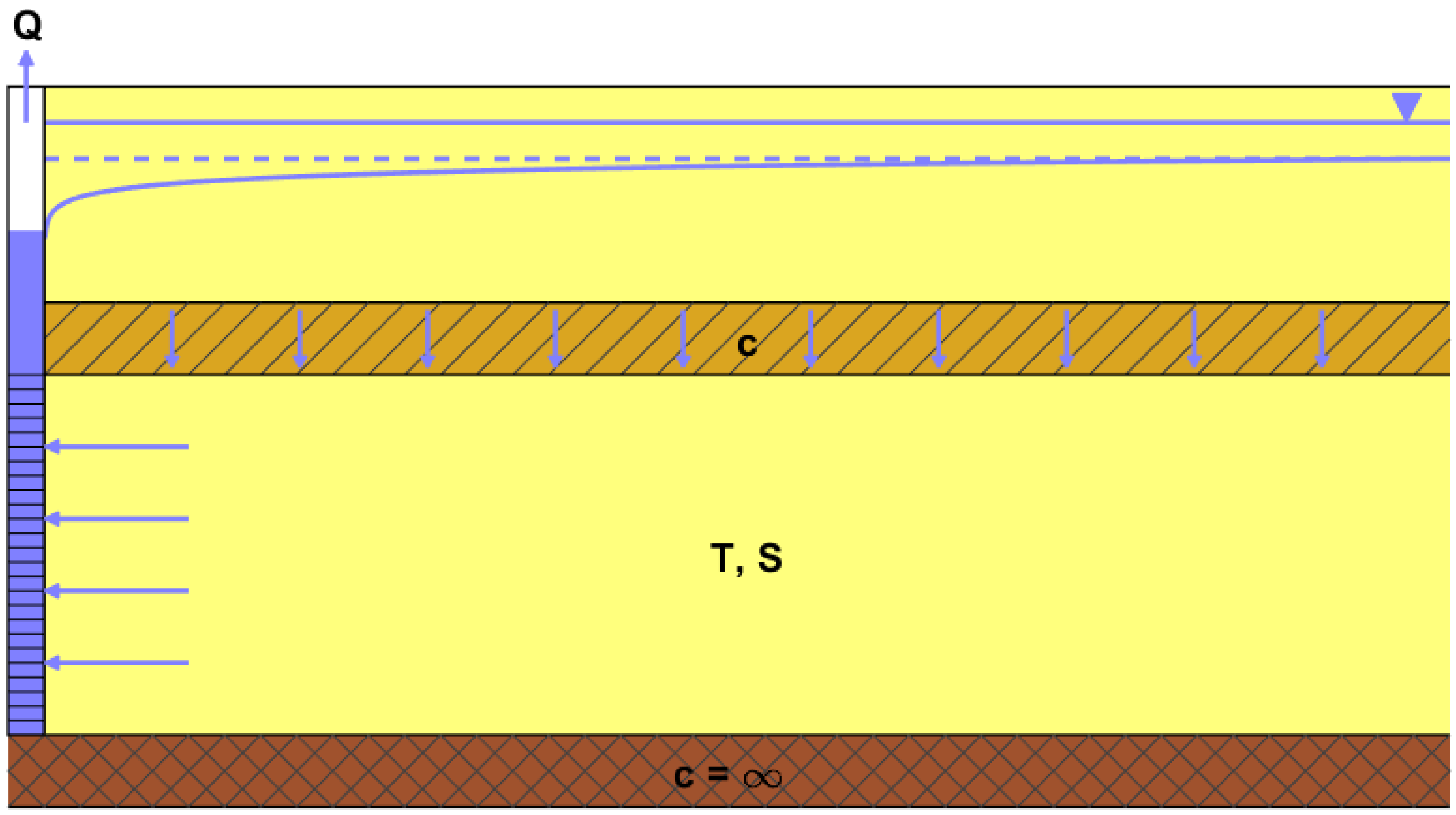
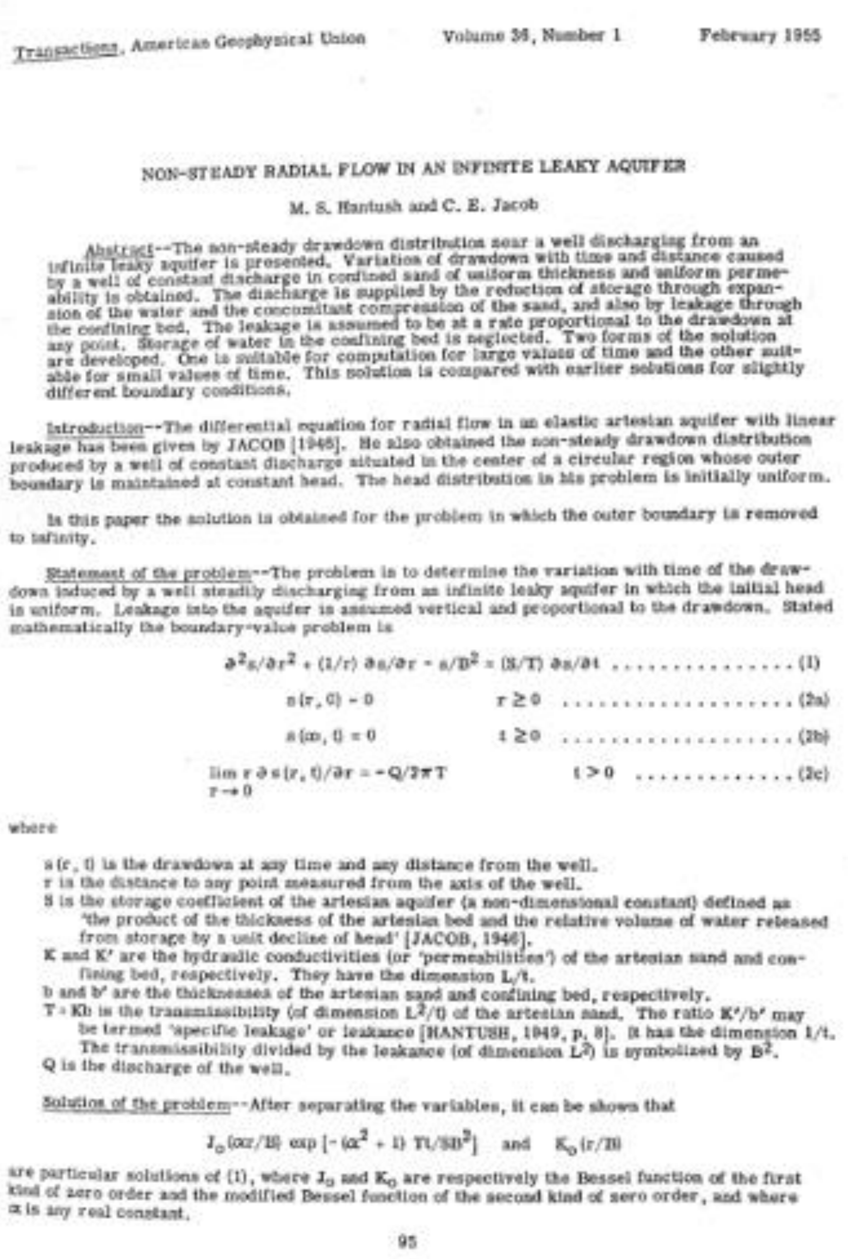
THE HANTUSH-JACOB MODEL



leakage


Transactions, American Geophysical Union Volume 36, Number 1 February 1955

NON-STEADY RADIAL FLOW IN AN INFINITE LEAKY AQUIFER

M. S. Hantush and C. E. Jacob

Abstract--The non-steady drawdown distribution near a well discharging from an infinite leaky aquifer is presented. Variation of drawdown with time and distance caused by a well of constant discharge in confined sand of uniform thickness and uniform permeability is obtained. The discharge is supplied by the reduction of storage through expansion of the water and the concomitant compression of the sand, and also by leakage through the confining bed. The leakage is assumed to be at a rate proportional to the drawdown at any point. Storage of water in the confining bed is neglected. Two forms of the solution are developed. One is suitable for computation for large values of time and the other suitable for small values of time. This solution is compared with earlier solutions for slightly different boundary conditions.

Introduction--The differential equation for radial flow in an elastic artesian aquifer with linear leakage has been given by JACOB [1946]. He also obtained the non-steady drawdown distribution produced by a well of constant discharge situated in the center of a circular region whose outer boundary is maintained at constant head. The head distribution in his problem is initially uniform.

In this paper the solution is obtained for the problem in which the outer boundary is removed to infinity.

Statement of the problem--The problem is to determine the variation with time of the drawdown induced by a well steadily discharging from an infinite leaky aquifer in which the initial head is uniform. Leakage into the aquifer is assumed vertical and proportional to the drawdown. Stated mathematically the boundary-value problem is

$$\partial^2 s/\partial r^2 + (1/r)\,\partial s/\partial r - s/B^2 = (S/T)\,\partial s/\partial t \quad \ldots\ldots (1)$$

$$s(r, 0) = 0 \qquad r \geq 0 \quad \ldots\ldots (2a)$$

$$s(\infty, t) = 0 \qquad t \geq 0 \quad \ldots\ldots (2b)$$

$$\lim_{r \to 0} r\,\partial s(r, t)/\partial r = -Q/2\pi T \qquad t > 0 \quad \ldots\ldots (2c)$$

where

$s(r, t)$ is the drawdown at any time and any distance from the well.
r is the distance to any point measured from the axis of the well.
S is the storage coefficient of the artesian aquifer (a non-dimensional constant) defined as 'the product of the thickness of the artesian bed and the relative volume of water released from storage by a unit decline of head' [JACOB, 1946].
K and K' are the hydraulic conductivities (or 'permeabilities') of the artesian sand and confining bed, respectively. They have the dimension L/t.
b and b' are the thicknesses of the artesian sand and confining bed, respectively.
$T = Kb$ is the transmissibility (of dimension L^2/t) of the artesian sand. The ratio K'/b' may be termed 'specific leakage' or leakance [HANTUSH, 1949, p. 8]. It has the dimension $1/t$. The transmissibility divided by the leakance (of dimension L^2) is symbolized by B^2.
Q is the discharge of the well.

Solution of the problem--After separating the variables, it can be shown that

$$J_0(\alpha r/B)\exp[-(\alpha^2 + 1)\,Tt/SB^2] \quad \text{and} \quad K_0(r/B)$$

are particular solutions of (1), where J_0 and K_0 are respectively the Bessel function of the first kind of zero order and the modified Bessel function of the second kind of zero order, and where α is any real constant.

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Mahdi S. Hantush

Charles E. Jacob

Transient leaky flow (Hantush & Jacob, 1955)

$$s(r,t) = \frac{Q}{4\pi KD}\mathrm{W}\left(\frac{r^2S}{4tKD}, r\sqrt{\frac{1}{cKD}}\right)$$

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HANTUSH-JACOB: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Transient-state
  - Strictly horizontal
- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded
  - Leaky top
- Well:
  - Fully penetrating
  - Constant pumping rate
  - Infinitesimal radius

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HANTUSH-JACOB: PROBLEM STATEMENT

Continuity of transient 1D leaky flow:

$$T\left(\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}\right)=\underbrace{\boxed{S\frac{\partial h}{\partial t}}}_{\text{storage change}}+\underbrace{\boxed{\frac{h-h_0}{c}}}_{\text{leakage}} \qquad (1)$$

Partial differential equation (PDE):

- head *h* is function of *r* and *t*
- apply Laplace transform w.r.t. *t*

Inner boundary condition at zero:

$$Q=\lim_{r\to 0}\left(2\pi r T\frac{\partial h}{\partial r}\right) \qquad (2)$$

Outer boundary condition at infinity:

$$h(\infty,t)=h_0 \qquad (3)$$

Initial condition at *t* = 0:

$$h(r,0)=h_0 \qquad (4)$$

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HANTUSH-JACOB: LAPLACE TRANSFORM

Laplace transform of PDE (1):

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r}\frac{d\bar{h}}{dr} = \left(\omega^2 + \frac{1}{\lambda^2}\right)\left(\bar{h} - \frac{h_0}{p}\right) \tag{5}$$

Laplace transform of inner BC (2):

$$\frac{Q}{p} = \lim_{r\to 0}\left(2\pi r T\frac{d\bar{h}}{dr}\right) \tag{6}$$

Laplace transform of outer BC (3):

$$\bar{h}(\infty, p) = \frac{h_0}{p} \tag{7}$$

Modified Bessel differential equation:

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r}\frac{d\bar{h}}{dr} = a\bar{h} - b$$

General solution:

$$\bar{h} = \alpha \mathrm{I}_0(r\sqrt{a}) + \beta \mathrm{K}_0(r\sqrt{a}) + \frac{b}{a}$$

with:

- $\mathrm{I}_0, \mathrm{K}_0$: the zero order modified Bessel functions of the first and second kind, resp.
- α, β: integration constants

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HANTUSH-JACOB: LAPLACE SOLUTION

General solution of ODE (5):

$$\bar{h} = \alpha \mathrm{I}_0(r\vartheta) + \beta \mathrm{K}_0(r\vartheta) + \frac{h_0}{p}$$

with $\vartheta = \sqrt{\omega^2 + \frac{1}{\lambda^2}}$

(8)

Applying BC (7): $\mathrm{I}_0(x) \rightarrow \infty \text{ if } x \rightarrow \infty$

$$\alpha = 0$$

(9)

Introducing (9) in (8):

$$\bar{h}(r,p) = \frac{h_0}{p} + \beta \mathrm{K}_0(r\vartheta)$$

(10)

Applying BC (6): $x\mathrm{K}_1(x) \rightarrow 1 \text{ if } x \rightarrow 0$

$$\beta = \frac{-Q}{2\pi T p}$$

(11)

Introducing (11) in (10):

$$\bar{h}(r,p) = \frac{h_0}{p} - \frac{Q}{2\pi T p}\mathrm{K}_0(r\vartheta)$$

(12)

Applying definition of drawdown to (12):

$$\bar{s}(r,p) = \frac{h_0}{p} - \bar{h} = \frac{Q}{2\pi T p}\mathrm{K}_0(r\vartheta)$$

(13)

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HANTUSH-JACOB: LAPLACE INVERSION

Inverting (13) analytically:

$$s(r,t) = \frac{Q}{4\pi T} \mathrm{W}(u,v)$$

with $u = \frac{r^2 S}{4tT}$ and $v = \frac{r}{\sqrt{Tc}}$ (14)

Hantush' well function W:

$$\mathrm{W}(u,v) = \int_u^{\infty} \frac{e^{-x-v^2/4x}}{x} dx$$ (15)

	$\bar{f}(p) = L\{f(t)\}$	$f(t) = L^{-1}\{\bar{f}(p)\}$
1.	$1/p$	1
2.	$1/p^n$	$t^{n-1}/(n-1)\,!$
3.	$1/p(p+a)$	$(1/a)[1-\exp(-at)]$
4.	$1/\sqrt{p}$	$1/\sqrt{\pi t}$
5.	$1/p^k$	$t^{k-1}/\Gamma(k)$
6.	$(1/\sqrt{p})\exp(-k\sqrt{p})$	$(1/\sqrt{\pi t})\exp(-k^2/4t)$
7.	$(1/p)\exp(-k\sqrt{p})$	$\mathrm{erfc}(k/\sqrt{4t})$
8.	$(1/p^{1+n/2})\exp(-k\sqrt{p})$	$(4t)^{n/2} i^n\,\mathrm{erfc}(k/\sqrt{4t})$
9.	$K_0(k\sqrt{p})$	$(1/2t)\exp(-k^2/4t)$
10.	$(1/p)K_0(k\sqrt{p})$	$(1/2)W(k^2/4t)$
11.	$K_0(k\sqrt{p+a})$	$(1/2t)\exp(-at-k^2/4t)$
12.	$(1/p)K_0(k\sqrt{p+a})$	$(1/2)W(k^2/4t,\, k\sqrt{a})$
13.	$(1/p)K_0(k\sqrt{p+a\sqrt{p}})$	$(1/2)H(k^2/4t,\, ka/4)$
14.	$K_0(k\sqrt{p})/pK_0(k_1\sqrt{p})$	$A(t/k_1^2, k/k_1)$
15.	$K_0(k\sqrt{p})/p(k_1\sqrt{p})K_1(k_1\sqrt{p})$	$(1/2)S(t/k_1^2,\, k/k_1)$
16.	$K_0(k\sqrt{p+a})/pK_0(k_1\sqrt{p+a})$	$Z(t/k_1^2,\, k/k_1,\, k_1\sqrt{a})$

p.303 of “Hydraulics of Wells” (Hantush, 1964)

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INVERTING NUMERICALLY: STEHFEST

- Semi-analytical solution:
  - Analytical closed-form solution in Laplace space
  - Numerical inversion to the real time domain
- Several inversion algorithms available:
  - de Hoog et al. (1982)
  - Gaver (1966) - Stehfest (1970):

$$\boxed{s(r,t) = \frac{\ln 2}{t} \sum_{k=1}^{N} \omega_k \bar{s}(r, k \ln 2/t)}$$

with *N* the Stehfest number and

$$\omega_k = (-1)^{N/2+k} \sum_{j=\frac{k+1}{2}}^{\min(k,N/2)} \frac{j^{N/2}(2j)!}{(N/2-j)!\,(j)!\,(j-1)!\,(k-j)!\,(2j-k)!}$$

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HANTUSH-JACOB VS THEIS VS DE GLEE

Confined: $c \rightarrow \infty$ and $v \rightarrow 0$

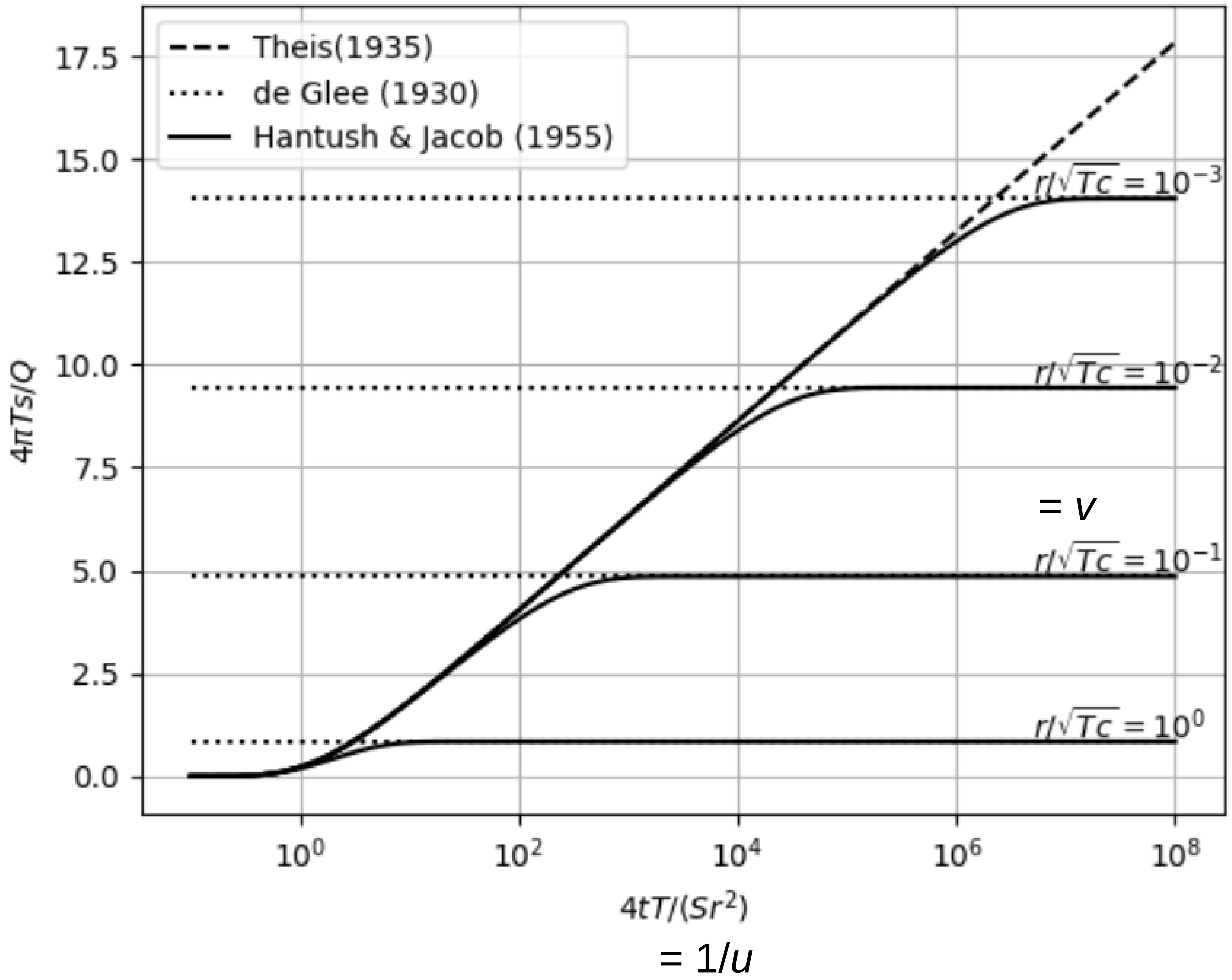
$$\lim_{v \rightarrow 0} \mathrm{W}(u, v) = \mathrm{W}(u)$$

Hantush-Jacob → Theis

Steady-state: $t \rightarrow \infty$ and $u \rightarrow 0$

$$\lim_{u \rightarrow 0} \mathrm{W}(u, v) = 2\mathrm{K}_0(v)$$

Hantush-Jacob → de Glee



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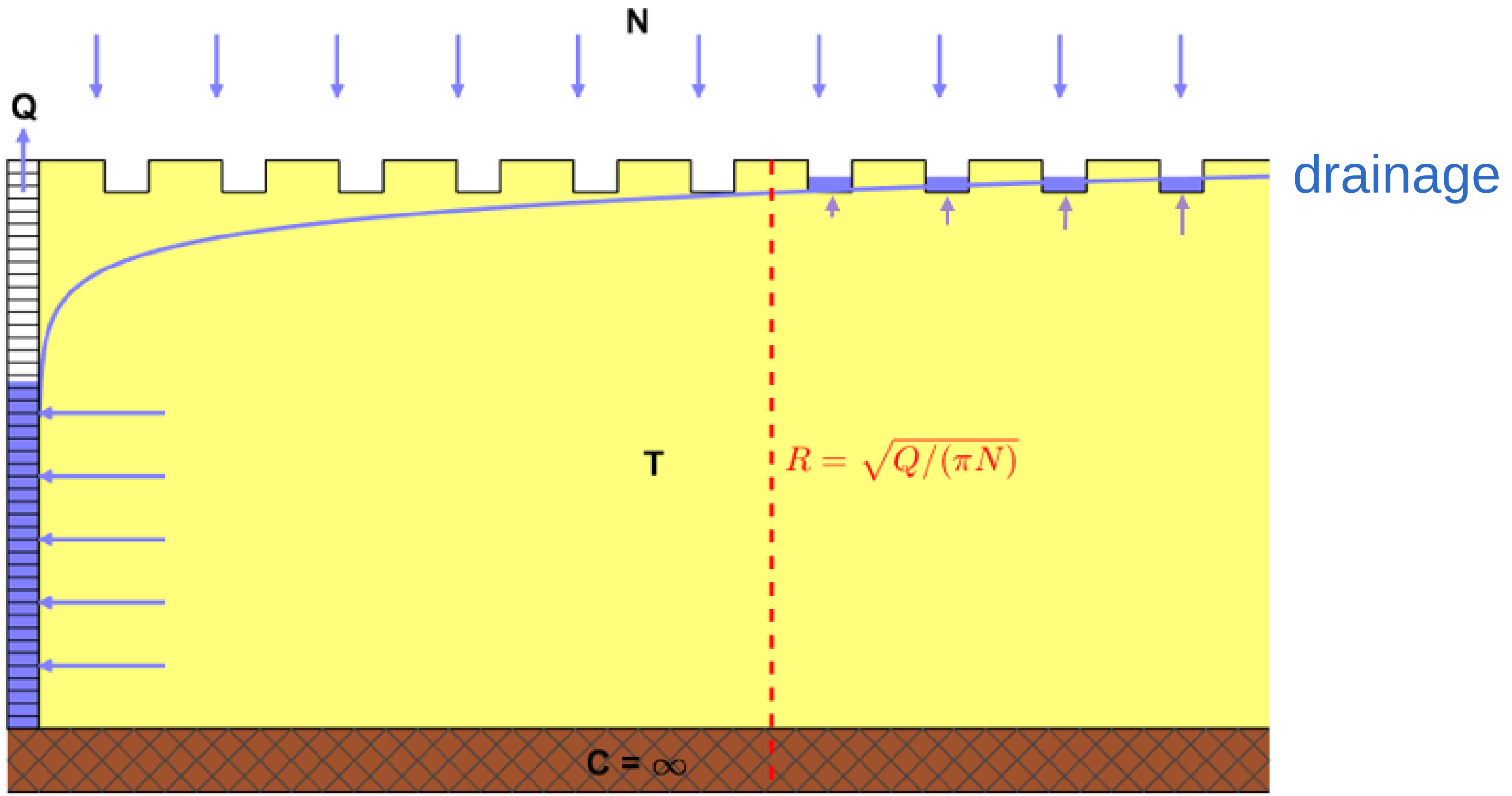
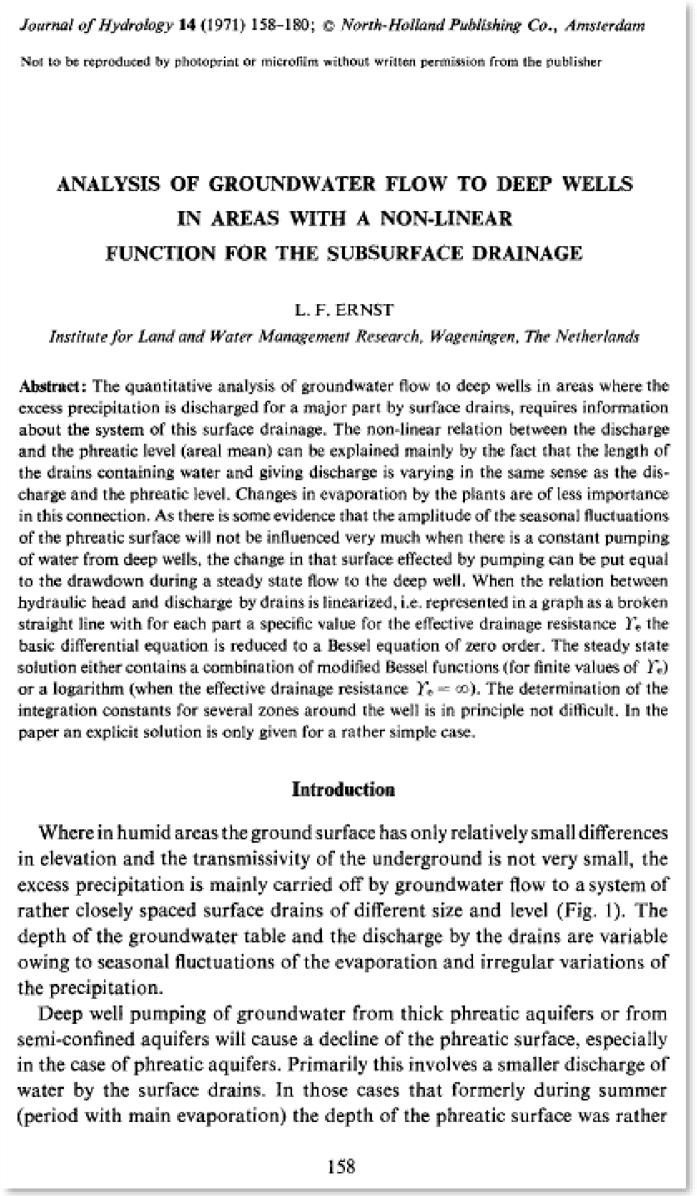
THE ERNST MODEL




Journal of Hydrology **14** (1971) 158–180; © *North-Holland Publishing Co., Amsterdam*

ANALYSIS OF GROUNDWATER FLOW TO DEEP WELLS IN AREAS WITH A NON-LINEAR FUNCTION FOR THE SUBSURFACE DRAINAGE

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Abstract: The quantitative analysis of groundwater flow to deep wells in areas where the excess precipitation is discharged for a major part by surface drains, requires information about the system of this surface drainage. The non-linear relation between the discharge and the phreatic level (areal mean) can be explained mainly by the fact that the length of the drains containing water and giving discharge is varying in the same sense as the discharge and the phreatic level. Changes in evaporation by the plants are of less importance in this connection. As there is some evidence that the amplitude of the seasonal fluctuations of the phreatic surface will not be influenced very much when there is a constant pumping of water from deep wells, the change in that surface effected by pumping can be put equal to the drawdown during a steady state flow to the deep well. When the relation between hydraulic head and discharge by drains is linearized, i.e. represented in a graph as a broken straight line with for each part a specific value for the effective drainage resistance Υ_e the basic differential equation is reduced to a Bessel equation of zero order. The steady state solution either contains a combination of modified Bessel functions (for finite values of Υ_e) or a logarithm (when the effective drainage resistance $\Upsilon_e = \infty$). The determination of the integration constants for several zones around the well is in principle not difficult. In the paper an explicit solution is only given for a rather simple case.

Introduction

Where in humid areas the ground surface has only relatively small differences in elevation and the transmissivity of the underground is not very small, the excess precipitation is mainly carried off by groundwater flow to a system of rather closely spaced surface drains of different size and level (Fig. 1). The depth of the groundwater table and the discharge by the drains are variable owing to seasonal fluctuations of the evaporation and irregular variations of the precipitation.

