



Mohammed VI Polytechnic University

College of Agriculture and Environmental Sciences

International Water Research Institute (IWRI)

Organizes a workshop:
Mathematical Models and
Artificial Intelligence for
Solving Groundwater Flow Problems

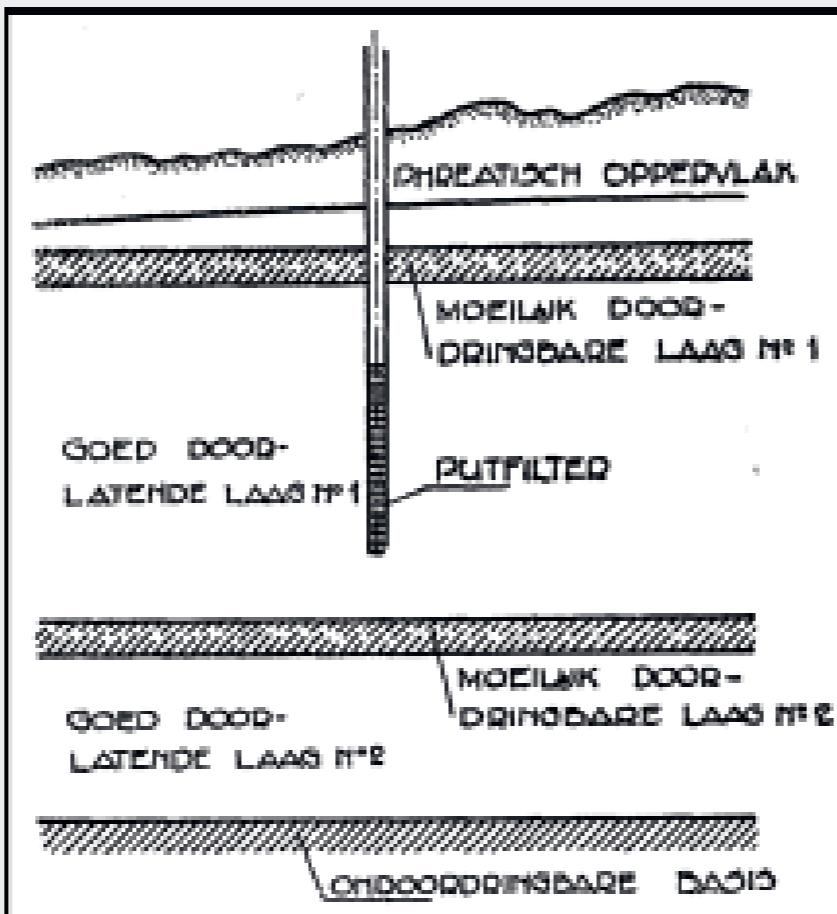
by Dr. Andy Louwyck

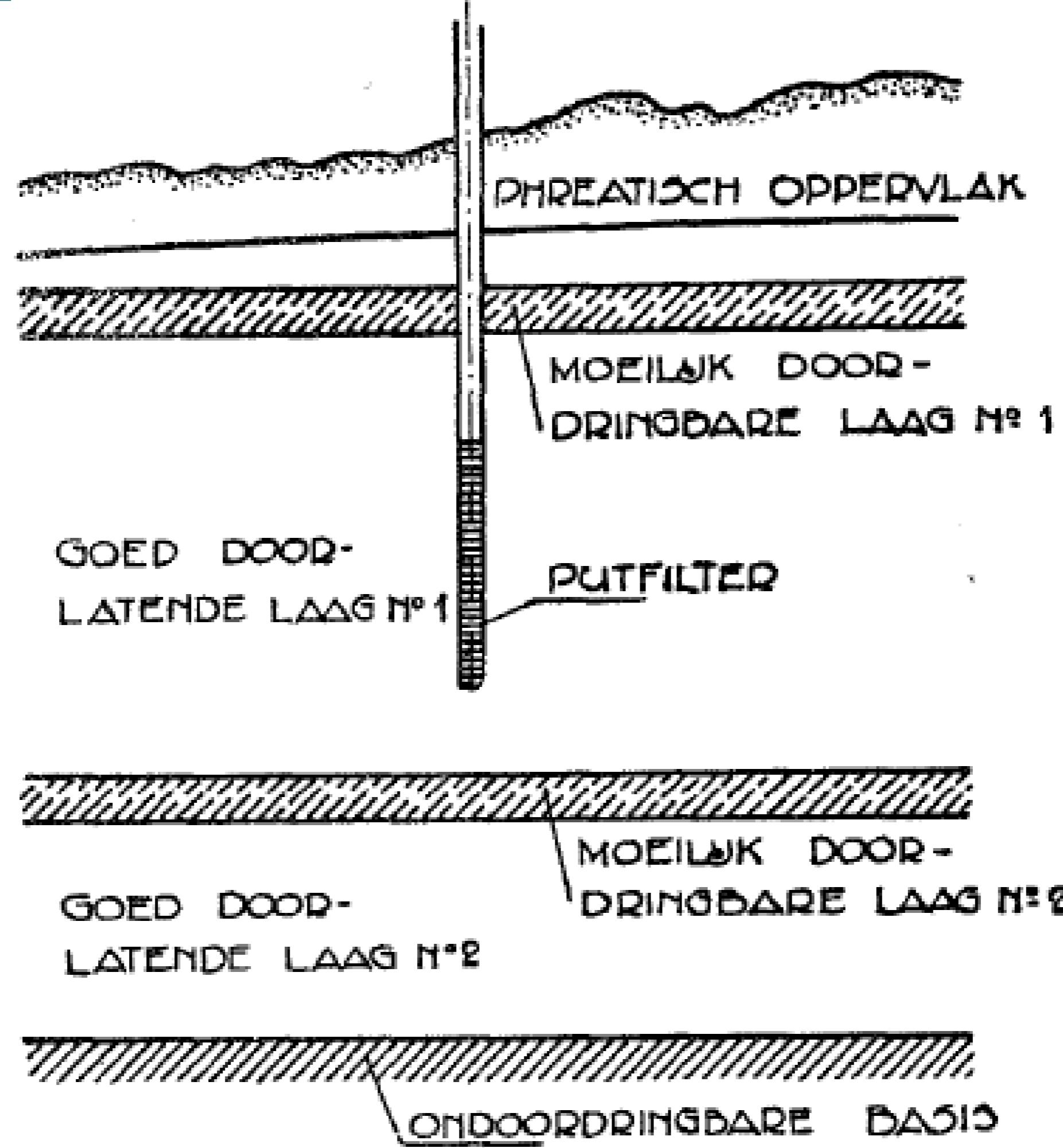
06 and 07 January 2025



DAY 1

Axisymmetric Flow in Multilayer Aquifer Systems: Solutions and Theoretical Considerations





OVER GRONDWATERSTROOMINGEN
BIJ WATERONTTREKKING DOOR
MIDDEL VAN PUTTEN.

PROEFSCHRIFT

TER VERKRIJVING VAN DEN GRAAD VAN
DOCTOR IN DE TECHNISCHE WETENSCHAP
AAN DE TECHNISCHE HOOGESCHOOL TE
DELFTE, OP GEZAG VAN DEN RECTOR MAG-
NIFICUS IR. F. WESTENDORP, HOOGLEERAAR
IN DE AFDEELING DER WERKTUIGBOUWKUNDE
EN SCHEEPSBOUWKUNDE, VOOR
EENE COMMISSIE UIT DEN SENAAT TE
VERDEDIGEN OP WOENSDAG 2 APRIL 1930,
DES NAMIDDAGS TE 3 UUR,

DOOR

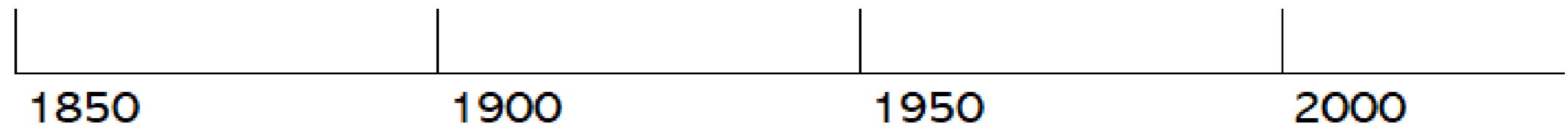
GERRIT JAN DE GLEE,
CIVIEL-INGENIEUR,
GEBOREN TE ASSEN.

SOLUTION METHODS

analytical models

numerical models

artificial intelligence



RELEVANCE

≡ Google Scholar "analytical model" groundwater 🔍

Articles About 1.950 results (0,31 sec)

Any time [Comprehensive two-dimensional analytical modeling of groundwater levels in bi-directional sloping heterogeneous aquifers under variable recharge conditions](#)
Since 2024
Since 2023
Since 2020
Custom range...
PC Hsieh, MC Wu - Journal of Hydrology, 2024 - Elsevier
... of groundwater ... **analytical model** effectively captures varying recharge dynamics, both spatially and temporally, contributing to a better understanding and management of groundwater ...
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≡ Google Scholar "artificial intelligence" groundwater 🔍

Articles About 8.730 results (0,13 sec)

Any time [\[HTML\] Artificial Intelligence Advancements for Accurate Groundwater Level Modelling: An Updated Synthesis and Review](#)
Since 2024
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Custom range...
S Pourmorad, M Kabolizade, LA Dimuccio - Applied Sciences, 2024 - mdpi.com
... , have proven to be important tools for accurate groundwater level (GWL) modelling. Through an ... complex and nonlinear relationships in groundwater data, providing more accurate and ...
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THE AI HYPE

Artificial intelligence curve matching method for pumping tests in leaky systems

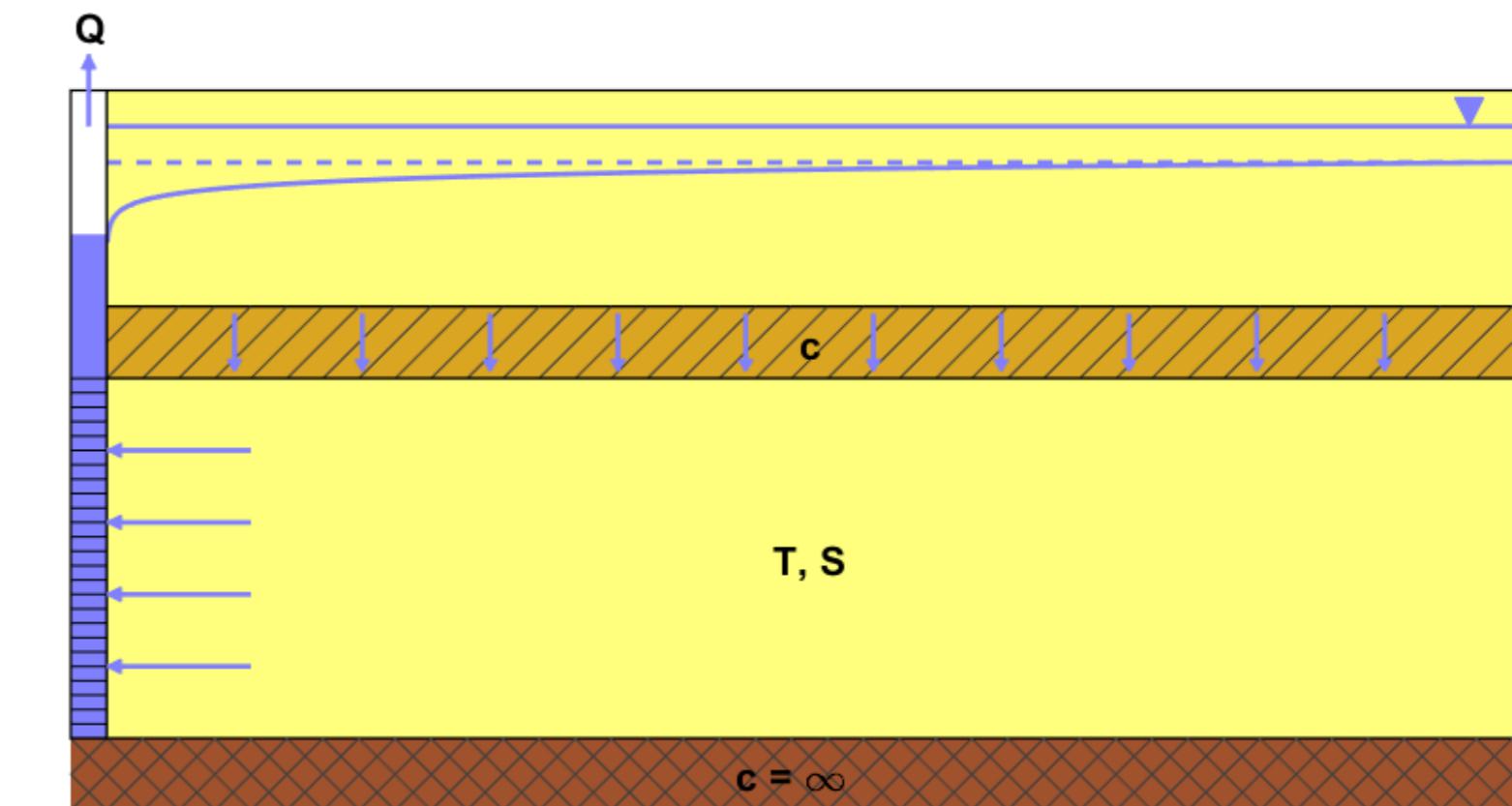
Yunyun Zhao* Yong Zhang,
Hebei University of Engineering, Handan 056038, Hebei, China
*Corresponding author.
E-mail address: zhang_yong001@163.com



Abstract

For the unsteady-state flow pumping test, the standard curve fitting method is generally used to calculate the hydrogeological parameters. Because of differences from person to person, there will be visual errors in artificial curve fitting, especially when the well function is a family of curves, and there are countless standard curves in theory. It is impossible to draw all the standard curves in the standard curve template for curve fitting; thus, the manual curve fitting error will be very large. Therefore, the curve fitting method has been greatly limited in practical application. In this paper, the random weight particle swarm optimization algorithm (RandWPSO) is applied to the curve matching calculation for the unsteady-state flow pumping test in a leaky system, and intelligent optimization curve fitting of the curve family well function is performed. The calculation results show that the curve fitting parameters selected by artificial intelligence can be as accurate as $r/B=0.3579764$, and the curve fitting accuracy is much higher than that of manual selection, $r/B=0.35$. Artificial intelligence curve fitting avoids observation error due to manual curve fitting, solves the problem that it is difficult to apply the standard curve fitting method of the curve family in practice, and makes the curve fitting method more practical.