Deep well pumping of groundwater from thick phreatic aquifers or from semi-confined aquifers will cause a decline of the phreatic surface, especially in the case of phreatic aquifers. Primarily this involves a smaller discharge of water by the surface drains. In those cases that formerly during summer (period with main evaporation) the depth of the phreatic surface was rather

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Steady flow with recharge and **nonlinear** drainage (Ernst, 1971)

$$s(r) = \frac{Q}{2\pi KD}\ln\left(\frac{R}{r}\right) - \frac{N}{4KD}\left(R^2 - r^2\right) \quad \text{if } c \to 0$$

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ERNST: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Steady-state
  - Strictly horizontal

- Well:
  - Fully penetrating
  - Constant pumping rate
  - Infinitesimal radius

- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded
  - Uniform areal infiltration + drainage

- Constant drainage resistance ***c***
- Constant drainage level equal to zero

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ERNST: PROBLEM STATEMENT

R

Steady flow with infiltration:

$$T\left(\frac{d^2h_1}{dr^2}+\frac{1}{r}\frac{dh_1}{dr}\right)=-\boxed{N}$$

infiltration

(1)

Inner boundary condition at zero:

$$Q=\lim_{r\to 0}\left(2\pi r T\frac{dh_1}{dr}\right)$$

(2)

Outer boundary condition at *R*:

$$h_1(R)=0$$

(3)

Proximal zone 1: $h_1 \leq 0$ → no drainage

Steady flow with infiltration & drainage:

$$T\left(\frac{d^2h_2}{dr^2}+\frac{1}{r}\frac{dh_2}{dr}\right)=-\boxed{N}+\boxed{\frac{h_2}{c}}$$

infiltration drainage

(4)

Inner boundary condition at *R*:

$$h_2(R)=0$$

(5)

Outer boundary condition at infinity:

$$h_2(\infty)=0$$

(6)

Distal zone 2: $h_2 \geq 0$ → drainage

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ERNST: SOLUTION

General solution of (1):

$$h_1 = \frac{-Nr^2}{4T} + \alpha_1 \ln r + \beta_1 (7)$$

(7)

Applying inner BC (2) and outer BC (3):

$$\alpha_1 = \frac{Q}{2\pi T}$$

$$\beta_1 = \frac{NR^2}{4T} - \frac{Q}{2\pi T} \ln R$$

(8)

Introducing (8) in (7):

$$h_1 = \frac{N}{4T}(R^2 - r^2) + \frac{Q}{2\pi T} \ln \frac{r}{R}$$

(9)

zone 1: $r \leq R$

R

General solution of (4):

$$h_2 = \alpha_2 \mathrm{I}_0\left(\frac{r}{\lambda}\right) + \beta_2 \mathrm{K}_0\left(\frac{r}{\lambda}\right) + Nc$$

(10)

Applying outer BC (6) and inner BC (5):

$$\alpha_2 = 0$$

$$\beta_2 = \frac{-Nc}{\mathrm{K}_0\left(\frac{R}{\lambda}\right)}$$

(11)

Introducting (11) in (10):

$$h_2 = Nc\left[1 - \frac{\mathrm{K}_0\left(\frac{r}{\lambda}\right)}{\mathrm{K}_0\left(\frac{R}{\lambda}\right)}\right]$$

zone 2: $r \geq R$

(12)

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ERNST: DETERMINING R

Continuity of flow at boundary *R*:

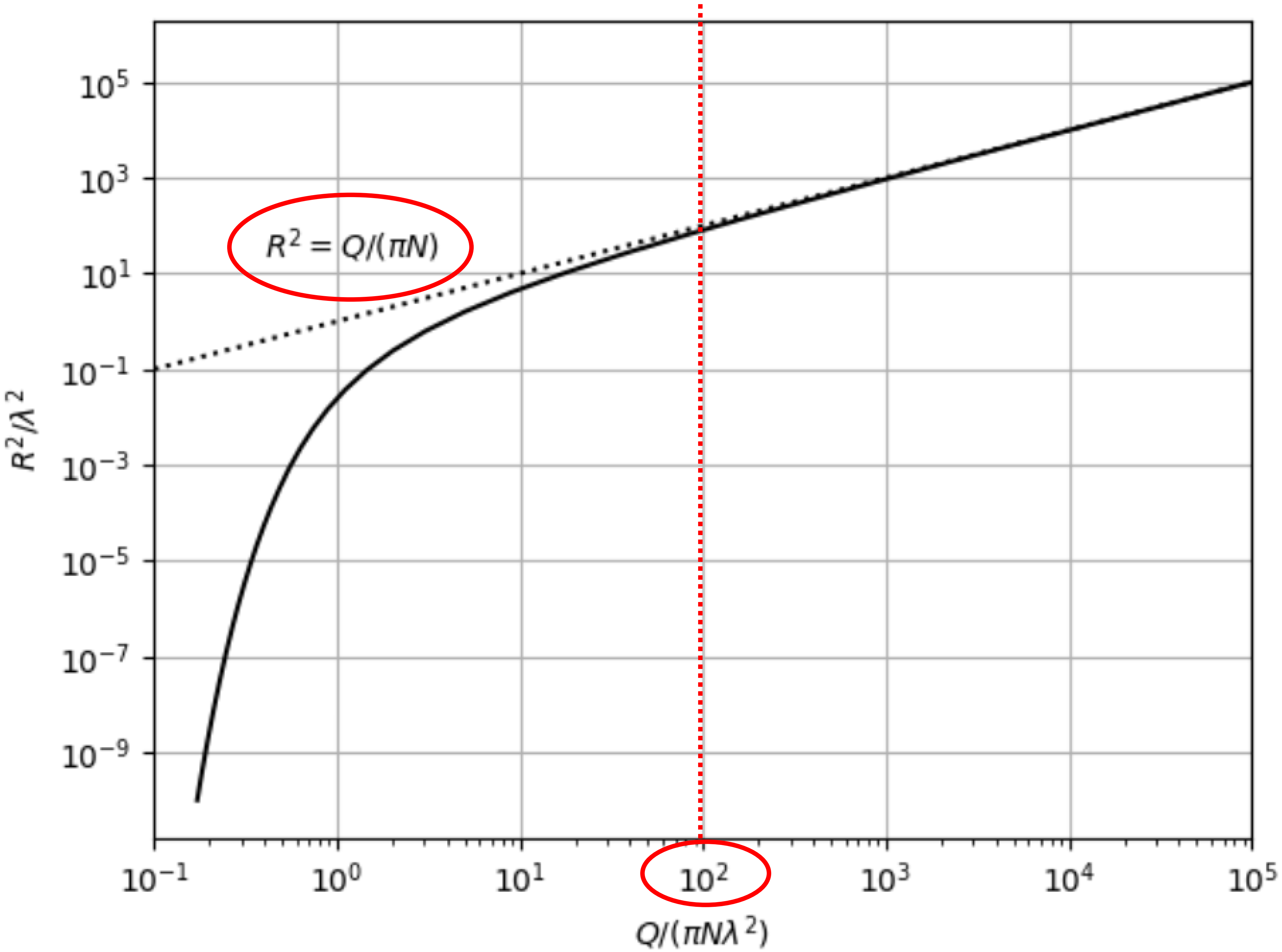
$$\frac{dh_1}{dr} = \frac{dh_2}{dr} \qquad (r = R) \tag{13}$$

Solving (13) using (9) and (12) and rearranging:

$$\frac{R}{\lambda}\left[\frac{R}{\lambda} + 2\frac{\mathrm{K}_1\left(\frac{R}{\lambda}\right)}{\mathrm{K}_0\left(\frac{R}{\lambda}\right)}\right] - \frac{Q}{\pi N \lambda^2} = 0 \tag{14}$$

Expression (14) simplifies to:

$$\boxed{Q = \pi R^2 N} \quad \text{if } Q > 100\pi N \lambda^2$$



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ERNST: ASYMPTOTIC SOLUTIONS

Low drainage resistance: $\lambda \to 0$

$$\boxed{R = \sqrt{Q/(\pi N)}} \quad (15)$$

Radius of influence!

$$h_1 = \frac{N}{4T}(R^2 - r^2) + \frac{Q}{2\pi T}\ln\frac{r}{R} \quad (16)$$

$$h_2 = 0 \quad (17)$$

High drainage resistance: $\lambda \to \infty$

$$R = 0 \quad (18)$$

$$\cancel{h_1 = -\infty} \quad (19)$$

$$h_2 = Nc - \frac{Q}{2\pi T}\mathrm{K}_0\left(\frac{r}{\lambda}\right) \quad (20)$$

$Q = 0$

Initial head h_0 is equal to Nc

Applying definition of drawdown s to (16):

$$s = -h_1 = \frac{Q}{2\pi T}\ln\frac{R}{r} - \frac{N}{4T}(R^2 - r^2)$$

= Thiem + circular infiltration pond

Applying definition of drawdown s to (20):

$$s = Nc - h_2 = \frac{Q}{2\pi T}\mathrm{K}_0\left(\frac{r}{\lambda}\right)$$

= de Glee

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ERNST VS THIEM VS DE GLEE

Relatively low resistance:

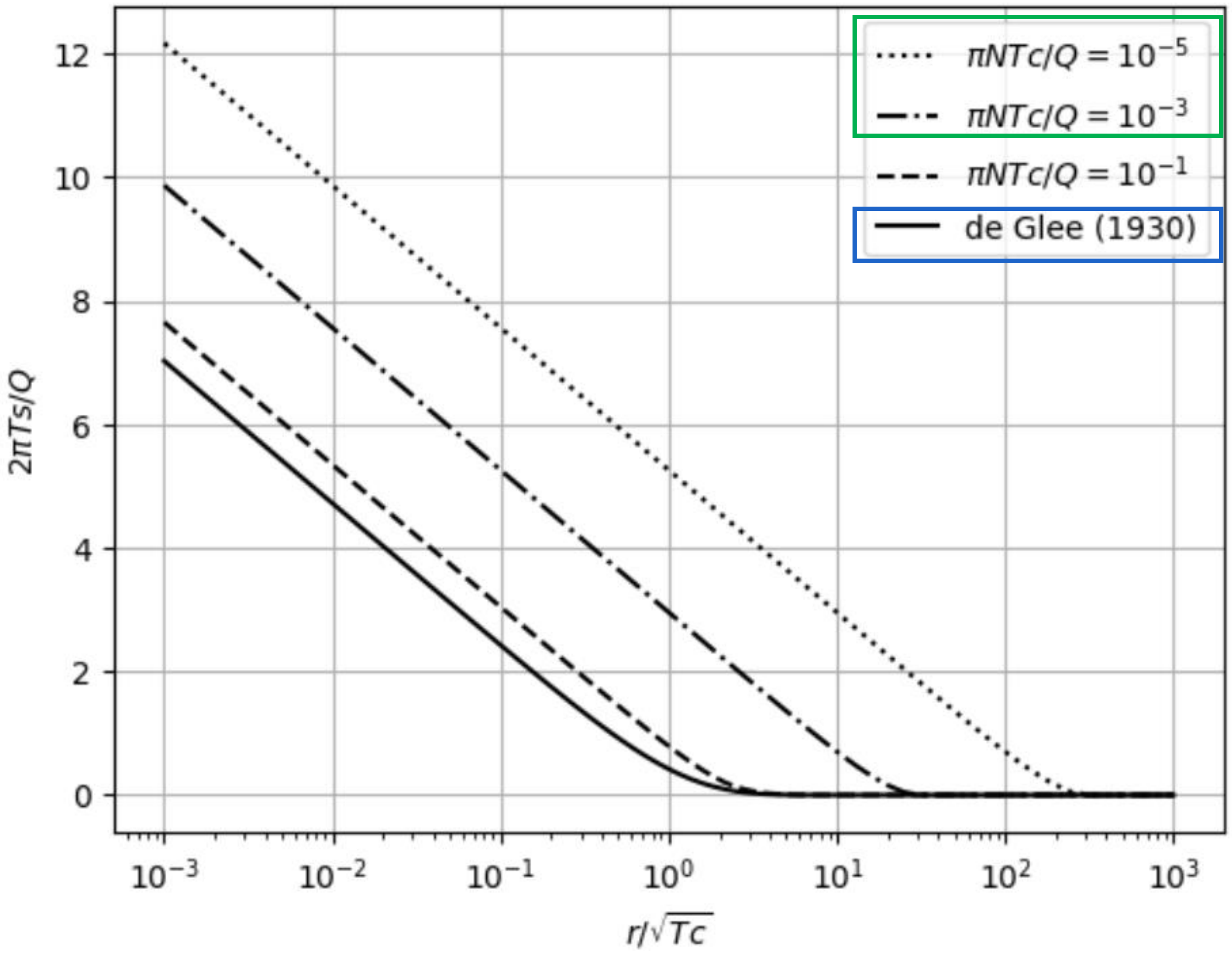
$$Q > 100\pi N\lambda^2$$

Ernst → Thiem + Infiltration pond

Relatively high resistance:

$$Q < \pi N\lambda^2$$

Ernst → de Glee



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THE RADIUS OF INFLUENCE MYTH

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THE RADIUS OF INFLUENCE

= the radial distance from the pumping well where there is no lowering of the head or beyond which drawdown is negligible

- Thiem-Dupuit equations: outer boundary
- How to determine?
  - empirical formulas
  - models defining more realistic boundary conditions

Author	Reference	Formula Radius of Influence
Lembke	(1886, 1887)	$R = h_o \times \sqrt{\frac{K}{2N}}$
Weber	[Kyrieleis-Sichardt,1930]	$R = 3 \times \sqrt{\frac{h_o \times K \times t}{n_e}}$
Kusakin	Chertusov, 1949	$R = 575 \times s_w \times \sqrt{K \times h_o}$
Kusakin	Aravin and Numerov, 1953	$R = 1{,}9 \times \sqrt{\frac{h_o \times K \times t}{n_e}}$
Sichardt	[Kirieleis-Sichardt, 1930]	$R = 3000 \times s_w \times \sqrt{K}$

https://hatariwater.tumblr.com/post/138690184404/overview-of-the-radius-of-influence

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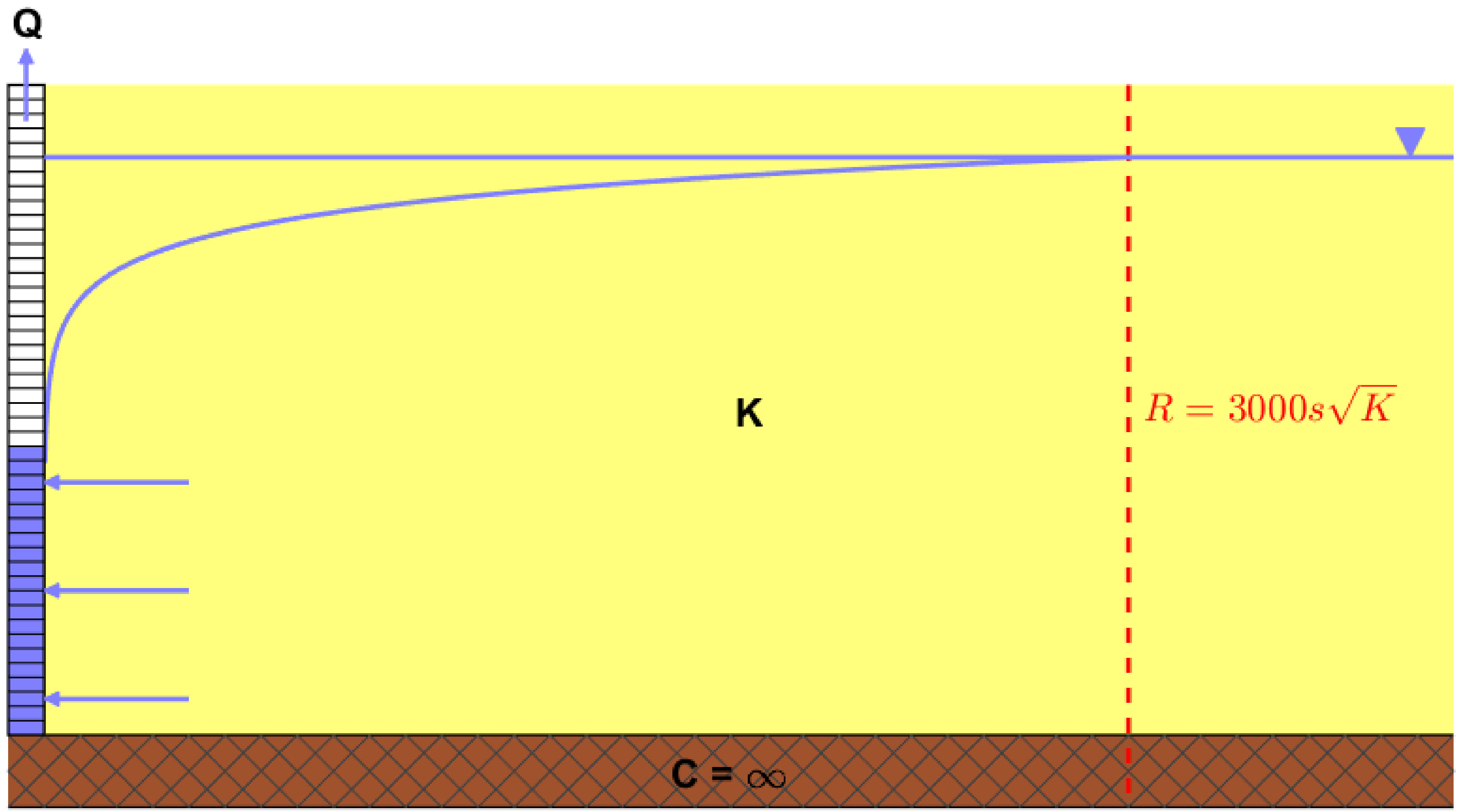
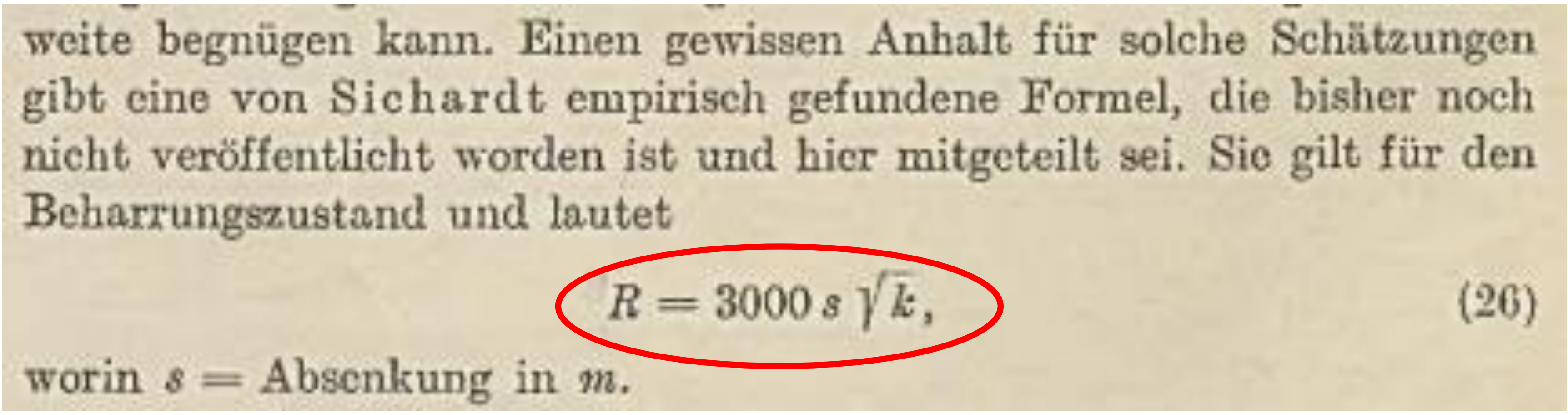
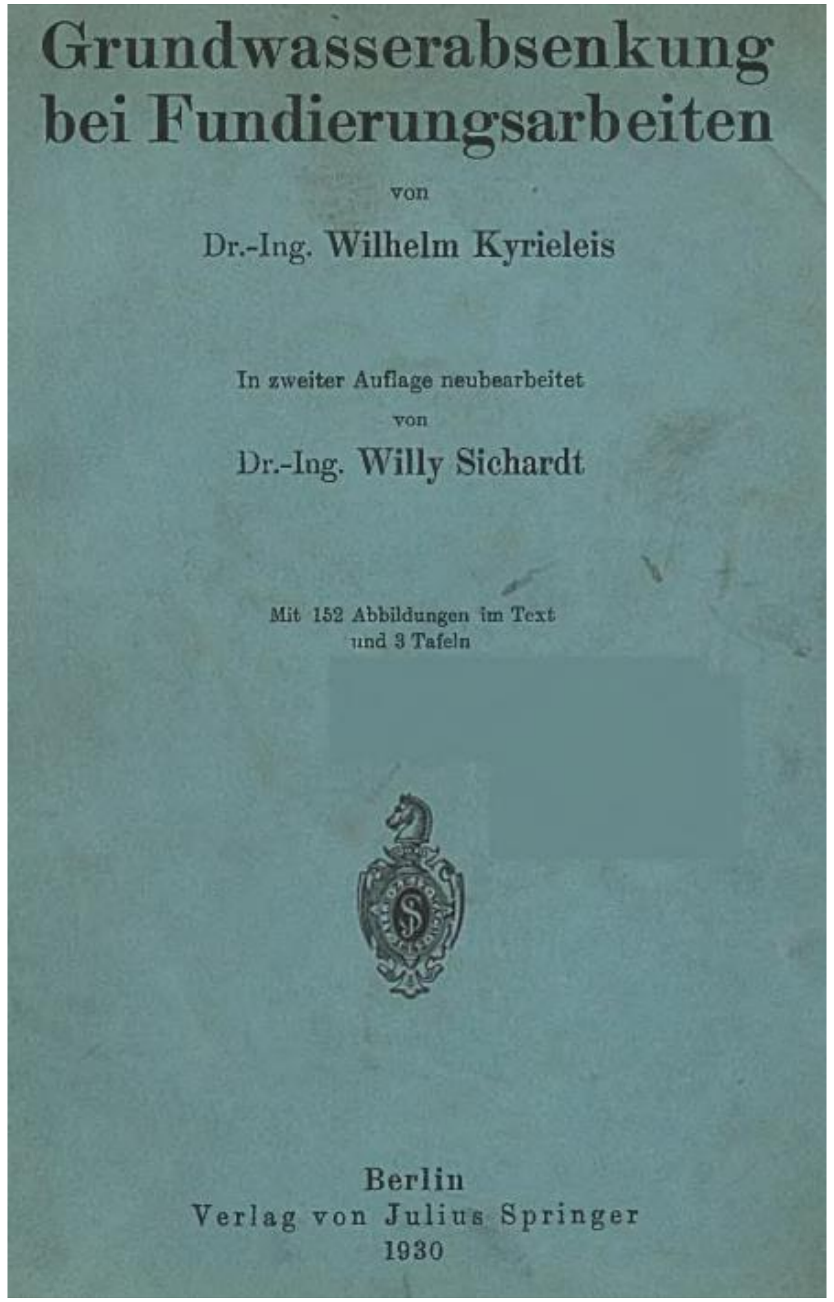
EMPIRICAL SICHARDT FORMULA




weite begnügen kann. Einen gewissen Anhalt für solche Schätzungen gibt eine von Sichardt empirisch gefundene Formel, die bisher noch nicht veröffentlicht worden ist und hier mitgeteilt sei. Sie gilt für den Beharrungszustand und lautet

$$R = 3000\, s \sqrt{k}, \tag{26}$$

worin s = Absenkung in m.



FORWARD AND INVERSE PROBLEMS

- **forward problem**
  - simulate head *h* or drawdown *s*
  - e.g. assessing the environmental impact of extractions

$$s = \frac{Q}{2\pi KD} \ln\left(\frac{R}{r}\right)$$

- **inverse problem type I**
  - derive transmissivity *KD*
  - e.g. pumping test interpretation

$$KD = \frac{Q}{2\pi} \frac{\ln r_2 - \ln r_1}{s_1 - s_2}$$

- **inverse problem type II**
  - derive pumping rate *Q*
  - e.g. construction dewatering

$$Q = 2\pi KD \frac{s_w}{\ln R - \ln r_w}$$

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DIFFERENT PERSPECTIVES

– **Well** performance and efficiency

– Hydraulic characteristics of **aquifers**

– The groundwater **basin** as part of the hydrological system

– Groundwater **sustainability** which also considers water quality, ecological and socio-economic aspects.

THE RADIUS OF INFLUENCE MYTH

Applying the Sichardt formula:

- is inconsistent with fundamental hydrogeological principles
- may underestimate the extent of the cone of depression
- is not recommended to assess the impact of extractions

water MDPI

Article

The Radius of Influence Myth

by Andy Louwyck [1,*], Alexander Vandenbohede [2], Dirk Libbrecht [3], Marc Van Camp [1] and Kristine Walraevens [1]

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Water **2022**, *14*(2), 149; **https://doi.org/10.3390/w14020149**

ALTERNATIVES

1D axisymmetric models consistent with fundamental principles:

– **de Glee (1930)**: steady well-flow in a leaky aquifer

– **Theis (1935)**: transient well-flow in a confined aquifer

– **Hantush-Jacob (1955)**: transient well-flow in a leaky aquifer

– **Ernst (1971)**: steady well-flow in a phreatic aquifer subject to uniform infiltration and drainage

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OVERVIEW

water

Article

The Radius of Influence Myth

Table 1. Summary of the analytical models discussed in the paper applied to simulate axisymmetric flow towards a fully penetrating well with infinitesimal radius and constant pumping rate in a homogeneous aquifer with impervious base. From the solutions of these models, equations and rules of thumb are derived to estimate the radius of influence *R*, with *KD* the transmissivity, *c* the resistance, *S* the storage coefficient, *N* the infiltration flux, *Q* the pumping rate, and *t* the time. See text for explanation and definitions.

Model	Flow Regime	Outer Boundary	Upper Boundary	Initial Flow	Super-Position	Radius of Influence R
Dupuit [76]	Steady	Finite	Water table	None	No [4]	Outer boundary (=input parameter)
Thiem [70]	Steady	Finite	Impervious [1]	Steady	Yes	Outer boundary (=input parameter)
de Glee [71,86]	Steady	Infinite	Leaky [2]	Steady	Yes	$R = 4\sqrt{cKD}$
Theis [72]	Transient	Infinite	Impervious [1]	Steady	Yes	$R = 1.5\sqrt{\frac{tKD}{S}}$
Hantush-Jacob [73]	Transient	Infinite	Leaky [2]	Steady	Yes	$R = 1.5\sqrt{\frac{tKD}{S}}$ if $t < 0.01Sc$ $R = 4\sqrt{cKD}$ if $t > 10Sc$
Ernst [74]	Steady	Infinite	Drainage + Recharge	None [3]	No [4]	$R = \sqrt{\frac{Q}{\pi N}}$ if $\frac{Q}{\pi NKDc} > 100$ $R = 4\sqrt{cKD}$ if $\frac{Q}{\pi NKDc} < 1$
Transient Ernst (Appendix A)	Transient	Infinite	Drainage + Recharge	None [3]	No [4]	See Figure 5

[1] Or water table if drawdown is less than 10% of initial saturated thickness. [2] Leakage through incompressible aquitard or linear surface water interaction (cfr. MODFLOW river). [3] Initial heads equal to *Nc* are relative to the steady drainage levels, which are set to zero for convenience. [4] Unless the solution may be approximated by its corresponding linear equation.

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THE WATER BUDGET MYTH AND THE SUPERPOSITION PRINCIPLE

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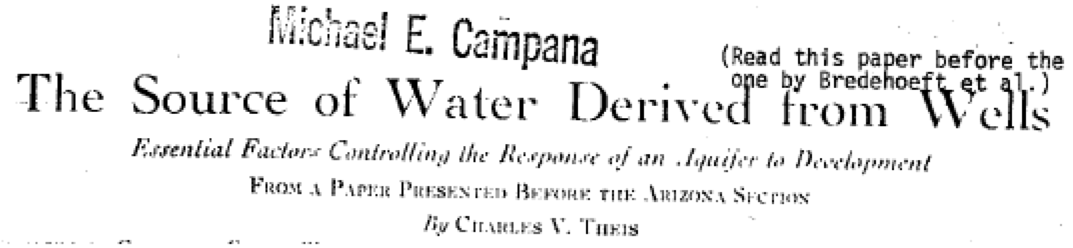
THE WATER BUDGET MYTH

– **Safe yield:** $\cancel{Q = R_0}$

– **Capture equation:** $Q = \Delta R - \Delta D - \frac{dV}{dt}$

(Theis, 1940; Bredehoeft el al., 1982; Bredehoeft, 2002)


Michael E. Campana (Read this paper before the one by Bredehoeft et al.)

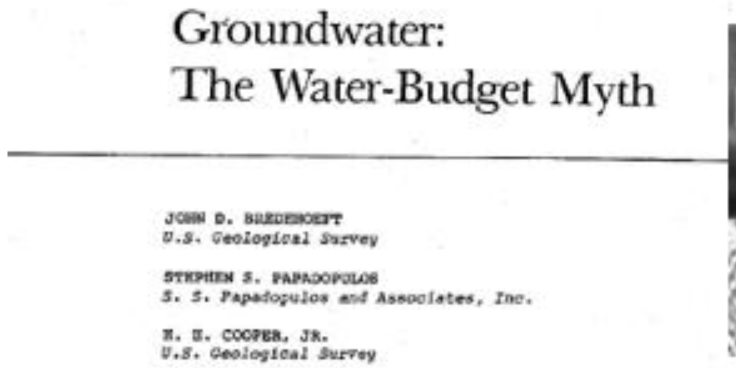
The Source of Water Derived from Wells

Essential Factors Controlling the Response of an Aquifer to Development

From a Paper Presented Before the Arizona Section

By Charles V. Theis

Bredehoeft, J.D., S.S. Papadopulos and H.H. Cooper. 1982. Groundwater: the Water-Budget Myth. In *Scientific Basis of Water-Resource Management*, Studies in Geophysics, Washington, DC: National Academy Press, pp. 51-57.

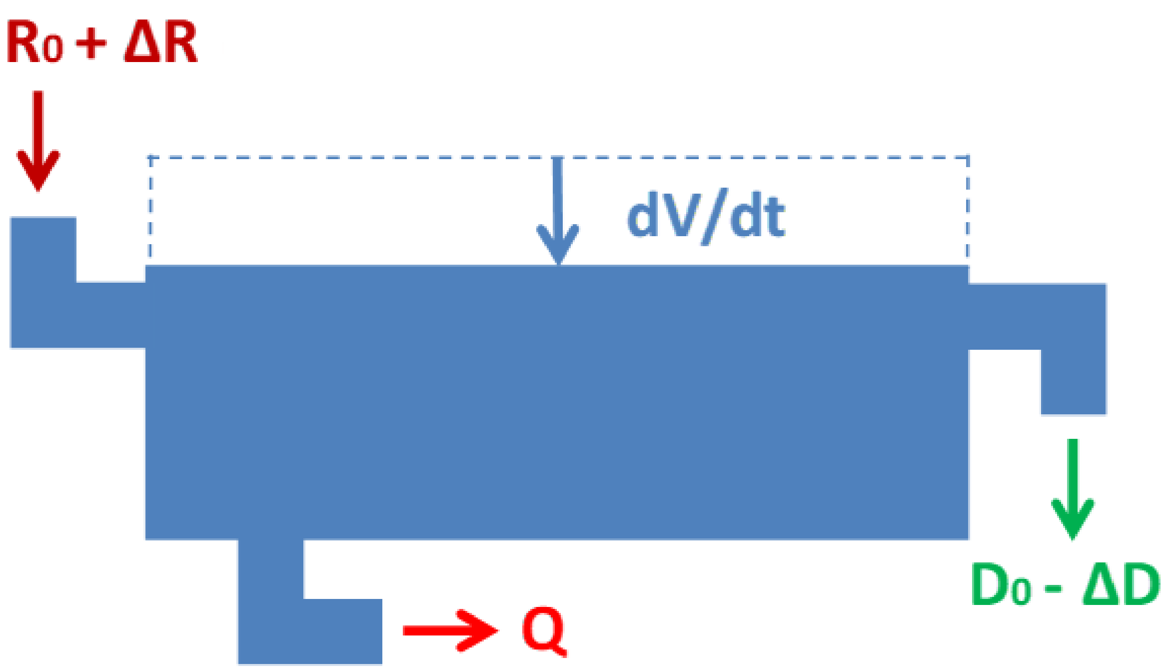

Groundwater:
The Water-Budget Myth

JOHN D. BREDEHOEFT
U.S. Geological Survey

STEPHEN S. PAPADOPULOS
S. S. Papadopulos and Associates, Inc.

H. H. COOPER, JR.
U.S. Geological Survey


John D. Bredehoeft



DIFFERENT PERSPECTIVES

- **Sustainable pumping:**

  = the well does not go dry

  only considers well performance

- **Sustainability:**

  = much broader concept

  also considers water quality,

  socio-economic and

  ecological aspects

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LINEAR VS NONLINEAR MODELS

- Linear model: $Q = \Delta R - \Delta D - \frac{dV}{dt}$
  - superposition: recharge is canceled out
  - implicit assumption of infinite sources of water
  - e.g.: Thiem, de Glee, Theis, Hantush-Jacob
- Nonlinear model: $Q = [R_t - R_0] - [D_t - D_0] - \frac{dV}{dt}$
  - initial conditions are relevant
  - so is recharge!

  - Ernst model: $Q = [D_t - D_0]$

    $= \pi R^2 N$

groundwater NGWA The Groundwater Association

Research Paper/

The Water Budget Myth and Its Recharge Controversy: Linear vs. Nonlinear Models

Andy Louwyck, Alexander Vandenbohede, Griet Heuvelmans, Marc Van Camp, Kristine Walraevens

First published: 15 August 2022 | https://doi.org/10.1111/gwat.13245 | Citations: 1

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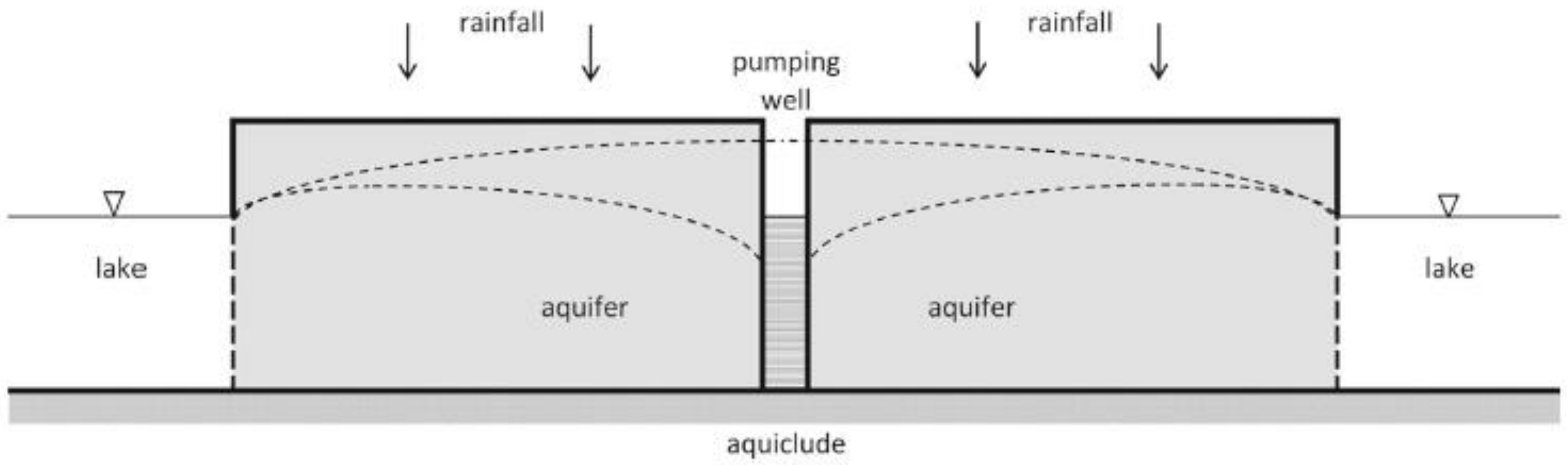
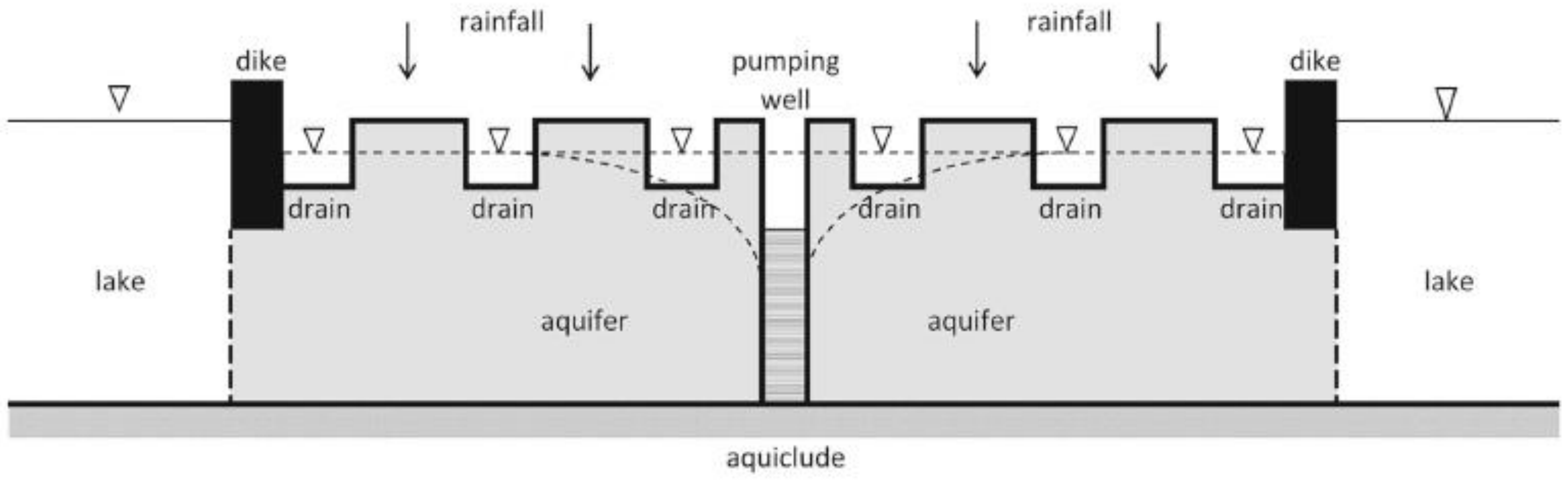
BREDEHOEFT'S VS POLDER ISLAND





Thiem formula

Ernst model

THE SUPERPOSITION PRINCIPLE

- Property of linear models:

$$s = \sum_i Q_i \sigma_i \quad \text{with } \sigma_i \text{ drawdown according to unit discharge}$$

- Linear model:
  - linear differential equation
  - linear boundary conditions
  - homogeneous differential equation (mostly)
    1. model before pumping: $\nabla^2 h_0 = -N$
    2. model during pumping: $\nabla^2 h = -N$
    3. drawdown model: $\nabla^2 s = \nabla^2 h_0 - \nabla^2 h = 0$

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= traditional approach to groundwater modeling

SUPERPOSITION IN TIME: EXAMPLE

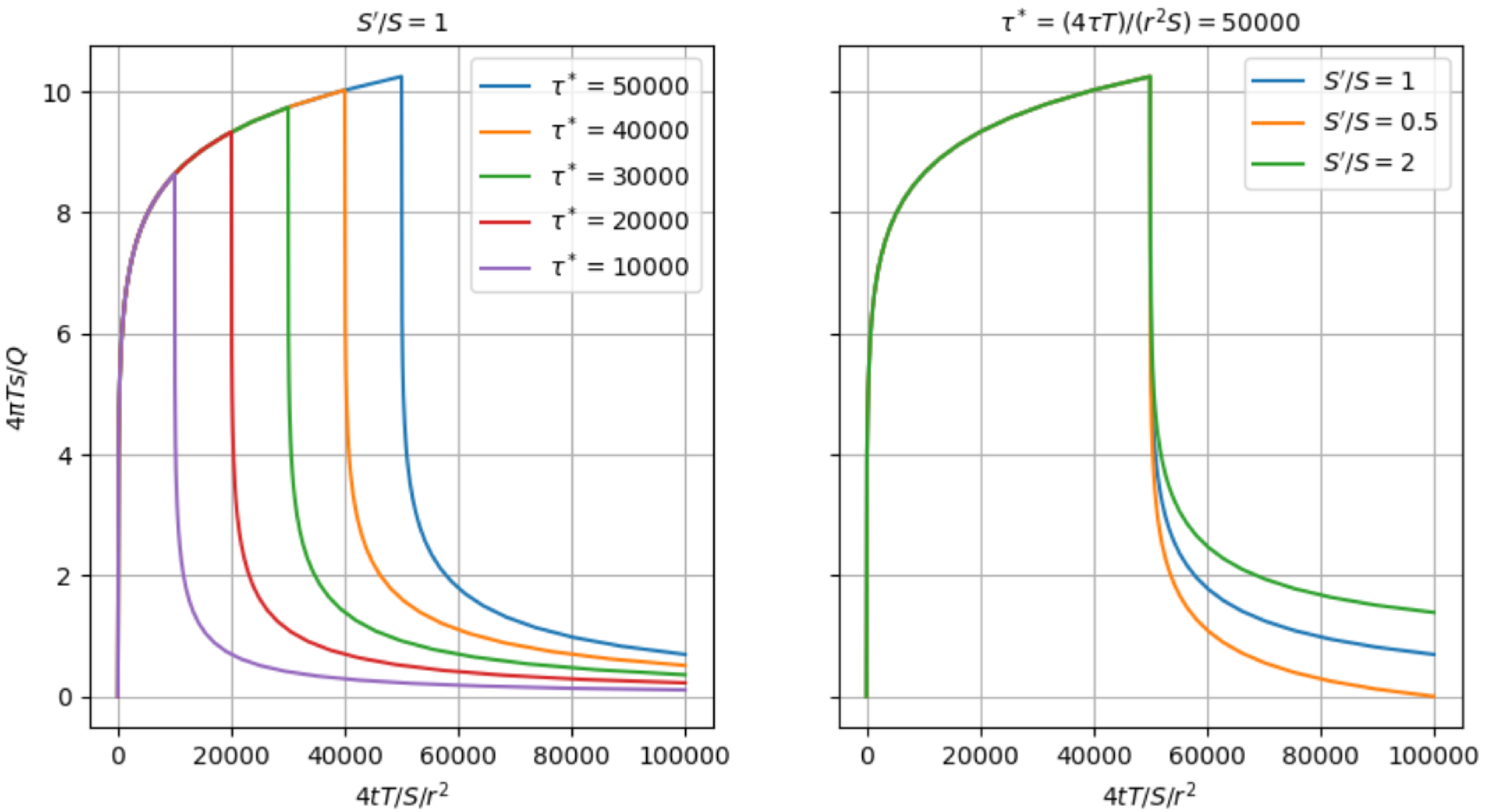
residual drawdown *s'*

during recovery (Theis, 1935):

- pump shuts down at time τ

  = start of injection -*Q*

- storativity S' during recovery



$$s'(r,t) = \frac{Q}{4\pi T}\left[\boxed{W\left(\frac{r^2 S}{4tT}\right)} \boxed{- W\left(\frac{r^2 S'}{4\Delta t T}\right)}\right] \quad \text{with } \Delta t = t - \tau$$

pumping + injection

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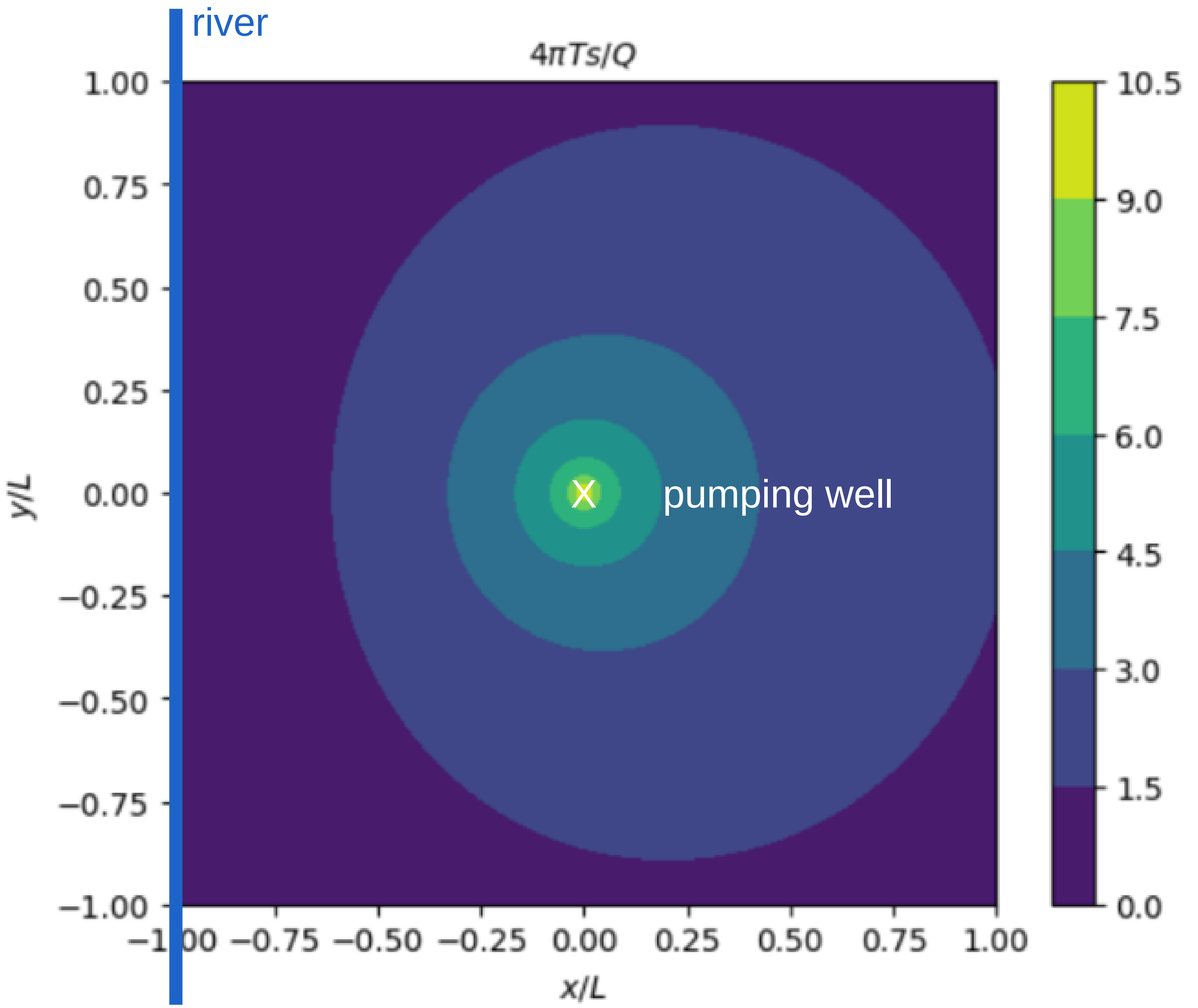
SUPERPOSITION IN SPACE: EXAMPLE

pumping well near straight constant-level river (Theis, 1941):

- well at position (0,0)
- straight river: $x = -L$
  = constant-head boundary
- **method of images:**
  add injection well at position ($-2L$, 0)

injection well X



$$s(x,y,t) = \frac{Q}{4\pi T}\left[\boxed{\mathrm{W}\left(\frac{[x^2+y^2]S}{4tT}\right)}\boxed{-\mathrm{W}\left(\frac{[(x+2L)^2+y^2]S}{4tT}\right)}\right]$$

pumping well + imaginary injection well

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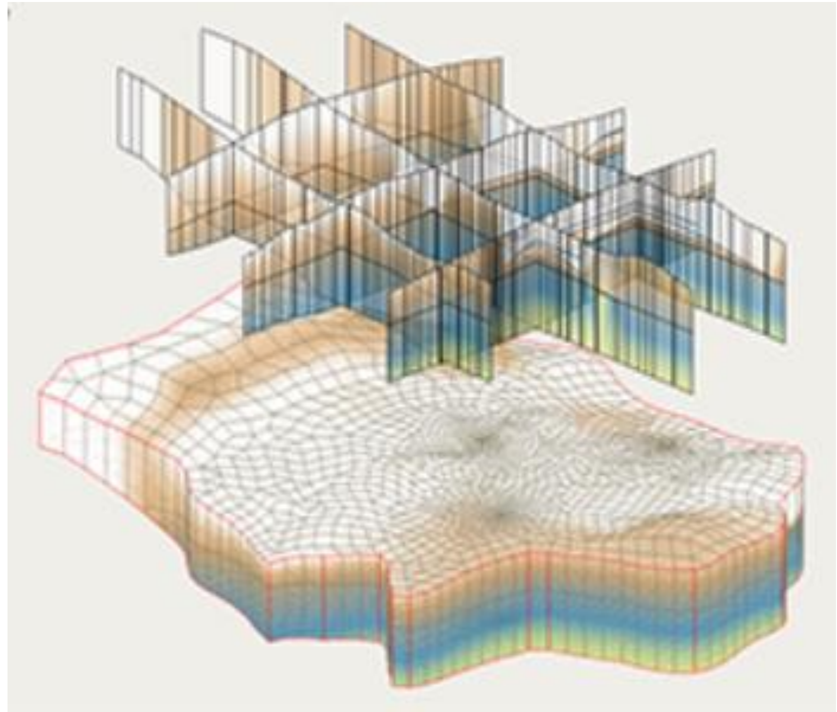
CONCLUSIONS

– assessing sustainability and impact of extractions requires **advanced numerical modeling**



– **analytical models** may be useful:
  – time and budget constraints
  – lack of data
  – *they offer insight!*

groundwater NGWA The Groundwater Association

The Role of Hand Calculations in Ground Water Flow Modeling

Henk Haitjema

First published: 08 March 2006 | https://doi.org/10.1111/j.1745-6584.2006.00189.x | Citations: 66

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MORE ADVANCED AXISYMMETRIC MODELS

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EVOLUTION OF AXISYMMETRIC MODELS

- **1 layer**
- incompressible aquitards
- well:
  - fully penetrating (mostly)
  - infinitesimal radius (mostly)

1856	Darcy
1857 & 1863	Dupuit
1870	A. Thiem
1906	G. Thiem
1914	Kooper
1930	de Glee
1935	Theis
1946	Jacob
1955	Hantush & Jacob

EVOLUTION OF AXISYMMETRIC MODELS

- **1, 2 or 3 layers**
- compressible aquitards
- anisotropy
- well:
  - partially penetrating
  - multi-aquifer
  - finite diameter (wellbore storage)
  - instantaneous head change (slug test)
  - finite-thickness skin
- water table conditions:
  - delayed yield
  - infiltration and drainage
  - confined-unconfined flow

1951	Huisman & Kemperman
1954 & 1963	Boulton
1964 & 1967	Hantush
1966	Papadopulos
1967	Papadopulos & Cooper
1967	Cooper et al.
1969	Neuman & Witherspoon
1971	Ernst
1972	Moench & Prickett
1972	Bruggeman
1972 & 1974	Neuman
1983	Javandel & Witherspoon
1984	Moench
1984	Wikramaratna
1988	Butler
1994	Hyder et al.
1995 & 1996	Moench
2012	Mishra et al.
2022	Louwyck et al.

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TWO EXAMPLES

Extensions/modifications of the Theis (1935) model:

- **Cooper et al. (1967):**

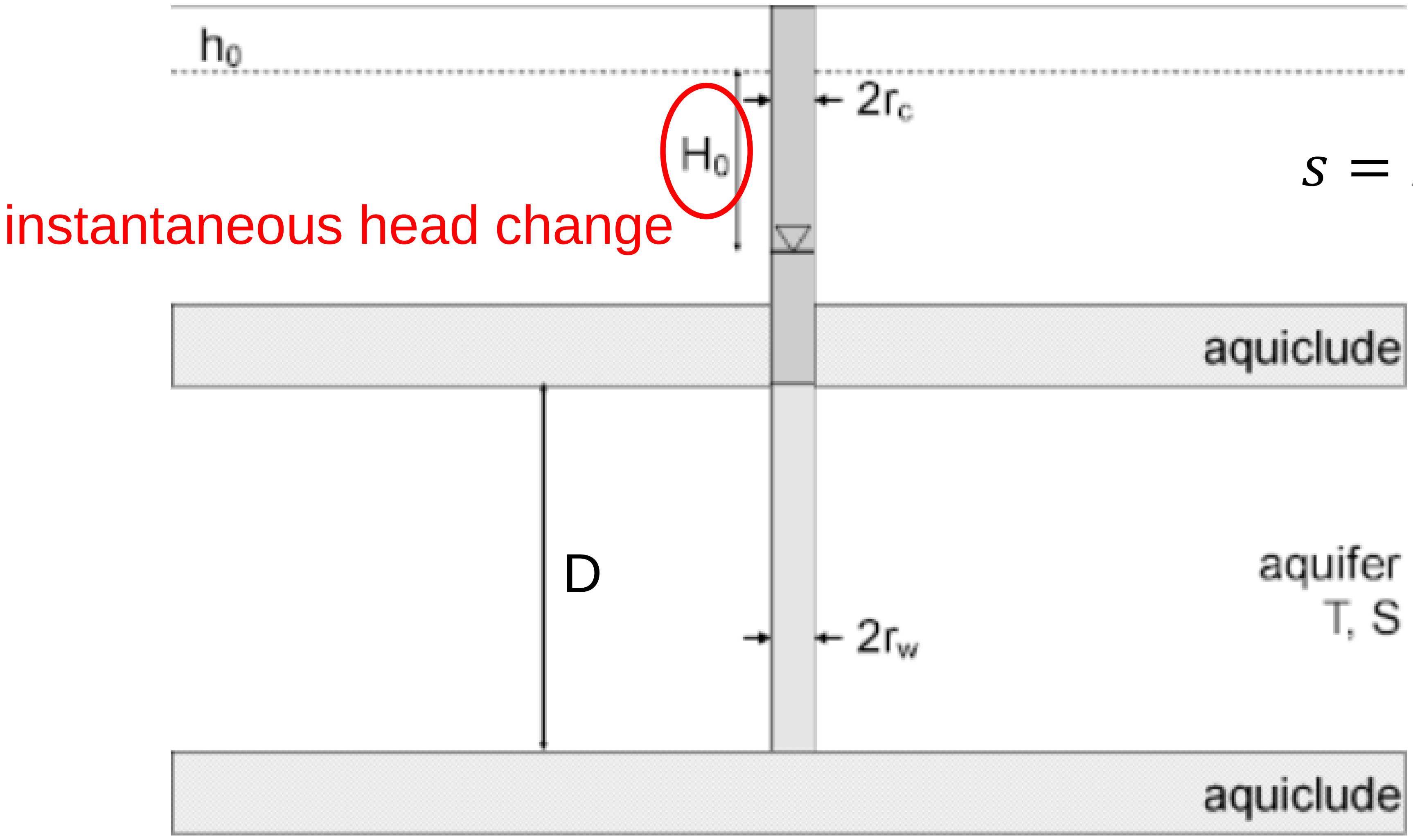
  Instantaneous head change inside the well

  (= slug test)

- **Butler (1988):**

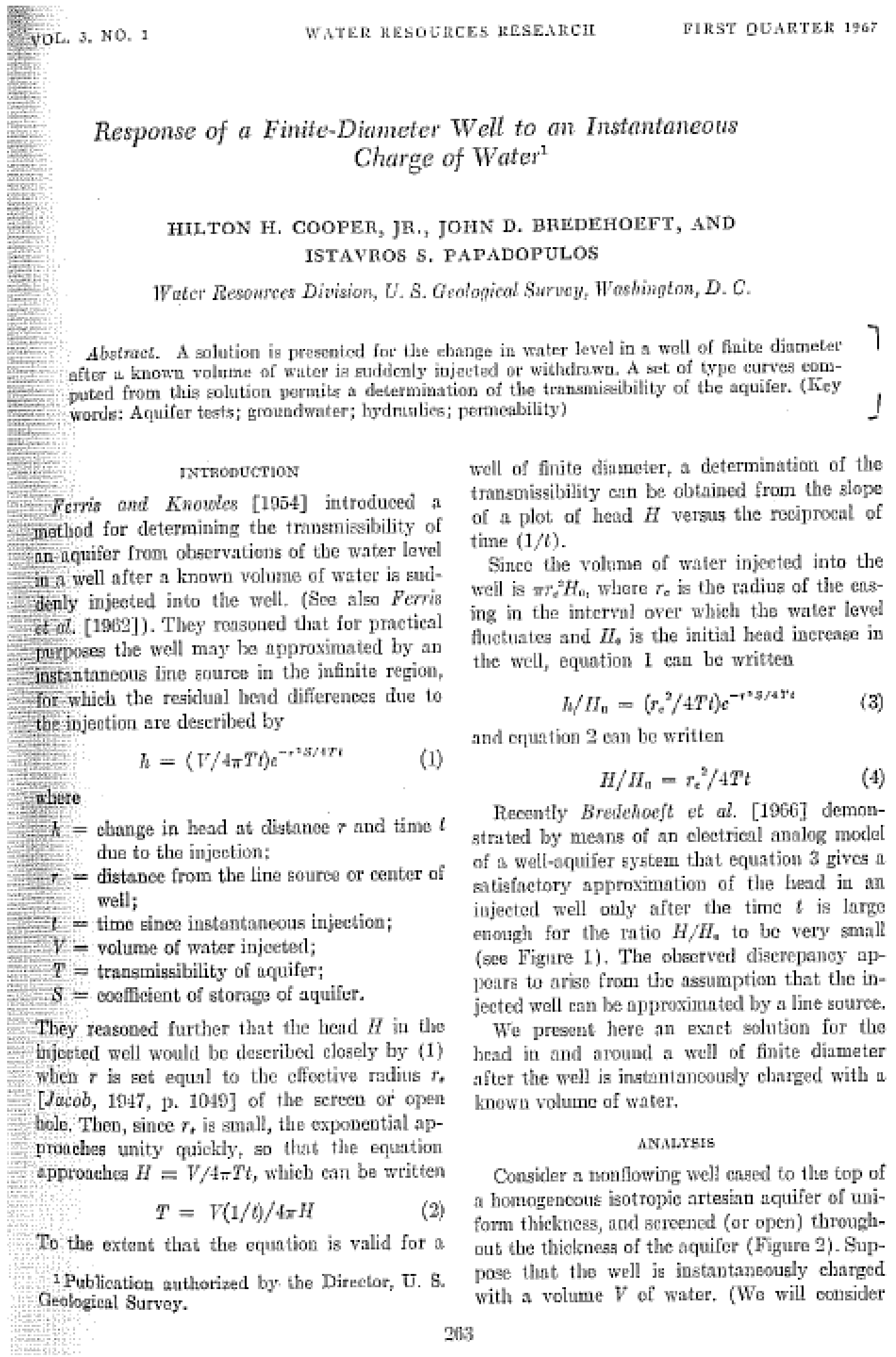
  Pumping well with finite-thickness skin

COOPER ET AL. MODEL



$$s = h_0 - h$$

Slug test in confined aquifer (Cooper et al., 1967)


VOL. 3, NO. 1 WATER RESOURCES RESEARCH FIRST QUARTER 1967

Response of a Finite-Diameter Well to an Instantaneous Charge of Water[1]

HILTON H. COOPER, JR., JOHN D. BREDEHOEFT, AND ISTAVROS S. PAPADOPULOS

Water Resources Division, U. S. Geological Survey, Washington, D. C.

Abstract. A solution is presented for the change in water level in a well of finite diameter after a known volume of water is suddenly injected or withdrawn. A set of type curves computed from this solution permits a determination of the transmissibility of the aquifer. (Key words: Aquifer tests; groundwater; hydraulics; permeability)

INTRODUCTION

Ferris and Knowles [1954] introduced a method for determining the transmissibility of an aquifer from observations of the water level in a well after a known volume of water is suddenly injected into the well. (See also *Ferris et al.* [1962]). They reasoned that for practical purposes the well may be approximated by an instantaneous line source in the infinite region, for which the residual head differences due to the injection are described by

$$h = (V/4\pi Tt)e^{-r^2S/4Tt} \quad (1)$$

where

h = change in head at distance r and time t due to the injection;
r = distance from the line source or center of well;
t = time since instantaneous injection;
V = volume of water injected;
T = transmissibility of aquifer;
S = coefficient of storage of aquifer.

They reasoned further that the head H in the injected well would be described closely by (1) when r is set equal to the effective radius r_s [*Jacob*, 1947, p. 1049] of the screen or open hole. Then, since r_s is small, the exponential approaches unity quickly, so that the equation approaches $H = V/4\pi Tt$, which can be written

$$T = V(1/t)/4\pi H \quad (2)$$

To the extent that the equation is valid for a well of finite diameter, a determination of the transmissibility can be obtained from the slope of a plot of head H versus the reciprocal of time $(1/t)$.

Since the volume of water injected into the well is $\pi r_c^2 H_0$, where r_c is the radius of the casing in the interval over which the water level fluctuates and H_0 is the initial head increase in the well, equation 1 can be written

$$h/H_0 = (r_c^2/4Tt)e^{-r^2S/4Tt} \quad (3)$$

and equation 2 can be written

$$H/H_0 = r_c^2/4Tt \quad (4)$$

Recently *Bredehoeft et al.* [1966] demonstrated by means of an electrical analog model of a well-aquifer system that equation 3 gives a satisfactory approximation of the head in an injected well only after the time t is large enough for the ratio H/H_0 to be very small (see Figure 1). The observed discrepancy appears to arise from the assumption that the injected well can be approximated by a line source.

We present here an exact solution for the head in and around a well of finite diameter after the well is instantaneously charged with a known volume of water.

ANALYSIS

Consider a nonflowing well cased to the top of a homogeneous isotropic artesian aquifer of uniform thickness, and screened (or open) throughout the thickness of the aquifer (Figure 2). Suppose that the well is instantaneously charged with a volume V of water. (We will consider

[1] Publication authorized by the Director, U. S. Geological Survey.

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$$\bar{s}(r,p) = \frac{H_0 r_w S K_0(r\omega)}{T\omega[r_w\omega K_0(r_w\omega) + 2\alpha K_1(r_w\omega)]} \qquad \text{with } \alpha = \frac{r_w^2}{r_c^2}S$$

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COOPER ET AL.: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Transient-state
  - Strictly horizontal
- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded
- Well:
  - Fully penetrating
  - **Instantaneous initial head change**
  - **Finite radius → wellbore storage!**

COOPER ET AL.: PROBLEM STATEMENT

Continuity of transient 1D confined flow:

$$T\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right) = \boxed{S\frac{\partial s}{\partial t}}$$

storage change (1)

Partial differential equation (PDE):

- drawdown *s* is function of *r* and *t*
- apply Laplace transform w.r.t. *t*

Inner boundary condition at zero:

wellbore storage

$$2\pi r T\frac{\partial s}{\partial r} = \boxed{\pi r_c^2\frac{dH}{\partial t}} \qquad (r = r_w)$$

with H the head change in the well

(2)

Outer boundary condition at infinity:

$$s(\infty, t) = 0$$

(3)

Initial condition at *t* = 0:

$$s(r,0) = \begin{cases} 0 & (r > r_w) \\ \boxed{H_0 \quad (r = r_w)} \end{cases}$$

instantaneous head change in the well (4)

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COOPER ET AL.: LAPLACE TRANSFORM

Laplace transform of PDE (1):

$$\frac{d^2\bar{s}}{dr^2} + \frac{1}{r}\frac{d\bar{s}}{dr} = \frac{S}{T}p\bar{s}$$

(5)

Laplace transform of inner BC (2):

initial condition!

$$2\pi rT\frac{d\bar{s}}{dr} = \pi r_c^2(p\bar{H} - \boxed{H_0})$$

(6)

Laplace transform of outer BC (3):

$$\bar{s}(\infty, p) = 0$$

(7)

Modified Bessel differential equation:

$$\frac{d^2\bar{s}}{dr^2} + \frac{1}{r}\frac{d\bar{s}}{dr} = a\bar{s}$$

General solution:

$$\bar{s} = \alpha \mathrm{I}_0(r\sqrt{a}) + \beta \mathrm{K}_0(r\sqrt{a})$$

with:

- $\mathrm{I}_0, \mathrm{K}_0$: the zero order modified Bessel functions of the first and second kind, resp.
- α, β: integration constants

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COOPER ET AL.: LAPLACE SOLUTION

General solution of (5) after applying BC (7) which yields $\alpha = 0$:

$$\bar{s}(r,p) = \beta \mathrm{K}_0(r\omega)$$

$$\text{with } \omega = \sqrt{Sp/T} \tag{8}$$

Applying BC (6) with $\overline{H}(p) = \bar{s}(r_w, p)$

$$-2\pi r_w \omega T \beta \mathrm{K}_1(r_w\omega) = \pi r_c^2 [p\beta \mathrm{K}_0(r_w\omega) - H_0] \tag{9}$$

From (9) it follows that:

$$\beta = \frac{r_c^2 H_0}{r_c^2 p \mathrm{K}_0(r_w\omega) + 2 r_w \omega T \mathrm{K}_1(r_w\omega)} \tag{10}$$

Introducing (10) in (8) and rearranging:

$$\bar{s}(r,p) = \frac{H_0 r_w S \mathrm{K}_0(r\omega)}{T\omega[r_w \omega \mathrm{K}_0(r_w\omega) + 2\alpha \mathrm{K}_1(r_w\omega)]}$$

$$\text{with } \alpha = \frac{r_w^2}{r_c^2} S \tag{11}$$

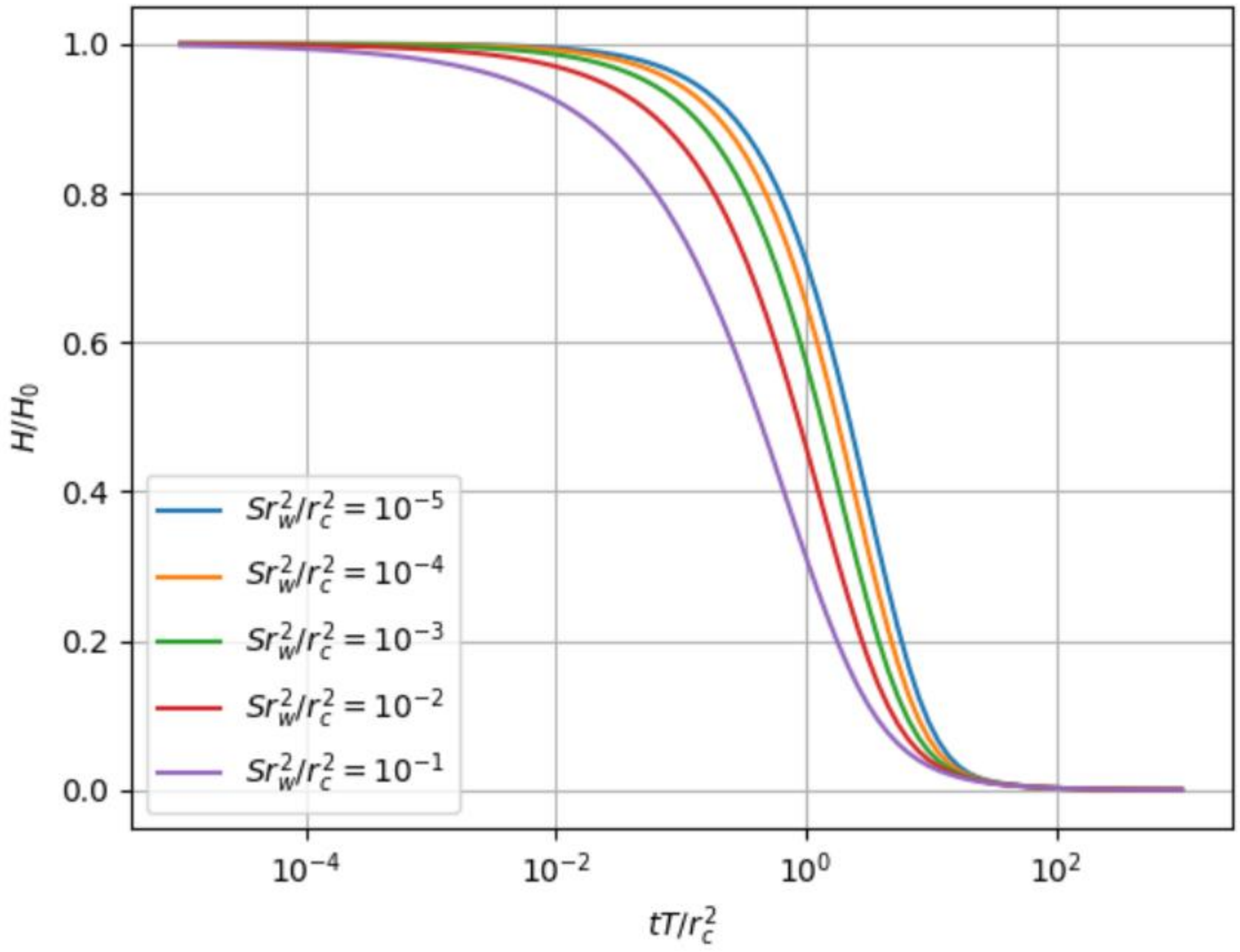
COOPER ET AL.: NUMERICAL INVERSION

Applying the Stehfest algorithm

to Laplace solution (11)

with $r = r_w$

as $H(t) = s(r_w, t)$



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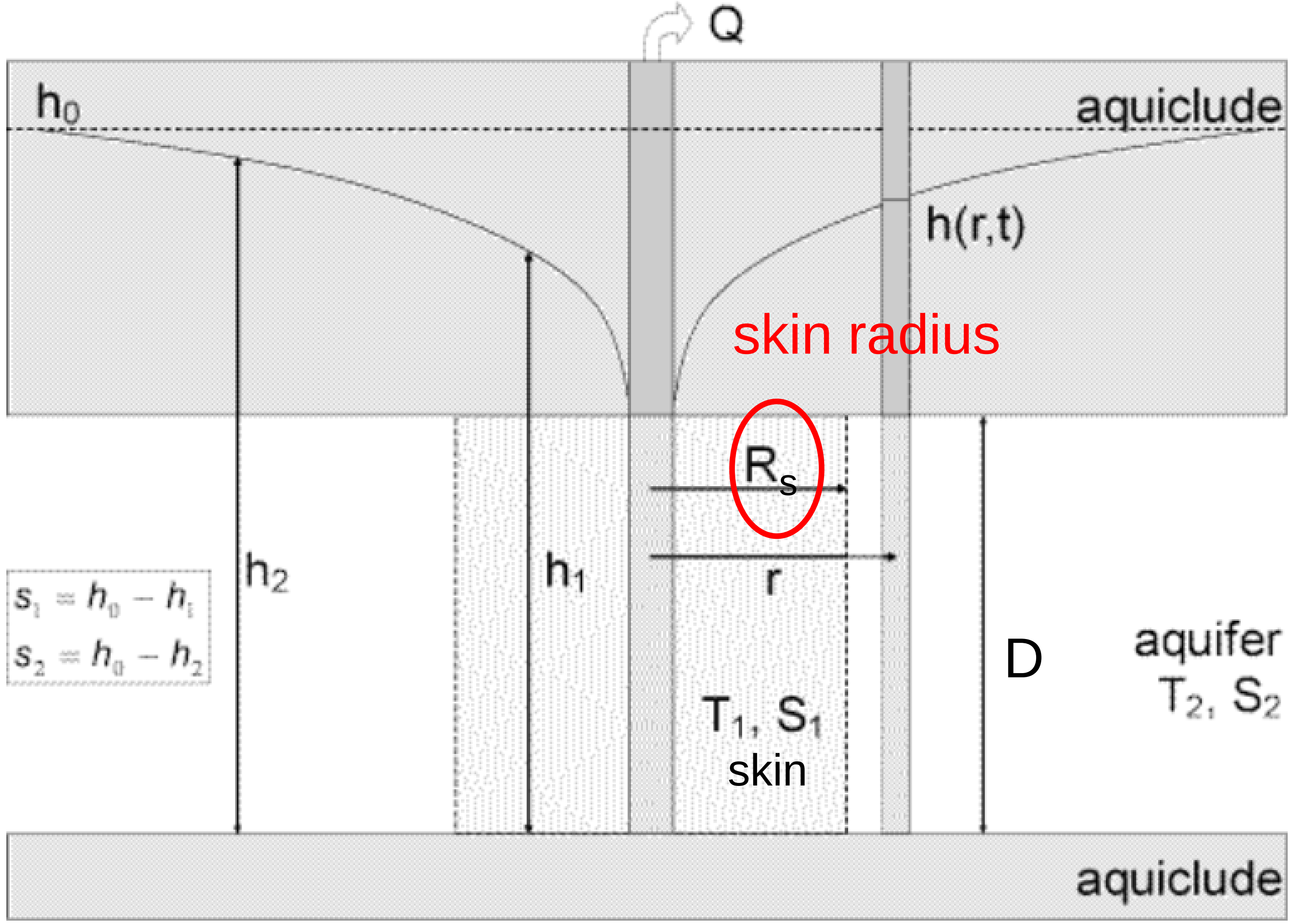
BUTLER MODEL



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[2]

PUMPING TESTS IN NONUNIFORM AQUIFERS — THE RADIALLY SYMMETRIC CASE

JAMES J. BUTLER, Jr.