Keywords: leaky aquifers; curve fitting; artificial intelligence optimization; pumping test; RandWPSO



Hantush-Jacob (1955) model:

$$s(r, t) = \frac{Q}{4\pi K D} W \left(\frac{r^2 S}{4t K D}, r \sqrt{\frac{1}{c K D}} \right)$$

Table 2 Comparative analysis of the calculation results of different curve fitting methods

Calculation parameters	Calculation results of different curve fitting methods			
	Composite Simpson's rule; set $m=1500$ and $h=1/3000$	Composite Simpson's rule; set $m=5000$ and $h=1/10000$	Directly call built-in MATLAB numerical integral	Manual curve fitting
$W\left(\frac{u}{B}\right)$ calculation method				
r/B	0.3579764	0.3578755	3.578325	0.35
T (m^2/d)	403.9988961794492	404.080110307088	404.1146770792042	414.7
S	$1.39784369448563 \times 10^{-4}$	$1.39764226866348 \times 10^{-4}$	$1.39755658961577 \times 10^{-4}$	1.469×10^{-4}
K_1/M_1	0.001334001987695	0.001333518091093	0.001333311883692	0.0013
f_v	0.011562541890477	0.01156480806024	0.011565926829623	\
Computation time (min)	4.5	15.3	30.5	\

DON'T BE DECEIVED

Artificial intelligence curve matching method for pumping tests in leaky systems

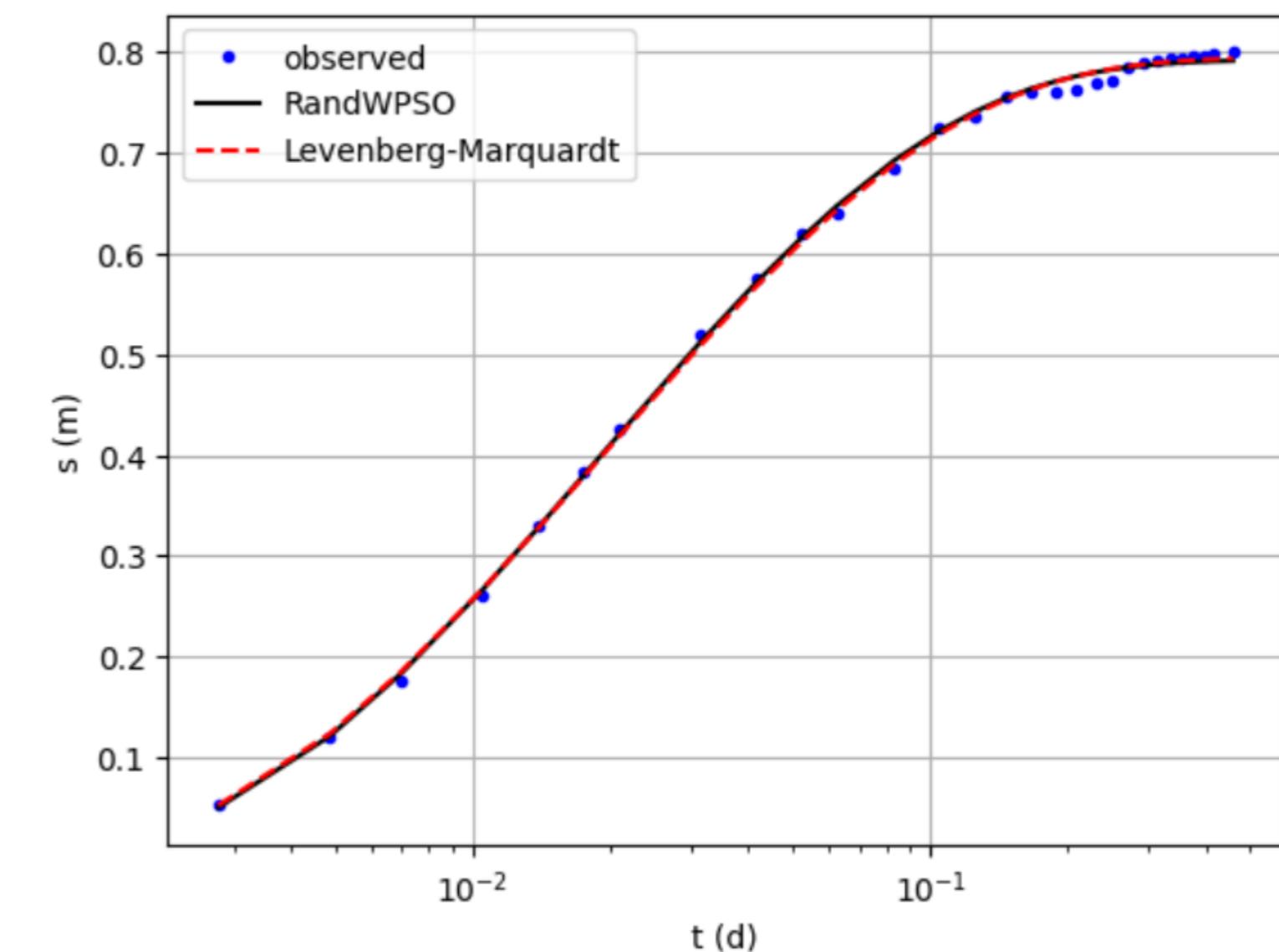
Yunyun Zhao* Yong Zhang,
Hebei University of Engineering, Handan 056038, Hebei, China
*Corresponding author.
E-mail address: zhang_yong001@163.com

Abstract

For the unsteady-state flow pumping test, the standard curve fitting method is generally used to calculate the hydrogeological parameters. Because of differences from person to person, there will be visual errors in artificial curve fitting, especially when the well function is a family of curves, and there are countless standard curves in theory. It is impossible to draw all the standard curves in the standard curve template for curve fitting; thus, the manual curve fitting error will be very large. Therefore, the curve fitting method has been greatly limited in practical application. In this paper, the random weight particle swarm optimization algorithm (RandWPSO) is applied to the curve matching calculation for the unsteady-state flow pumping test in a leaky system, and intelligent optimization curve fitting of the curve family well function is performed. The calculation results show that the curve fitting parameters selected by artificial intelligence can be as accurate as $r/B=0.3579764$, and the curve fitting accuracy is much higher than that of manual selection, $r/B=0.35$. Artificial intelligence curve fitting avoids observation error due to manual curve fitting, solves the problem that it is difficult to apply the standard curve fitting method of the curve family in practice, and makes the curve fitting method more practical.

Keywords: leaky aquifers; curve fitting; artificial intelligence optimization; pumping test; RandWPSO

REJECTED



A model should be as simple as it
can be but no simpler
— Albert Einstein —

```
start = time()
result = least_squares(fun=residuals, x0=x0, method='lm',
                      ftol=1e-12, xtol=1e-12, gtol=1e-12)
print(f"elapsed time: {time() - start} sec")
```

elapsed time: 0.034934043884277344 sec

$r/B = 0.33966118442493454$
 $T = 418.38884431293474$
 $S = 0.0001391052948927095$
 $c = 804.0082714958618$
 $K/M = 0.0012437682987259504$

MY RESUME

FLANDERS
ENVIRONMENT AGENCY



Education:

- 2023: Doctor of Science: Geology (Ghent University)
- 2020: Micro Degree: AI & Data Science (KdG University Antwerp)
- 2015: Associate Degree: IT & Programming (CVO Brussels)
- 2001: Master of Science: Geology (Ghent University)
- 1999: Bachelor of Science: Geology (Catholic University of Leuven)



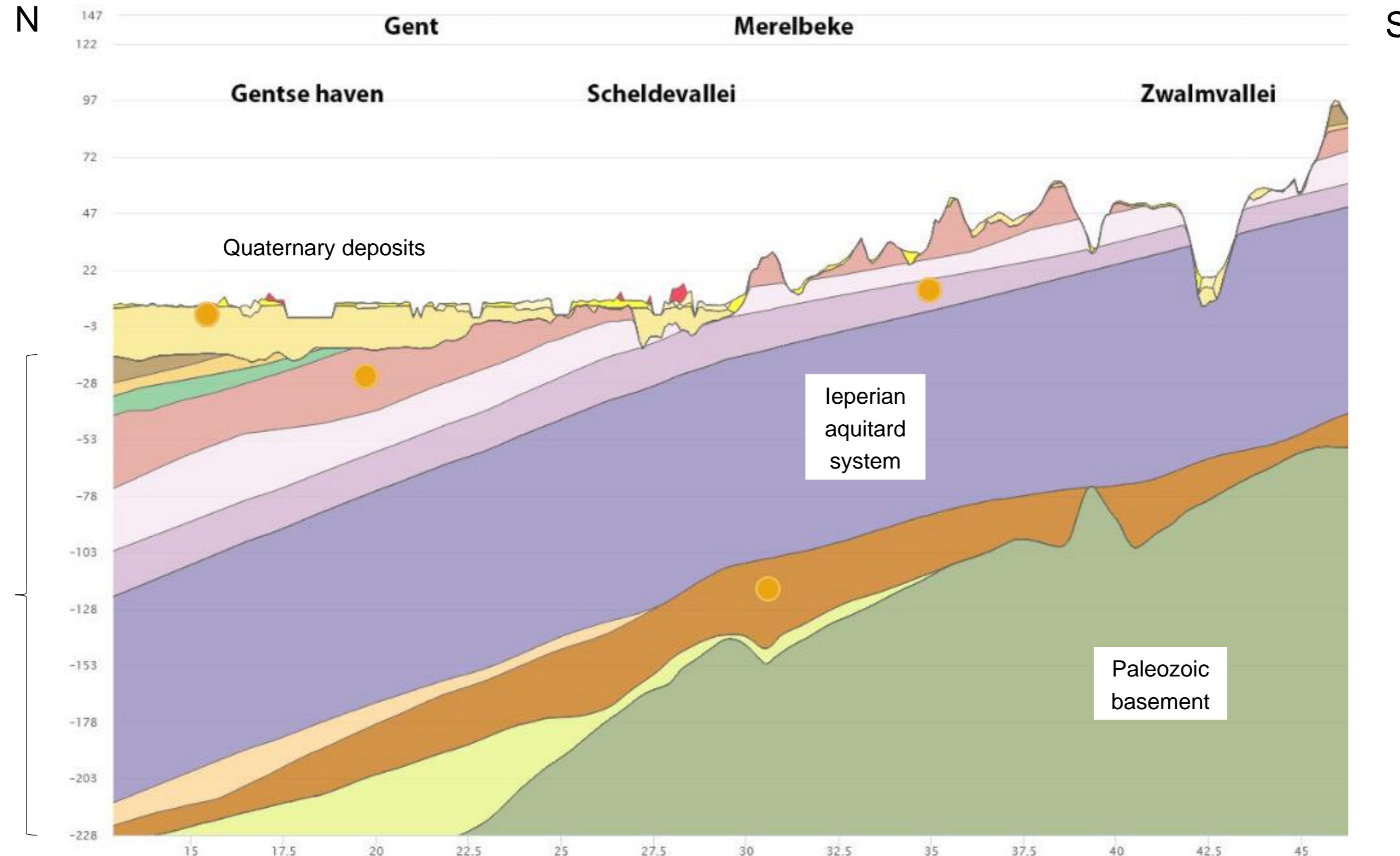
Professional Experience:

- 2023 - ... Voluntary Post-Doctoral Researcher in Hydrogeology (Ghent University)
- 2023 - ... AI Expert-Coordinator (Flanders Environment Agency)
- 2020 - ... Lecturer in AI (Vives University)
- 2020 - 2022 Research Associate in AI (Vives University)
- 2008 - 2020 Groundwater Modeler (Flanders Environment Agency)
- 2007 - 2008 Project Engineer Water Management (IMDC)
- 2006 Science Teacher (HH Ninove Secondary School)
- 2002 - 2005 PhD Fellow in Hydrogeology (Ghent University)

WHERE DO I LIVE?



HYDROGEOLOGICAL STRATIGRAPHY



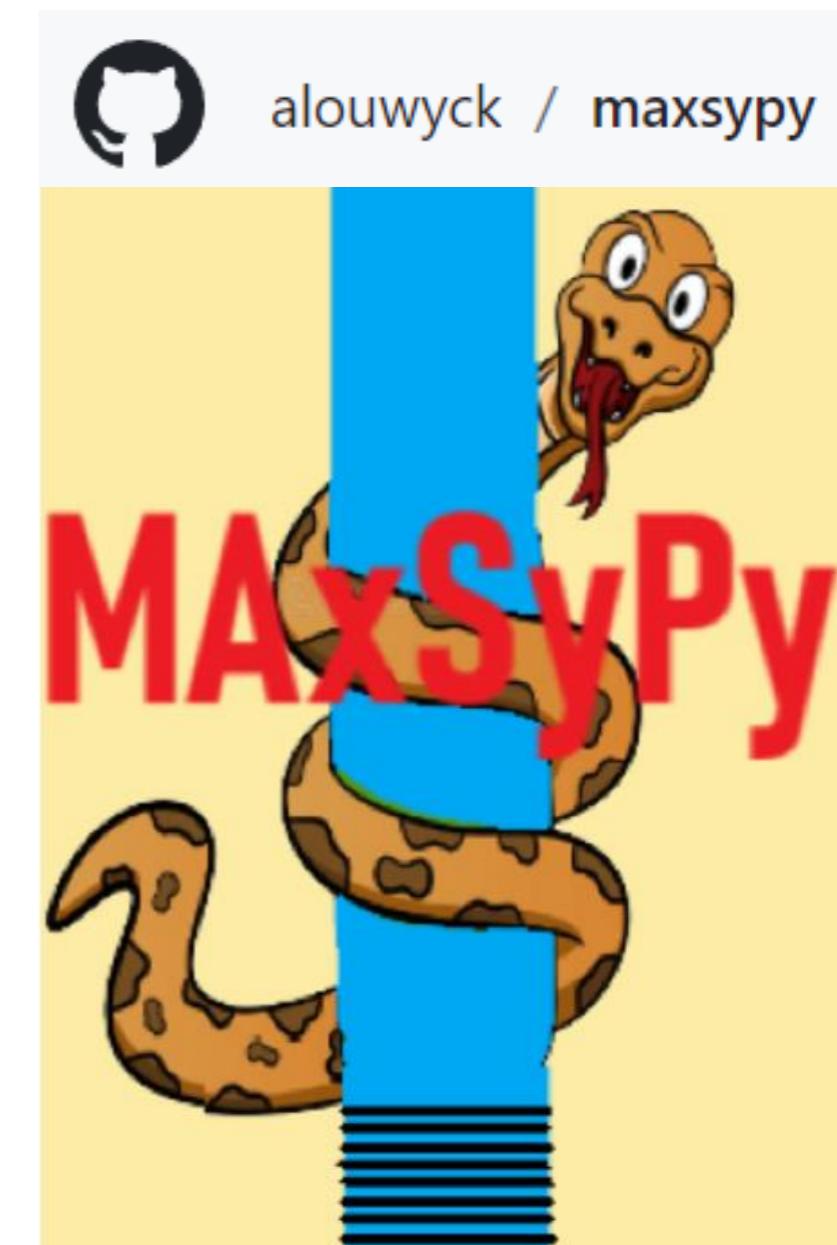
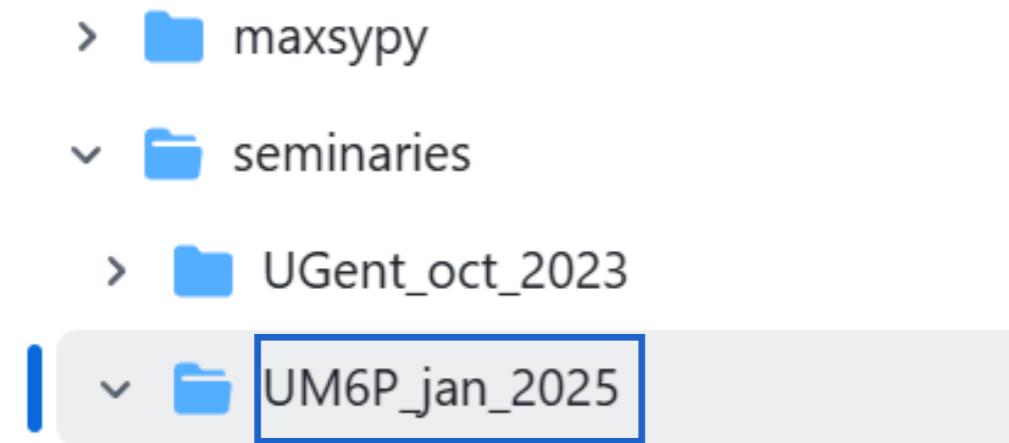
DAY 1: OVERVIEW

0. Axisymmetric flow
1. The very first axisymmetric models
2. Well-known 1D axisymmetric models
3. The radius of influence myth
4. The water budget myth and the superposition principle
5. More advanced axisymmetric models
6. Axisymmetric flow in multilayer aquifer systems
7. Aquifer tests
8. A theoretical case study
9. A practical case study