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(Received May 22, 1987; revised and accepted December 12, 1987)

ABSTRACT

Butler, J.J., Jr., 1988. Pumping tests in nonuniform aquifers – The radially symmetric case. J. Hydrol., 101: 15–30.

Traditionally, pumping-test-analysis methodology has been limited to applications involving aquifers whose properties are assumed uniform in space. This work attempts to assess the applicability of analytical methodology to a broader class of units with spatially varying properties. An examination of flow behavior in a simple configuration consisting of pumping from the center of a circular disk embedded in a matrix of differing properties is the basis for this investigation. A solution describing flow in this configuration is obtained through Laplace-transform techniques using analytical and numerical inversion schemes. Approaches for the calculation of flow properties in conditions that can be roughly represented by this simple configuration are proposed. Possible applications include a wide variety of geologic structures, as well as the case of a well skin resulting from drilling or development. Of more importance than the specifics of these techniques for analysis of water-level responses is the insight into flow behavior during a pumping test that is provided by the large-time form of the derived solution. The solution reveals that drawdown during a pumping test can be considered to consist of two components that are dependent and independent of near-well properties, respectively. Such an interpretation of pumping-test drawdown allows some general conclusions to be drawn concerning the relationship between parameters calculated using analytical approaches based on curve-matching and those calculated using approaches based on the slope of a semilog straight line plot. The infinite-series truncation that underlies the semilog analytical approaches is shown to remove further contributions of near-well material to total drawdown. In addition, the semilog distance–drawdown approach is shown to yield an expression that is equivalent to the Thiem equation. These results allow some general recommendations to be made concerning observation-well placement for pumping tests in nonuniform aquifers. The relative diffusivity of material on either side of a discontinuity is shown to be the major factor in controlling flow behavior during the period in which the front of the cone of depression is moving across the discontinuity. Though resulting from an analysis of flow in an idealized configuration, the insights of this work into flow behavior during a pumping test are applicable to a wide class of nonuniform units.

INTRODUCTION

The pumping test has traditionally been the standard method used to evaluate the transmissive and storage properties of subsurface material for

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James J. Butler, Jr.

Pumping well with finite-thickness skin in confined aquifer (Butler, 1988)

$$s(r,t) \approx \begin{cases} \dfrac{Q}{2\pi T_2}\ln\dfrac{R}{R_s} + \dfrac{Q}{2\pi T_1}\ln\dfrac{R_s}{r} & (r \leq R_s) \\ \dfrac{Q}{2\pi T_2}\ln\dfrac{R}{r} & (r \geq R_s) \end{cases} \quad \text{with } R = \sqrt{\frac{4tT_2}{e^{\gamma}S_2}} \text{ and } t \to \infty$$

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BUTLER (1988) MODEL: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Transient-state
  - Strictly horizontal
- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded
- Well:
  - Fully penetrating
  - Constant pumping rate
  - Infinitesimal radius
  - **Finite-thickness skin**

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BUTLER: PROBLEM STATEMENT

R_s

Transient flow in the skin zone:

$$T_1\left(\frac{\partial^2 s_1}{\partial r^2}+\frac{1}{r}\frac{\partial s_1}{\partial r}\right)=S_1\frac{\partial s_1}{\partial t} \quad (1)$$

Inner boundary condition at zero:

$$Q=-\lim_{r\to 0}\left(2\pi r T_1\frac{\partial s_1}{\partial r}\right) \quad (2)$$

Proximal zone 1: $r \le R_s \rightarrow$ skin: T_1, S_1

Transient flow in the aquifer:

$$T_2\left(\frac{\partial^2 s_2}{\partial r^2}+\frac{1}{r}\frac{\partial s_2}{\partial r}\right)=S_2\frac{\partial s_2}{\partial t} \quad (3)$$

Outer boundary condition at infinity:

$$s_2(\infty)=0 \quad (4)$$

Distal zone 2: $r \ge R_s \rightarrow$ aquifer: T_2, S_2

Continuity of flow at the common boundary:

$$s_1(R_s,t)=s_2(R_s,t) \quad (5)$$

$$2\pi R_s T_1\frac{\partial s_1}{\partial r}=2\pi R_s T_2\frac{\partial s_2}{\partial r} \quad (6)$$

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BUTLER: LARGE-TIME APPROXIMATION

Pseudo-steady flow in the skin zone:

$$\frac{d^2 s_1}{dr^2} + \frac{1}{r}\frac{ds_1}{dr} \approx 0 \qquad (7)$$

Condition (2) is true for all distances *r* :

$$Q \approx -2\pi r T_1 \frac{ds_1}{dr} \qquad (8)$$

Proximal zone 1: $r \leq R_s$ → skin: T_1

R_s

Laplace transform of PDE (3):

$$\frac{d^2 \bar{s}_2}{dr^2} + \frac{1}{r}\frac{d\bar{s}_2}{dr} = \frac{S_2}{T_2} p \bar{s}_2 \qquad (9)$$

Laplace transform of BC (4):

$$\bar{s}_2(\infty) = 0 \qquad (10)$$

Distal zone 2: $r \geq R_s$ → aquifer: T_2, S_2

Laplace transform of conditions (5) and (6):

$$\bar{s}_1(R_s, p) = \bar{s}_2(R_s, p) \qquad (11)$$

$$2\pi R_s T_2 \frac{d\bar{s}_2}{dr} \approx \frac{-Q}{p} \qquad (12)$$

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BUTLER: LARGE-TIME APPROXIMATION

Solution of (7) subject to (8) and (5):

$$s_1(r,t) \approx s_2(R_s,t) + \frac{Q}{2\pi T_1}\ln\frac{R_s}{r} \tag{13}$$

General solution of (9) after applying BC (10) which yields $\alpha = 0$:

$$\bar{s}_2(r,p) = \beta \mathrm{K}_0(r\omega)$$

$$\text{with } \omega = \sqrt{S_2 p/T_2} \tag{14}$$

Applying BC (12):

$$2\pi R_s T_2 \beta \omega \mathrm{K}_1(R_s\omega) \approx \frac{Q}{p} \tag{15}$$

From (14) and (15):

$$\bar{s}_2(r,p) \approx \frac{Q}{2\pi T_2 p}\frac{\mathrm{K}_0(r\omega)}{\boxed{R_s\omega \mathrm{K}_1(R_s\omega)}} \tag{16}$$

If $t \to \infty$, then $p \to 0$ and $\boxed{R_s\omega \mathrm{K}_1(R_s\omega)} \to 1$:

$$\bar{s}_2(r,p) \approx \frac{Q}{2\pi T_2 p}\mathrm{K}_0(r\omega) \tag{17}$$

Analytically inverting (17):

$$s_2(r,t) \approx \frac{Q}{4\pi T_2}\mathrm{W}\left(\frac{r^2 S_2}{4tT_2}\right) \tag{18}$$

If $t \to \infty$, then (18) is approximated as:

$$s_2(r,t) \approx \frac{Q}{2\pi T_2}\ln\left(\frac{1}{r}\sqrt{\frac{4tT_2}{e^{\gamma} S_2}}\right) \tag{19}$$

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BUTLER: LARGE-TIME SOLUTION

Combining (13) and (19):

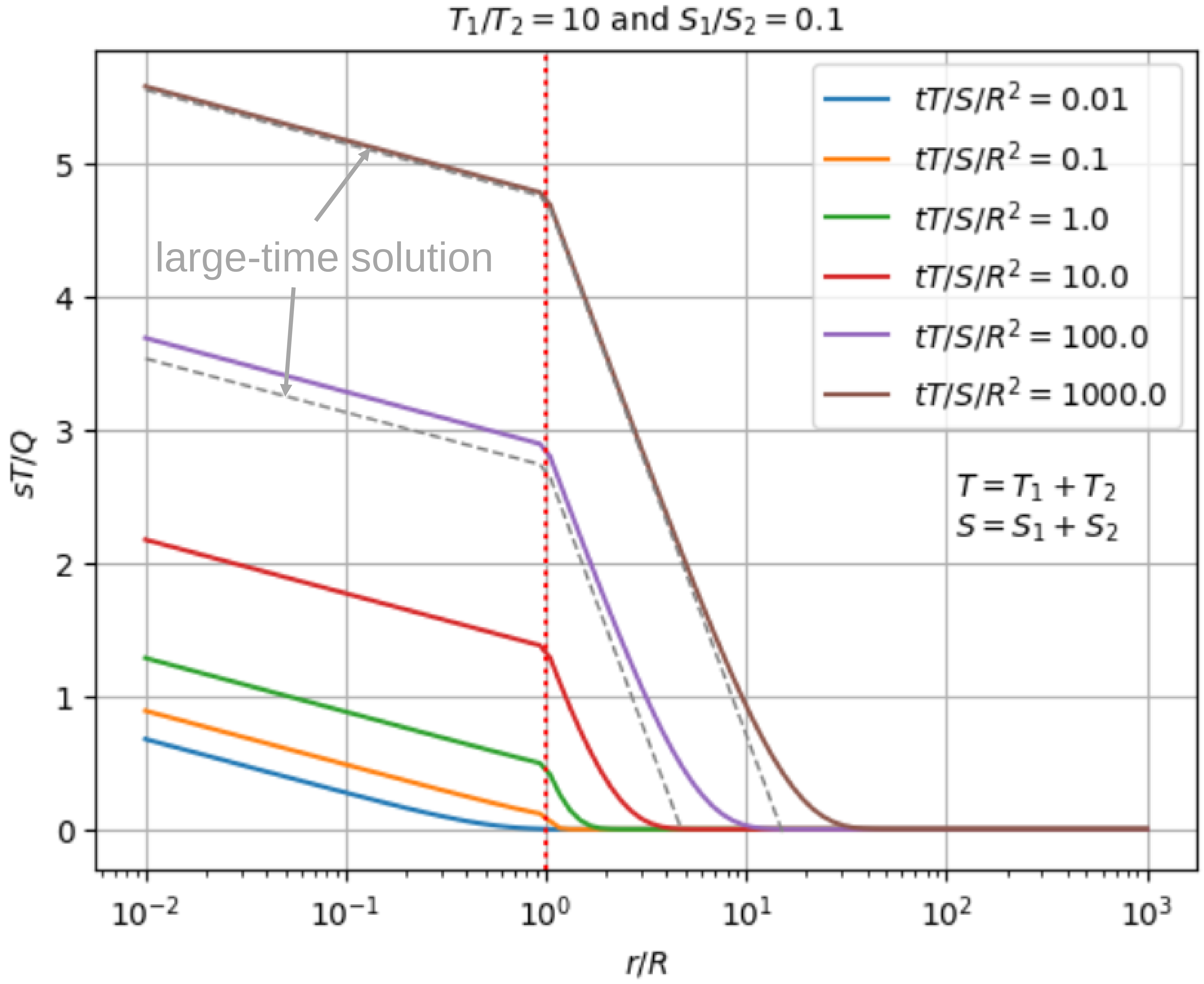
$$s_1(r,t) \approx \boxed{\frac{Q}{2\pi T_2}\ln\frac{R}{R_s}} + \boxed{\frac{Q}{2\pi T_1}\ln\frac{R_s}{r}}$$

Cooper-Jacob Thiem

$$s_2(r,t) \approx \boxed{\frac{Q}{2\pi T_2}\ln\frac{R}{r}}$$

with $\boxed{R = \sqrt{\frac{4tT_2}{e^{\gamma}S_2}}}$ (20)

radius of influence



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SKIN FACTOR

Definition of dimensionless skin factor F :

$$F = \frac{T_2}{T_1} \ln \frac{R_s}{r_w} \tag{21}$$

Drawdown s_w in pumping well:

$$s_w(t) = \boxed{s(R_s,t)} + \boxed{\frac{Q}{2\pi T_2} F} \tag{22}$$

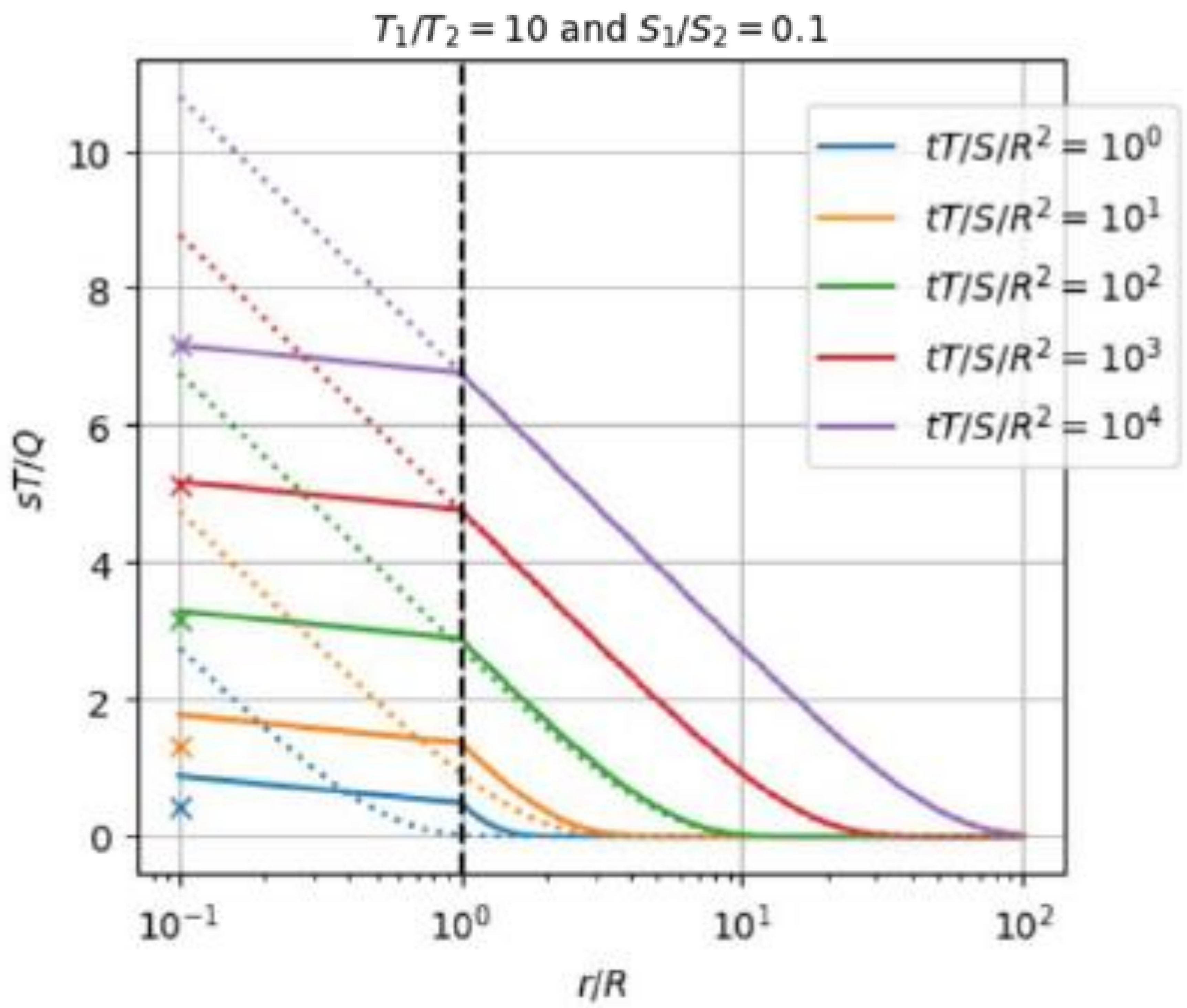
(22) corresponds to (20) as:

$$s_w(t) = s_1(r_w, t)$$

$$s(R_s,t) = \boxed{\frac{Q}{2\pi T_2} \ln \frac{R}{R_s}}$$ Cooper-Jacob

$$\frac{Q}{2\pi T_2} F = \boxed{\frac{Q}{2\pi T_1} \ln \frac{R_s}{r_w}}$$ Thiem

- solid lines: Butler (1988)
- dotted lines: Theis (1935)
- crosses: Theis + skin factor



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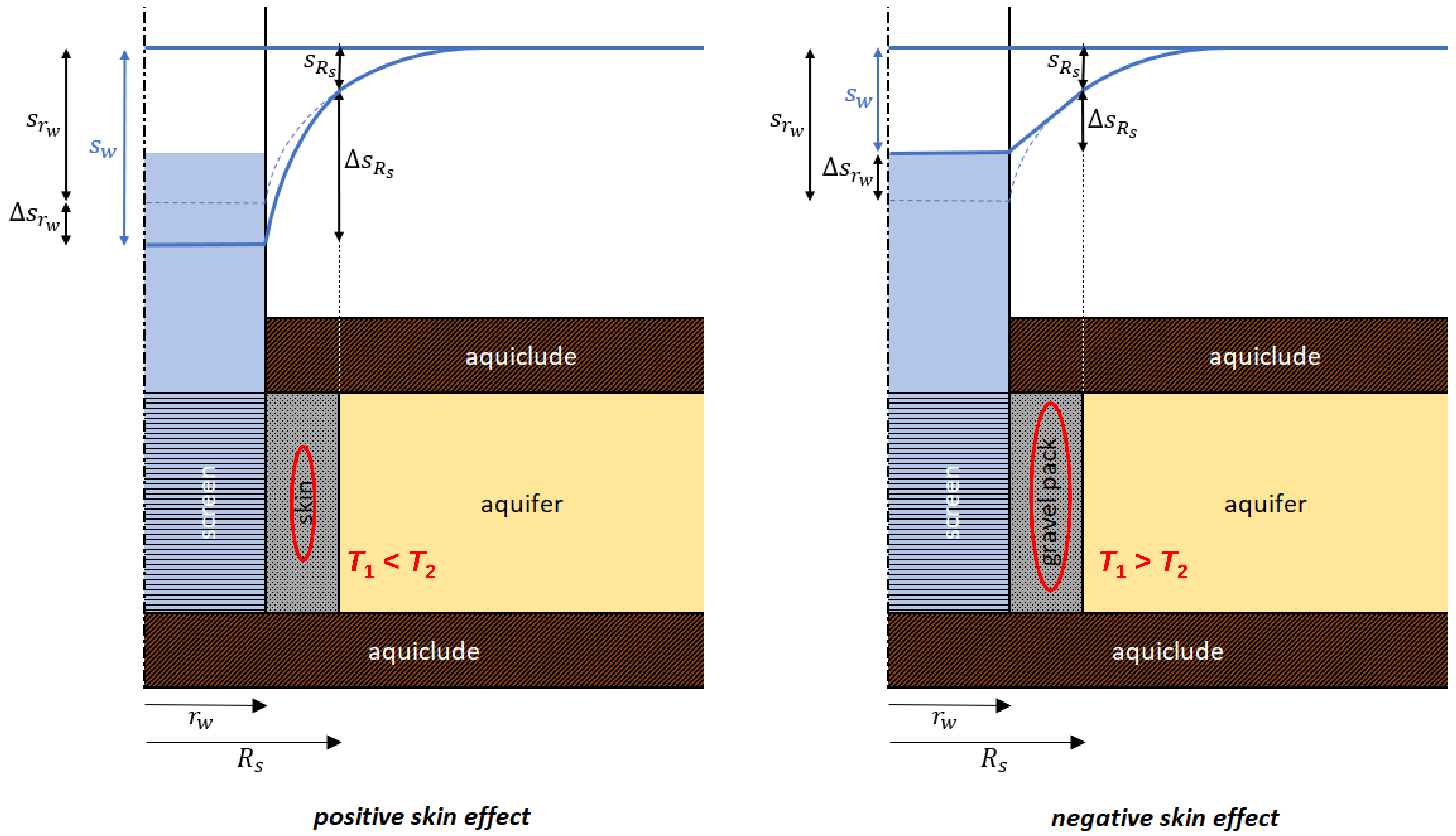
SKIN EFFECT



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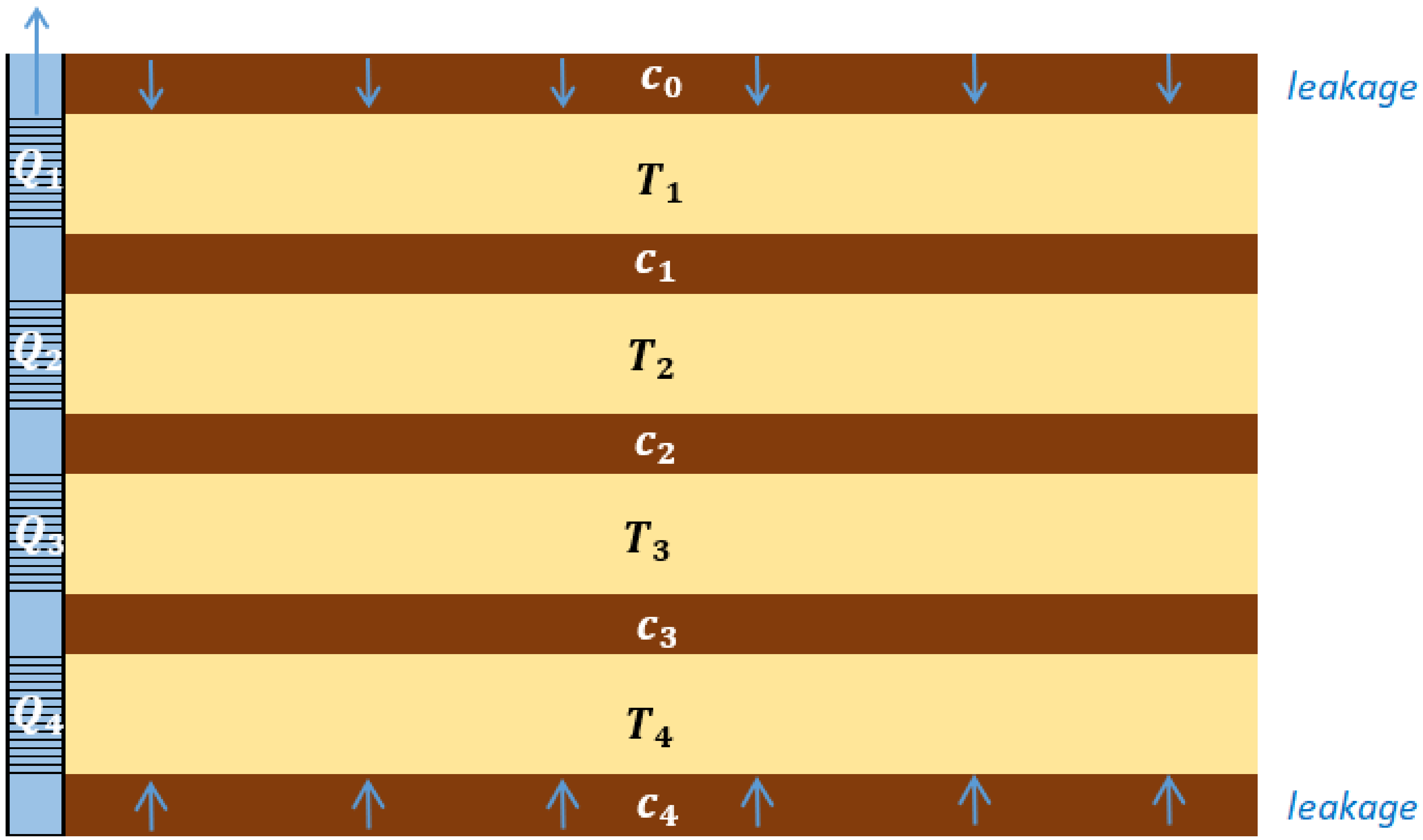
AXISYMMETRIC FLOW IN MULTILAYER AQUIFER SYSTEMS

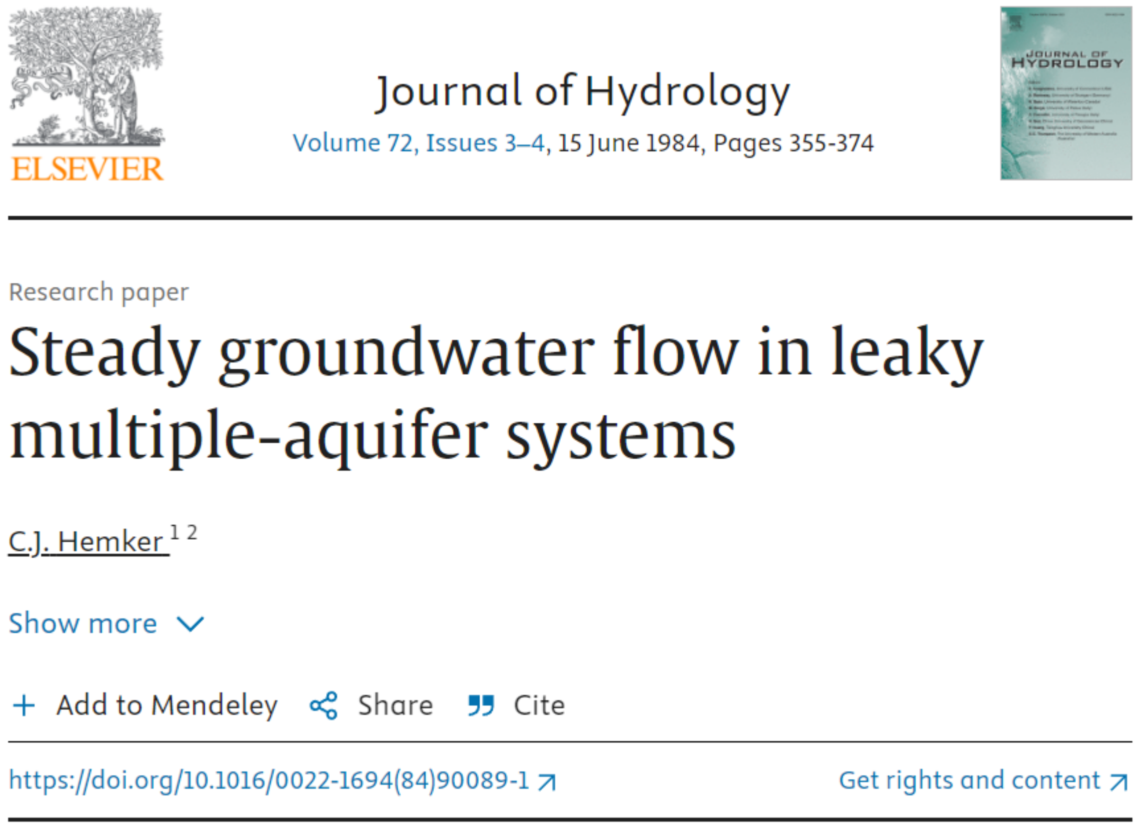
EVOLUTION OF AXISYMMETRIC MODELS

- **N layers**
- compressible aquitards
- anisotropy
- well:
  - partially penetrating
  - multi-aquifer
  - finite diameter (wellbore storage)
  - instantaneous head change (slug test)
  - finite-thickness skin
- water table conditions:
  - delayed yield
  - infiltration and drainage
  - confined-unconfined flow

1984 & 1985	**Hemker**
1985 & 1986	Hunt
1986 & 1987	Maas
1987	Hemker & Maas
1987	Yu
1993	Cheng & Morohunfola
1999	**Hemker**
2001	Bakker
2002 & 2004	Bakker & Hemker
2003	**Bakker & Strack**
2004	Meesters et al.
2006	Bakker & Hemker
2007	**Hunt & Scott**
2009	Veling & Maas
2023	**Louwyck**

HEMKER: STEADY MULTI-AQUIFER FLOW





ELSEVIER
Journal of Hydrology
Volume 72, Issues 3–4, 15 June 1984, Pages 355-374

Research paper

Steady groundwater flow in leaky multiple-apuifer systems

C.J. Hemker [1,2]

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Steady flow to a well in a leaky multi-aquifer system (Hemker, 1984)

$$s = V^{-1}KVQ$$

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HEMKER (1984) MODEL: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - **Steady-state**
  - Aquifers: strictly horizontal
  - Aquitards: strictly vertical

- Well:
  - Fully penetrating screens
  - Screens are not connected
  - Constant pumping rates
  - Infinitesimal radius

- Aquifer system:
  - Homogeneous aquifers and aquitards
  - Aquifers have constant saturated thickness
  - Incompressible aquitards → zero-thickness resistance layers
  - Laterally unbounded
  - Leaky top and bottom **(both top and bottom impervious is not possible!)**

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HEMKER (1984): PROBLEM STATEMENT

Steady flow in each layer *i* :

$$T_i\left(\frac{d^2 s_i}{dr^2}+\frac{1}{r}\frac{ds_i}{dr}\right)=\boxed{\frac{s_i-s_{i-1}}{c_{i-1}}+\frac{s_i-s_{i+1}}{c_i}} \quad (1)$$

leakage from adjacent layers

Matrix notation:

$$\frac{d^2\boldsymbol{s}}{dr^2}+\frac{1}{r}\frac{d\boldsymbol{s}}{dr}=\boldsymbol{As} \quad (4)$$

Inner boundary condition at zero:

$$Q_i=-\lim_{r\to 0}\left(2\pi r T_i\frac{ds_i}{dr}\right) \quad (2)$$

$$\boldsymbol{Q}=-\lim_{r\to 0}\left(r\frac{d\boldsymbol{s}}{dr}\right) \quad (5)$$

Outer boundary condition at infinity:

$$s_i(\infty)=0 \quad (3)$$

$$\boldsymbol{s}(\infty)=\boldsymbol{0} \quad (6)$$

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HEMKER (1984): MATRICES

Vector ***s*** :

$$\boldsymbol{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_{n_l} \end{bmatrix}$$