JUPYTER NOTEBOOKS WITH CODE EXAMPLES

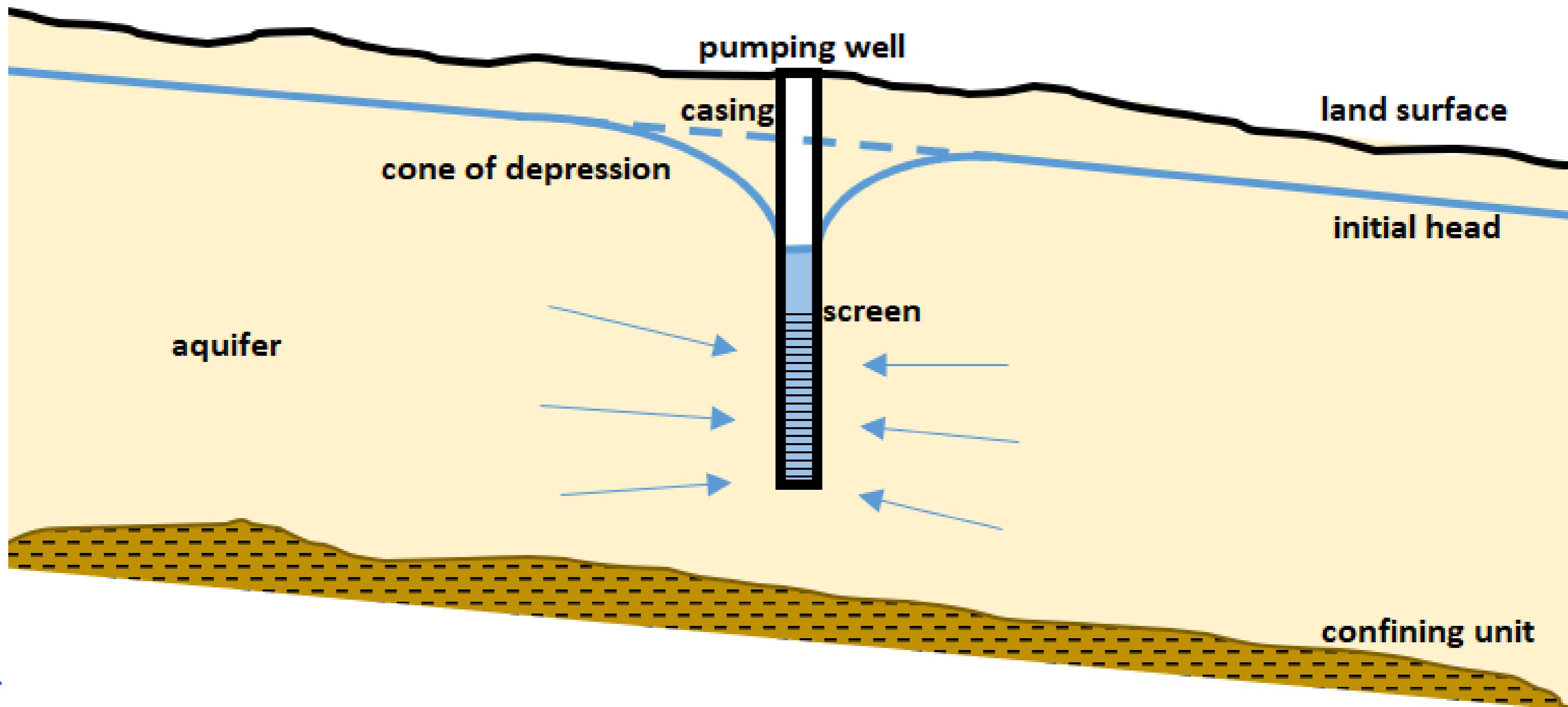
<https://github.com/alouwyck/maxsypy>



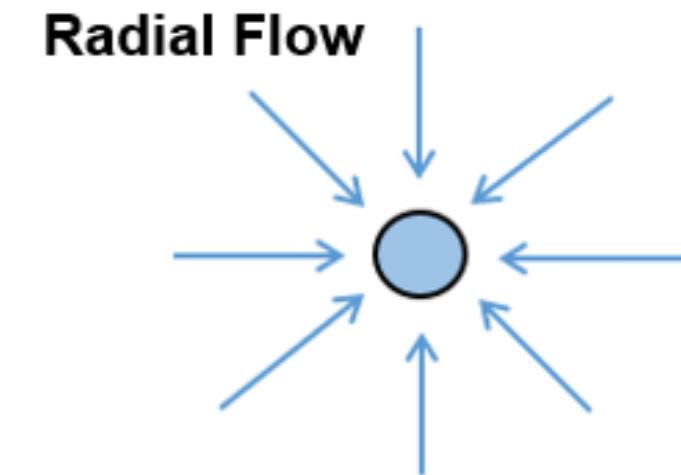
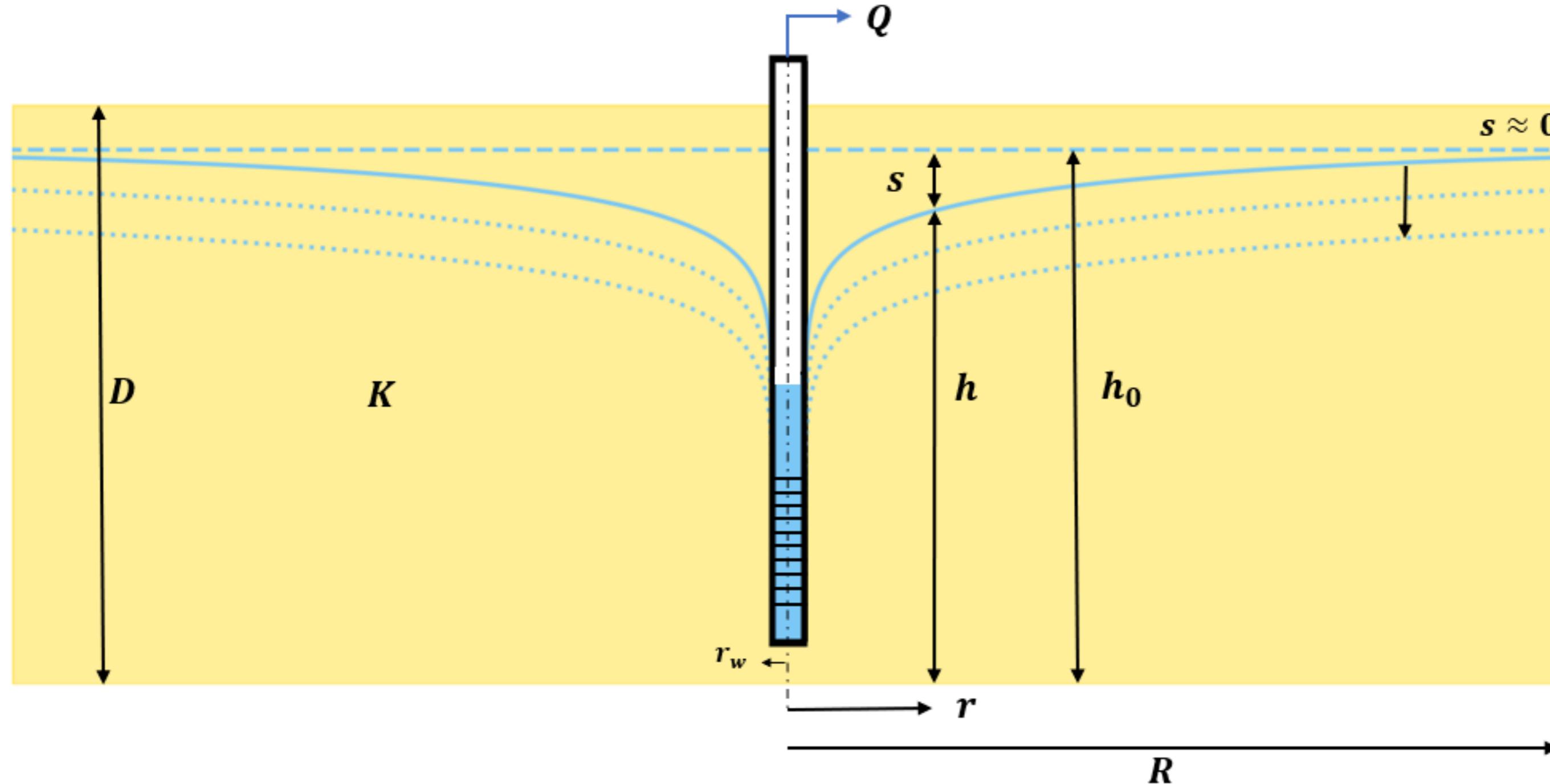
+ popular Python packages for groundwater modeling
TimML, TTim, Pastas, and FloPy

AXISYMMETRIC FLOW

FLOW TO A PUMPING WELL



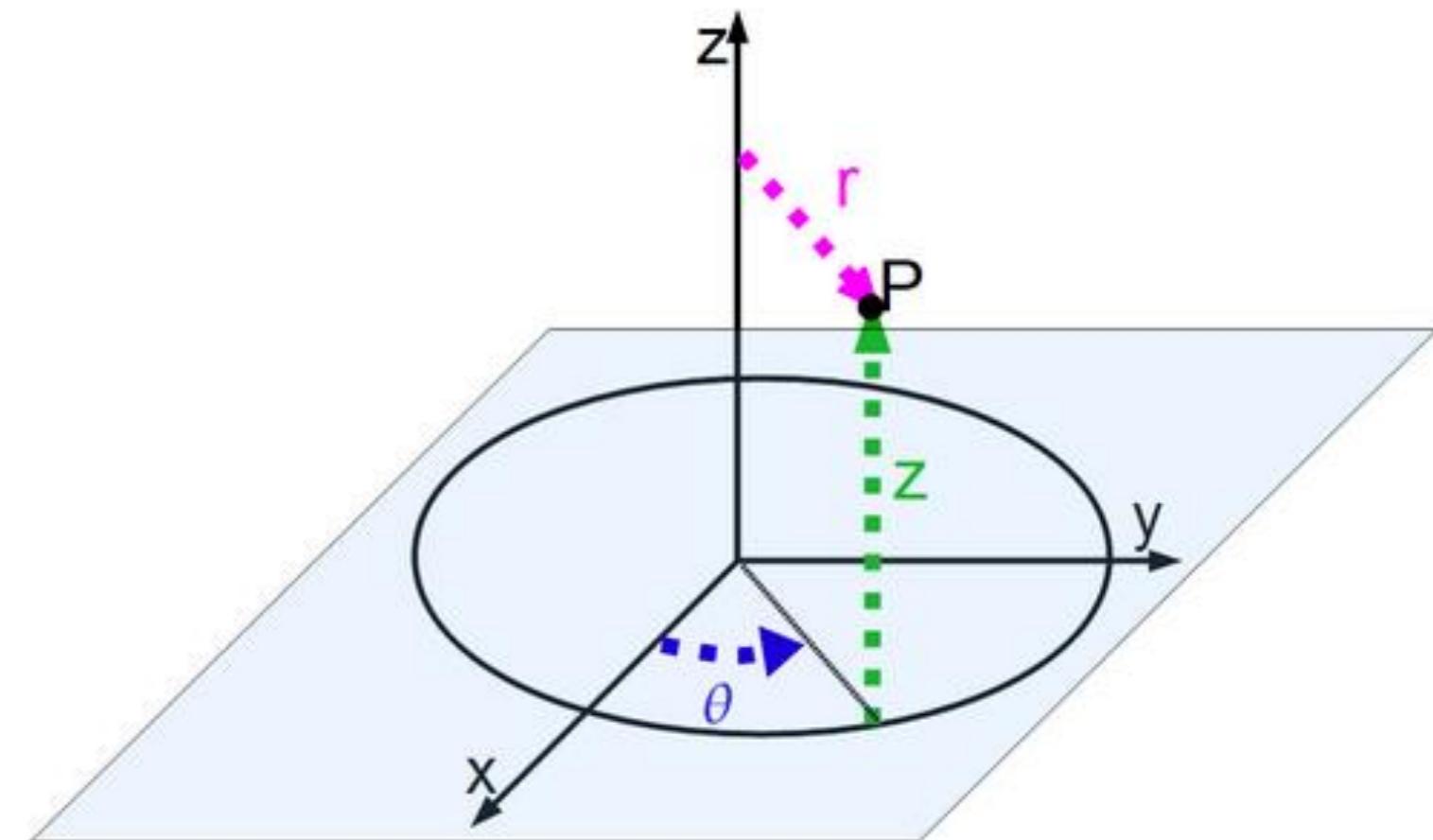
AXISYMMETRIC MODEL



pumping rate	Q
aquifer thickness	D
aquifer conductivity	K
aquifer transmissivity	$T = KD$
hydraulic head	h
initial head	h_0
drawdown	s
radial distance	r
well radius	r_w
radius of influence	R

CYLINDRICAL COORDINATES

- **Cartesian coordinates:** (x, y, z)
- **Cylindrical coordinates:** (r, θ, z)
 - Polar coordinates: (r, θ)
- **Axial symmetry:** (r, z)
 - No θ dimension!
 - 1D flow: only r



$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$

$$r = \sqrt{x^2 + y^2}$$

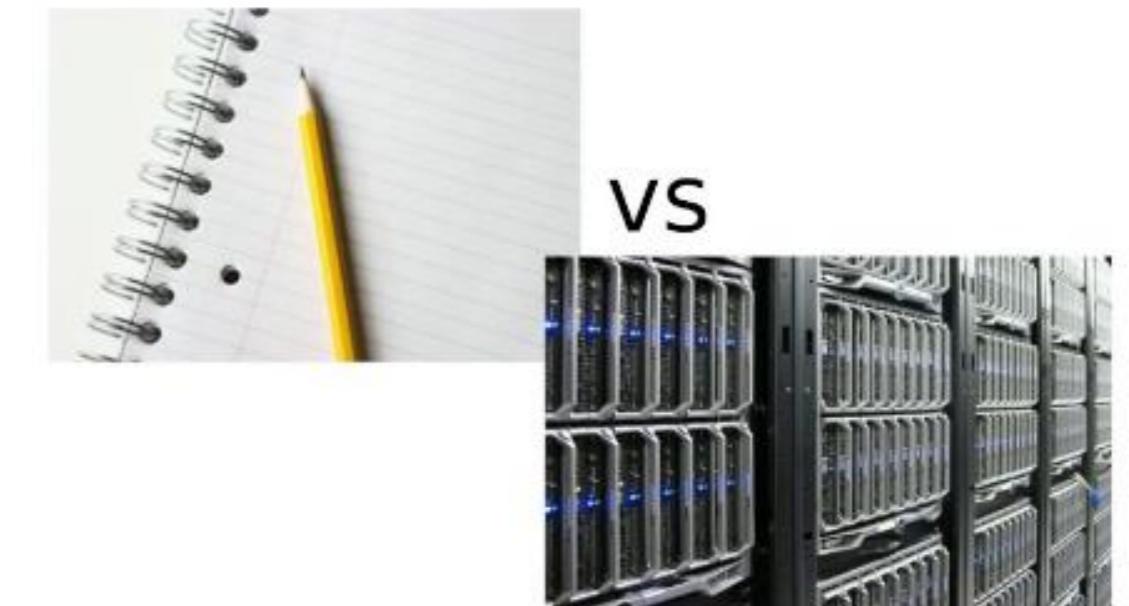
PARAMETERS AND UNITS

parameter	symbol	dimension
hydraulic head	h	L
initial head	h_0	L
drawdown	s	L
well drawdown	s_w	L
head change in well	H	L
initial head change in well	H_0	L
radial distance	r	L
time	t	T

parameter	symbol	dimension
pumping rate	Q	L^3/T
aquifer thickness	D	L
aquifer conductivity	K	L/T
aquifer transmissivity	$T = KD$	L^2/T
aquifer storativity	S	-
resistance	c	T
infiltration flux	N	L/T
radius of influence	R	L
well-screen radius	r_w	L
well-casing radius	r_c	L
well-skin radius	R_s	L

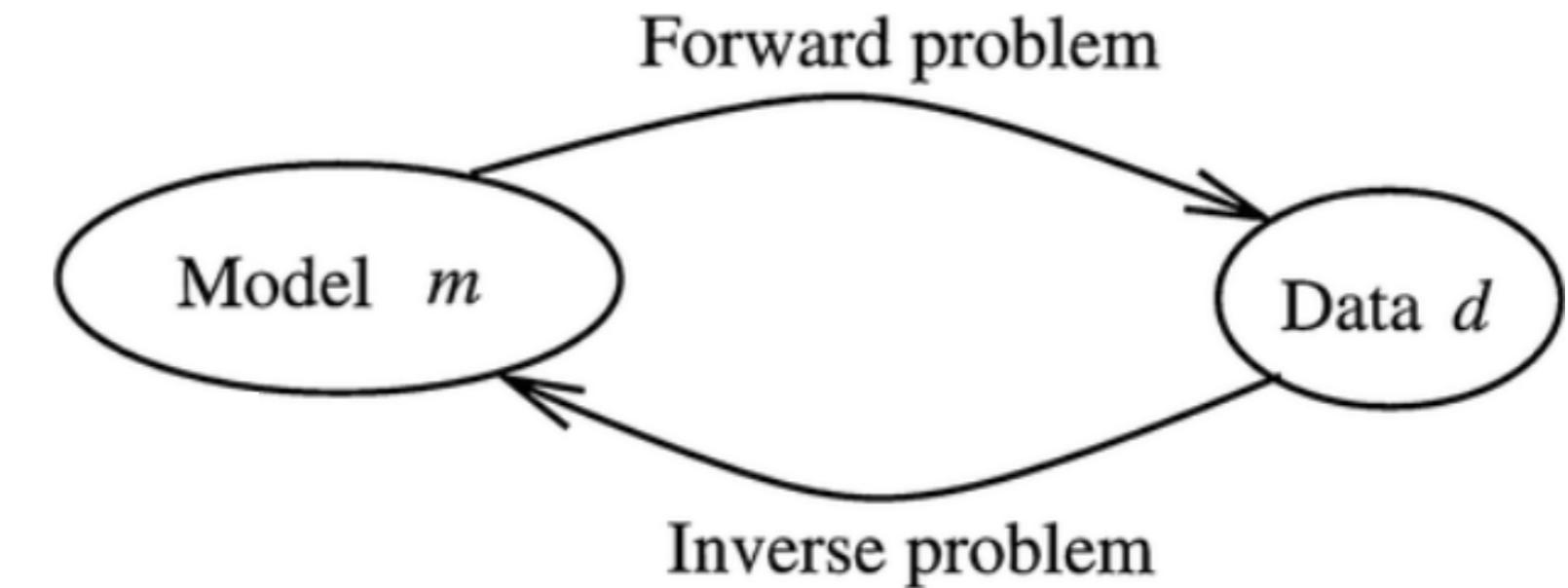
ANALYTICAL VS NUMERICAL MODELS

- Analytical solutions
 - exact
 - closed-form equations
 - methods from calculus
 - e.g. integral transforms
- Numerical solutions
 - approximate
 - discretization of the model domain
 - iterative methods
 - e.g. finite differences, finite elements, ...