Vector ***Q*** :

$$\boldsymbol{Q} = \frac{1}{2\pi} \begin{bmatrix} Q_1/T_1 \\ \vdots \\ Q_{n_l}/T_{n_l} \end{bmatrix}$$

with n_l the number of layers

System matrix ***A*** :

$$\boldsymbol{A} = \begin{bmatrix} a_1 + b_1 & -b_1 & 0 & \cdots & & \\ -a_2 & a_2 + b_2 & -b_2 & & & \\ 0 & & \ddots & & & \\ \vdots & & & & & \\ & & & & & \\ & & & & & \vdots \\ & & & \cdots & -a_{n_l} & a_{n_l} + b_{n_l} \end{bmatrix}$$

with

$$a_i = \frac{1}{c_{i-1} T_i}$$

$$b_i = \frac{1}{c_i T_i}$$

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HEMKER (1984): EIGENDECOMPOSITION

Eigendecomposition of matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{V}^{-1}\boldsymbol{D}\boldsymbol{V} \tag{7}$$

Applying (7) to ODE (4) with $\boldsymbol{g} = \boldsymbol{V}\boldsymbol{s}$:

$$\frac{d^2\boldsymbol{g}}{dr^2} + \frac{1}{r}\frac{d\boldsymbol{g}}{dr} = \boldsymbol{D}\boldsymbol{g} \tag{8}$$

Multiplying each side of (5) and (6) by $\boldsymbol{V}$:

$$\boldsymbol{q} = \boldsymbol{V}\boldsymbol{Q} = -\lim_{r \to 0}\left(r\frac{d\boldsymbol{g}}{dr}\right) \tag{9}$$

$$\boldsymbol{g}(\infty) = \boldsymbol{0} \tag{10}$$

General solution of (8):

$$\boldsymbol{g} = \boldsymbol{I}\boldsymbol{\alpha} + \boldsymbol{K}\boldsymbol{\beta} \tag{11}$$

Diagonal matrices $\boldsymbol{I}$ and $\boldsymbol{K}$:

$$\boldsymbol{I}_{ii} = \mathrm{I}_0\left(r\sqrt{d_i}\right) \text{ and } \boldsymbol{K}_{ii} = \mathrm{K}_0\left(r\sqrt{d_i}\right)$$

with d_i the *i*-th eigenvalue of $\boldsymbol{A}$

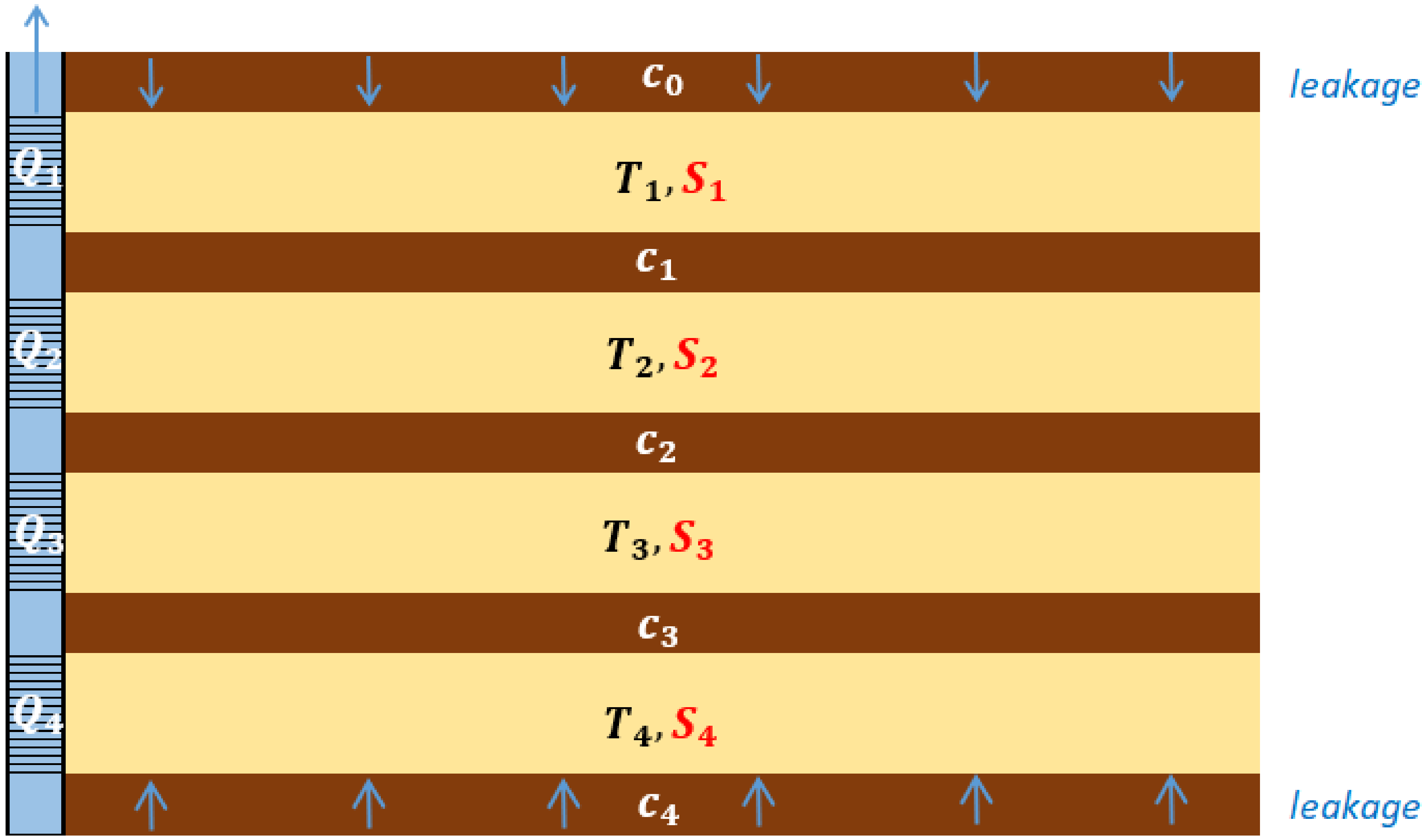
From (10): $\boldsymbol{\alpha} = \boldsymbol{0}$

From (9): $\boldsymbol{\beta} = \boldsymbol{q}$

Introducing $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (11):

$$\boxed{\boldsymbol{s} = \boldsymbol{V}^{-1}\boldsymbol{K}\boldsymbol{V}\boldsymbol{Q}}$$

HEMKER: TRANSIENT MULTI-AQUIFER FLOW



Transient flow to a well in a leaky multi-aquifer system (Hemker, 1985)

$$\bar{s} = \boldsymbol{V}^{-1}\boldsymbol{K}\boldsymbol{V}\boldsymbol{Q}$$

= Laplace solution which is numerically inverted

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HEMKER (1985) MODEL: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - **Transient-state**
  - Aquifers: strictly horizontal
  - Aquitards: strictly vertical
- Well:
  - Fully penetrating screens
  - Screens are not connected
  - Constant pumping rates
  - Infinitesimal radius
- Aquifer system:
  - Homogeneous aquifers and aquitards
  - Aquifers have constant saturated thickness
  - Incompressible aquitards → zero-thickness resistance layers
  - Laterally unbounded
  - Leaky top and bottom **(both top and bottom impervious is possible!)**

HEMKER (1985): PROBLEM STATEMENT

Transient flow in each layer *i* :

$$T_i\left(\frac{\partial^2 s_i}{\partial r^2}+\frac{1}{r}\frac{\partial s_i}{\partial r}\right)=\underbrace{\boxed{S_i\frac{\partial s_i}{\partial r}}}_{\text{storage change}}+\underbrace{\boxed{\frac{s_i-s_{i-1}}{c_{i-1}}+\frac{s_i-s_{i+1}}{c_i}}}_{\text{leakage}} \qquad (1)$$

Matrix notation + Laplace transform:

$$\frac{d^2\bar{\mathbf{s}}}{dr^2}+\frac{1}{r}\frac{d\bar{\mathbf{s}}}{dr}=\boldsymbol{A}\bar{\mathbf{s}} \qquad (4)$$

Inner boundary condition at zero:

$$Q_i=-\lim_{r\to 0}\left(2\pi r T_i\frac{\partial s_i}{\partial r}\right) \qquad (2)$$

$$\boldsymbol{Q}=-\lim_{r\to 0}\left(r\frac{d\bar{\mathbf{s}}}{dr}\right) \qquad (5)$$

Outer boundary condition at infinity:

$$s_i(\infty)=0 \qquad (3)$$

$$\bar{\mathbf{s}}(\infty)=\mathbf{0} \qquad (6)$$

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HEMKER (1985): MATRICES

Vector $\bar{\boldsymbol{s}}$:

$$\bar{\boldsymbol{s}} = \begin{bmatrix} \bar{s}_1 \\ \vdots \\ \bar{s}_{n_l} \end{bmatrix}$$

Vector $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \frac{1}{2\pi p} \begin{bmatrix} Q_1/T_1 \\ \vdots \\ Q_{n_l}/T_{n_l} \end{bmatrix}$$

with n_l the number of layers

System matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} a_1 + b_1 + \omega_1^2 & -b_1 & 0 & \cdots & & \\ -a_2 & a_2 + b_2 + \omega_2^2 & -b_2 & & & \\ 0 & & \ddots & & & \\ \vdots & & & & & \\ & & & & & \vdots \\ & & & \cdots & -a_{n_l} & a_{n_l} + b_{n_l} + \omega_{n_l}^2 \end{bmatrix}$$

with

$$a_i = \frac{1}{c_{i-1}T_i}$$

$$b_i = \frac{1}{c_i T_i}$$

$$\omega_i = \sqrt{pS_i/T_i}$$

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HEMKER (1985): EIGENDECOMPOSITION

Eigendecomposition of matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{V}^{-1}\boldsymbol{D}\boldsymbol{V} \tag{7}$$

Applying (7) to ODE (4) with $\boldsymbol{g} = \boldsymbol{V}\bar{\boldsymbol{s}}$:

$$\frac{d^2\boldsymbol{g}}{dr^2} + \frac{1}{r}\frac{d\boldsymbol{g}}{dr} = \boldsymbol{D}\boldsymbol{g} \tag{8}$$

Multiplying each side of (5) and (6) by $\boldsymbol{V}$:

$$\boldsymbol{q} = \boldsymbol{V}\boldsymbol{Q} = -\lim_{r\to 0}\left(r\frac{d\boldsymbol{g}}{dr}\right) \tag{9}$$

$$\boldsymbol{g}(\infty) = \boldsymbol{0} \tag{10}$$

General solution of (8):

$$\boldsymbol{g} = \boldsymbol{I}\boldsymbol{\alpha} + \boldsymbol{K}\boldsymbol{\beta} \tag{11}$$

Diagonal matrices $\boldsymbol{I}$ and $\boldsymbol{K}$:

$$\boldsymbol{I}_{ii} = \mathrm{I}_0\left(r\sqrt{d_i}\right) \text{ and } \boldsymbol{K}_{ii} = \mathrm{K}_0\left(r\sqrt{d_i}\right)$$

with d_i the i-th eigenvalue of $\boldsymbol{A}$

From (10): $\boldsymbol{\alpha} = \boldsymbol{0}$

From (9): $\boldsymbol{\beta} = \boldsymbol{q}$

Introducing $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in (11):

$$\boxed{\bar{\boldsymbol{s}} = \boldsymbol{V}^{-1}\boldsymbol{K}\boldsymbol{V}\boldsymbol{Q}}$$

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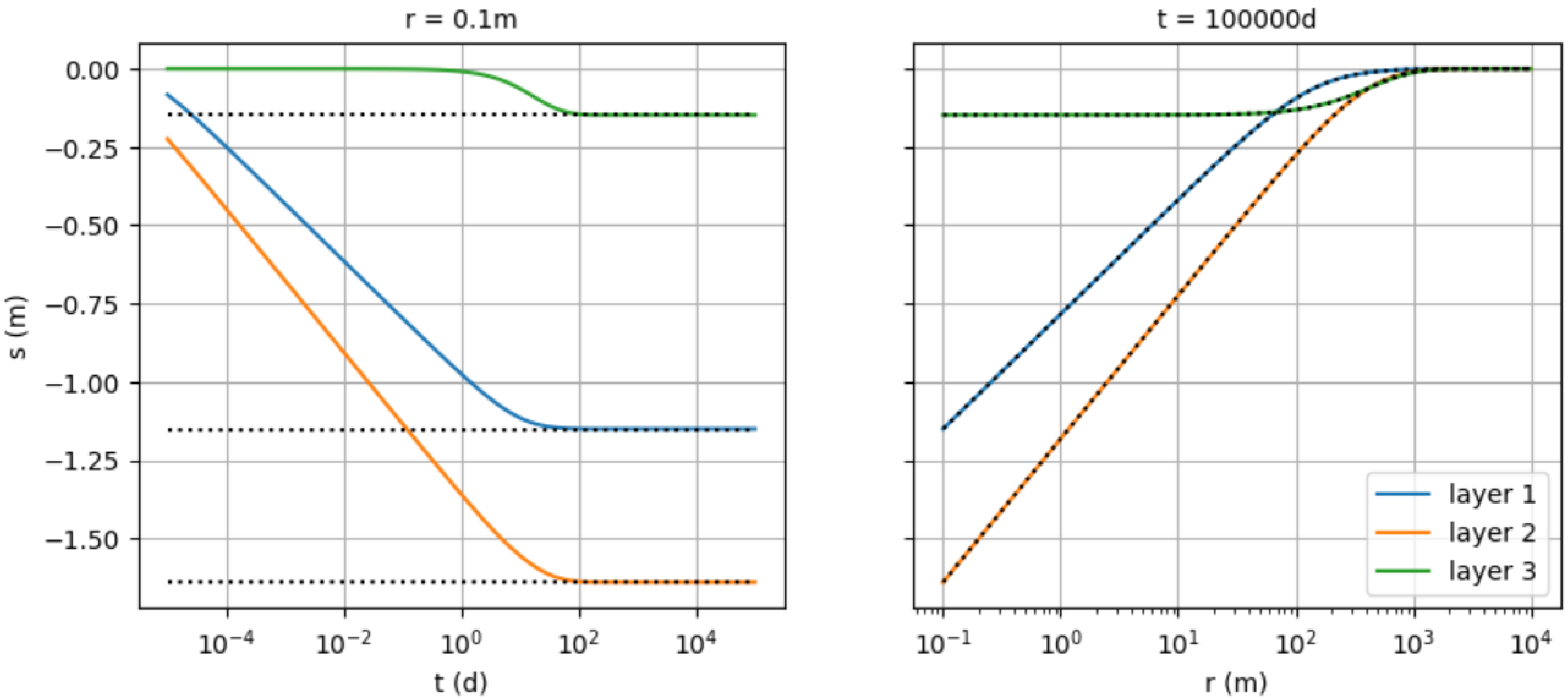
HEMKER: STEADY VS TRANSIENT

alouwyck / maxsypy

```
T = [100, 200, 50]              # aquifer transmissivities (m²/d)
S = [0.1, 0.05, 0.01]           # aquifer storativities (-)
c = [100, 500, 1000, np.inf]    # aquitard resistances (d)
Q = [-100, -250, 0]             # pumping rates (m³/d)

model1 = Steady(T=T, Q=Q, c_top=c[0], c=c[1:-1], c_bot=c[-1])
model2 = Transient(T=T, S=S, Q=Q, c_top=c[0], c=c[1:-1], c_bot=c[-1])
```

– solid lines: transient-state
– dotted lines: steady-state



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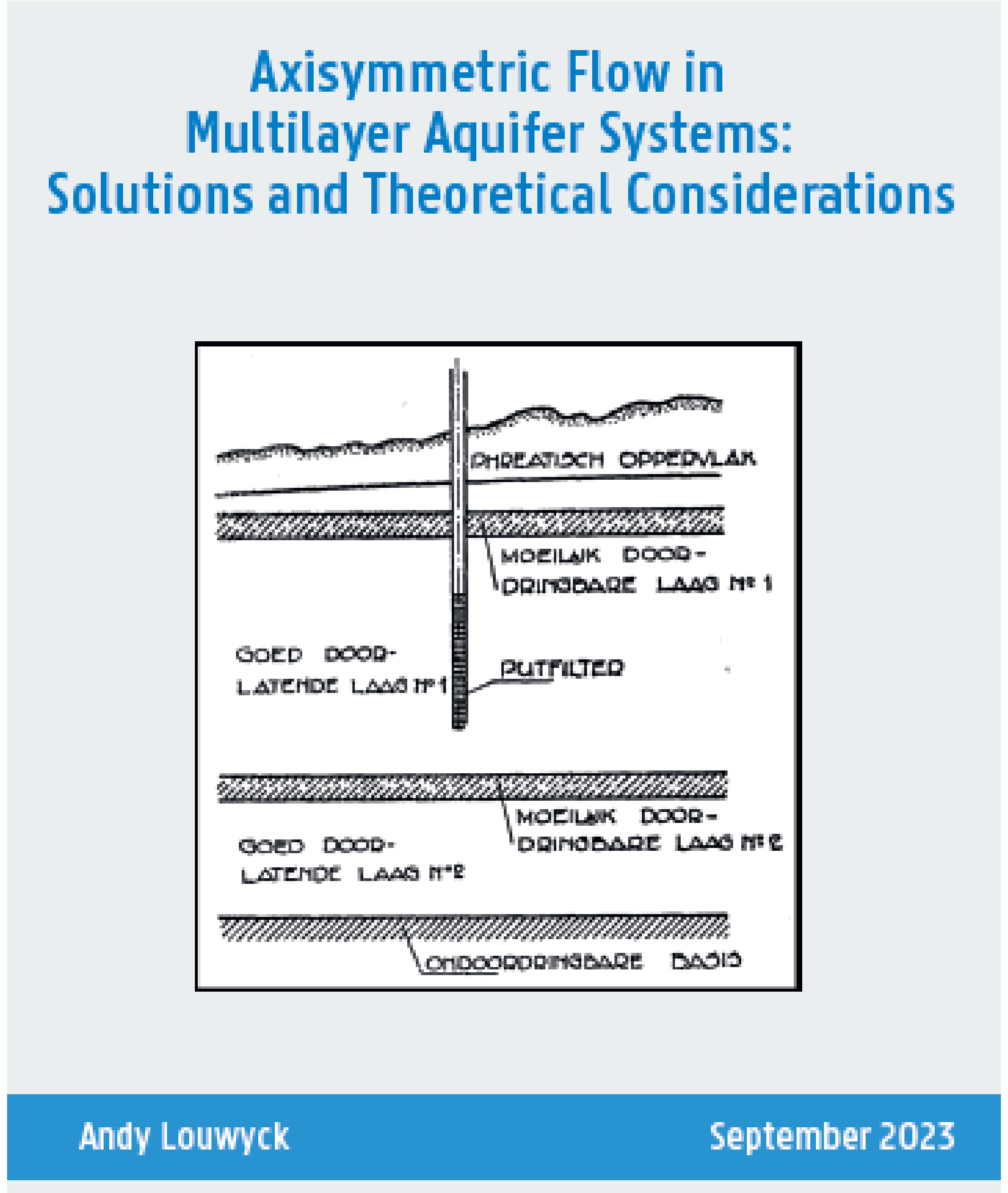
MY PHD THESIS

alouwyck / PhD

- **Semi-analytical method:**
  - Generalized semi-analytical solution for multilayer flow
  - Extension to multilayer-multizone flow
- **Finite-difference approach:**
  - Matlab tool MAxSym for multilayer flow
  - MODFLOW procedure for axisymmetric flow
  - Extension to multi-node wells
- **Comparing both solution methods**



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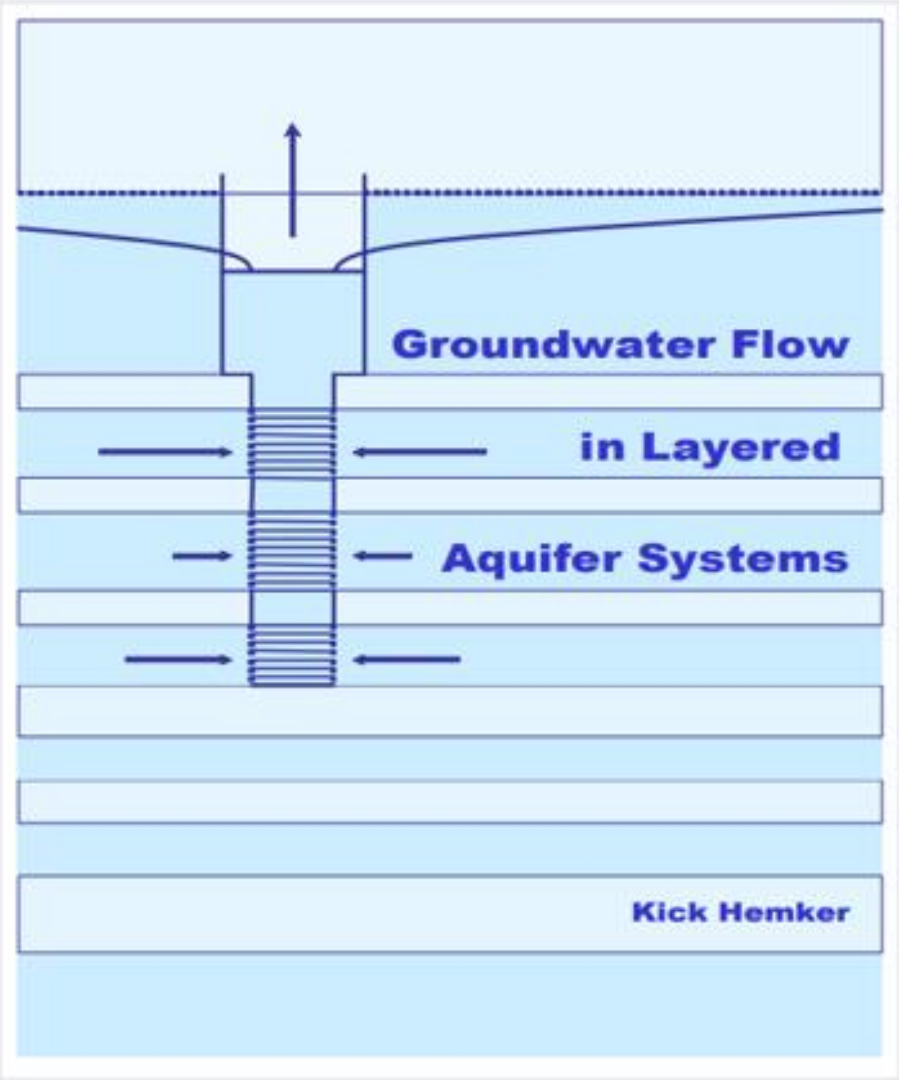
GENERALIZED SEMI-ANALYTICAL SOLUTION

- Python code:
  - axisymmetric or parallel flow
  - steady or transient state
  - specified discharge or head
  - laterally bounded or unbounded
  - confined or leaky + recharge
  - superposition in space and time

```
model = Transient(T=[100, 200, 50],      # transmissivities (m²/d)
                  S=[0.1, 0.05, 0.01],   # storativities (-)
                  Q=[-100, -250, 0],     # pumping rates (m³/d)
                  c=[500, 1000],         # resistances (d)
                  c_top=100)             # top resistance (d)
t = np.logspace(-5, 5, 100)  # simulation times (d)
r = 0.1  # well-radius (m)
s = model.h(r, t)  # drawdown s (m)
```

- based on earlier work
  - Hemker (1984, 1985, 1999, 2000)
  - Bakker & Strack (2003)
  - MLU app (Hemker & Post)
  - Python packages TimML and TTim (Bakker)



Kick Hemker

Mark Bakker

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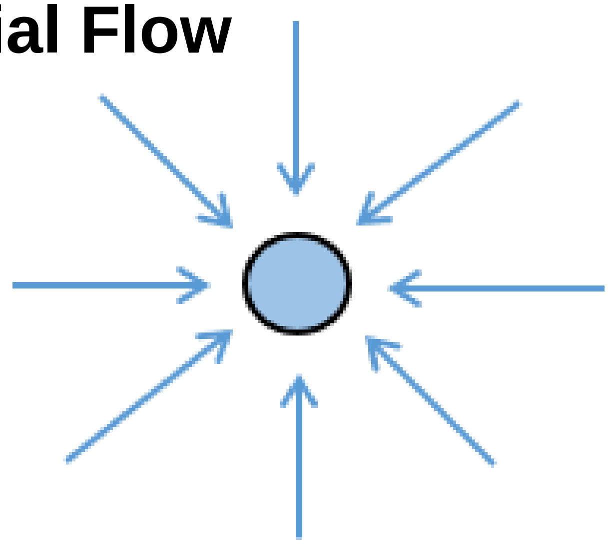
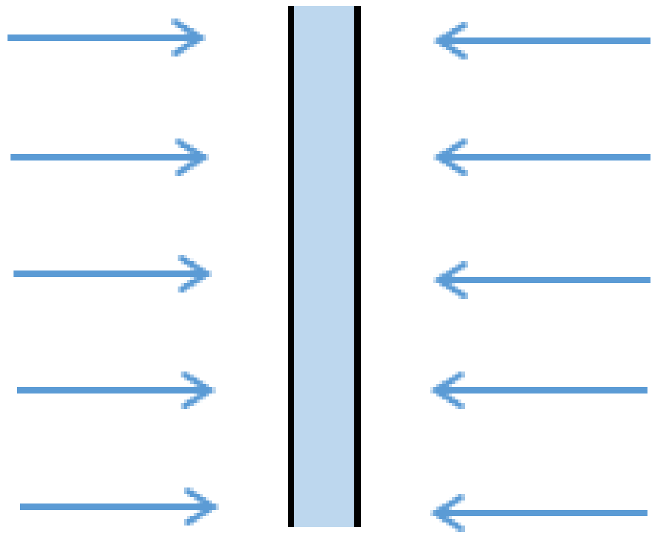
2D MULTILAYER FLOW