FORWARD AND INVERSE PROBLEMS

- **forward problem**
 - simulate head h or drawdown s
 - e.g. assessing the environmental impact of extractions
- **inverse problem type I**
 - derive transmissivity T
 - e.g. pumping test interpretation
- **inverse problem type II**
 - derive pumping rate Q
 - e.g. construction dewatering



THE VERY FIRST AXISYMMETRIC MODELS

THE THIEM-DUPUIT FORMULAS

- Steady confined flow (Thiem, 1870, 1906)

$$s(r) = \frac{Q}{2\pi K D} \ln \left(\frac{R}{r} \right)$$

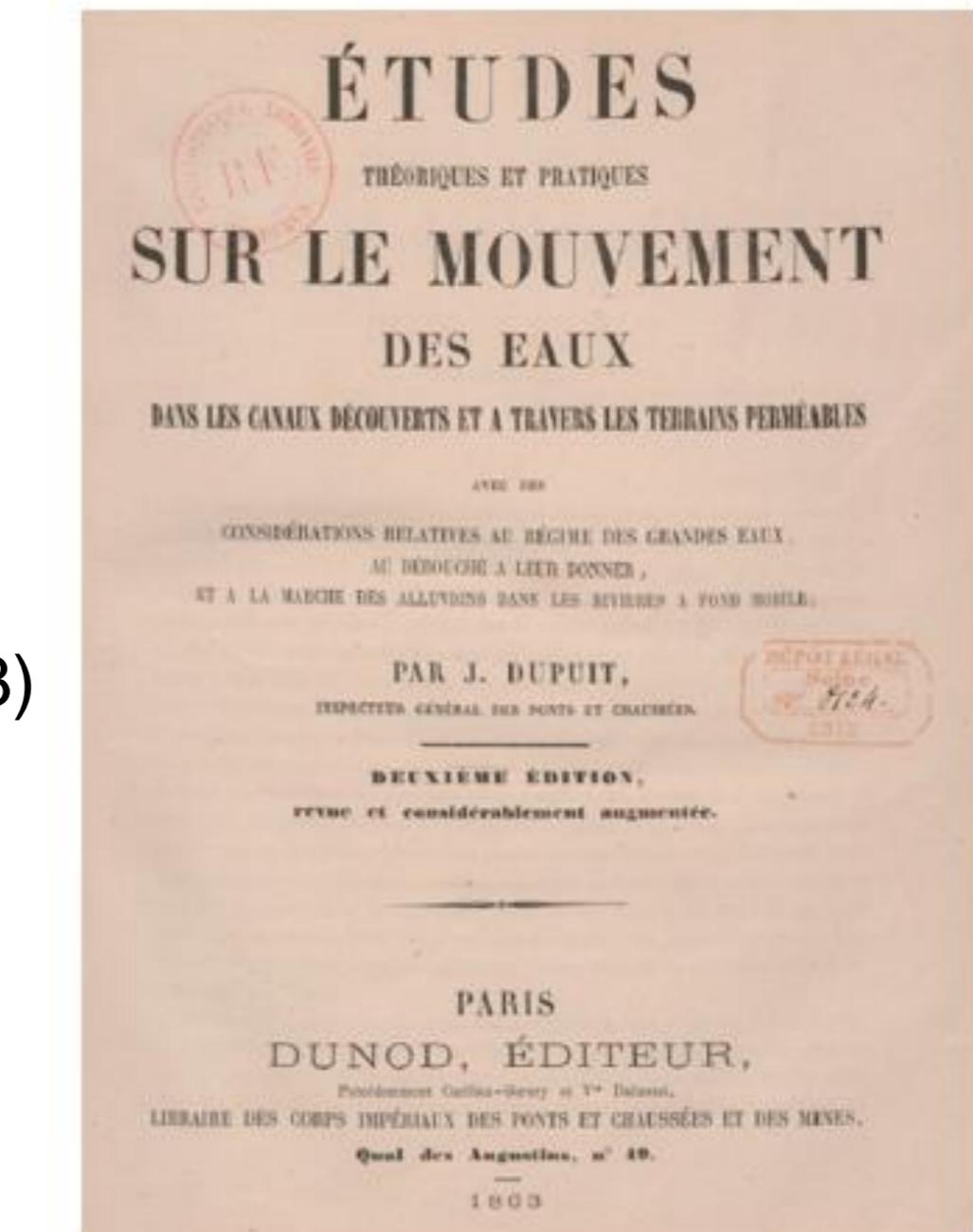
Initial head h_0 is not required

- Steady unconfined flow (Dupuit, 1857, 1863)

$$s(r) = h_0 - \sqrt{h_0^2 - \frac{Q}{\pi K} \ln \left(\frac{R}{r} \right)}$$

h

Initial head h_0 is required!



Jules Dupuit



Adolf Thiem



Günther Thiem

CONFINED FLOW

- Constant saturated thickness D
- If aquifer is homogeneous:
 - K is constant
 - T is constant
 - $T = KD$
- Linear problem

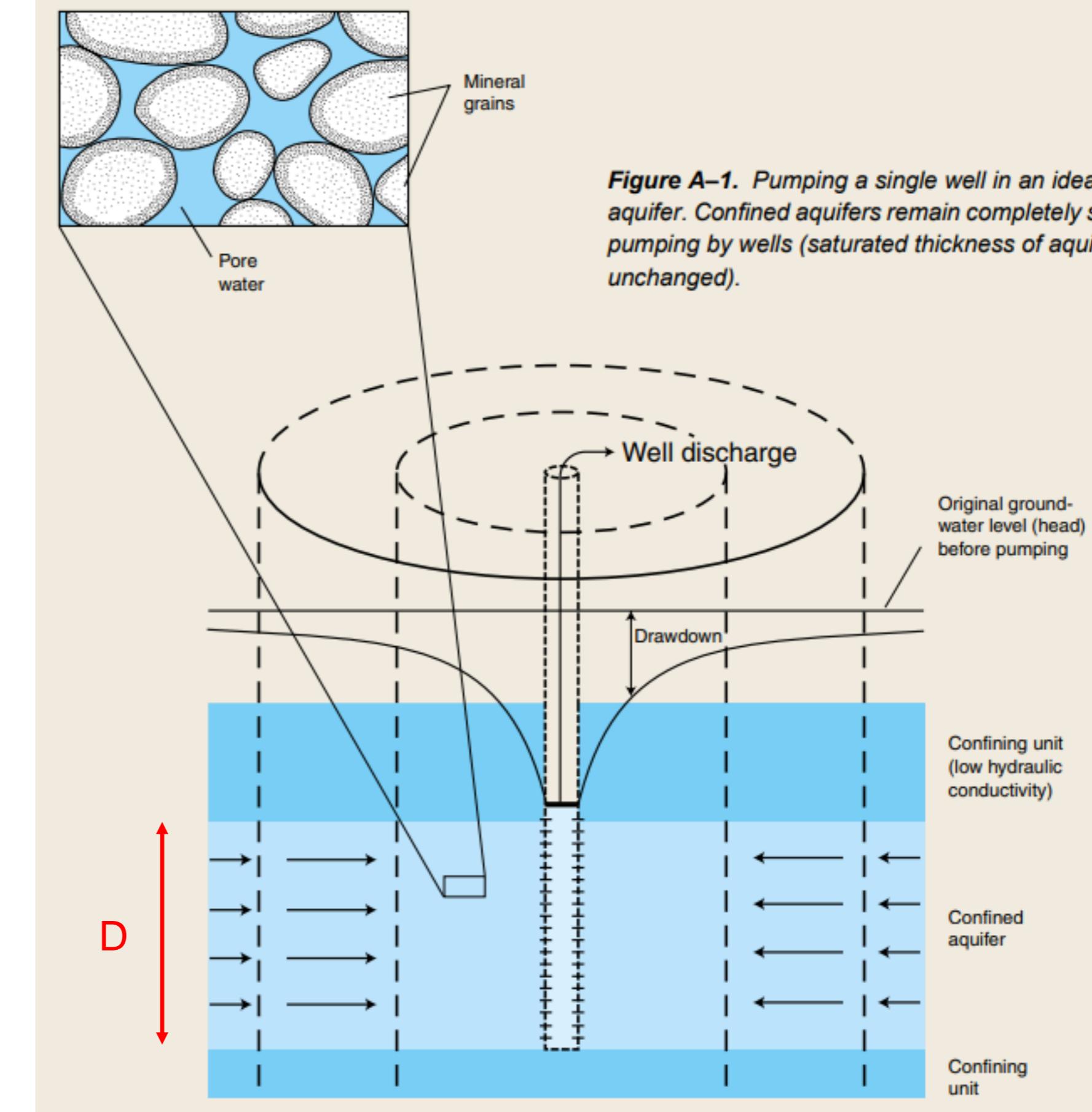
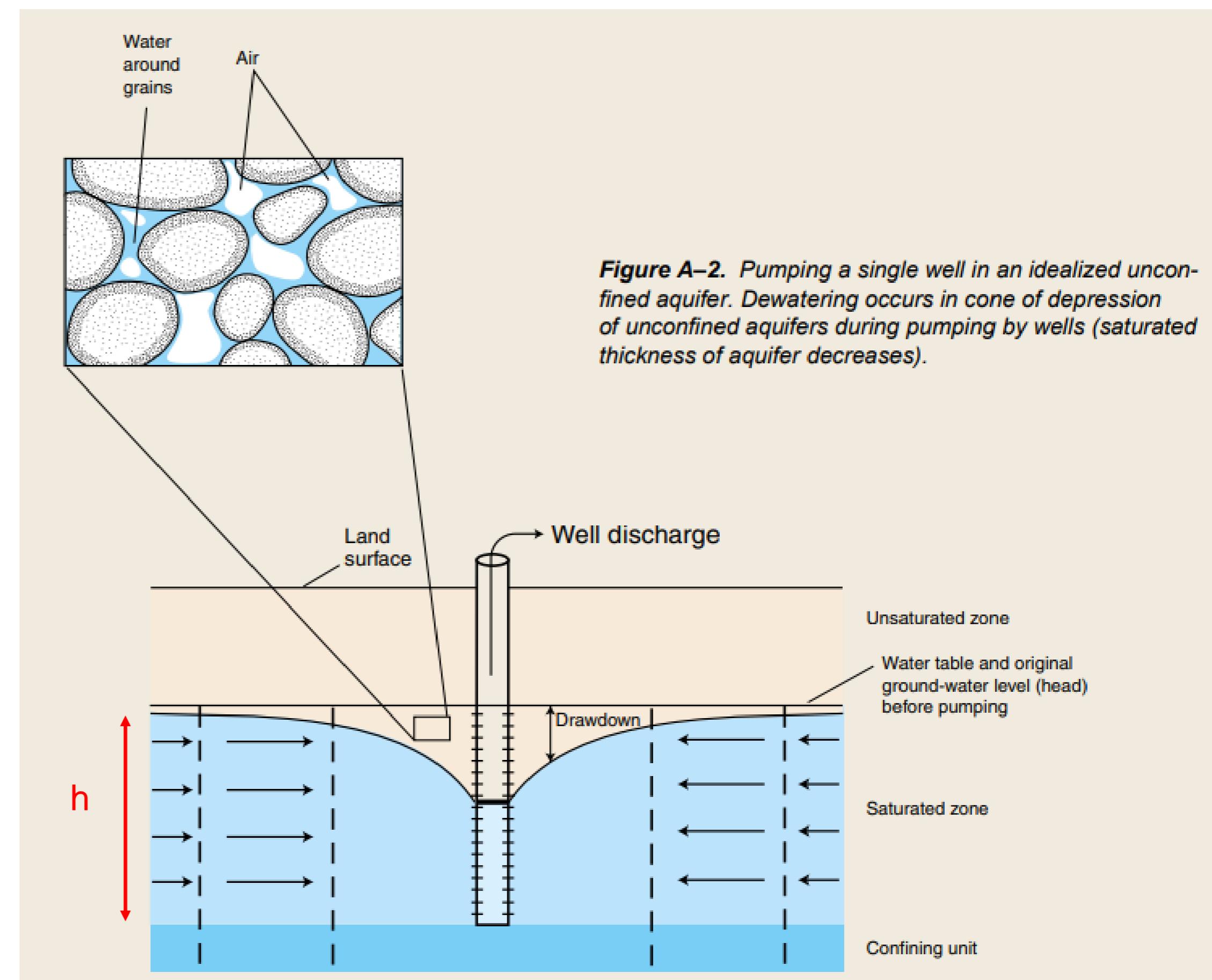


Figure A-1. Pumping a single well in an idealized confined aquifer. Confined aquifers remain completely saturated during pumping by wells (saturated thickness of aquifer remains unchanged).

UNCONFINED FLOW

- Saturated thickness = head h
- If aquifer is homogeneous:
 - K is constant
 - T is head-dependent
 - $T = Kh$
- **Nonlinear** problem



THIEM EQUATION: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Steady-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Constant pumping rate
- Aquifer:
 - Homogeneous
 - **Constant saturated thickness**
 - Laterally bounded

THIEM EQUATION: PROBLEM STATEMENT

Darcy's law:

$$Q = 2\pi r \mathbf{K} \mathbf{D} \frac{dh}{dr} \quad (1)$$

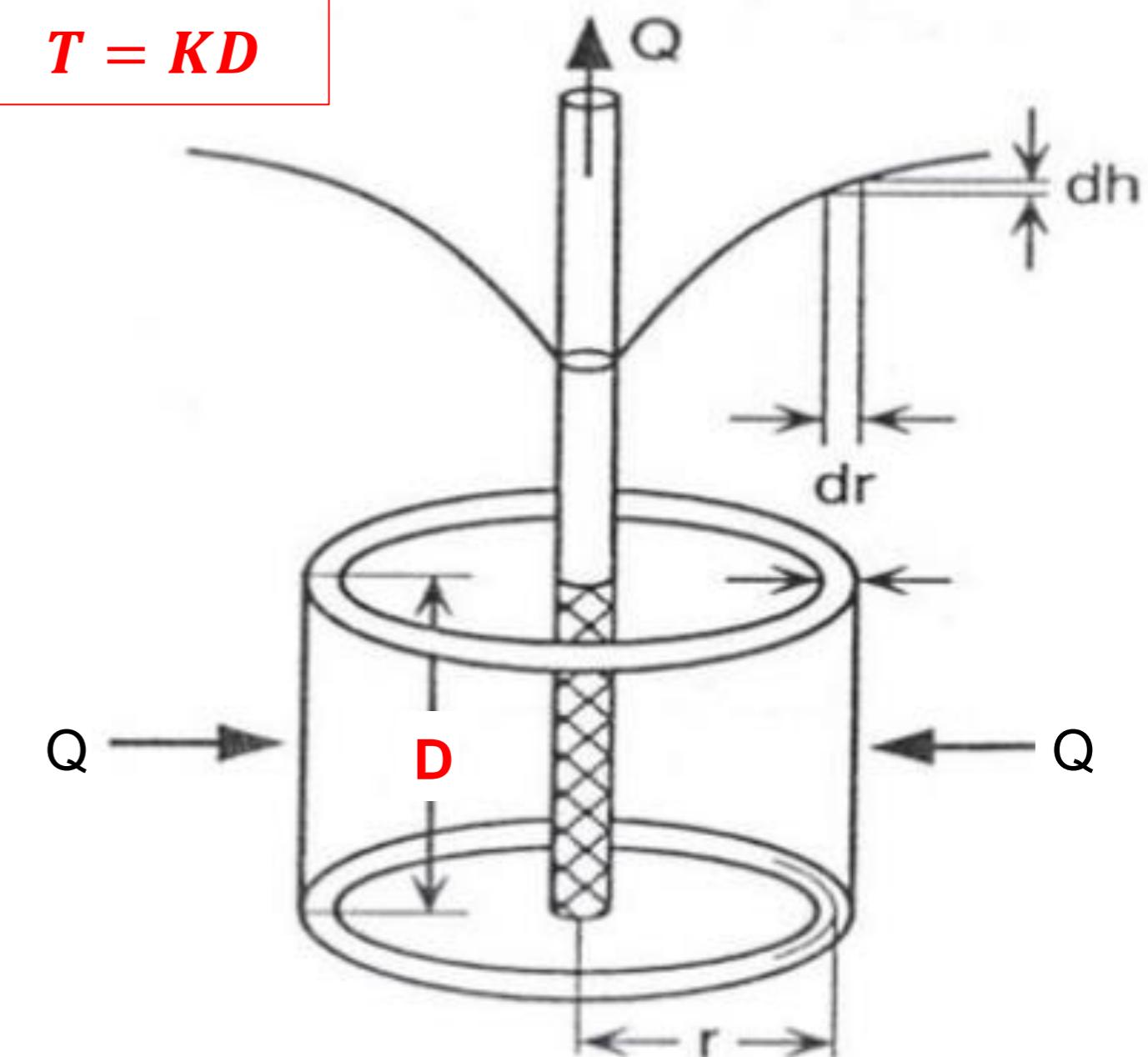
$$\mathbf{T} = \mathbf{K} \mathbf{D}$$

Continuity of steady 1D axisymmetric flow:

$$\frac{dQ}{dr} = 0 \quad \text{or} \quad \frac{d^2h}{dr^2} + \frac{1}{r} \frac{dh}{dr} = 0$$