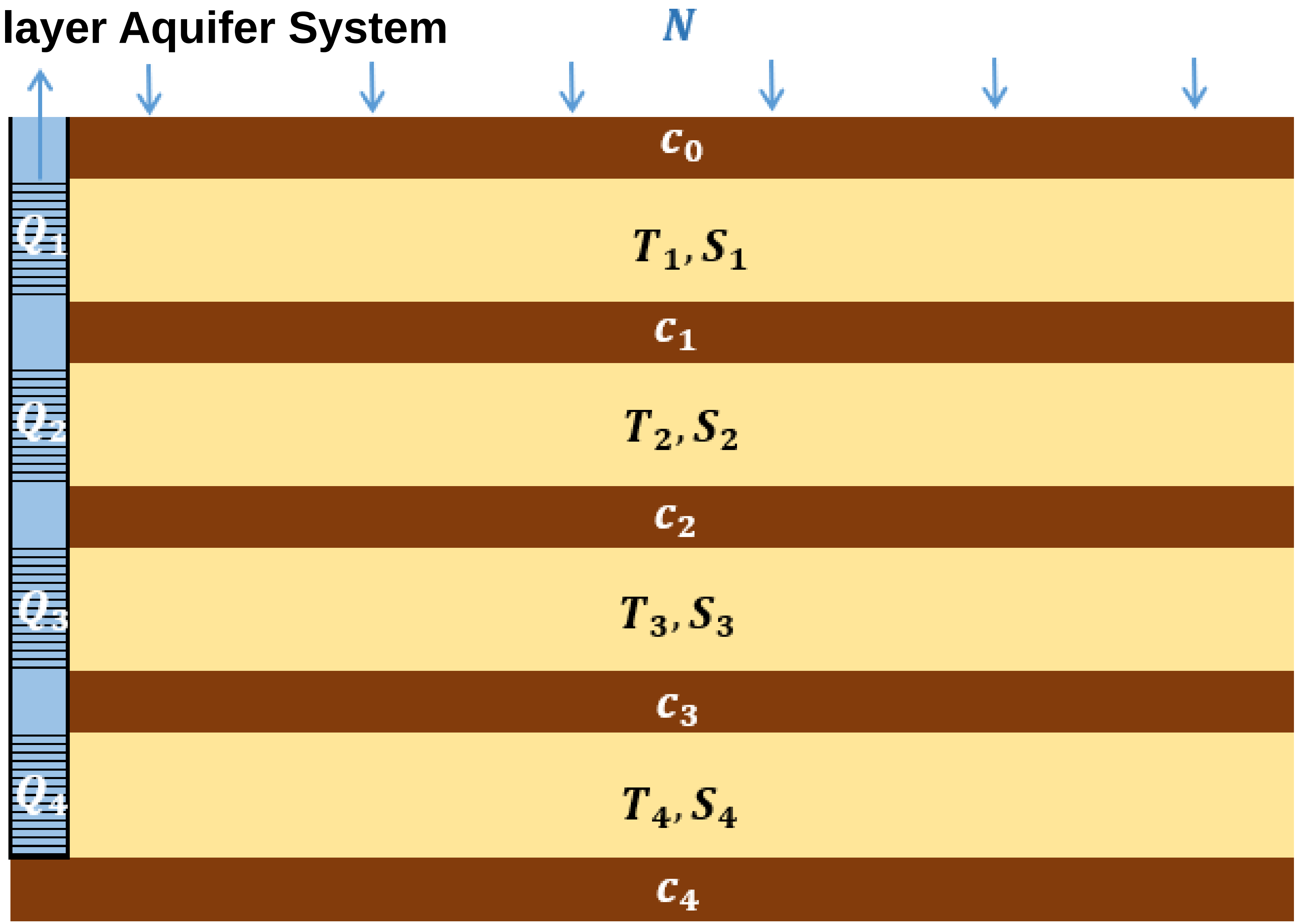
Radial Flow



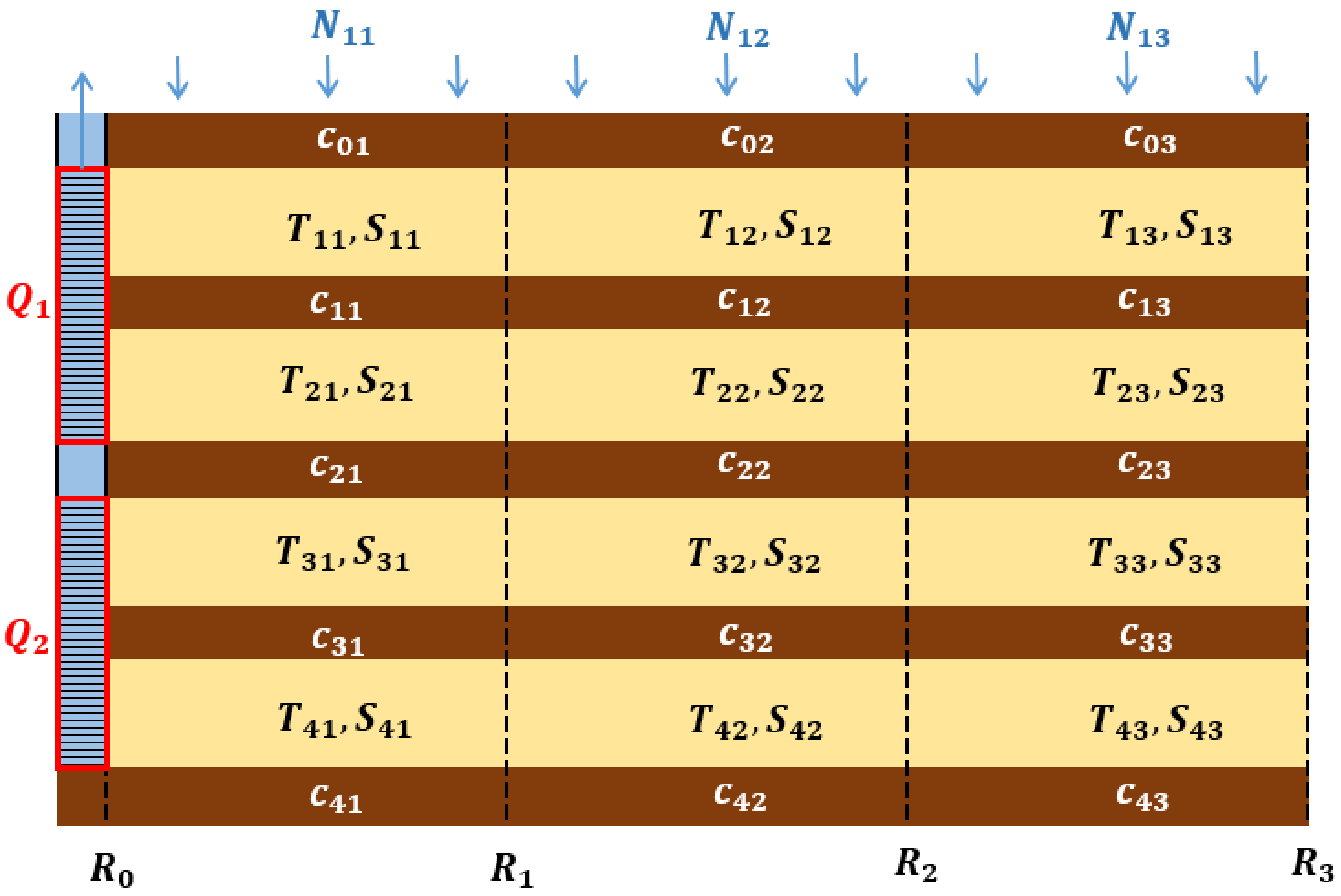
Parallel Flow

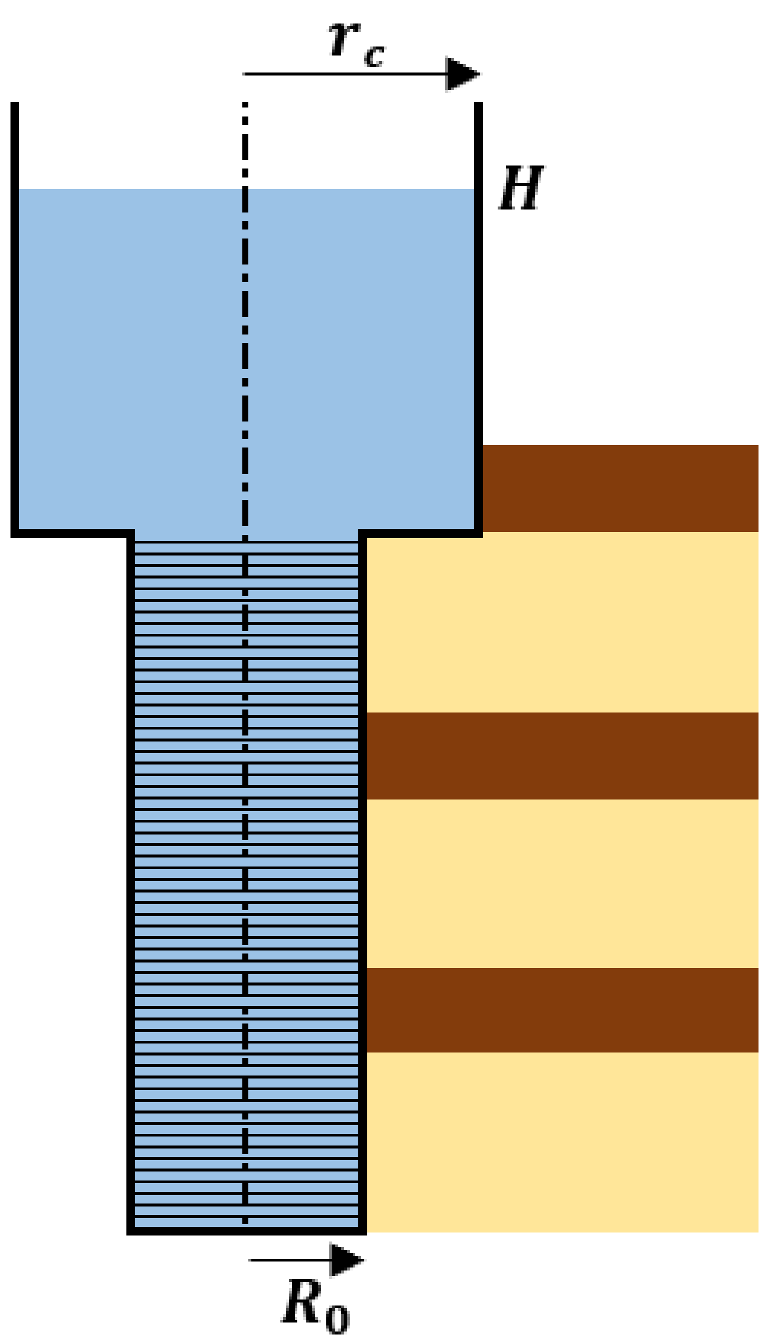


Multilayer Aquifer System



EXTENSION: MULTILAYER-MULTIZONE FLOW





EXAMPLE: MULTILAYER WELL

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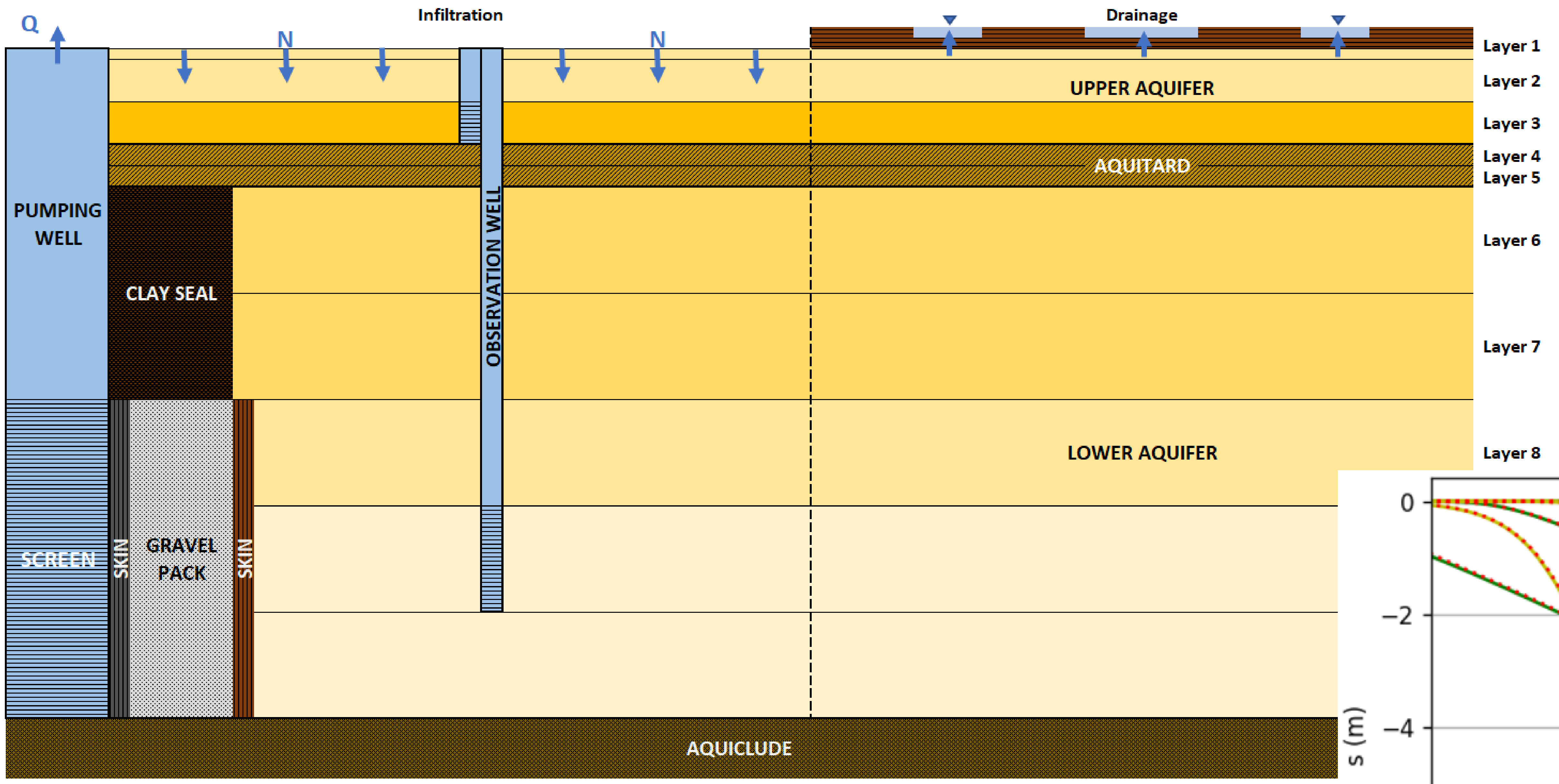

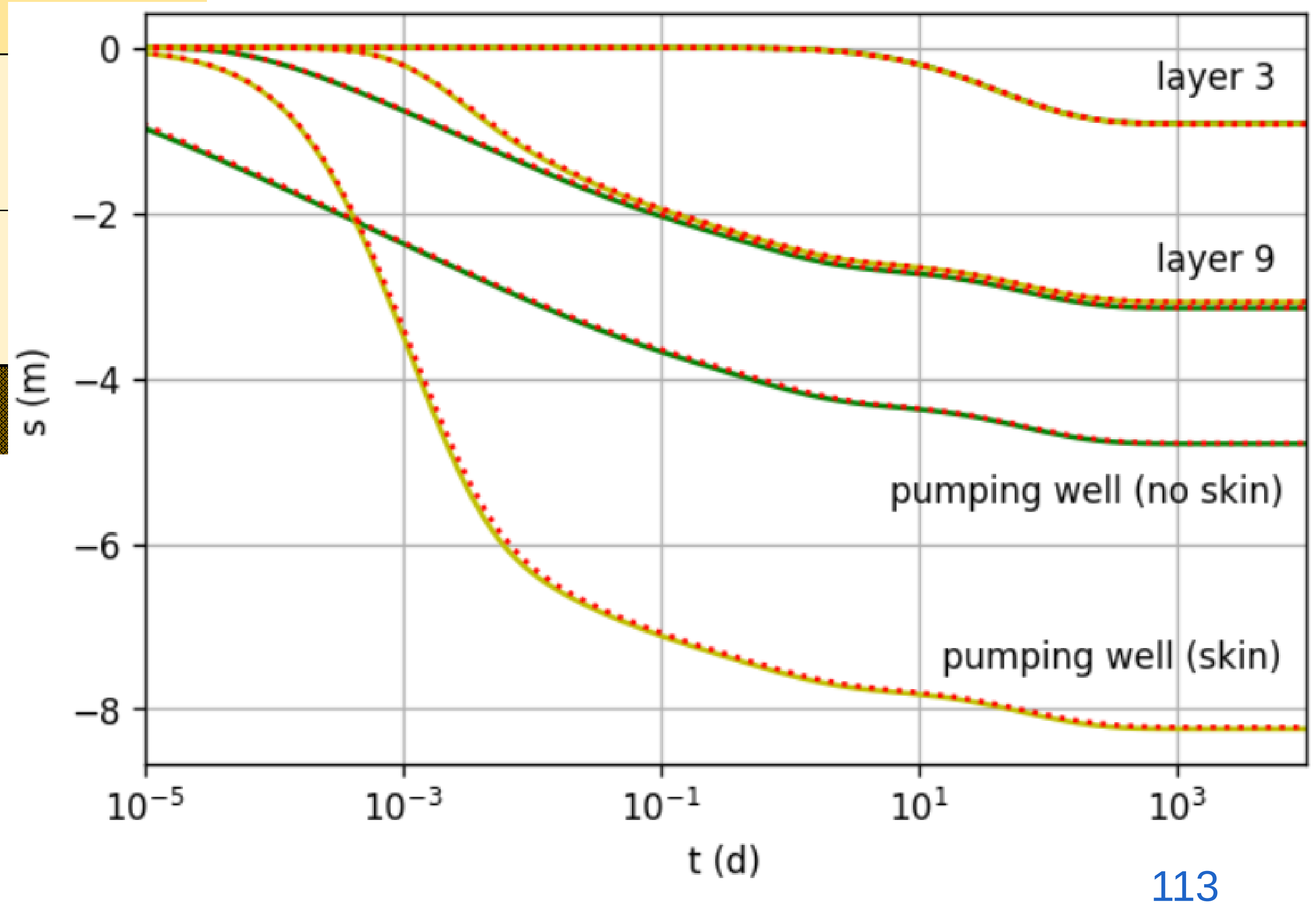

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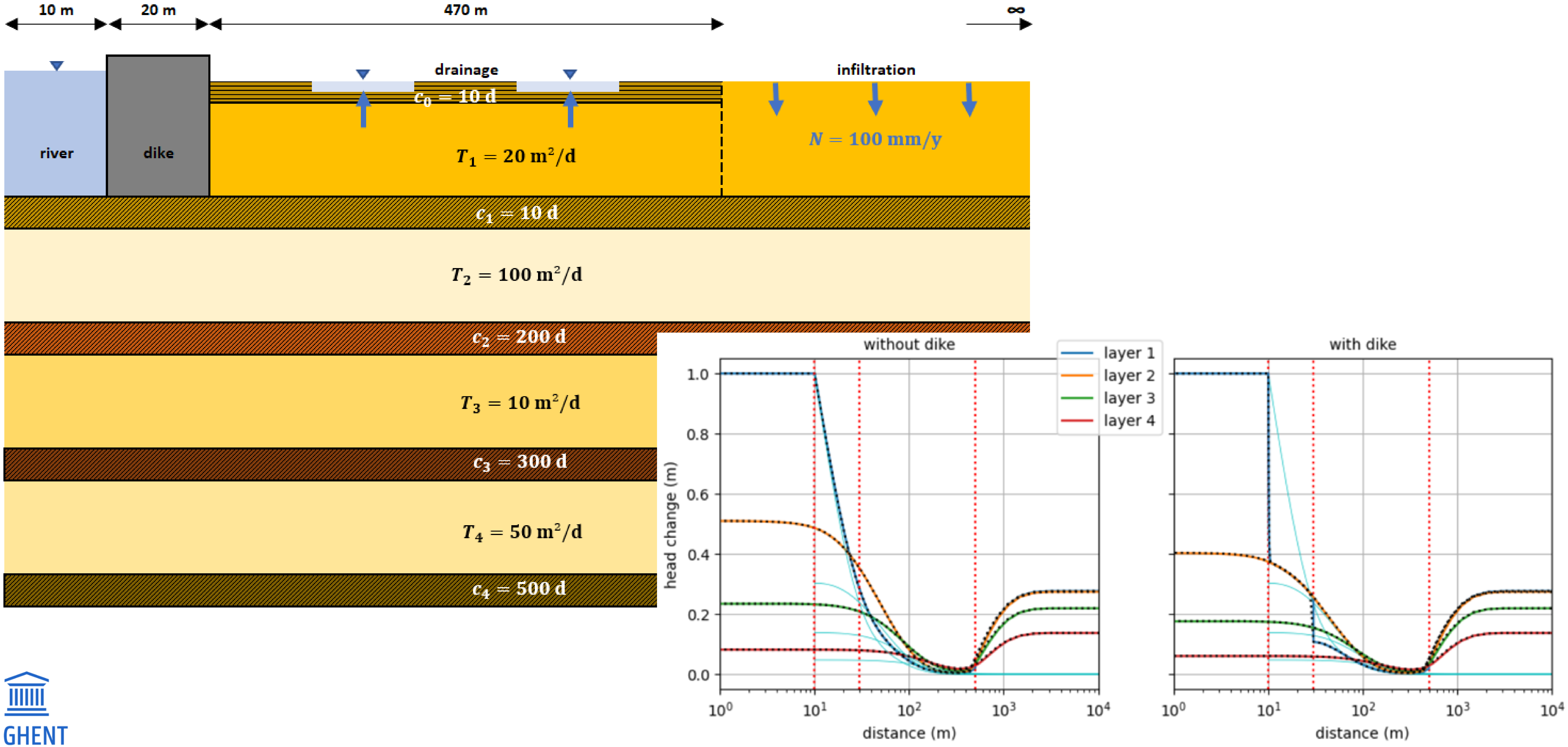
EXAMPLE: EMBANKED RIVER



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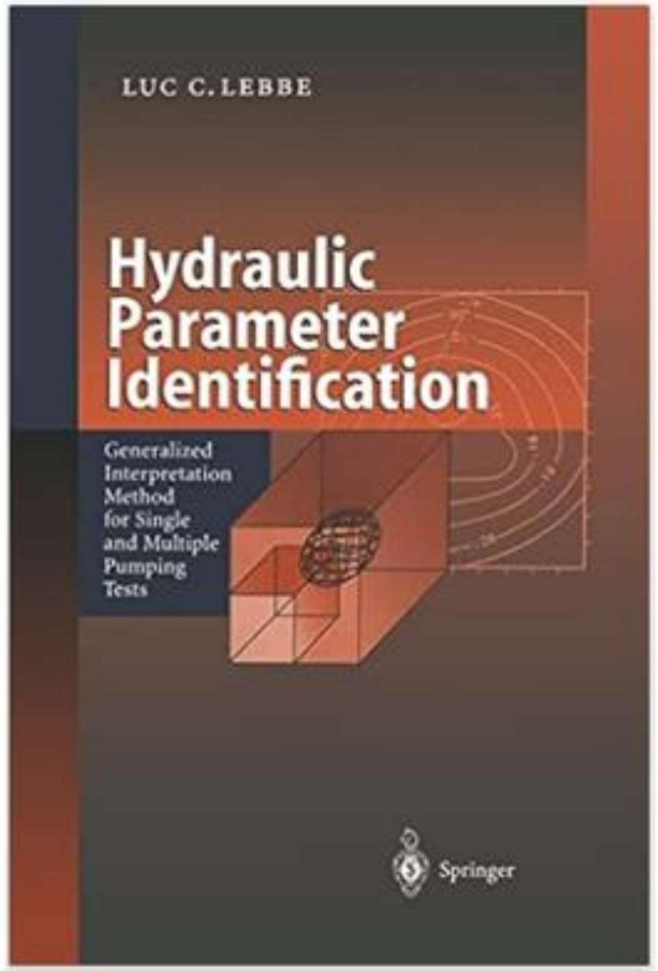
FINITE-DIFFERENCE APPROACH

alouwyck / MAxSym

- software implementations
  - AS2D Matlab wrapper
  - OGMA-RF (Louwyck et al., 2007, 2010; Vandenbohede et al., 2008, 2009)
  - MAxSym (Louwyck, 2011, 2015; Louwyck et al., 2012)
  - MODFLOW procedure (Louwyck et al., 2012, 2014)
  - Python version

- based on earlier work
  - AS2D (Lebbe, 1983, 1988, 1999)
  - MODFLOW (1984, 1988, 1996, 2000, 2005)
  - MODFLOW procedure (Langevin, 2008)



Luc Lebbe

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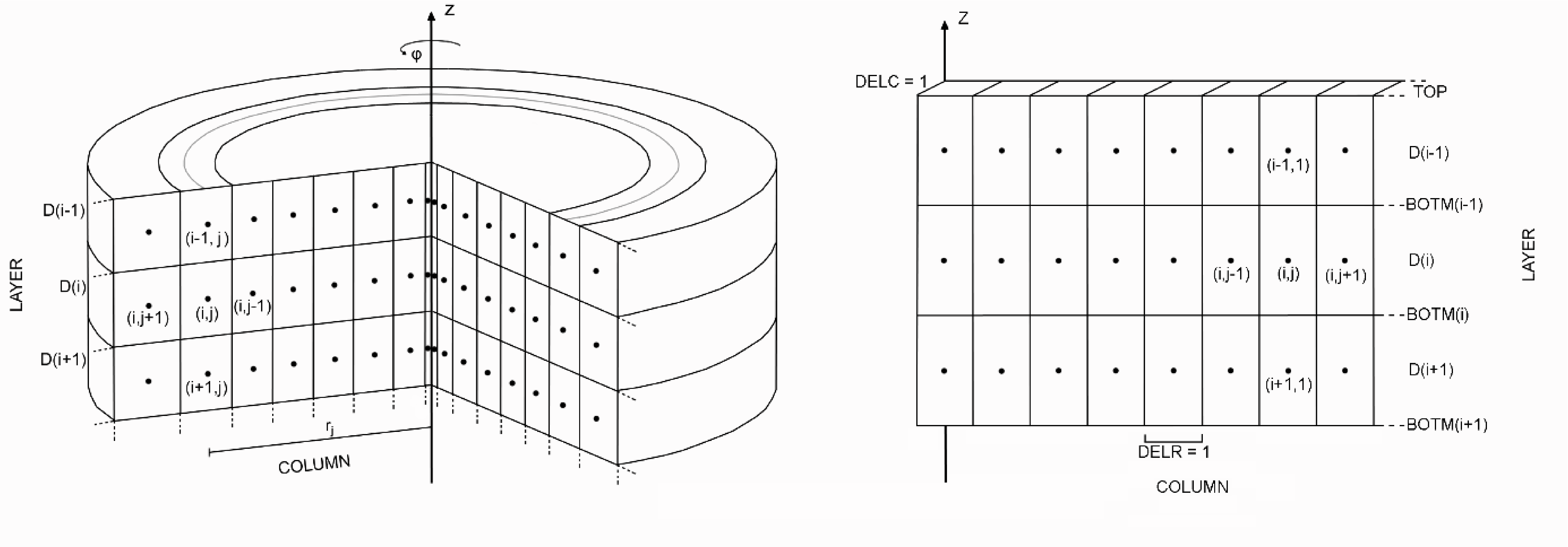
MODFLOW PROCEDURE



Hydrogeology Journal (2014) 22: 1217–1226
DOI 10.1007/s10040-014-1150-0

MODFLOW procedure to simulate axisymmetric flow in radially heterogeneous and layered aquifer systems

Andy Louwyck · Alexander Vandenbohede ·
Mark Bakker · Luc Lebbe

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CONCLUSIONS

Semi-Analytical (SA) vs Finite-Difference (FD):

- both very accurate and fast
- FD easier to implement in case of
  - heterogeneities
  - nonlinearities
- SA offers insight!

AQUIFER TESTS

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AQUIFER TESTING

- Used to characterize aquifer systems

  E.g., pumping test, step-drawdown test, slug test, recovery test, …

- Test conducted in the field:
  - Stimulate the aquifer by stressing a well
  - Measure drawdown in observation wells
- Interpretation of the test:
  - Fit the observed data
  - Derive hydraulic parameters
  - Inverse problem type I

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TRADITIONAL MANUAL CURVE FITTING

- Analytical solution produces type curves
- Test data are visually fit to these curves
- See Kruseman & de Ridder (2000)

E.g., the Theis curve-fitting method

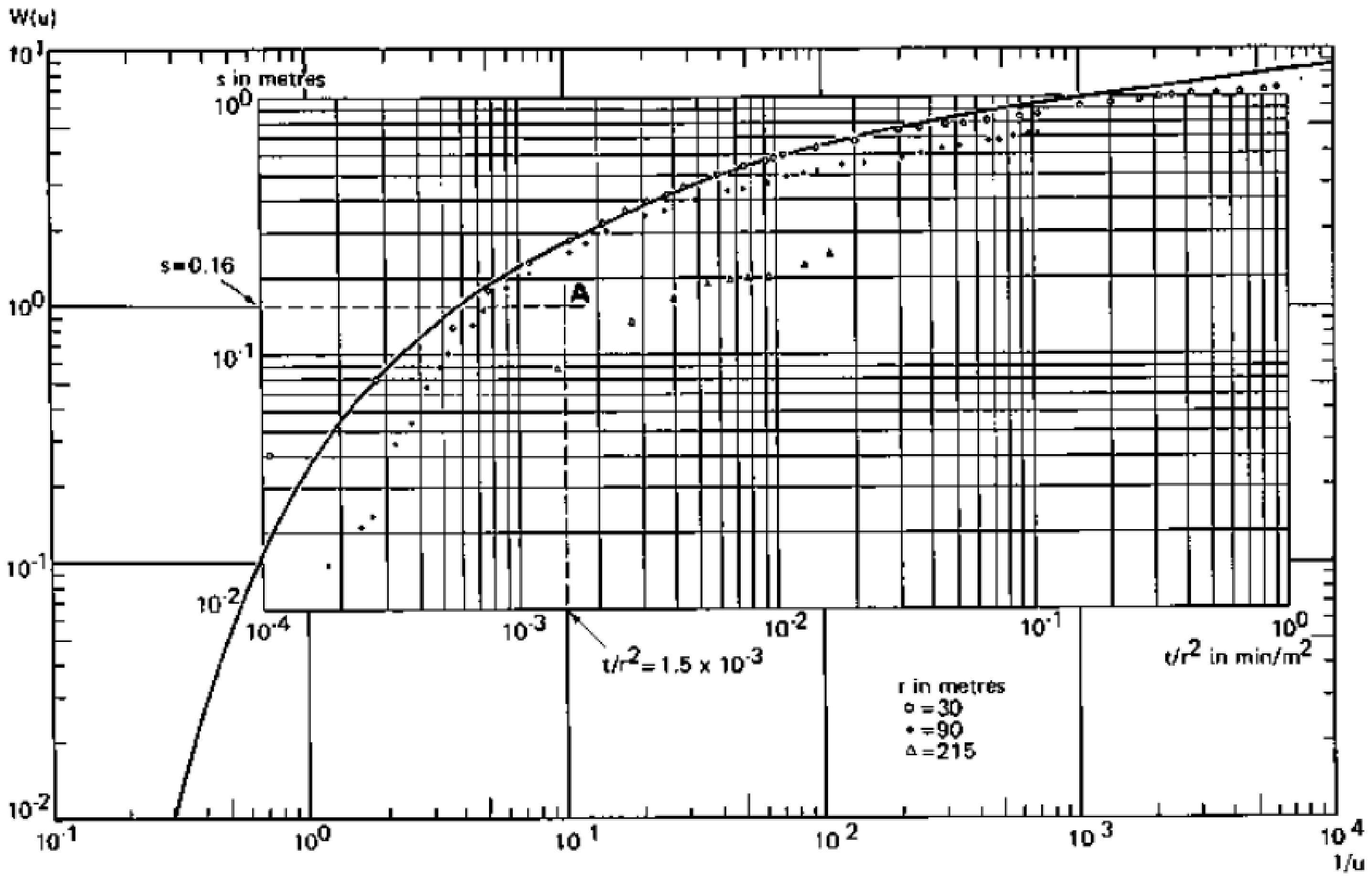
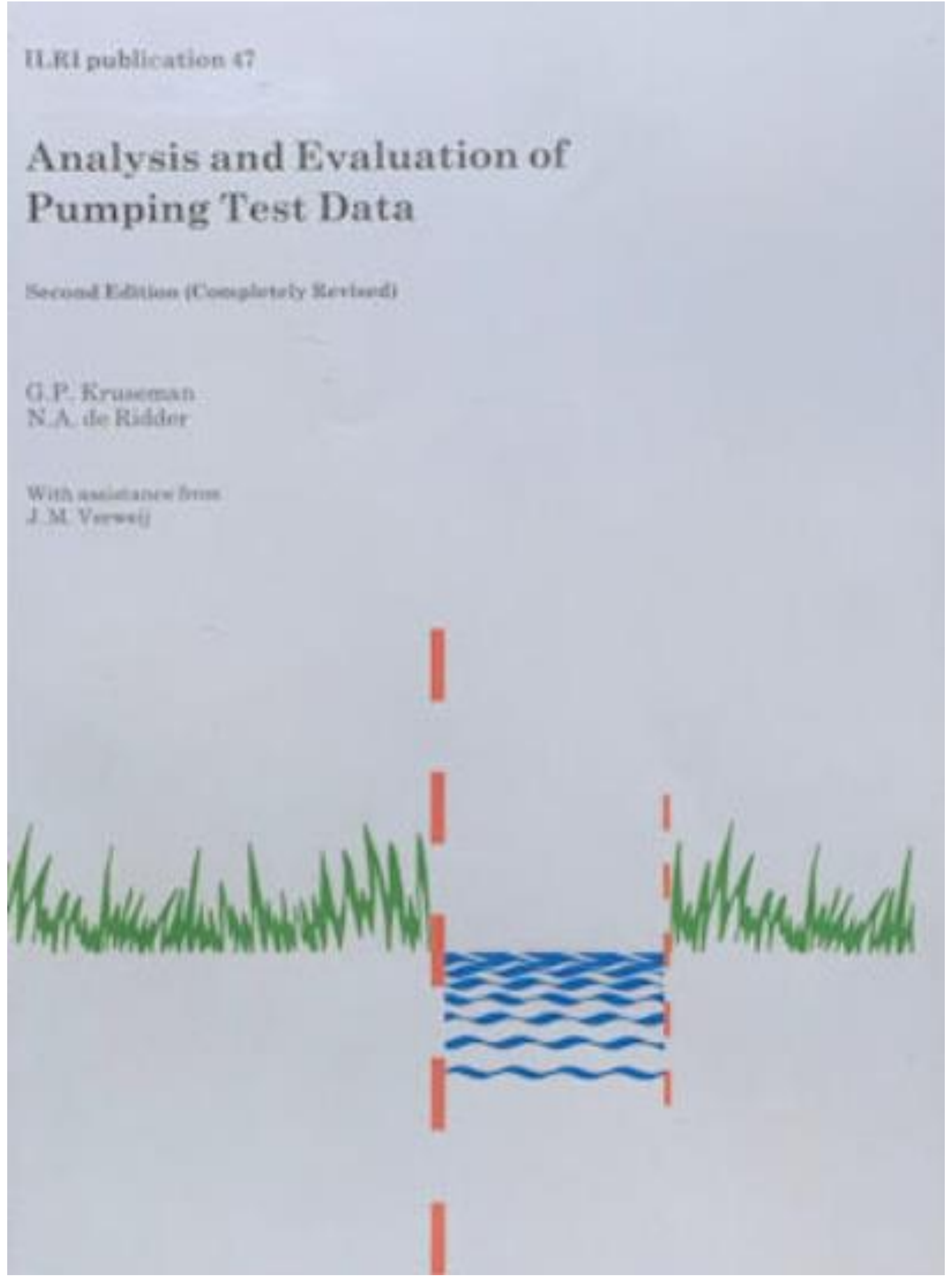


Figure 3.6 Analysis of data from pumping test 'Oude Korendijk' with the Theis method; Procedure 3.3



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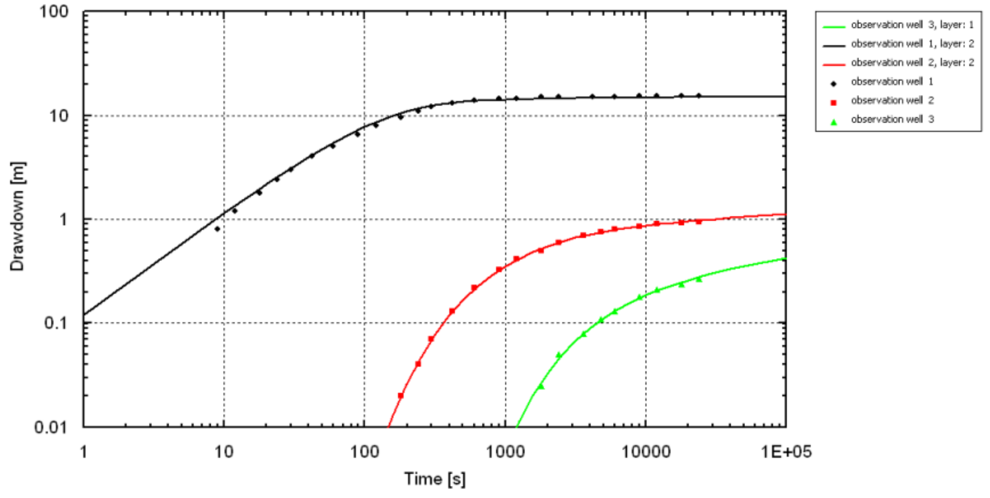
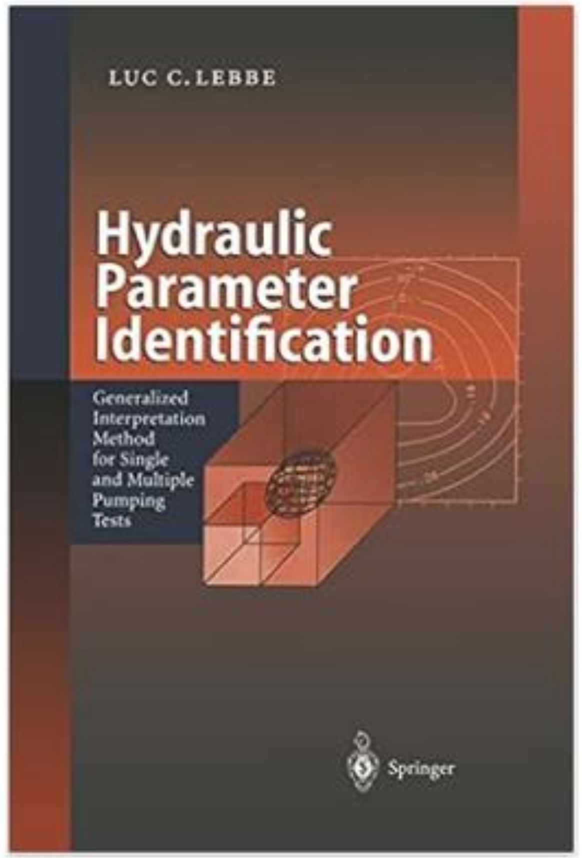
NONLINEAR REGRESSION

- Test data are fit through nonlinear regression

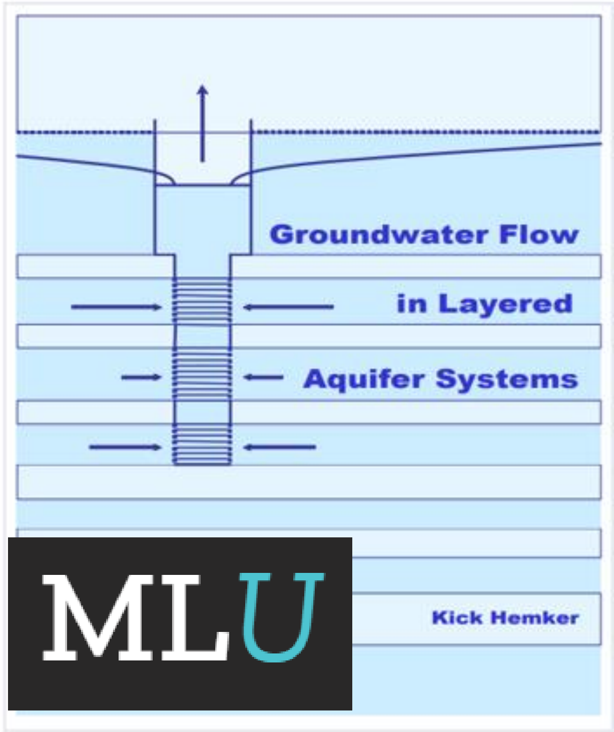
  E.g., Gauss-Newton, Levenberg-Marquardt