Boundary condition: constant head h_0 at distance R :

$$h(R) = h_0 \quad (2)$$



Source: Kresic, 1997

THIEM EQUATION: DERIVATION

Rearranging (1):

$$dh = \frac{Q}{2\pi K D} \frac{dr}{r} \quad (3)$$

Integrating both sides of (3):

$$h(r) = \frac{Q}{2\pi K D} \ln r + C \quad (4)$$

Introducing (2) in (4):

$$h(R) = h_0 = \frac{Q}{2\pi K D} \ln R + C \quad (5)$$

Deriving integration constant C from (5):

$$C = h_0 - \frac{Q}{2\pi K D} \ln R \quad (6)$$

Introducing (6) in general solution (4):

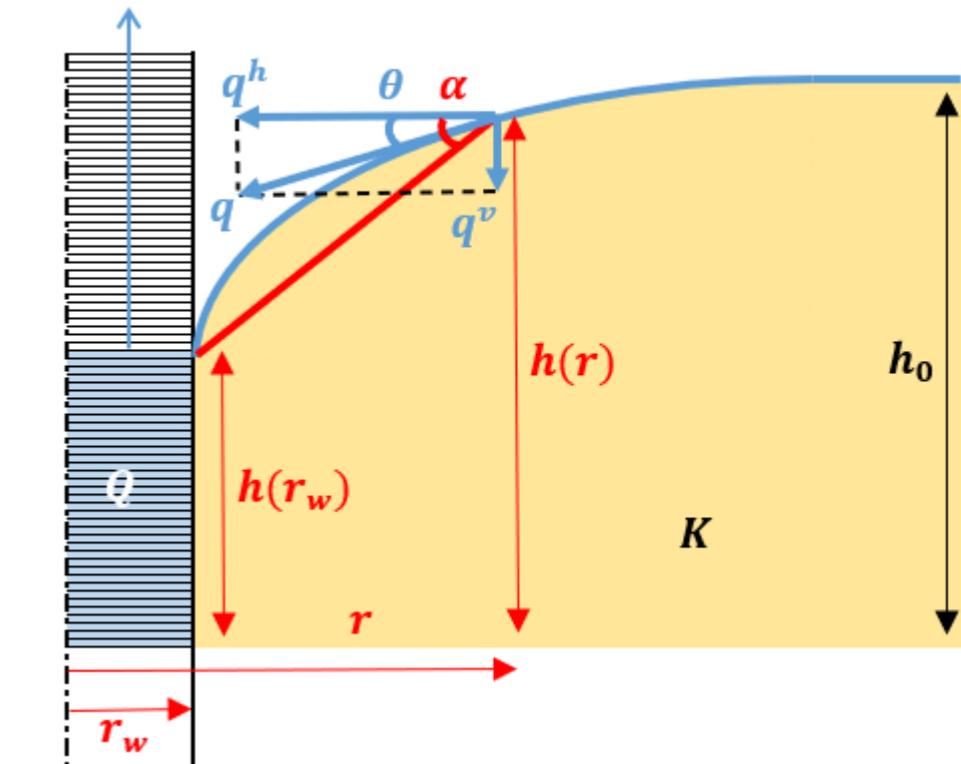
$$h(r) = h_0 - \frac{Q}{2\pi K D} \ln \frac{R}{r} \quad (7)$$

Applying definition of drawdown s to (7):

$$s(r) = h_0 - h(r) = \frac{Q}{2\pi K D} \ln \frac{R}{r} \quad (8)$$

DUPUIT EQUATION: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Steady-state
 - Strictly horizontal: $\theta < 30^\circ$
= the Dupuit-Forchheimer approximation!



- Aquifer:
 - Homogeneous
 - **Head-dependent saturated thickness**
 - Laterally bounded

- Well:
 - Fully penetrating
 - Constant pumping rate
 - No seepage face

DUPUIT EQUATION: PROBLEM STATEMENT

Darcy's law:

$$Q = 2\pi r \mathbf{K} h \frac{dh}{dr} \quad (1)$$

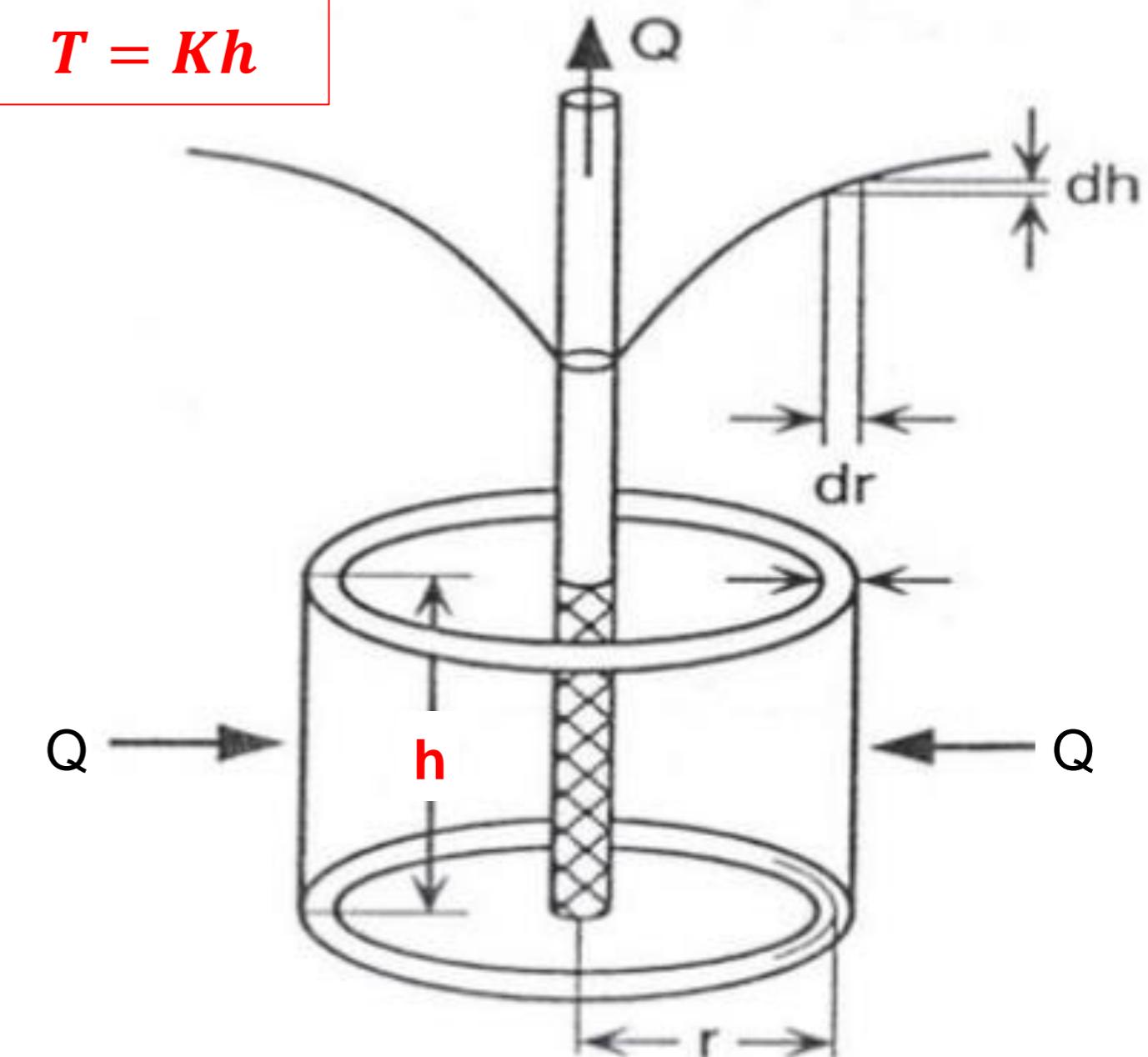
$$\mathbf{T} = \mathbf{K} h$$

Continuity of steady 1D axisymmetric flow:

$$\frac{dQ}{dr} = 0 \quad \text{or} \quad \frac{d}{dr} \left(rh \frac{dh}{dr} \right) = 0$$

Boundary condition: constant head h_0 at distance R :

$$h(R) = h_0 \quad (2)$$



Source: Kresic, 1997

DUPUIT EQUATION: DERIVATION

Rearranging (1):

$$h dh = \frac{Q}{2\pi K} \frac{dr}{r} \quad (3)$$

Integrating both sides of (3):

$$h^2(r) = \frac{Q}{\pi K} \ln r + C \quad (4)$$

Introducing (2) in (4):

$$h^2(R) = h_0^2 = \frac{Q}{\pi K} \ln R + C \quad (5)$$

Deriving integration constant C from (5):

$$C = h_0^2 - \frac{Q}{\pi K} \ln R \quad (6)$$

Introducing (6) in general solution (4):

$$h(r) = \sqrt{h_0^2 - \frac{Q}{\pi K} \ln \frac{R}{r}} \quad (7)$$

Applying definition of drawdown s to (7):

$$s(r) = h_0 - \sqrt{h_0^2 - \frac{Q}{\pi K} \ln \frac{R}{r}} \quad (8)$$

DUPUIT VS THIEM

Rearranging (8):

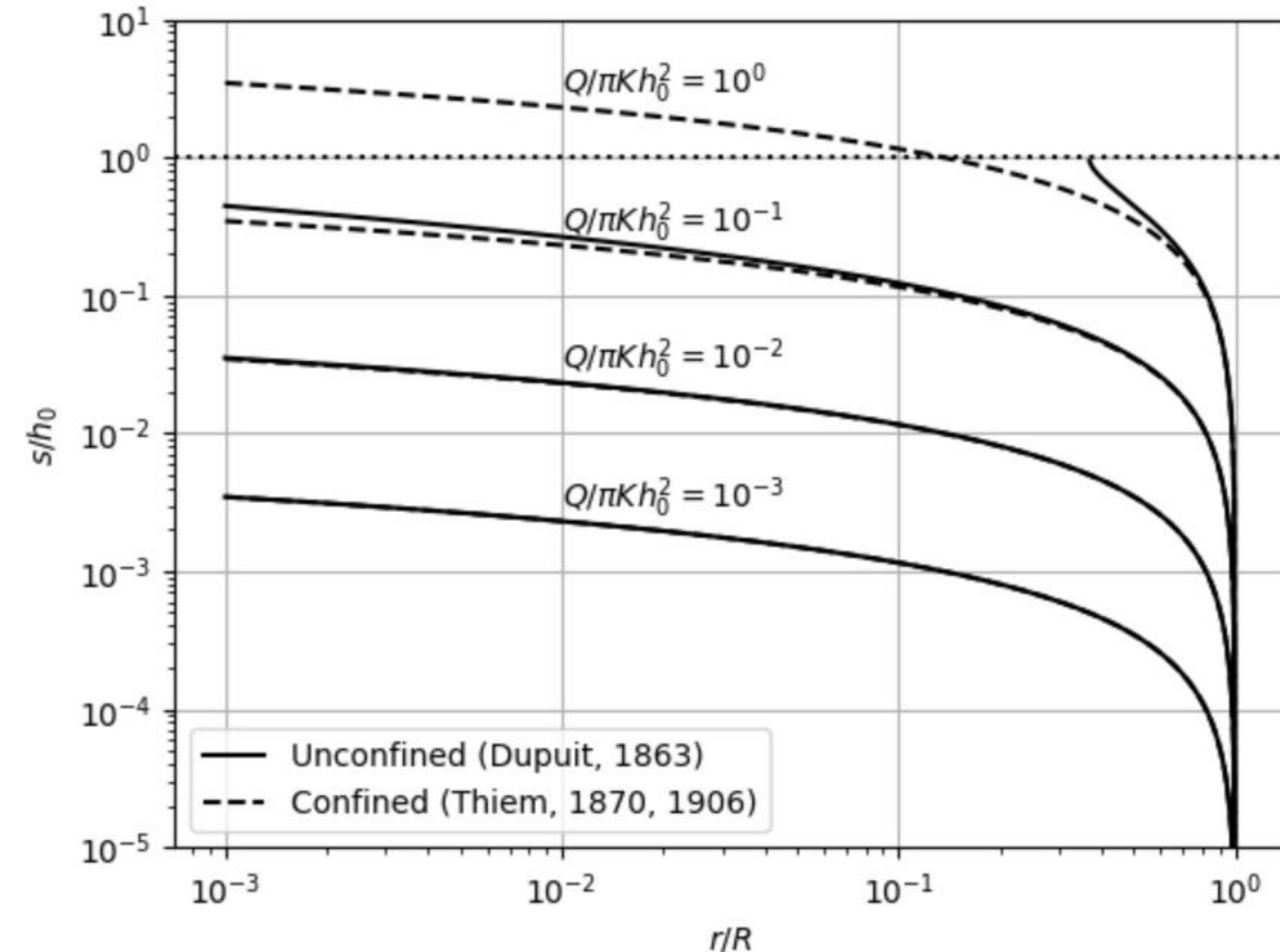
$$s(r) = h_0 \left(1 - \sqrt{1 - \frac{Q}{\pi K h_0^2} \ln \frac{R}{r}} \right) \quad (9)$$

Series expansion:

$$\sqrt{1-x} \approx 1 - \frac{x}{2} \quad (x \rightarrow 0) \quad (10)$$

Applying (10) to (9):

$$s(r) \approx \frac{Q}{2\pi K h_0} \ln \frac{R}{r} \quad (s < 0.1 h_0)$$

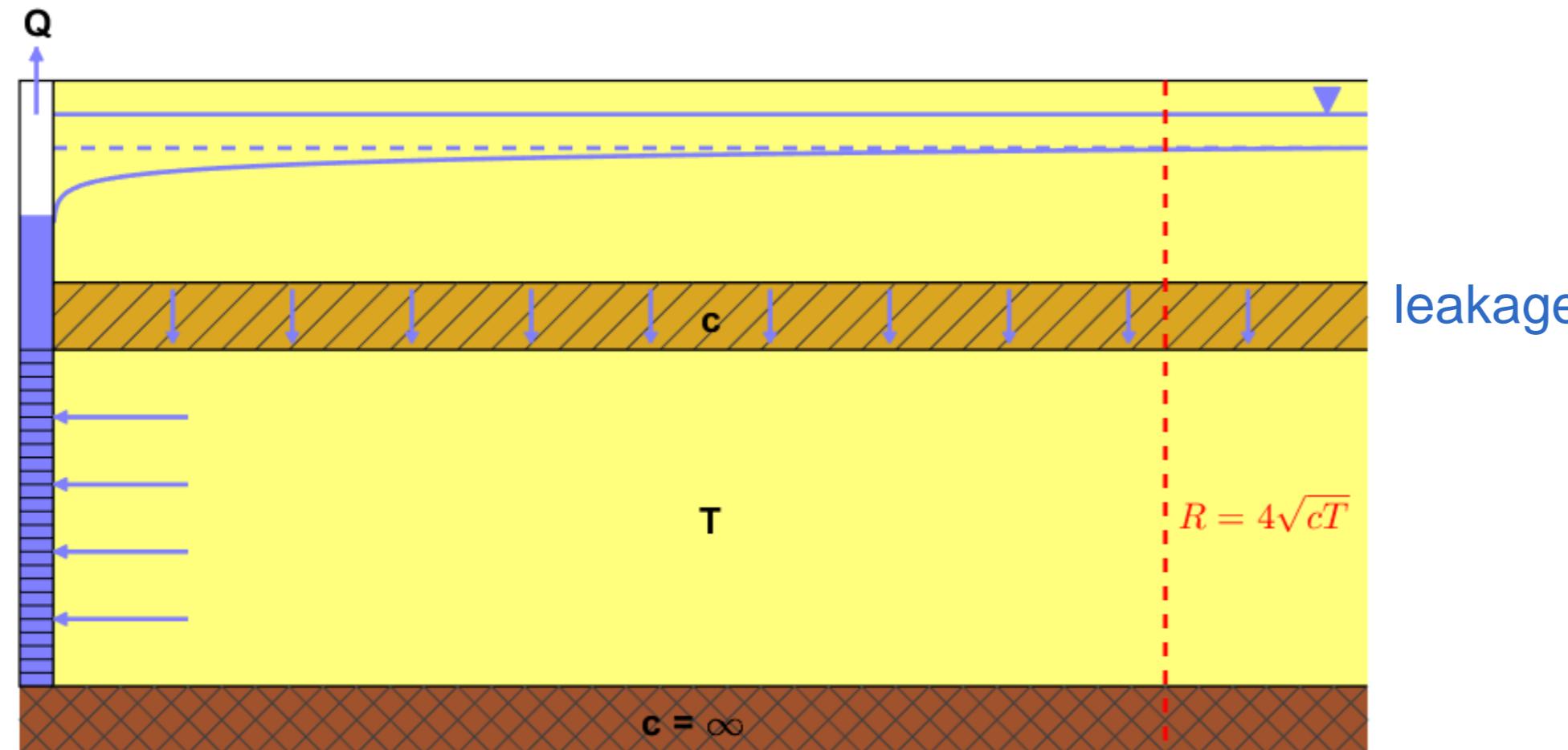


WELL-KNOWN 1D AXISYMMETRIC MODELS

OVERVIEW

- **de Glee (1930)**: steady well-flow in a leaky aquifer
- **Theis (1935)**: transient well-flow in a confined aquifer
- **Hantush-Jacob (1955)**: transient well-flow in a leaky aquifer
- **Ernst (1971)**: steady well-flow in a phreatic aquifer subject to uniform infiltration and drainage

THE DE GLEE FORMULA



OVER GRONDWATERSTROOMINGEN
BIJ WATERONTTREKKING DOOR
MIDDEL VAN PUTTEN.

PROEFSCHRIFT

TER VERKRIJGING VAN DEN GRAAD VAN
DOCTOR IN DE TECHNISCHE WETENSCHAP
AAN DE TECHNISCHE HOOGESCHOOL TE
DELFTH, OP GEZAG VAN DEN RECTOR MAG-
NIFICUS IR. F. WESTENDORP, HOOGLEERAAR
IN DE AFDEELING DER WERKTUIGBOUW-
KUNDE EN SCHEEPSBOUWKUNDE, VOOR
EENE COMMISSIE UIT DEN SENAAT TE
VERDEDIGEN OP WOENSDAG 2 APRIL 1930,
DES NAMIDDAGS TE 3 UUR,

DOOR

GERRIT JAN DE GLEE,
CIVIEL-INGENIEUR,
GEBOREN TE ASSEN.



Johan Kooper



Charles E. Jacob

GEDRUKT BIJ DE TECHNISCHE BOEKHANDEL EN DRUKKERIJ
J. WALTMAN JR. DELFT. — 1930.

Steady leaky flow (Kooper, 1914; de Glee, 1930; Jacob, 1946)

$$s(r) = \frac{Q}{2\pi K D} K_0 \left(r \sqrt{\frac{1}{c K D}} \right) \approx \frac{Q}{2\pi K D} \ln \left(\frac{2e^{-\gamma} \sqrt{K D c}}{r} \right)$$

DE GLEE EQUATION: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Steady-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Constant pumping rate
 - Infinitesimal radius
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded
 - Leaky top

DE GLEE EQUATION: PROBLEM STATEMENT

Continuity of steady 1D leaky flow:

$$T \left(\frac{d^2 h}{dr^2} + \frac{1}{r} \frac{dh}{dr} \right) = \frac{h - h_0}{c} \quad (1)$$

leakage

Inner boundary condition at zero:

$$Q = \lim_{r \rightarrow 0} \left(2\pi r T \frac{dh}{dr} \right) \quad (2)$$

Outer boundary condition at infinity:

$$h(\infty) = h_0 \quad (3)$$

Modified Bessel differential equation:



$$\frac{d^2 h}{dr^2} + \frac{1}{r} \frac{dh}{dr} = ah - b$$

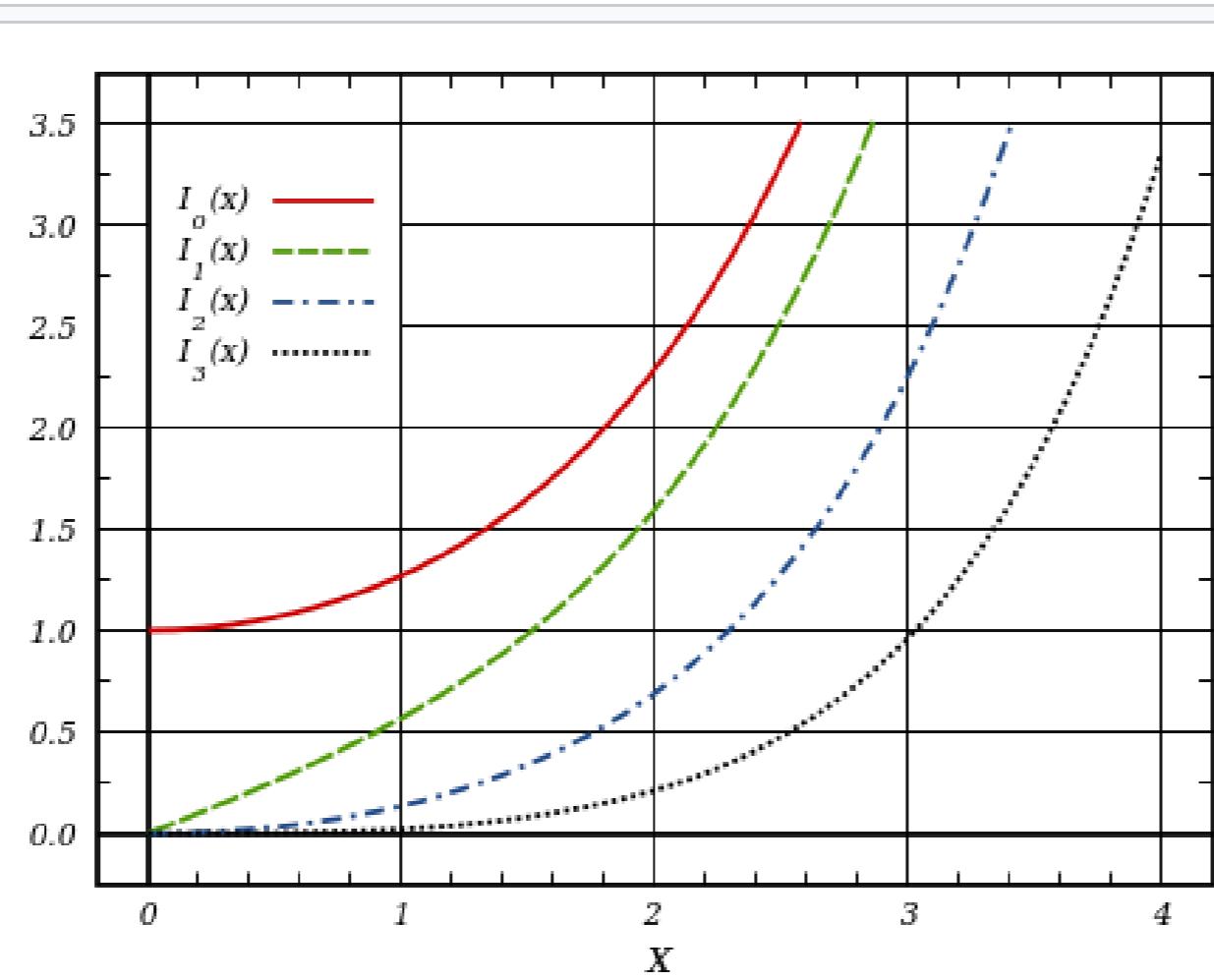
General solution:

$$h = \alpha I_0(r\sqrt{a}) + \beta K_0(r\sqrt{a}) + \frac{b}{a}$$

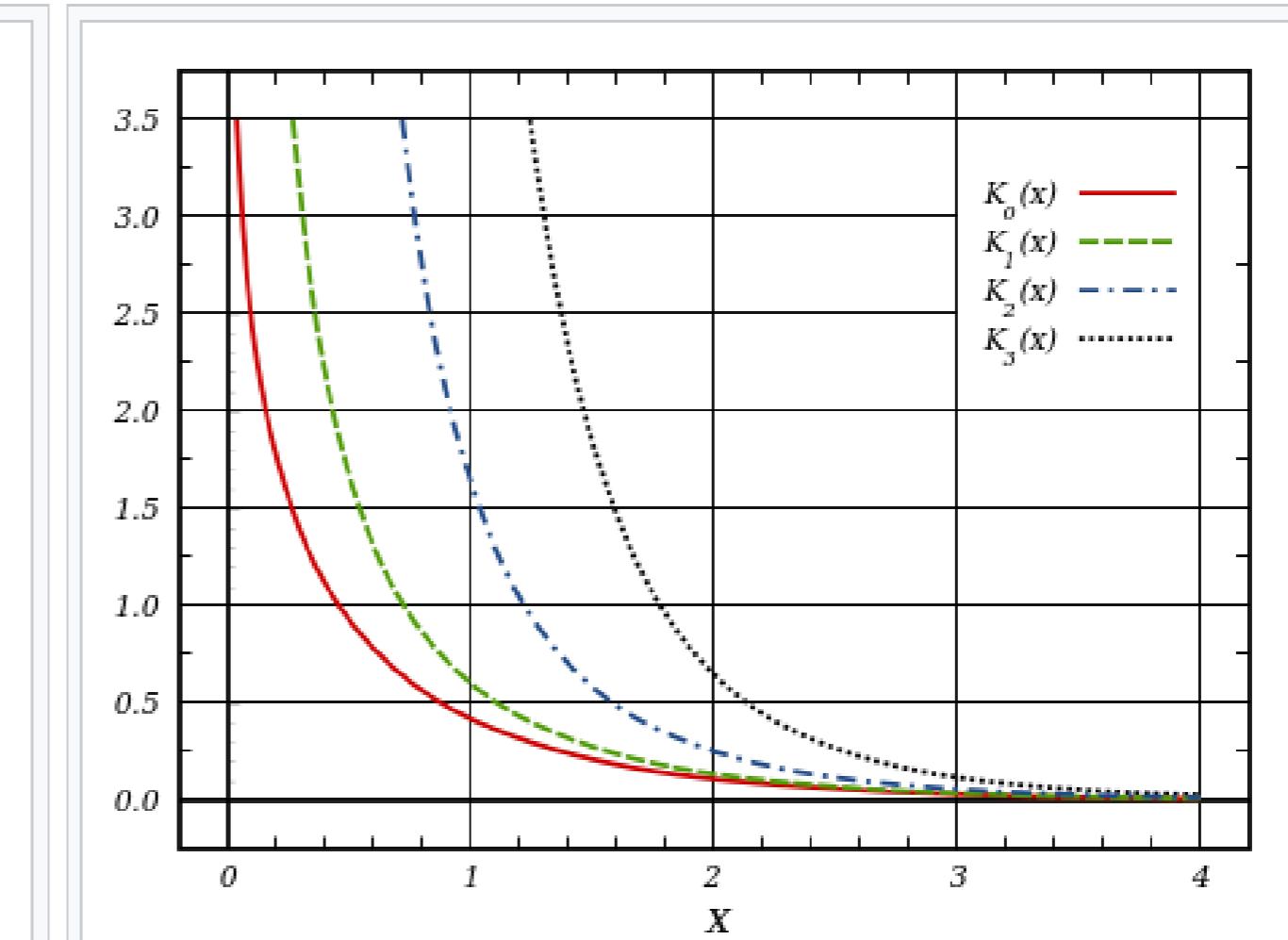
with:

- I_0, K_0 : the zero order modified Bessel functions of the first and second kind, resp.
- α, β : integration constants

MODIFIED BESSEL FUNCTIONS



Modified Bessel functions of the first kind, $I_\alpha(x)$,
for $\alpha = 0, 1, 2, 3$



Modified Bessel functions of the second kind,
 $K_\alpha(x)$, for $\alpha = 0, 1, 2, 3$

$$\frac{dI_0(ax)}{dx} = aI_1(ax)$$

$$\frac{dK_0(ax)}{dx} = -aK_1(ax)$$

	$I_0(x)$	$K_0(x)$	$xI_1(x)$	$xK_1(x)$
$x \rightarrow 0$	1	∞	0	1
$x \rightarrow \infty$	∞	0	∞	0

DE GLEE EQUATION: DERIVATION

General solution of (1):

$$h = \alpha I_0\left(\frac{r}{\lambda}\right) + \beta K_0\left(\frac{r}{\lambda}\right) + h_0 \quad (4)$$

with $\lambda = \sqrt{Tc}$ the leakage factor

Applying BC (3): $I_0(x) \rightarrow \infty$ if $x \rightarrow \infty$

$$\alpha = 0 \quad (5)$$

Introducing (5) in (4):

$$h(r) = h_0 + \beta K_0\left(\frac{r}{\lambda}\right) \quad (6)$$

Applying BC (2): $xK_1(x) \rightarrow 1$ if $x \rightarrow 0$

$$\beta = \frac{-Q}{2\pi T} \quad (7)$$

Introducing (7) in (6):

$$h(r) = h_0 - \frac{Q}{2\pi T} K_0\left(\frac{r}{\lambda}\right) \quad (8)$$

Applying definition of drawdown s to (8):

$$s(r) = h_0 - h(r) = \frac{Q}{2\pi T} K_0\left(\frac{r}{\lambda}\right) \quad (9)$$

DE GLEE VS THIEM

Series expansion:

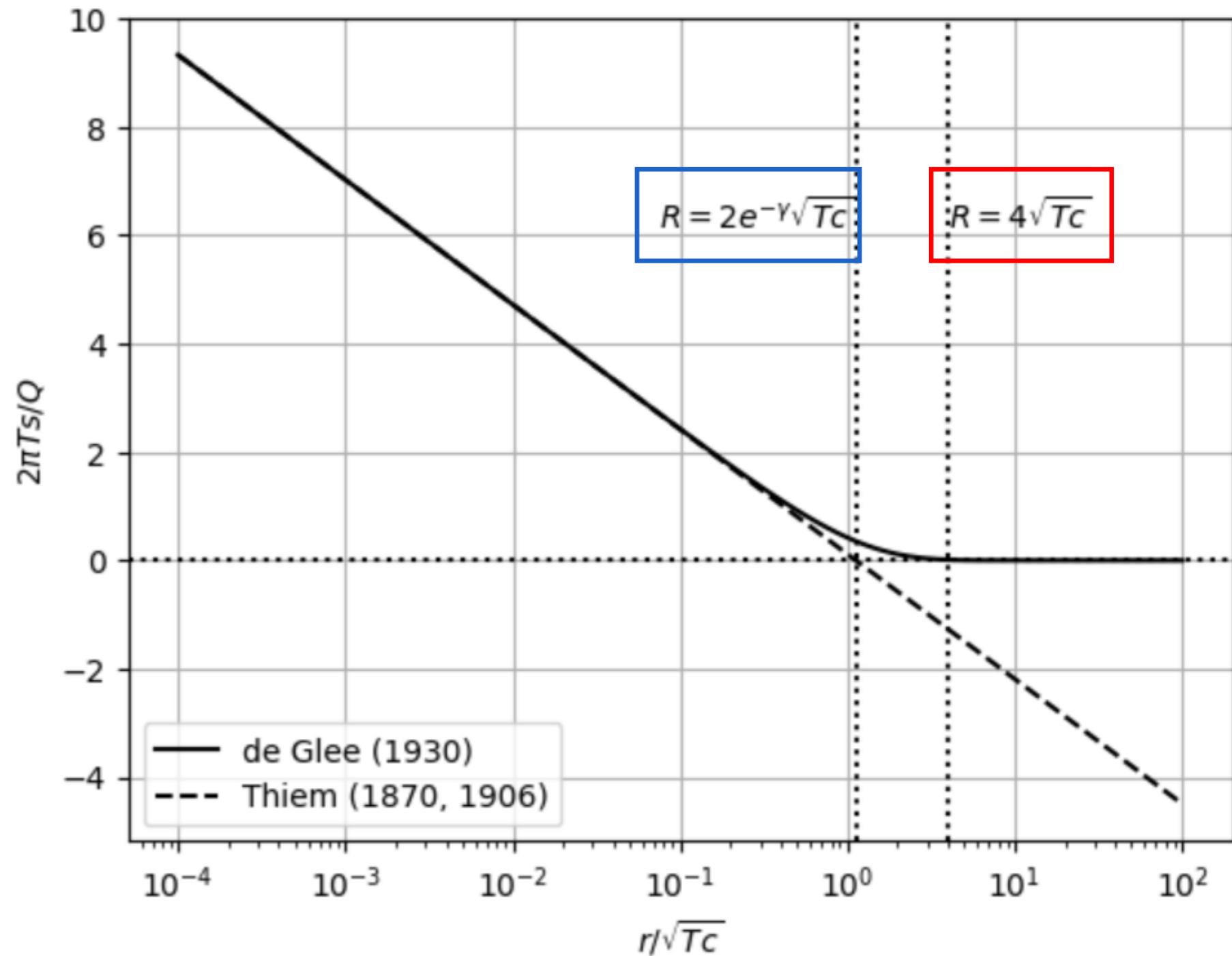
$$K_0(x) \approx -\gamma - \ln \frac{x}{2} \quad (x \rightarrow 0) \quad (10)$$