- Pros:
  - Automated curve-fitting
  - Reproducible results



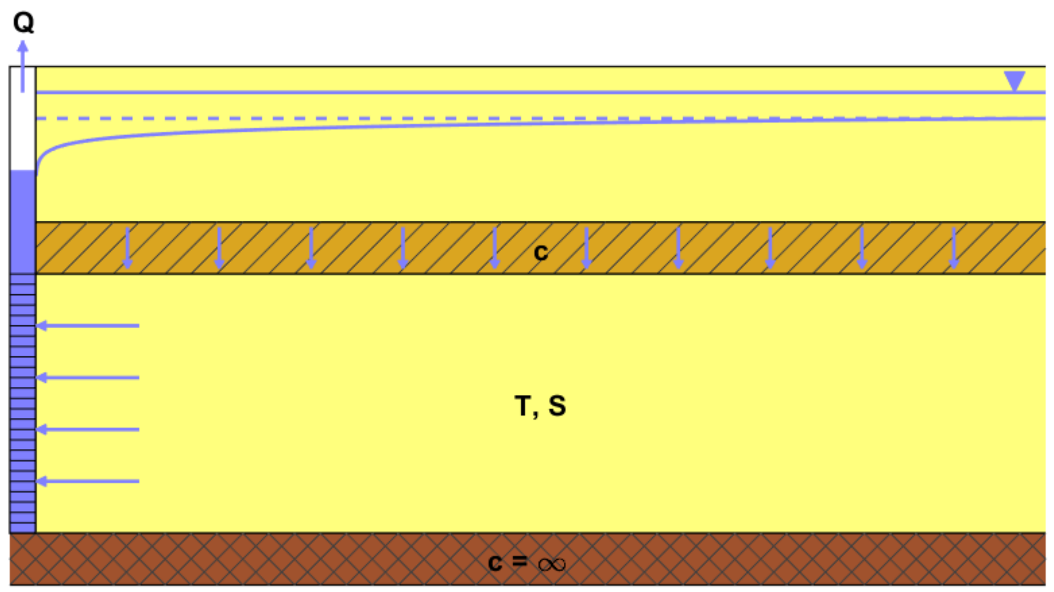


- Software:
  - HYPARIDEN (Lebbe, 1999)
  - MLU: https://mlu.app/
  - Python: SciPy



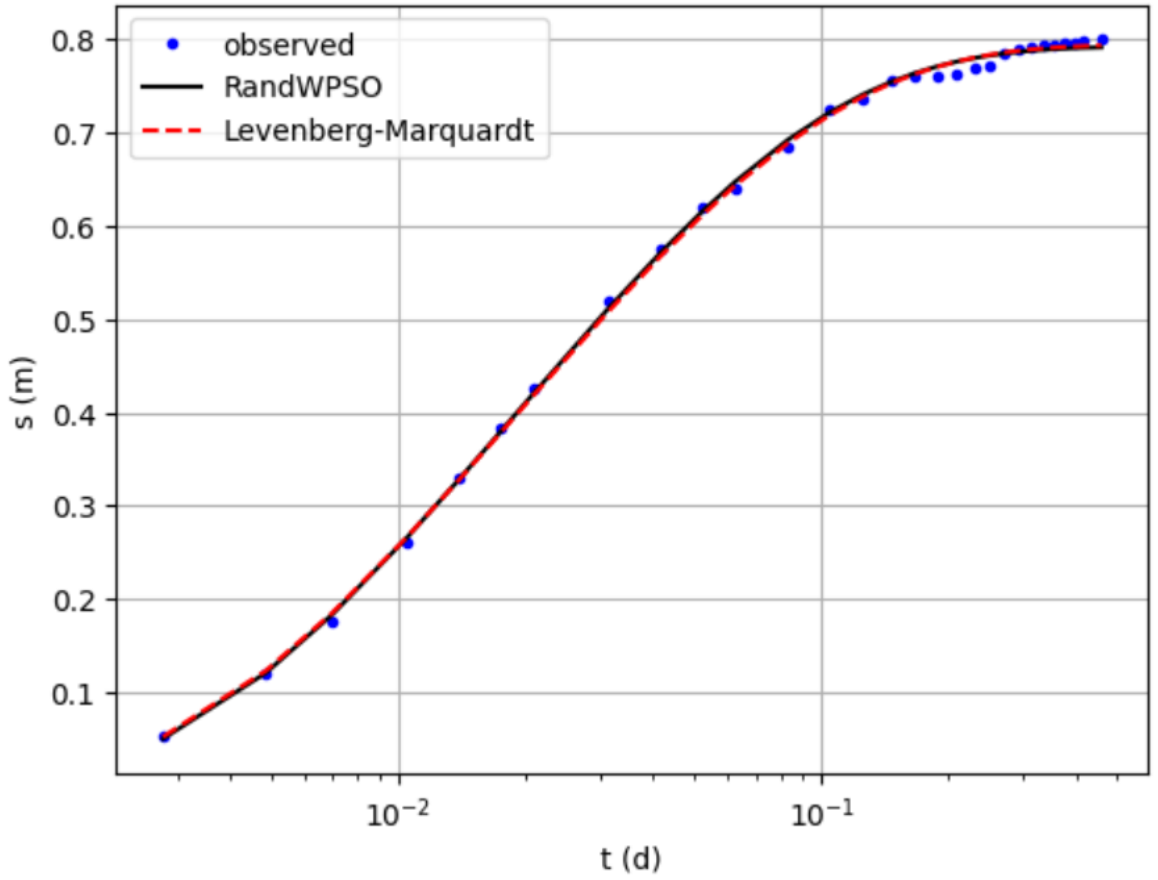
PUMPING TEST IN LEAKY AQUIFER

- Pumping test (Jiang Hui et al., 2009):
  - Leaky aquifer consisting of gravel
  - Fully penetrating pumping well: Q = 69.1 m³/h
  - Obervation well: r = 197 m
  - 30 drawdown observations



- Interpretation:
  - Forward model: Hantush-Jacob (MAxSyPy)
  - Curve fitting: Levenberg-Marquardt (SciPy)
  - Optimal parameter values:

```
T = 418.38884431293474
S = 0.0001391052948927095
c = 804.0082714958618
```



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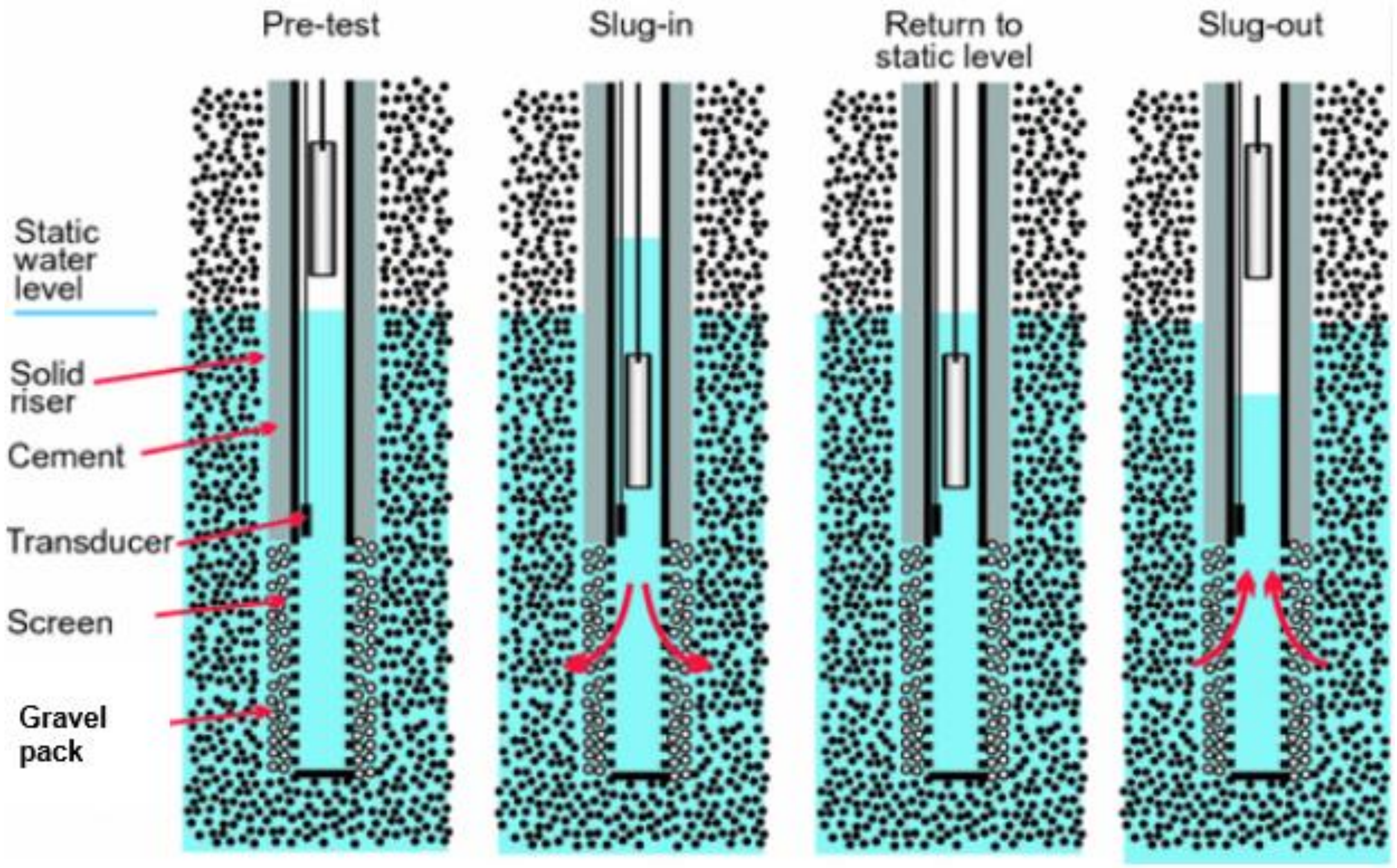
SLUG TEST IN PARTIALLY PENETRATING WELL

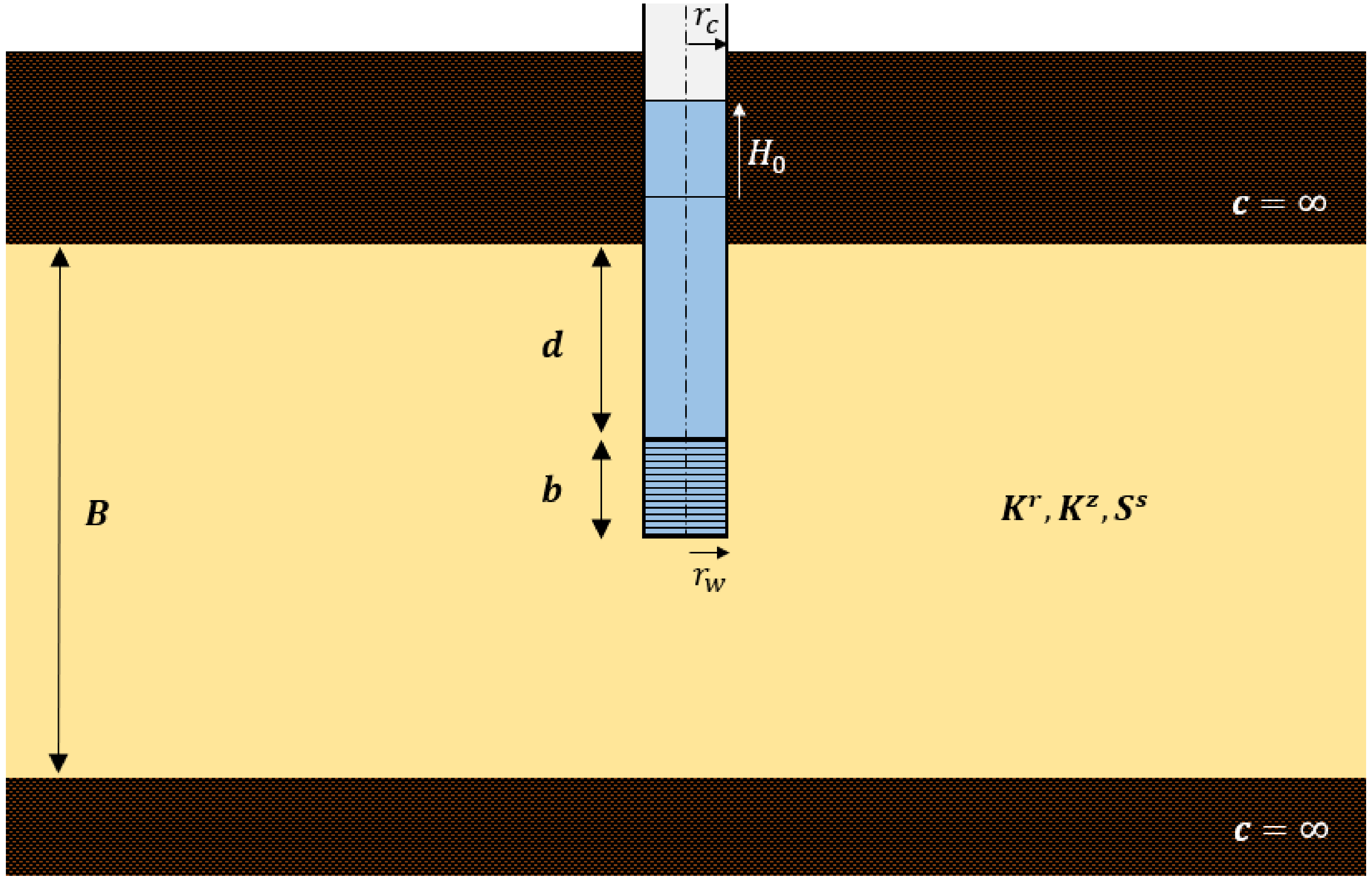
- Add/remove water (slug) to a well
- Monitor head change in well



KGS MODEL

- Simulation of a slug test in a partially penetrating well
- Semi-analytical solution applying integral transforms
- Developed at Kansas Geological Survey (KGS)
- Originally written in Fortran
- Python implementation available with MAxSyPy

```python
from maxsypy.kgs import KGS
```



Water Resources Research

Subsurface Hydrology

Slug tests in partially penetrating wells

Zafar Hyder, James J. Butler Jr., Carl D. McElwee, Wenzhi Liu

First published: November 1994 | https://doi.org/10.1029/94WR01670 | Citations: 139

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KGS MODEL: ASSUMPTIONS

- Flow:
  - Axisymmetric
  - Transient-state
  - Horizontal + **vertical**
- Well:
  - **Partially penetrating**
  - Instantaneous initial head change
  - Finite radius → wellbore storage!
  - Finite-thickness skin
- Aquifer:
  - Homogeneous
  - Constant saturated thickness
  - Laterally unbounded
  - Confined or leaky (= static water table)

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SLUG TESTS IN DAMME

- **Slug test campaign** conducted on wells of the groundwater monitoring network
- Multi-level **well 3-0523** with 3 screens in different geological formations

- Number of slug test performed:
  - Screen 1: 6 tests
  - Screen 2: 6 tests
  - Screen 3: 4 tests

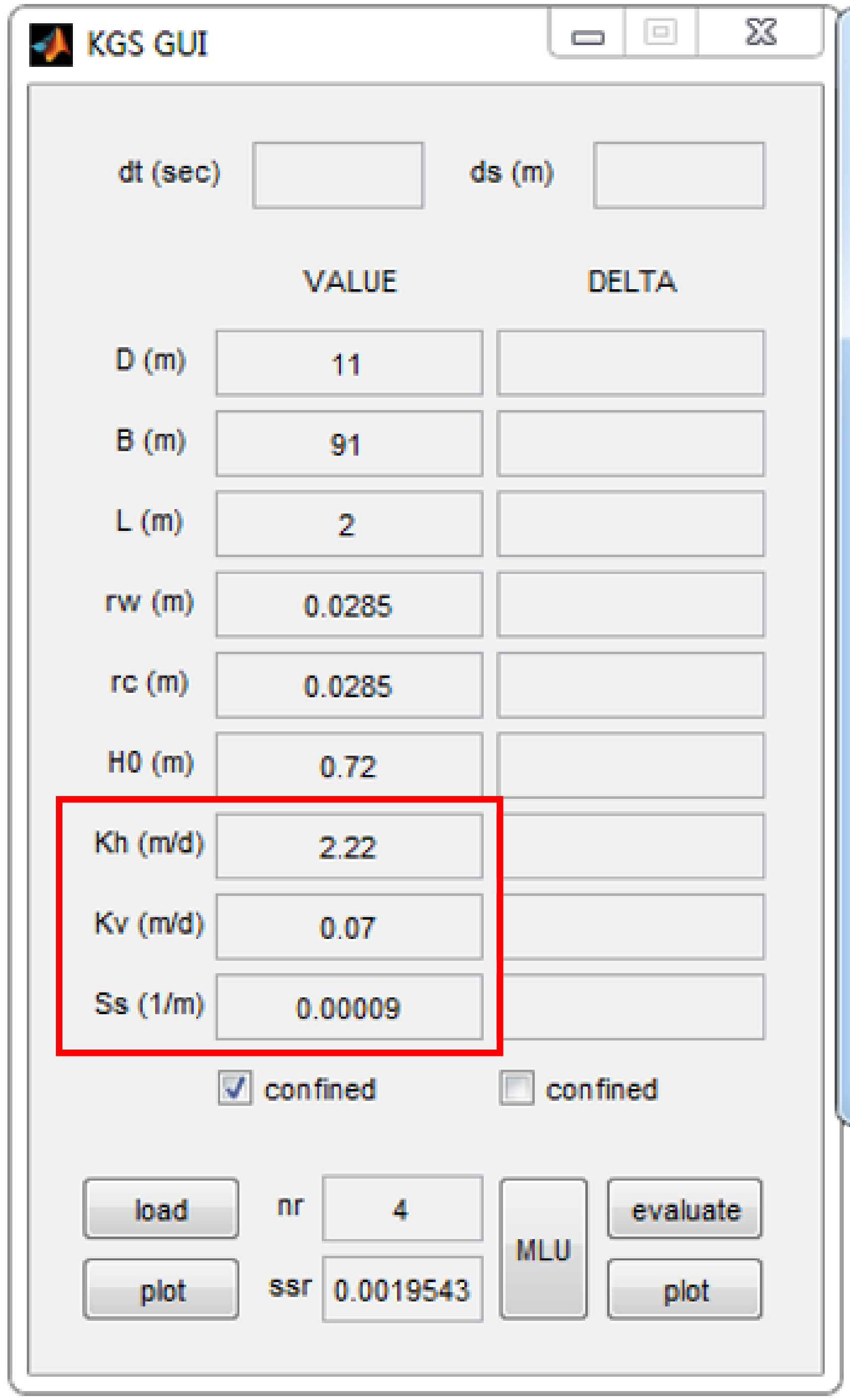
Depth (m)	Lithology	Well
0,0 13,5 15,5 20,0	fine sand	1
20,0 40,0 42,0 58,0	very fine sand	2
58,0 80,0	fine to medium sand	
80,0 87,0	clay	
87,0 91,0 93,0 98,0	clayey sand	3
98,0 200?	heavy clay	

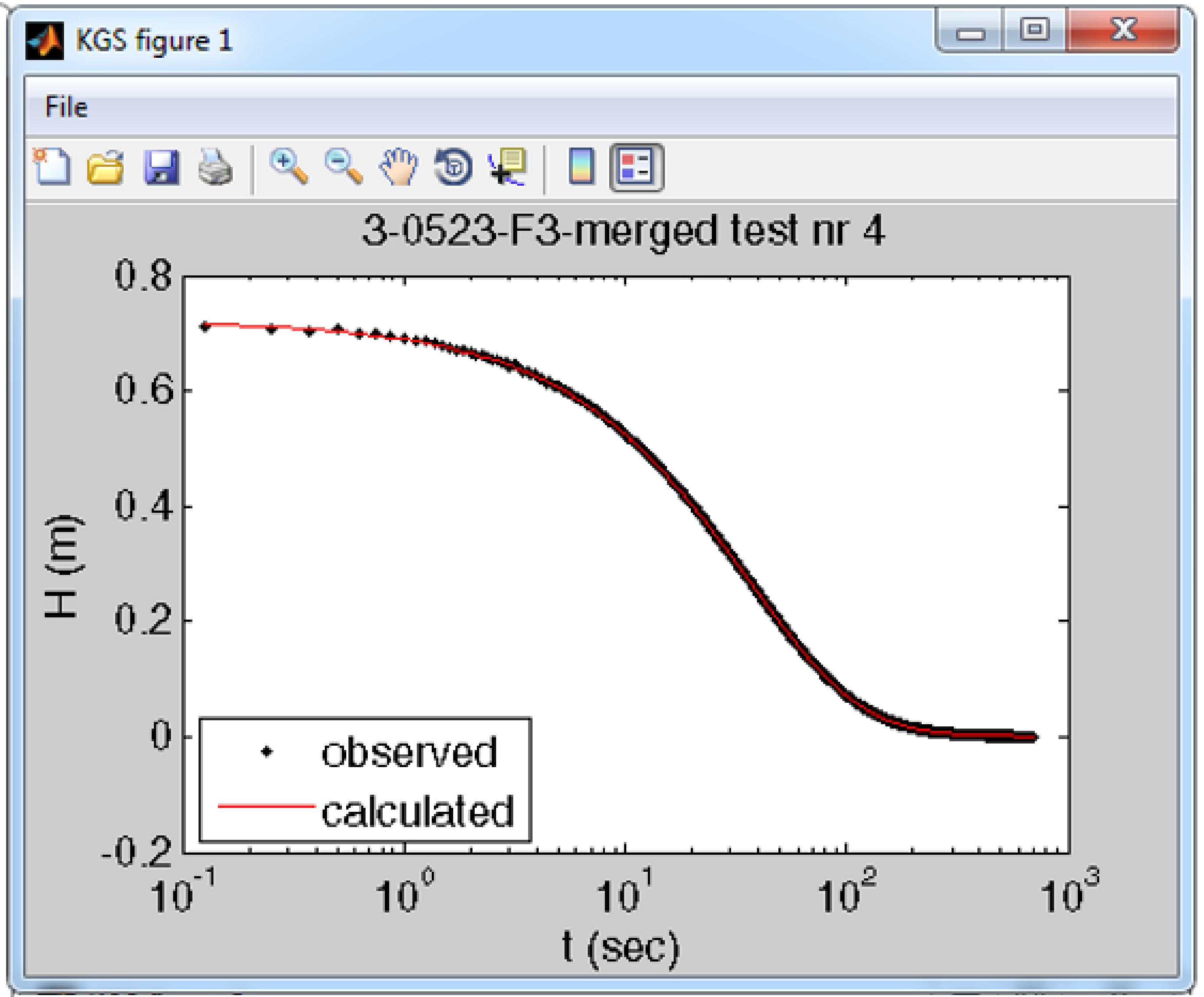
FLANDERS
ENVIRONMENT AGENCY

Flanders
State of the Art

INTERPRETATION OF TEST 4 IN SCREEN 3

Manual curve-fitting using KGS model:





INTERPRETATION OF TEST 4 IN SCREEN 3

KGS model +
Nelder-Mead method
to minimize MSE:

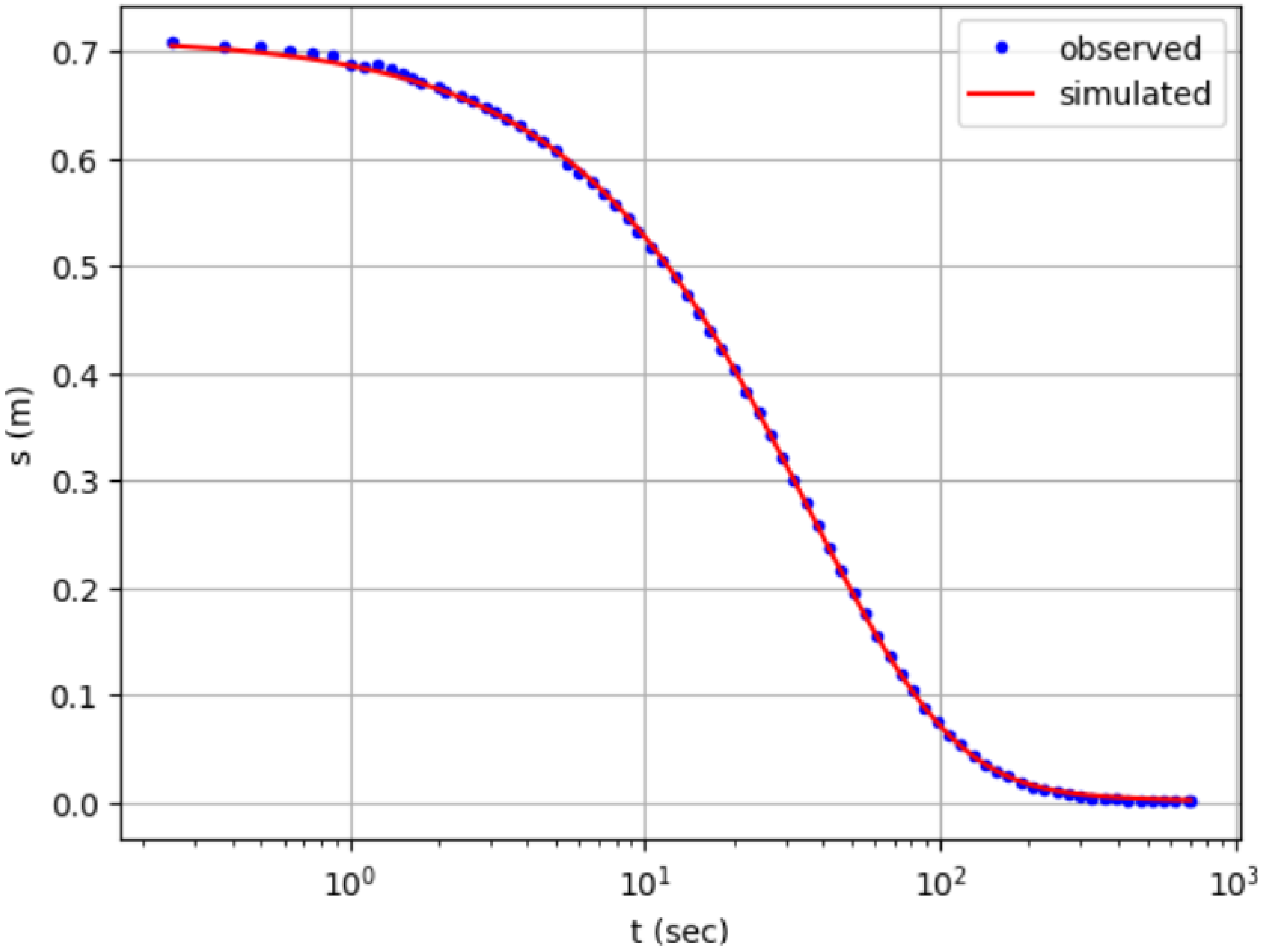
```
from scipy.optimize import fmin

x0 = np.array([0, 0, -5])  # starting values for logP
x = fmin(func=MSE, x0=x0)
print(f'\t MSE = {MSE(x)}')

Kr, Kz, Ss = tuple(10**x)
print('\nOptimal values:')
print(f'\t Kr = {Kr} m/d')
print(f'\t Kz = {Kz} m/d')
print(f'\t Ss = {Ss} m^-1')
```

```
Optimization terminated successfully.
         Current function value: 0.000005
         Iterations: 232
         Function evaluations: 412
         MSE = 4.58556435401376e-06
```

```
Optimal values:
         Kr = 3.2437663500170486 m/d
         Kz = 0.0001843799706562987 m/d
         Ss = 1.718446610208439e-06 m^-1
```



A THEORETICAL CASE STUDY

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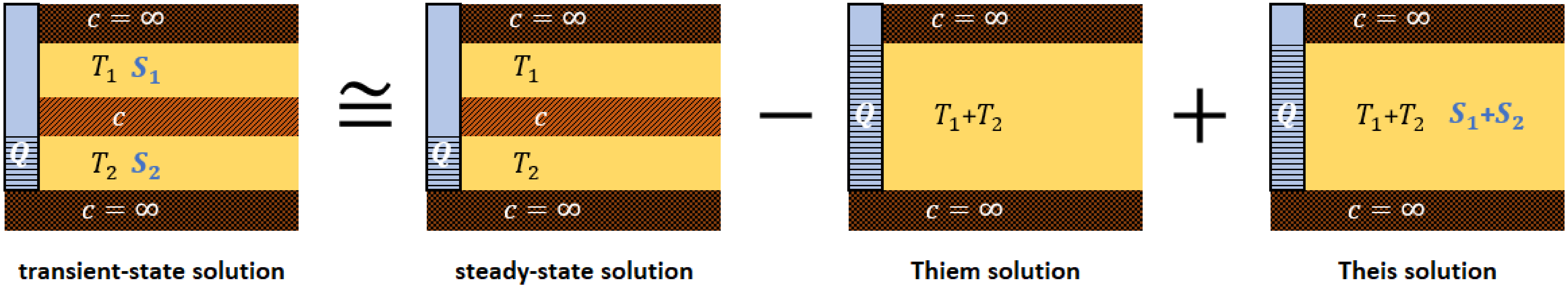
LARGE TIME APPROXIMATION

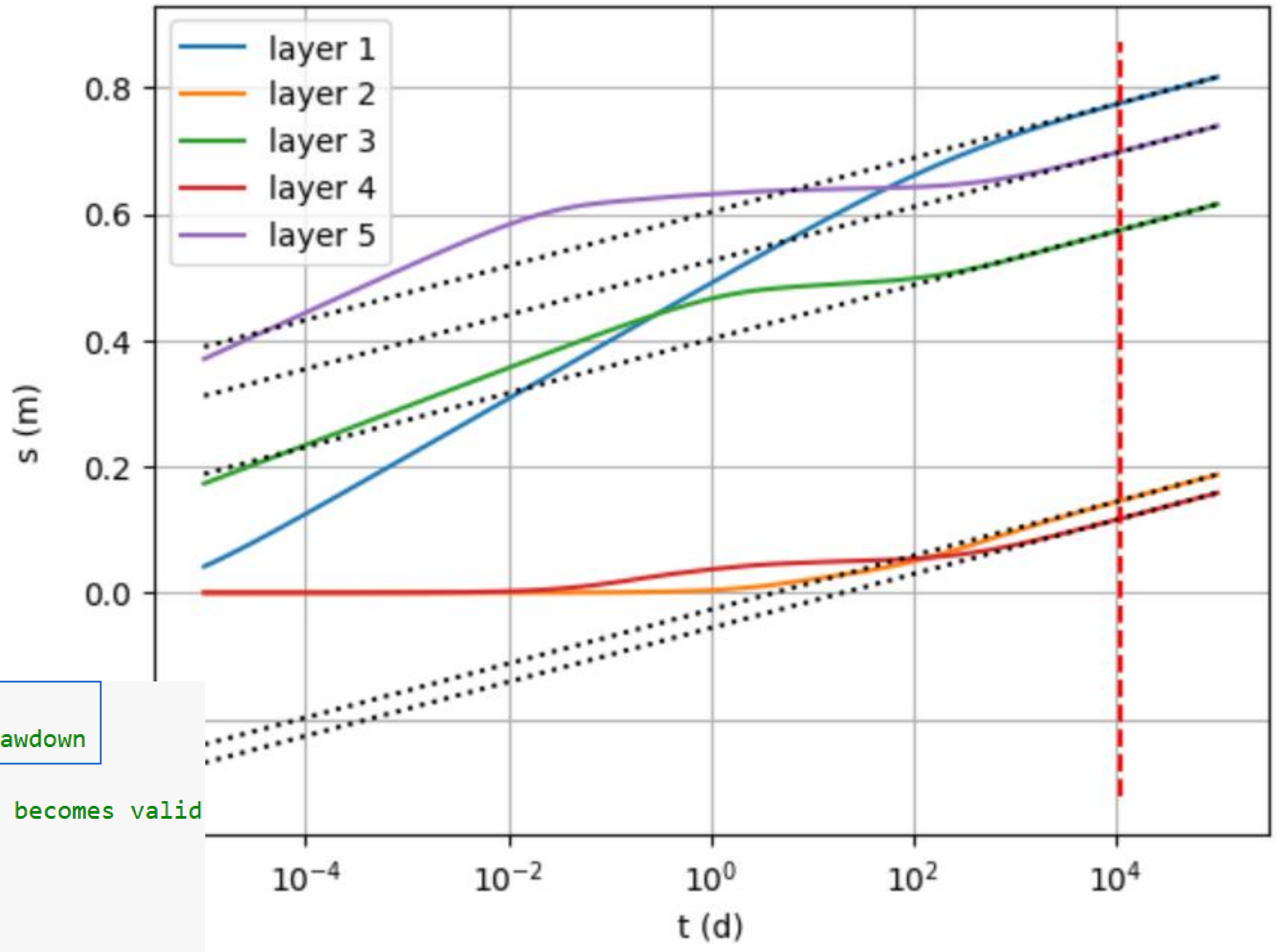
for confined multilayer well-flow:

$$\boldsymbol{s}(\boldsymbol{i},\boldsymbol{r},\boldsymbol{t}) \sim \boldsymbol{s}_{\boldsymbol{steady}}(\boldsymbol{i},\boldsymbol{r}) - \boldsymbol{s}_{\boldsymbol{thiem}}(\boldsymbol{r}) + \boldsymbol{s}_{\boldsymbol{theis}}(\boldsymbol{r},\boldsymbol{t}) \qquad (\boldsymbol{t} \to \infty)$$