Applying (10) to (9):

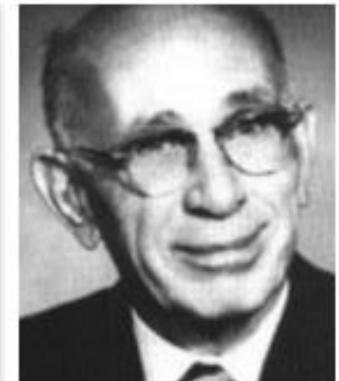
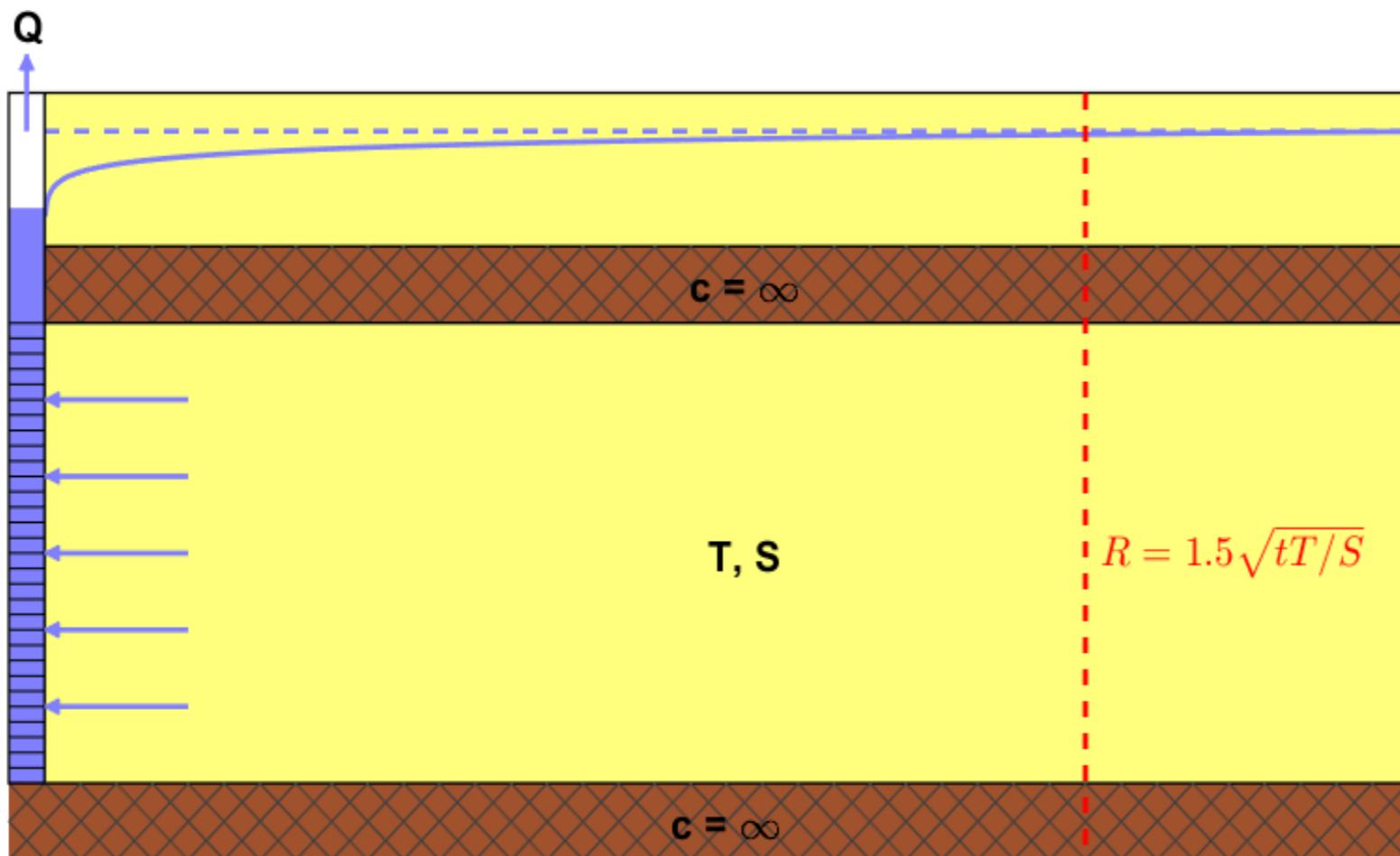
$$s(r) \approx \frac{Q}{2\pi T} \ln \left(\frac{2e^{-\gamma}\lambda}{r} \right) \quad (r < 0.1\lambda) \quad (11)$$

Comparing (11) to the Thiem equation:

$$R = 2e^{-\gamma}\lambda$$



THE THEIS EQUATION



Transient confined flow (Theis, 1935; Cooper & Jacob, 1946)

$$s(r, t) = \frac{Q}{4\pi K D} W\left(\frac{r^2 S}{4t K D}\right) \approx \frac{Q}{2\pi K D} \ln\left(\frac{1}{r} \sqrt{\frac{4t K D}{e^\gamma S}}\right)$$

THEIS EQUATION: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Transient-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Constant pumping rate
 - Infinitesimal radius
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded

THEIS EQUATION: PROBLEM STATEMENT

Continuity of transient 1D confined flow:

$$T \left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} \right) = S \frac{\partial h}{\partial t}$$

storage change

(1)



Partial differential equation (PDE):

- head h is function of r and t
- apply Laplace transform w.r.t. t

Inner boundary condition at zero:

$$Q = \lim_{r \rightarrow 0} \left(2\pi r T \frac{\partial h}{\partial r} \right)$$

(2)

Outer boundary condition at infinity:

$$h(\infty, t) = h_0$$

(3)

Initial condition at $t = 0$:

$$h(r, 0) = h_0$$

(4)

THE LAPLACE TRANSFORM

= convert PDE in ODE by eliminating derivative w.r.t. time

Definition:

$$\bar{h}(r, p) = \mathcal{L}\{h(r, t)\}(p) = \int_0^{\infty} h(r, t)e^{-pt} dt \quad (5)$$

Derivative w.r.t. time t :

$$\mathcal{L}\left\{\frac{\partial h}{\partial t}\right\}(p) = p\bar{h}(r, p) - \boxed{h(r, 0)} = p\bar{h}(r, p) - \boxed{h_0} \quad \text{Initial condition (4)} \quad (6)$$

Laplace transform of a constant:

$$\mathcal{L}\{Q\}(p) = \frac{Q}{p} \quad (7)$$

THEIS EQUATION: LAPLACE TRANSFORM

Laplace transform of PDE (1):

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r} \frac{d\bar{h}}{dr} = \frac{S}{T} (p\bar{h} - h_0) \quad (8)$$



Modified Bessel differential equation:

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r} \frac{d\bar{h}}{dr} = a\bar{h} - b$$

Laplace transform of inner BC (2):

$$\frac{Q}{p} = \lim_{r \rightarrow 0} \left(2\pi r T \frac{d\bar{h}}{dr} \right) \quad (9)$$

Laplace transform of outer BC (3):

$$\bar{h}(\infty, p) = \frac{h_0}{p} \quad (10)$$

General solution:

$$\bar{h} = \alpha I_0(r\sqrt{a}) + \beta K_0(r\sqrt{a}) + \frac{b}{a}$$

with:

- I_0, K_0 : the zero order modified Bessel functions of the first and second kind, resp.
- α, β : integration constants

THEIS EQUATION: LAPLACE SOLUTION

General solution of (8):

$$\bar{h} = \alpha I_0(r\omega) + \beta K_0(r\omega) + \frac{h_0}{p}$$

with $\omega = \sqrt{Sp/T}$

(11)

Applying BC (9): $I_0(x) \rightarrow \infty$ if $x \rightarrow \infty$

$$\alpha = 0$$

(12)

Introducing (12) in (11):

$$\bar{h}(r, p) = \frac{h_0}{p} + \beta K_0(r\omega)$$

(13)

Applying BC (10): $xK_1(x) \rightarrow 1$ if $x \rightarrow 0$

$$\beta = \frac{-Q}{2\pi T p}$$

(14)

Introducing (14) in (13):

$$\bar{h}(r, p) = \frac{h_0}{p} - \frac{Q}{2\pi T p} K_0(r\omega)$$

(15)

Applying definition of drawdown to (15):

$$\bar{s}(r, p) = \frac{h_0}{p} - \bar{h} = \frac{Q}{2\pi T p} K_0(r\omega)$$

(16)

THEIS EQUATION: LAPLACE INVERSION

Inverting (16) analytically:

$$s(r, t) = \frac{Q}{4\pi T} W(u)$$

$$\text{with } u = \frac{r^2 S}{4tT}$$

(17)

Theis' well function W :

$$W(u) = \int_u^\infty \frac{e^{-x}}{x} dx$$

(18)

$f(p) = L\{\bar{f}(t)\}$	$f(t) = L^{-1}\{\bar{f}(p)\}$
1. $1/p$	$\frac{1}{t^{n-1}/(n-1)}$
2. $1/p^n$	$(1/a)[1 - \exp(-at)]$
3. $1/p(p+a)$	$\frac{1}{\sqrt{\pi t}}$
4. $1/\sqrt{p}$	$t^{k-1}/\Gamma(k)$
5. $1/p^k$	$(1/\sqrt{\pi t}) \exp(-k^2/4t)$
6. $(1/\sqrt{p}) \exp(-k\sqrt{p})$	$\text{erfc}(k/\sqrt{4t})$
7. $(1/p) \exp(-k\sqrt{p})$	$(4t)^{n/2} i^n \text{erfc}(k/\sqrt{4t})$
8. $(1/p^{1+n/2}) \exp(-k\sqrt{p})$	$(1/2t) \exp(-k^2/4t)$
9. $K_0(k\sqrt{p})$	$(1/2)W(k^2/4t)$
10. $(1/p)K_0(k\sqrt{p})$	$(1/2t) \exp(-at - k^2/4t)$
11. $K_0(k\sqrt{p+a})$	$(1/2)W(k^2/4t, k\sqrt{a})$
12. $(1/p)K_0(k\sqrt{p+a})$	$(1/2)H(k^2/4t, ka/4)$
13. $(1/p)K_0(k\sqrt{p+a}\sqrt{p})$	$A(t/k_1^2, k/k_1)$
14. $K_0(k\sqrt{p})/pK_0(k_1\sqrt{p})$	$(1/2)S(t/k_1^2, k/k_1)$
15. $K_0(k\sqrt{p})/p(k_1\sqrt{p})K_1(k_1\sqrt{p})$	$Z(t/k_1^2, k/k_1, k_1\sqrt{a})$
16. $K_0(k\sqrt{p+a})/pK_0(k_1\sqrt{p+a})$	

p.303 of "Hydraulics of Wells" (Hantush, 1964)

THE COOPER-JACOB APPROXIMATION

Series expansion of (18):

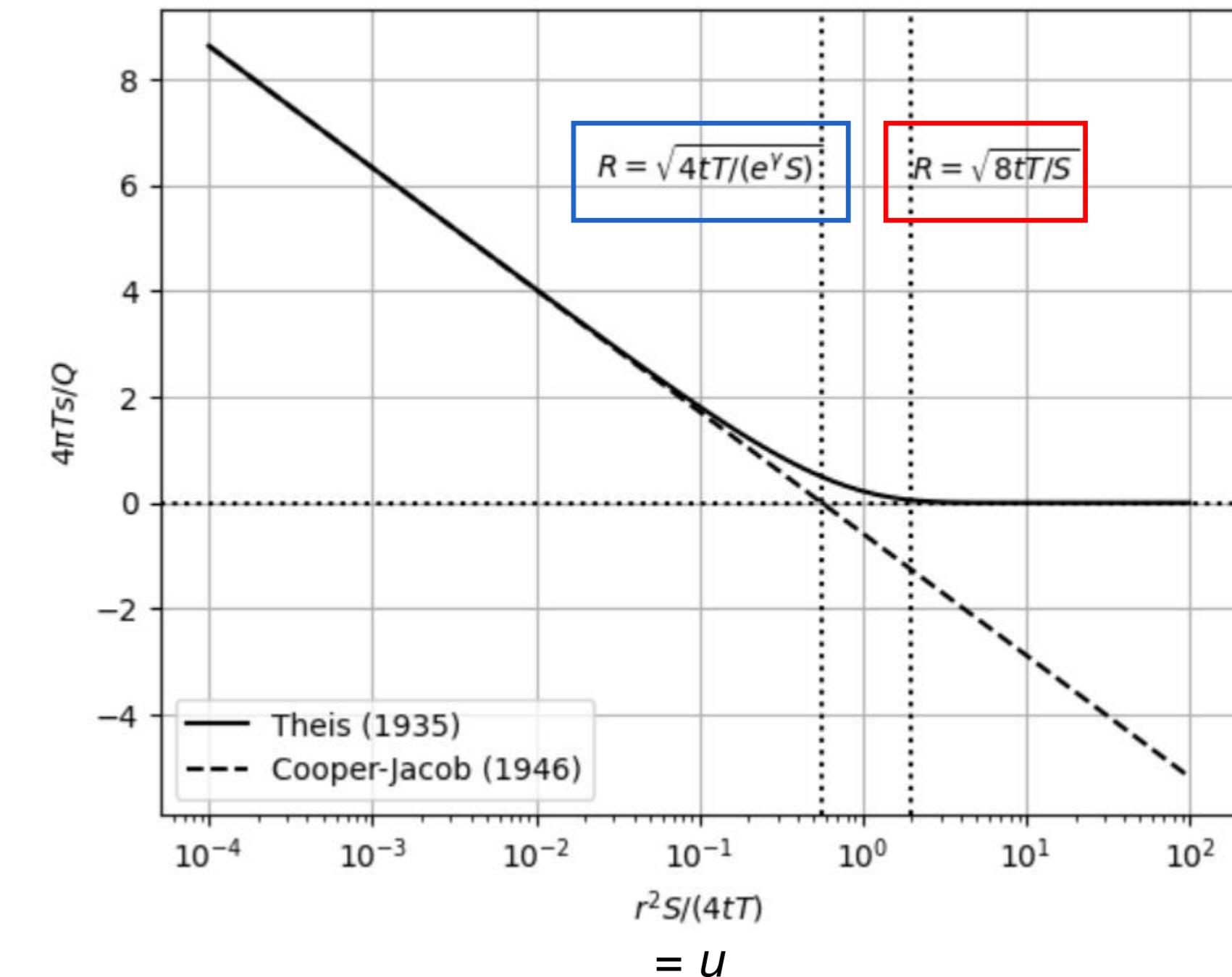
$$W(u) = -\gamma - \ln(u) - \sum_{n=1}^{\infty} \frac{(-u)^n}{n \cdot n!} \quad (19)$$

Truncating (19) and applying to (17):

$$s(r) \approx \frac{Q}{2\pi T} \ln \left(\frac{1}{r} \sqrt{\frac{4tT}{e^\gamma S}} \right) \quad (u < 0.1) \quad (20)$$