EXAMPLE

$$s_{trans} \sim s_{steady} - s_{thiem} + s_{theis}$$




```
plt.semilogx(t, s_trans);  # exact drawdown
plt.semilogx(t, s_steady - s_thiem + s_theis, 'k:');  # approximate drawdown

t_approx = 10 * S.sum() * c.sum()  # time from which the approximation becomes valid
plt.semilogx([t_approx, t_approx], plt.ylim(), 'r--');

plt.legend([f'layer {i+1}' for i in range(transient.nl)]);
plt.grid();
plt.xlabel('t (d)');
plt.ylabel('s (m)');
```

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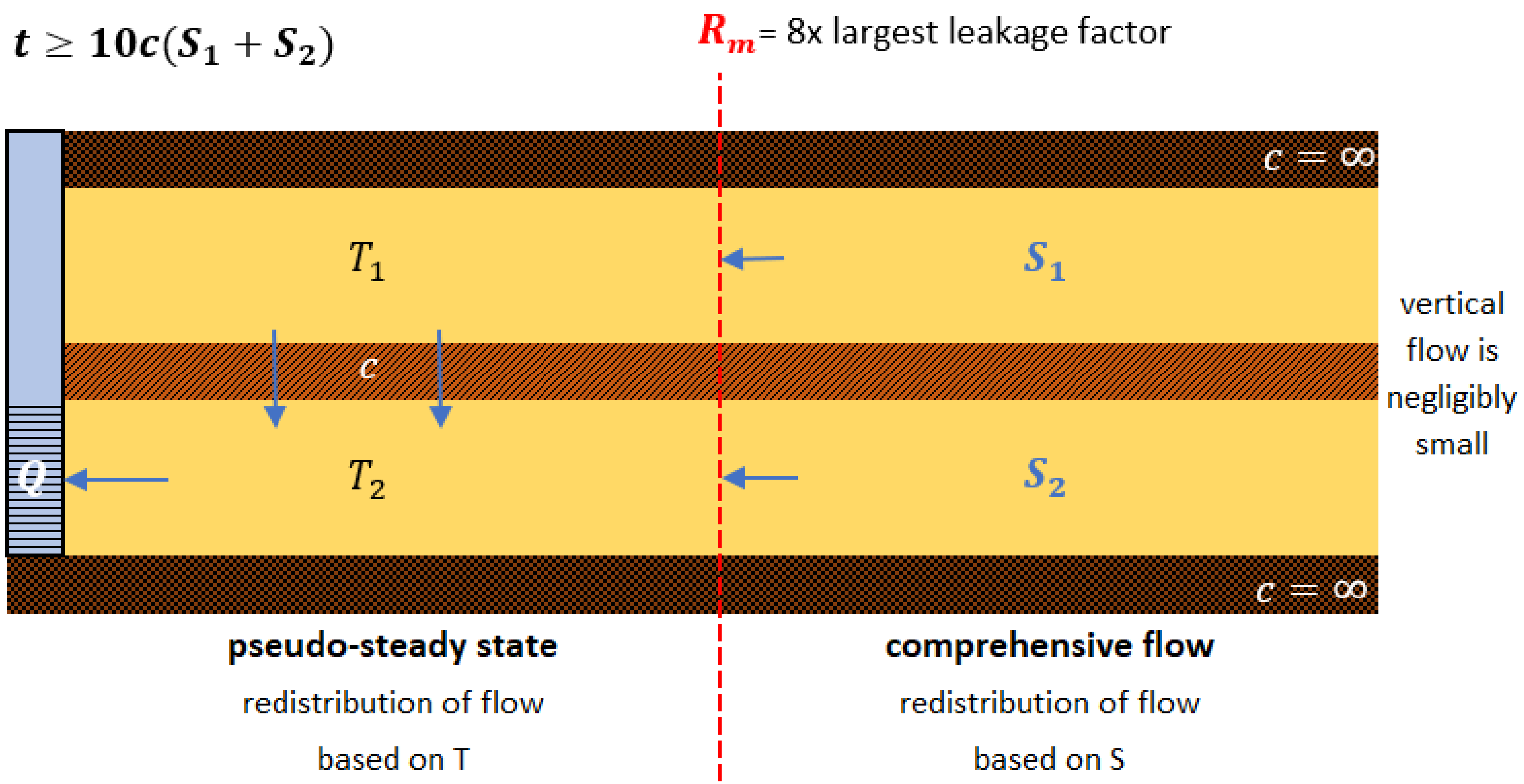
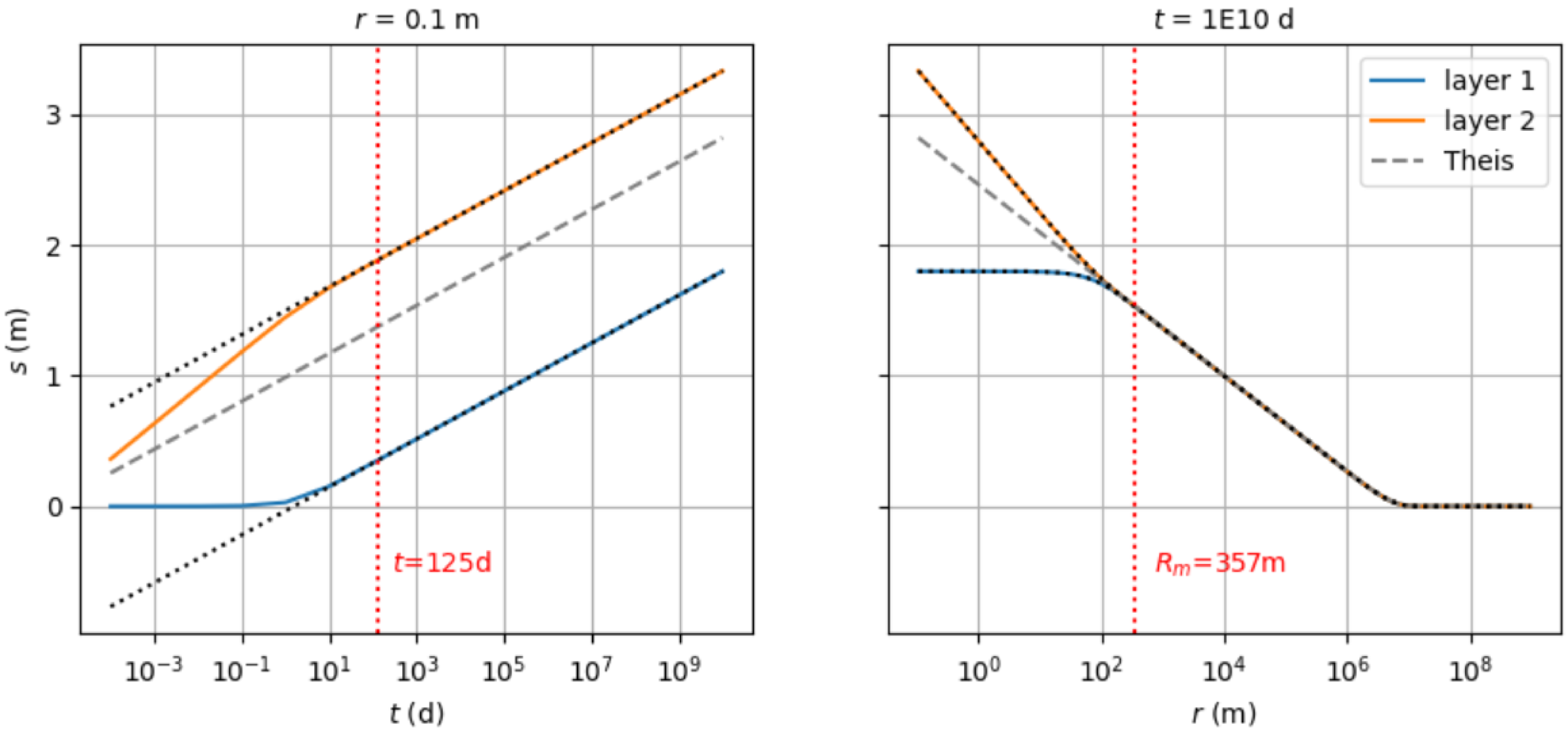
PSEUDO-STEADY STATE

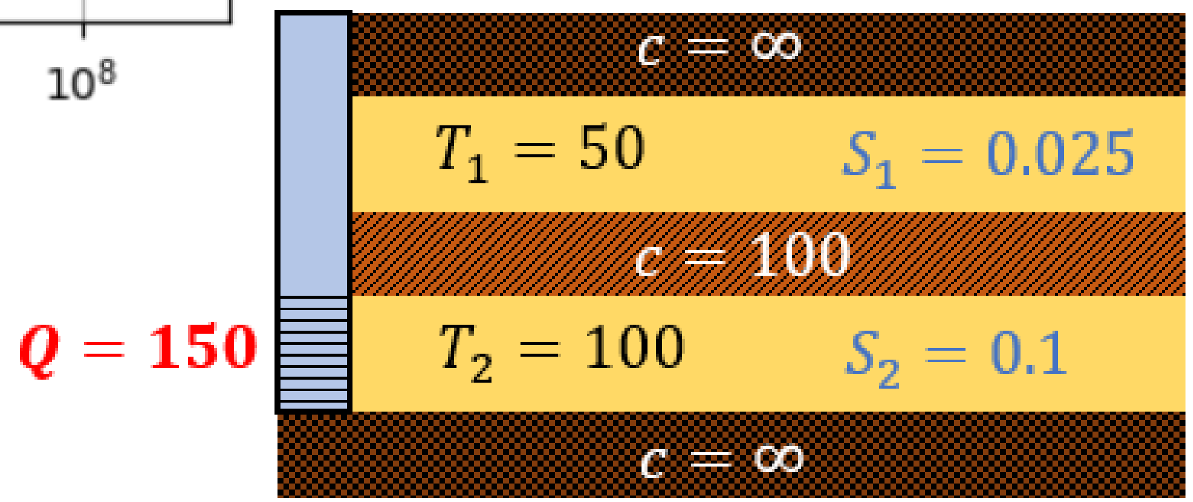
also called *steady shape* (Bohling et al., 2002)



EXAMPLE



- solid lines: exact
- dotted lines: approximate



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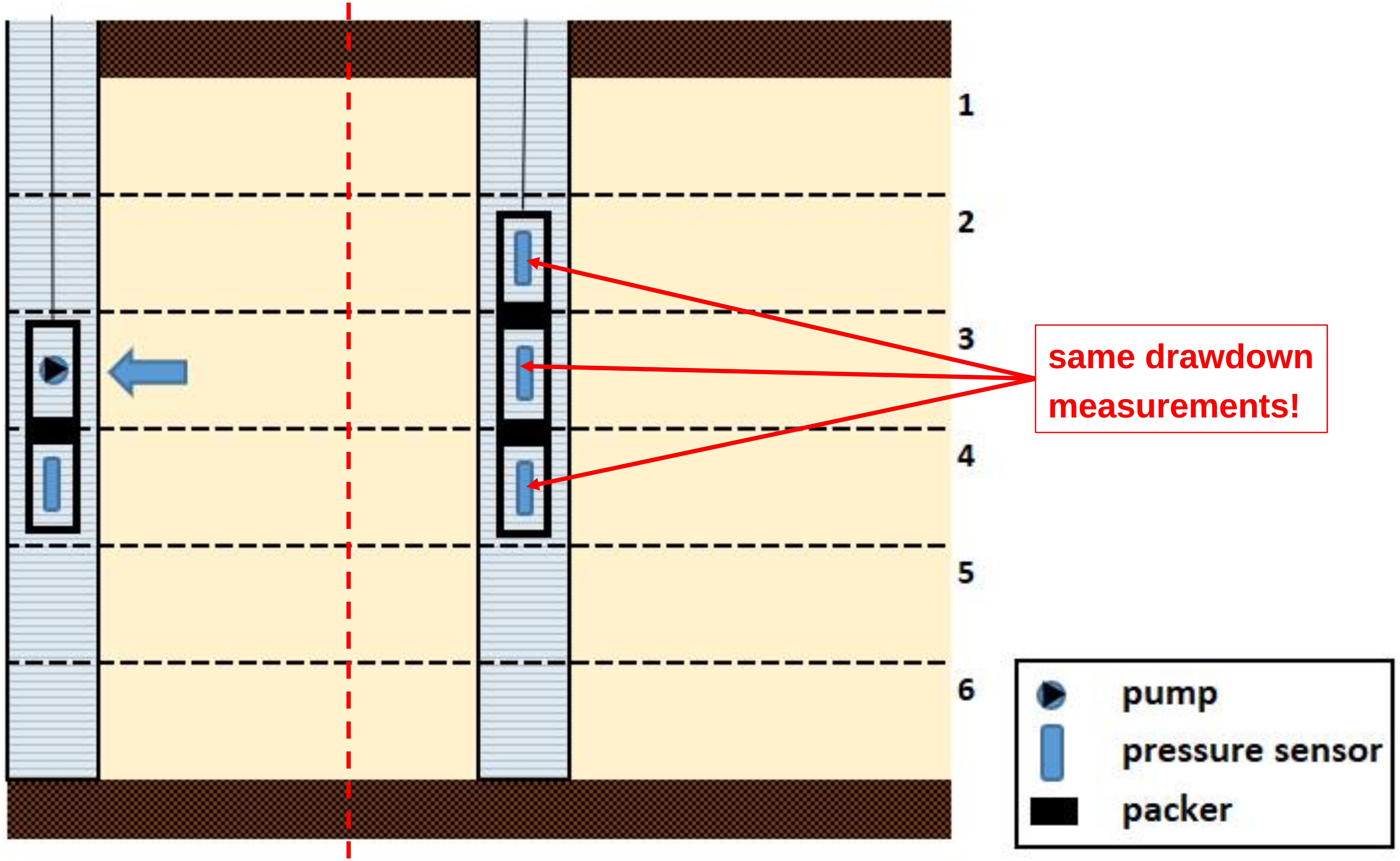
HYDRAULIC TOMOGRAPHY



R_m??

= sequence of small-scale short-term pumping tests

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CONCLUSIONS

- Theis curves (again) at large values of time
- inversion of flow in the distal zone possible
- spatial averaging in drawdown measurements
- multilevel pumping tests have inherent limitations

groundwater NGWA The Groundwater Association

Inherent Limitations of Hydraulic Tomography

Geoffrey C. Bohling, James J. Butler Jr.

First published: 22 September 2010 | https://doi.org/10.1111/j.1745-6584.2010.00757.x | Citations: 63

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A PRACTICAL CASE STUDY

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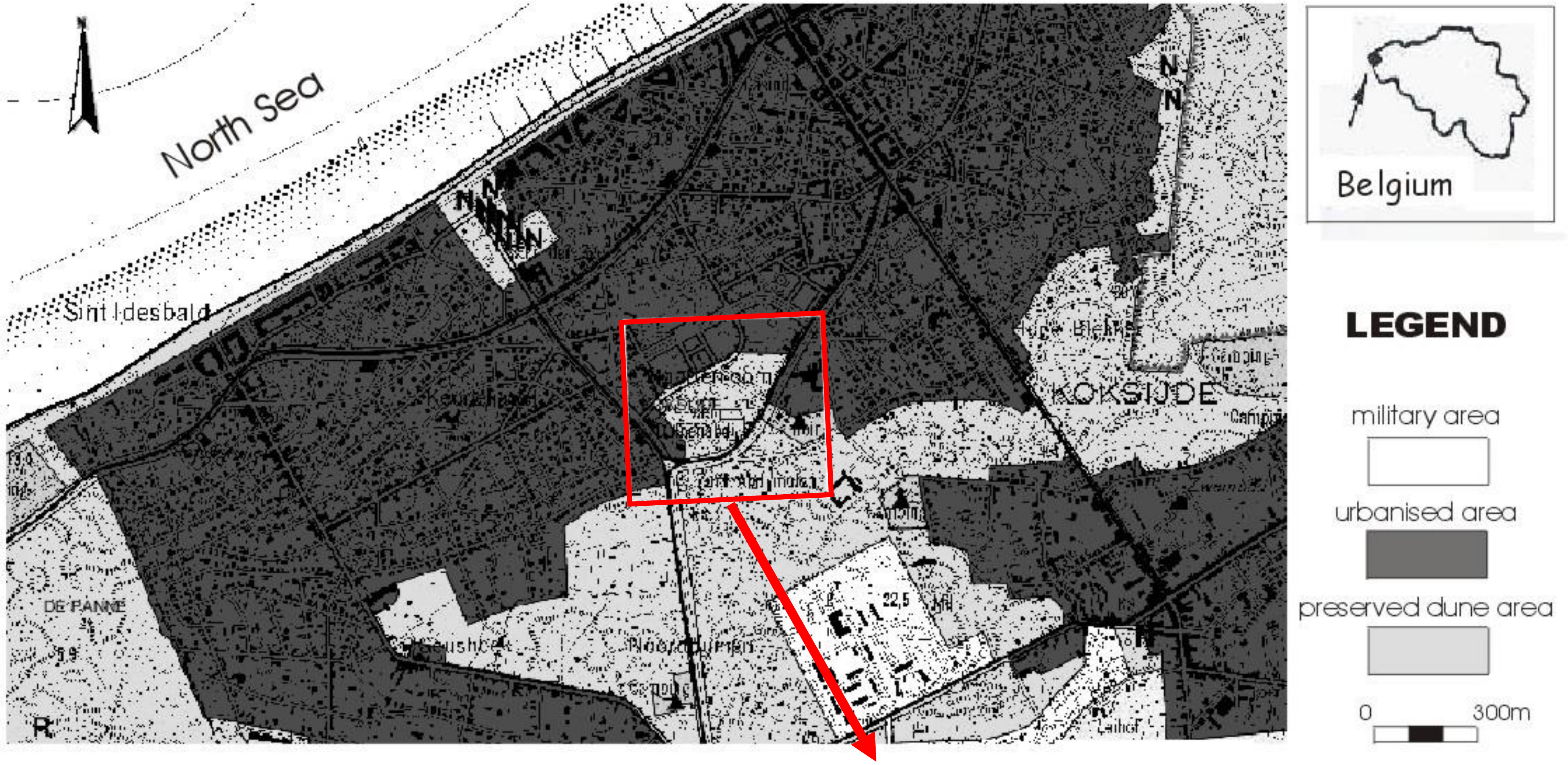
OPTIMIZING A DRAINAGE SYSTEM

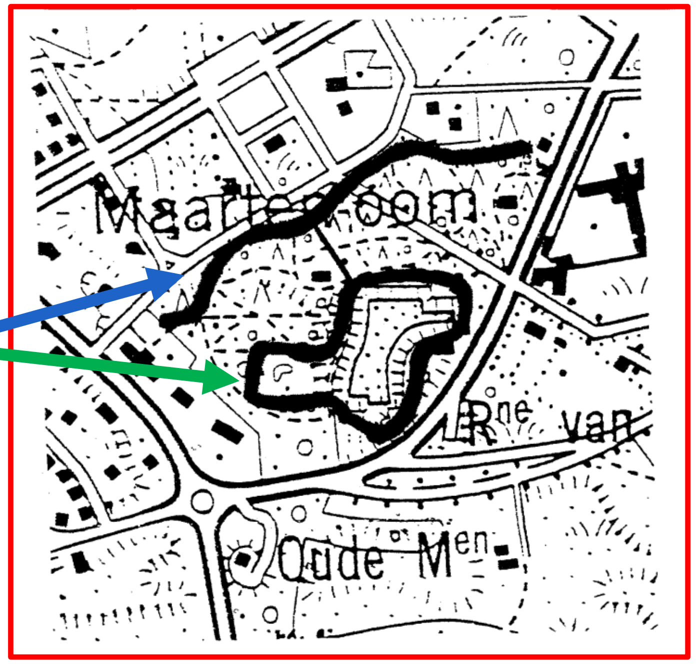
Excavation site “Duinenabdij”

(Koksijde, Belgium)

- Valuable dune area in the Belgian coastal plane
- Multilayer aquifer system
- Shallow semi-pervious layer caused flooding during the winter



- **Combined system of pumping and injection wells**
  - Pumping to drain excess groundwater
  - Re-injecting the extracted groundwater
  - Re-injecting to protect the dunes in the north



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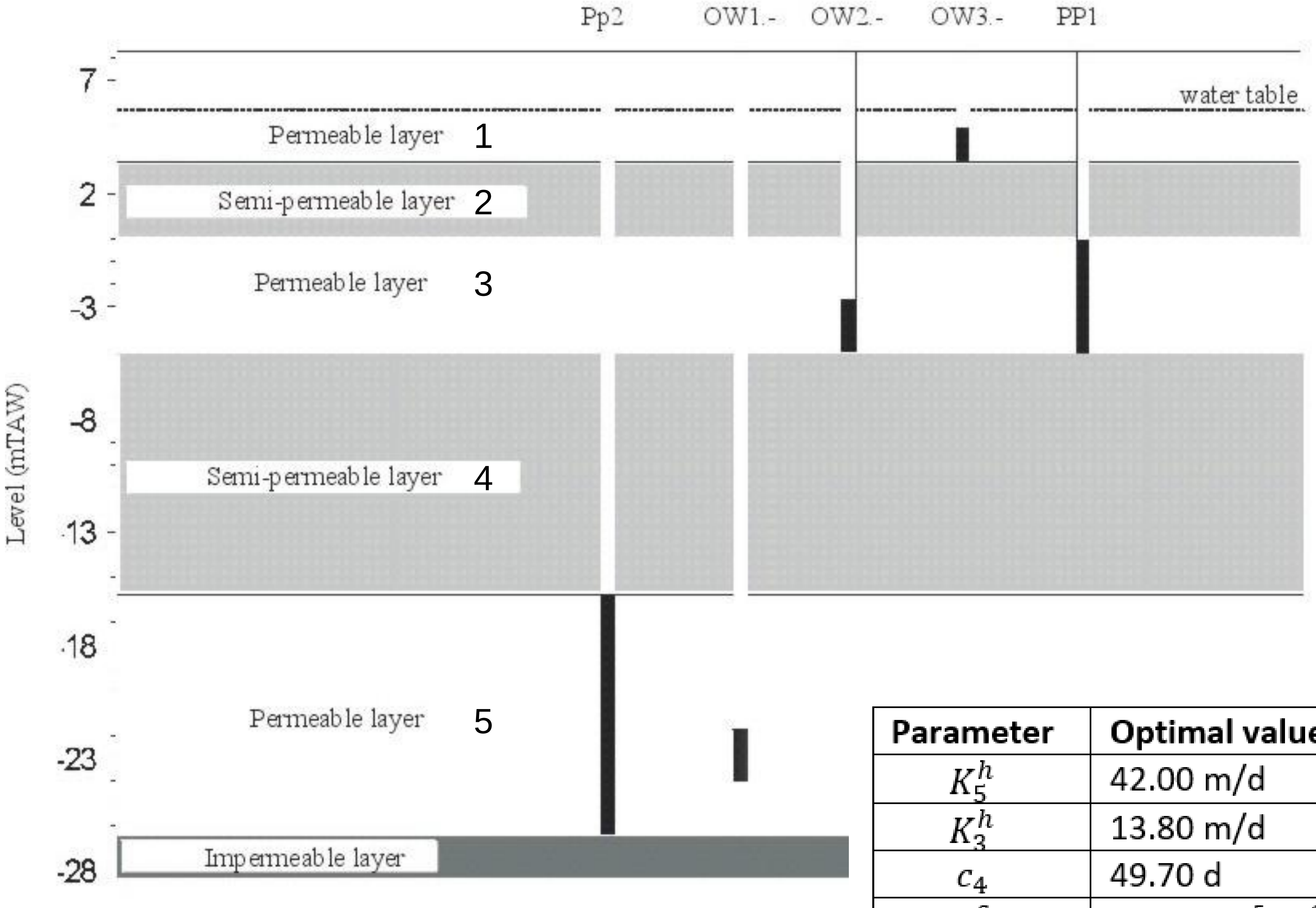
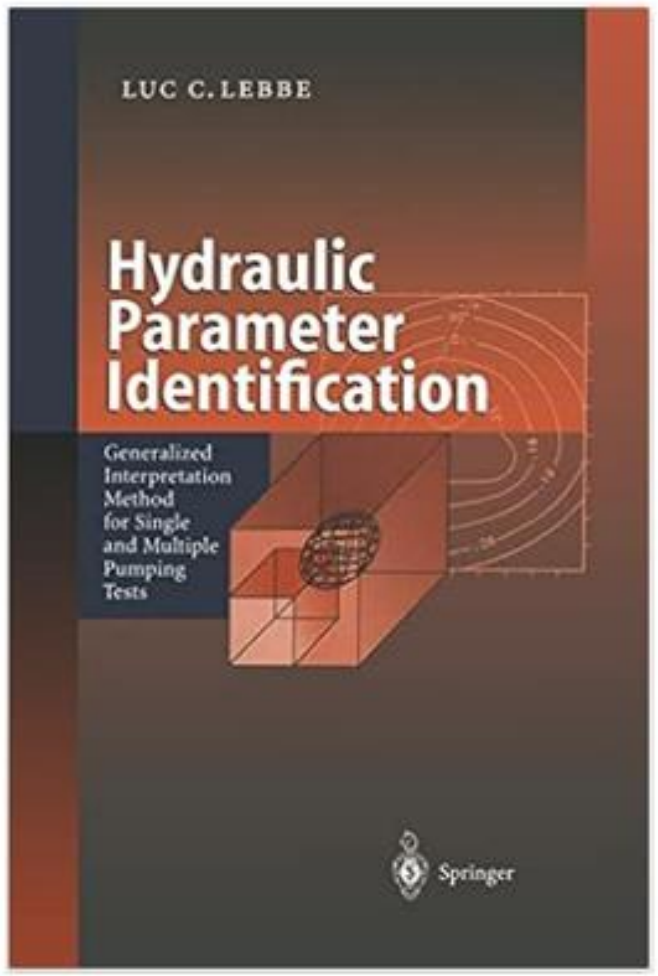
THE MULTILAYER AQUIFER SYSTEM



Double pumping test:

- Pumping test on layer 3
- Pumping test on layer 5
- Observations in all layers
- Simultaneous interpretation

Parameter	Optimal value
K_5^h	42.00 m/d
K_3^h	13.80 m/d
c_4	49.70 d
S_5^s	7.12 x 10^{-5} m^{-1}
$S_2^s = S_3^s$	7.80 x 10^{-5} m^{-1}
S_4^s	2.09 x 10^{-5} m^{-1}
c_2	735.00 d



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SIMULATING THE DRAINAGE SYSTEM

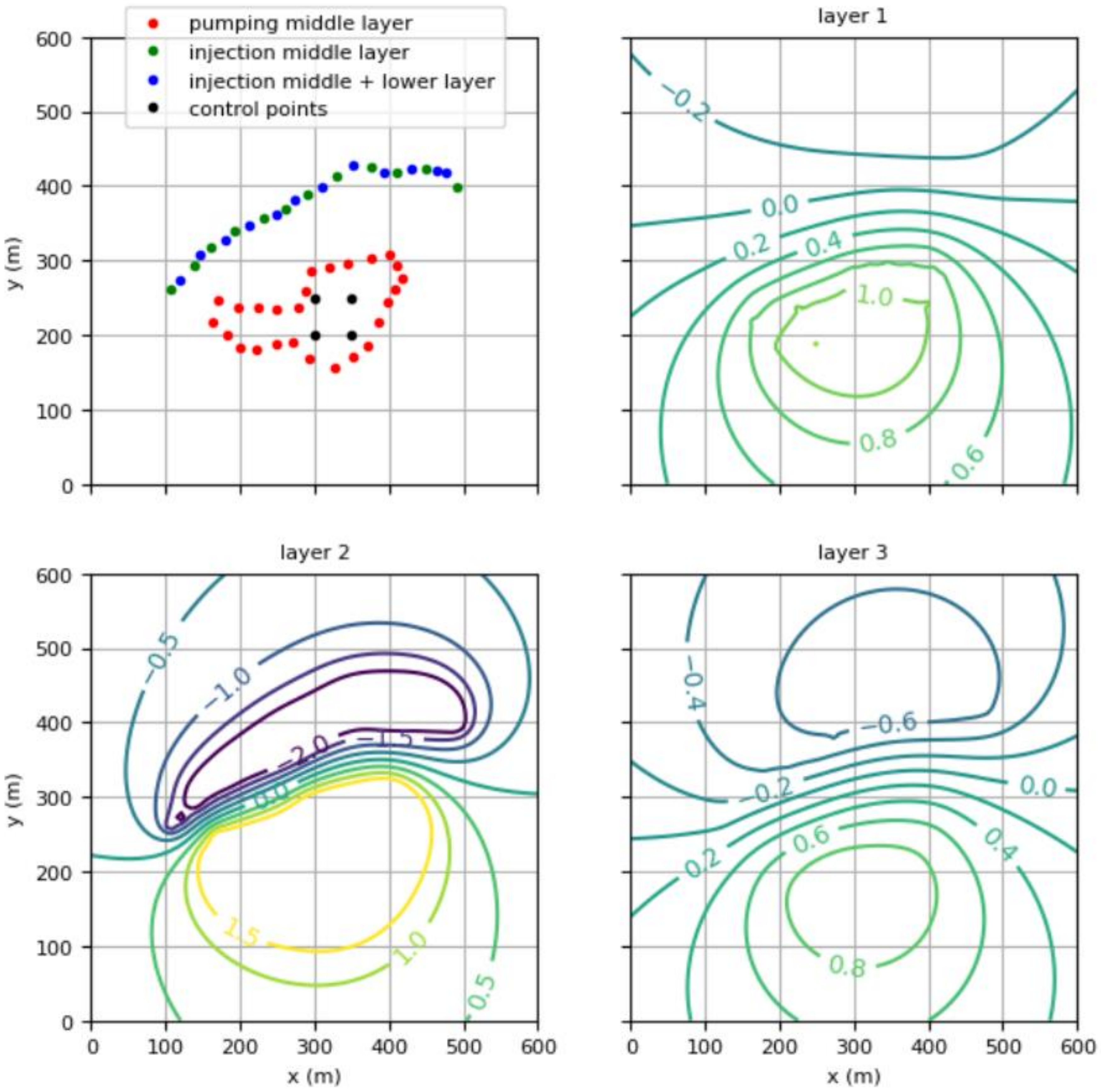
alouwyck / PhD

- Analytical solution:
  - steady-state
  - axisymmetric flow
  - confined system
  - 3 homogeneous layers
  - 2 separating resistances

- Superposition in space

$$s_i(x,y,t) = \sum_{p=1}^{n_w} \frac{Q_p}{Q_{ref}} s_i(r_p,t)$$

$$\text{with } r_p = \sqrt{(x_p - x)^2 + (y_p - y)^2}$$



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OPTIMIZING THE DRAINAGE SYSTEM

alouwyck / PhD

= Minimizing the total pumping rate:

- Linear programming
- Python package PuLP

```
!pip install pulp
from pulp import *
```

- Constraints:
  - 4 control points
  - drawdown min 1m

$s \geq 1$

```python
# instantiating LpProblem object (linear programming problem)
prob = LpProblem("Scenario_1", LpMinimize)  # it's a minimization problem

# defining the variables
Q_pump1 = LpVariable("Q_pump", lowBound=0)  # pumping rate middle layer (Q > 0)
Q_inj1 = LpVariable("Q_inj", upBound=0)  # injection rate middle layer (Q < 0)

# defining the objective function
prob += Q_pump1, "minimize total pumping rate"

# adding constraint Q_out == Q_in
prob += npw * (Q_grav + Q_pump1) + niw * Q_inj1 + niw / 2 * Q_deep == 0

# add constraint at the 4 control points
for i in range(len(xc)):
    # drawdown in top layer must be at least 1 m: P1*s1 + P2*s2 + P3*s3 + P4*s4 >= 1
    prob += Q_grav * s1[i] + Q_pump1 * s2[i] + Q_inj1 * s3[i] + Q_deep * s4[i] >= 1.0

# solving the problem
print(prob.solve())
print(LpStatus[prob.status])  # checking the status of the solution
print(Q_pump1.value(), Q_inj1.value(), Q_deep)  # checking the optimized variables
```

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CONCLUSIONS

- The **combined system of pumping and injection wells** is effective in creating local drawdown and protecting the surrounding dunes
- A **hydrogeological study including field tests** was necessary to reliably characterize the hydraulic properties of the aquifer system
- The **analytical multilayer solution** can be applied to efficiently solve a real-world problem without having to build a computationally expensive model
- **Linear programming** is an effective way of minimizing pumping rates subject to given drawdown constraints
- It is straightforward to implement both the analytical solution and the linear programming using **Python's packages for scientific computing**.

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