Comparing (20) to the Thiem equation:

$$R = \sqrt{\frac{4tT}{e^\gamma S}}$$



THE HANTUSH-JACOB MODEL

Transactions, American Geophysical Union

Volume 36, Number 1

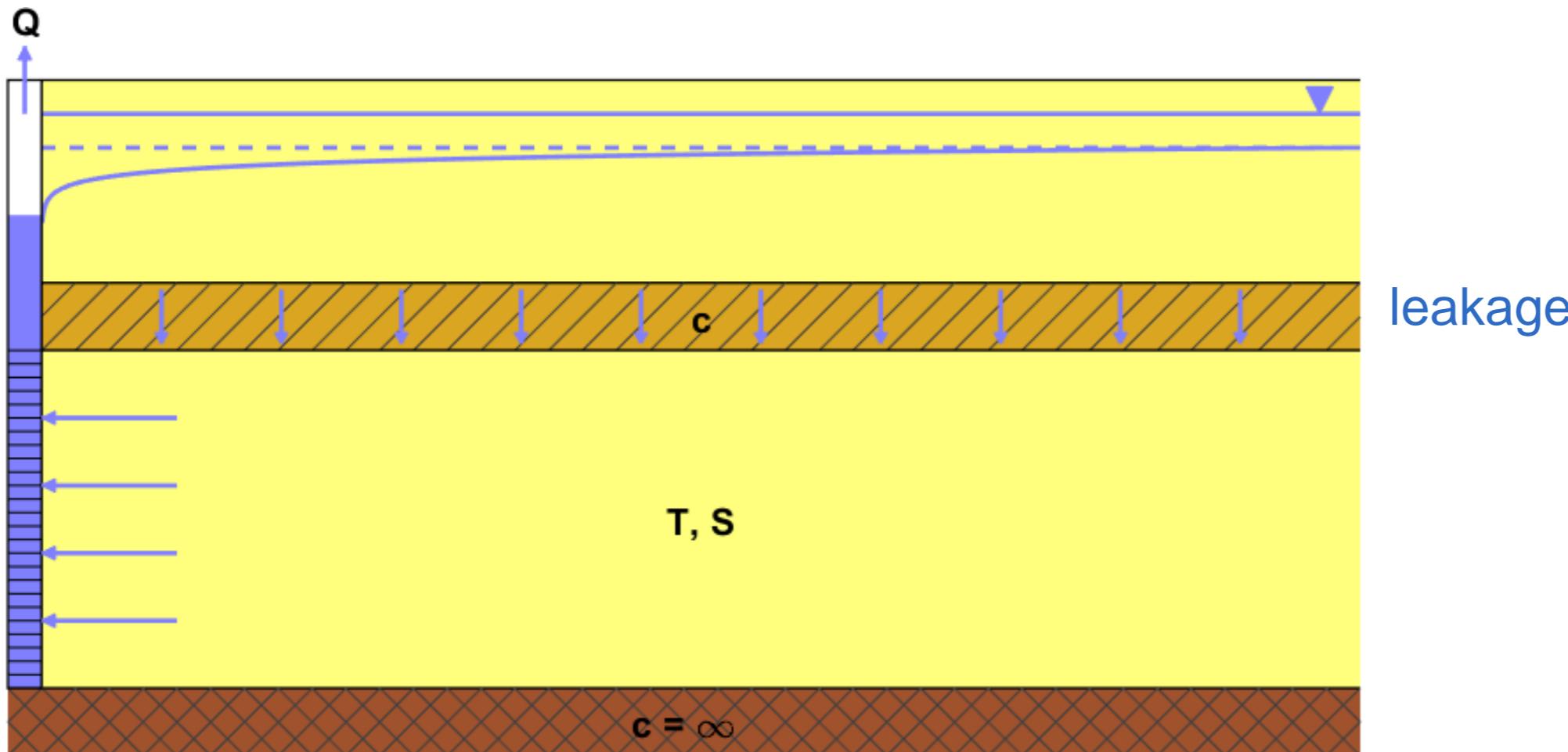
February 1955



Mahdi S. Hantush



Charles E. Jacob



NON-STADY RADIAL FLOW IN AN INFINITE LEAKY AQUIFER

M. S. Hantush and C. E. Jacob

Abstract--The non-steady drawdown distribution near a well discharging from an infinite leaky aquifer is presented. Variation of drawdown with time and distance caused by a well of constant discharge in confined sand of uniform thickness and uniform permeability is obtained. The discharge is supplied by the reduction of storage through expansion of the water and the concomitant compression of the sand, and also by leakage through the confining bed. The leakage is assumed to be at a rate proportional to the drawdown at any point. Storage of water in the confining bed is neglected. Two forms of the solution are developed. One is suitable for computation for large values of time and the other suitable for small values of time. This solution is compared with earlier solutions for slightly different boundary conditions.

Introduction--The differential equation for radial flow in an elastic artesian aquifer with linear leakage has been given by JACOB [1946]. He also obtained the non-steady drawdown distribution produced by a well of constant discharge situated in the center of a circular region whose outer boundary is maintained at constant head. The head distribution in his problem is initially uniform.

In this paper the solution is obtained for the problem in which the outer boundary is removed to infinity.

Statement of the problem--The problem is to determine the variation with time of the drawdown induced by a well steadily discharging from an infinite leaky aquifer in which the initial head is uniform. Leakage into the aquifer is assumed vertical and proportional to the drawdown. Stated mathematically the boundary-value problem is

$$\frac{\partial^2 s}{\partial r^2} + \left(\frac{1}{r}\right) \frac{\partial s}{\partial r} - \frac{s}{D^2} = \left(\frac{K}{T}\right) \frac{\partial s}{\partial t} \quad \dots \dots \dots (1)$$

$$s(r, 0) = 0 \quad r \geq 0 \quad \dots \dots \dots (2a)$$

$$s(\infty, 0) = 0 \quad t \geq 0 \quad \dots \dots \dots (2b)$$

$$\lim_{r \rightarrow 0} r \frac{\partial s}{\partial r} = -Q/2\pi T \quad t > 0 \quad \dots \dots \dots (2c)$$

where

$s(r, t)$ is the drawdown at any time and any distance from the well.

r is the distance to any point measured from the axis of the well.

S is the storage coefficient of the artesian aquifer (a non-dimensional constant) defined as "the product of the thickness of the artesian bed and the relative volume of water released from storage by a unit decline of head" [JACOB, 1946].

K and K' are the hydraulic conductivities (or 'permeabilities') of the artesian sand and confining bed, respectively. They have the dimension L/t .

b and b' are the thicknesses of the artesian sand and confining bed, respectively.

$T = Kb$ is the transmissibility (of dimension L^2/t) of the artesian sand. The ratio K'/b' may be termed 'specific leakage' or 'leakance' [HANTUSH, 1949, p. 8]. It has the dimension t/L .

The transmissibility divided by the leakance (of dimension L^3) is symbolized by B^2 .

Q is the discharge of the well.

Solution of the problem--After separating the variables, it can be shown that

$$J_0(\alpha r/B) \exp[-(\alpha^2 + 1) Tt/BD^2] \quad \text{and} \quad K_0(r/B)$$

are particular solutions of (1), where J_0 and K_0 are respectively the Bessel function of the first kind of zero order and the modified Bessel function of the second kind of zero order, and where α is any real constant.

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Transient leaky flow (Hantush & Jacob, 1955)

$$s(r, t) = \frac{Q}{4\pi K D} W \left(\frac{r^2 S}{4t K D}, r \sqrt{\frac{1}{c K D}} \right)$$

HANTUSH-JACOB: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Transient-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Constant pumping rate
 - Infinitesimal radius
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded
 - Leaky top

HANTUSH-JACOB: PROBLEM STATEMENT

Continuity of transient 1D leaky flow:

$$T \left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} \right) = S \frac{\partial h}{\partial t} + \frac{h - h_0}{c}$$

storage change leakage

(1)

Inner boundary condition at zero:

$$Q = \lim_{r \rightarrow 0} \left(2\pi r T \frac{\partial h}{\partial r} \right)$$

(2)

Outer boundary condition at infinity:

$$h(\infty, t) = h_0$$

(3)



Partial differential equation (PDE):

- head h is function of r and t
- apply Laplace transform w.r.t. t

Initial condition at $t = 0$:

$$h(r, 0) = h_0$$

(4)

HANTUSH-JACOB: LAPLACE TRANSFORM

Laplace transform of PDE (1):

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r} \frac{d\bar{h}}{dr} = \left(\omega^2 + \frac{1}{\lambda^2} \right) \left(\bar{h} - \frac{h_0}{p} \right) \quad (5)$$

Laplace transform of inner BC (2):

$$\frac{Q}{p} = \lim_{r \rightarrow 0} \left(2\pi r T \frac{d\bar{h}}{dr} \right) \quad (6)$$

Laplace transform of outer BC (3):

$$\bar{h}(\infty, p) = \frac{h_0}{p} \quad (7)$$

Modified Bessel differential equation:

$$\frac{d^2\bar{h}}{dr^2} + \frac{1}{r} \frac{d\bar{h}}{dr} = a\bar{h} - b$$

General solution:

$$\bar{h} = \alpha I_0(r\sqrt{a}) + \beta K_0(r\sqrt{a}) + \frac{b}{a}$$

with:

- I_0, K_0 : the zero order modified Bessel functions of the first and second kind, resp.
- α, β : integration constants

HANTUSH-JACOB: LAPLACE SOLUTION

General solution of ODE (5):

$$\bar{h} = \alpha I_0(r\vartheta) + \beta K_0(r\vartheta) + \frac{h_0}{p}$$

with $\vartheta = \sqrt{\omega^2 + \frac{1}{\lambda^2}}$

(8)

Applying BC (7): $I_0(x) \rightarrow \infty$ if $x \rightarrow \infty$

$$\alpha = 0$$

(9)

Introducing (9) in (8):

$$\bar{h}(r, p) = \frac{h_0}{p} + \beta K_0(r\vartheta)$$

(10)

Applying BC (6): $xK_1(x) \rightarrow 1$ if $x \rightarrow 0$

$$\beta = \frac{-Q}{2\pi T p}$$

(11)

Introducing (11) in (10):

$$\bar{h}(r, p) = \frac{h_0}{p} - \frac{Q}{2\pi T p} K_0(r\vartheta)$$

(12)

Applying definition of drawdown to (12):

$$\bar{s}(r, p) = \frac{h_0}{p} - \bar{h} = \frac{Q}{2\pi T p} K_0(r\vartheta)$$

(13)

HANTUSH-JACOB: LAPLACE INVERSION

Inverting (13) analytically:

$$s(r, t) = \frac{Q}{4\pi T} W(u, v)$$

$$\text{with } u = \frac{r^2 S}{4tT} \text{ and } v = \frac{r}{\sqrt{Tc}} \quad (14)$$

Hantush' well function W:

$$W(u, v) = \int_u^\infty \frac{e^{-x-v^2/4x}}{x} dx \quad (15)$$

$f(p) = L\{\bar{f}(t)\}$	$f(t) = L^{-1}\{\bar{f}(p)\}$
1. $1/p$	1
2. $1/p^n$	$t^{n-1}/(n-1) !$
3. $1/p(p+a)$	$(1/a)[1 - \exp(-at)]$
4. $1/\sqrt{p}$	$1/\sqrt{\pi t}$
5. $1/p^k$	$t^{k-1}/\Gamma(k)$
6. $(1/\sqrt{p}) \exp(-k\sqrt{p})$	$(1/\sqrt{\pi t}) \exp(-k^2/4t)$
7. $(1/p) \exp(-k\sqrt{p})$	$\text{erfc}(k/\sqrt{4t})$
8. $(1/p^{1+n/2}) \exp(-k\sqrt{p})$	$(4t)^{n/2} t^n \text{erfc}(k/\sqrt{4t})$
9. $K_0(k\sqrt{p})$	$(1/2t) \exp(-k^2/4t)$
10. $(1/p)K_0(k\sqrt{p})$	$(1/2)W(k^2/4t)$
11. $K_0(k\sqrt{p+a})$	$(1/2t) \exp(-at - k^2/4t)$
12. $(1/p)K_0(k\sqrt{p+a})$	$(1/2)W(k^2/4t, k\sqrt{a})$
13. $(1/p)K_0(k\sqrt{p+a}\sqrt{p})$	$(1/2)H(k^2/4t, ka/4)$
14. $K_0(k\sqrt{p})/p K_0(k_1\sqrt{p})$	$A(t/k_1^2, k/k_1)$
15. $K_0(k\sqrt{p})/p (k_1\sqrt{p}) K_1(k_1\sqrt{p})$	$(1/2)S(t/k_1^2, k/k_1)$
16. $K_0(k\sqrt{p+a})/p K_0(k_1\sqrt{p+a})$	$Z(t/k_1^2, k/k_1, k_1\sqrt{a})$

p.303 of "Hydraulics of Wells" (Hantush, 1964)

INVERTING NUMERICALLY: STEHFEST

- Semi-analytical solution:
 - Analytical closed-form solution in Laplace space
 - Numerical inversion to the real time domain
- Several inversion algorithms available:
 - de Hoog et al. (1982)
 - Gaver (1966) - Stehfest (1970):

$$s(r, t) = \frac{\ln 2}{t} \sum_{k=1}^N \omega_k \bar{s}(r, k \ln 2/t)$$

with N the Stehfest number and

$$\omega_k = (-1)^{N/2+k} \sum_{j=\frac{k+1}{2}}^{\min(k, N/2)} \frac{j^{N/2} (2j)!}{(N/2-j)! (j)! (j-1)! (k-j)! (2j-k)!}$$

HANTUSH-JACOB VS THEIS VS DE GLEE

Confined: $c \rightarrow \infty$ and $v \rightarrow 0$

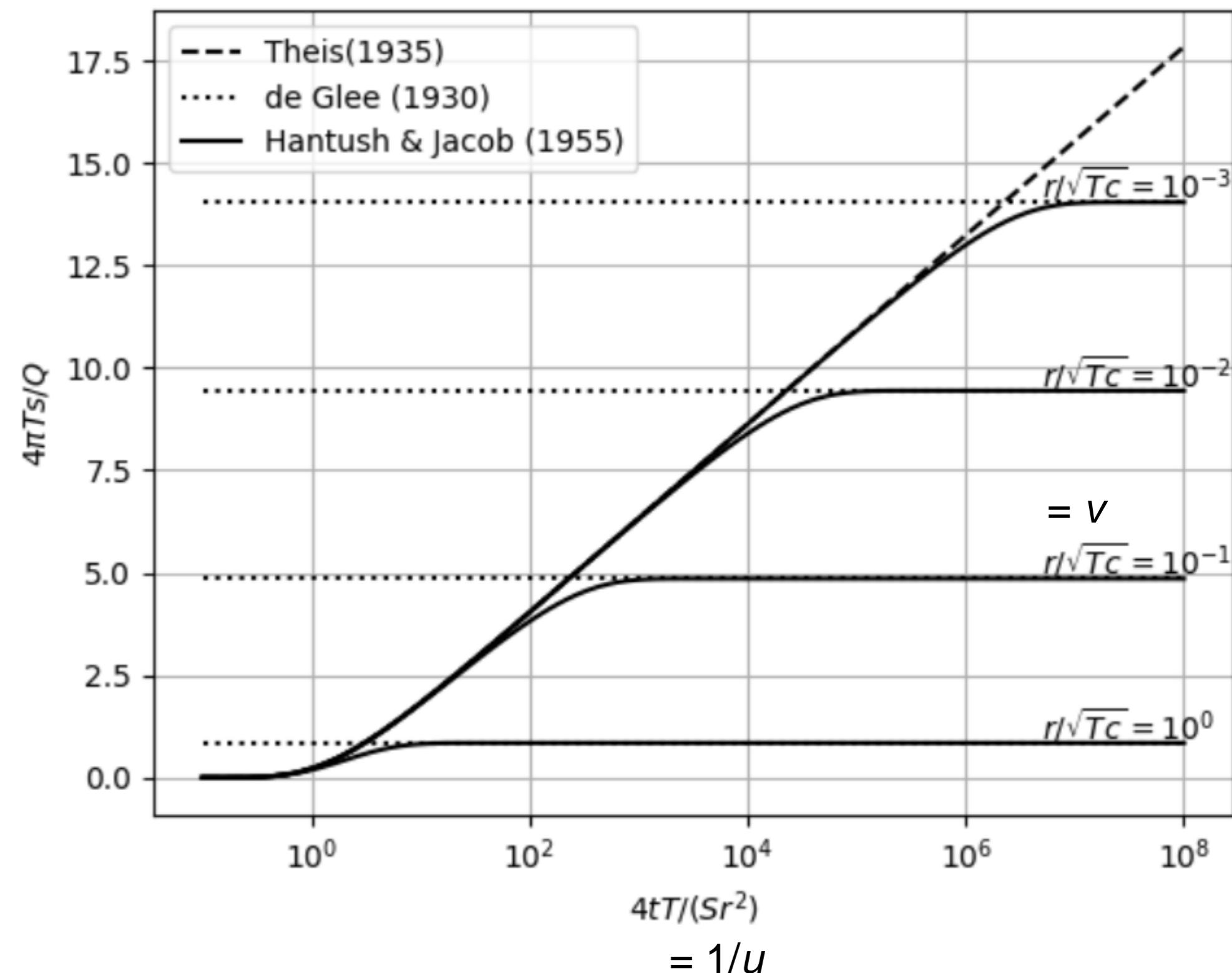
$$\lim_{v \rightarrow 0} W(u, v) = W(u)$$

Hantush-Jacob \rightarrow Theis

Steady-state: $t \rightarrow \infty$ and $u \rightarrow 0$

$$\lim_{u \rightarrow 0} W(u, v) = 2K_0(v)$$

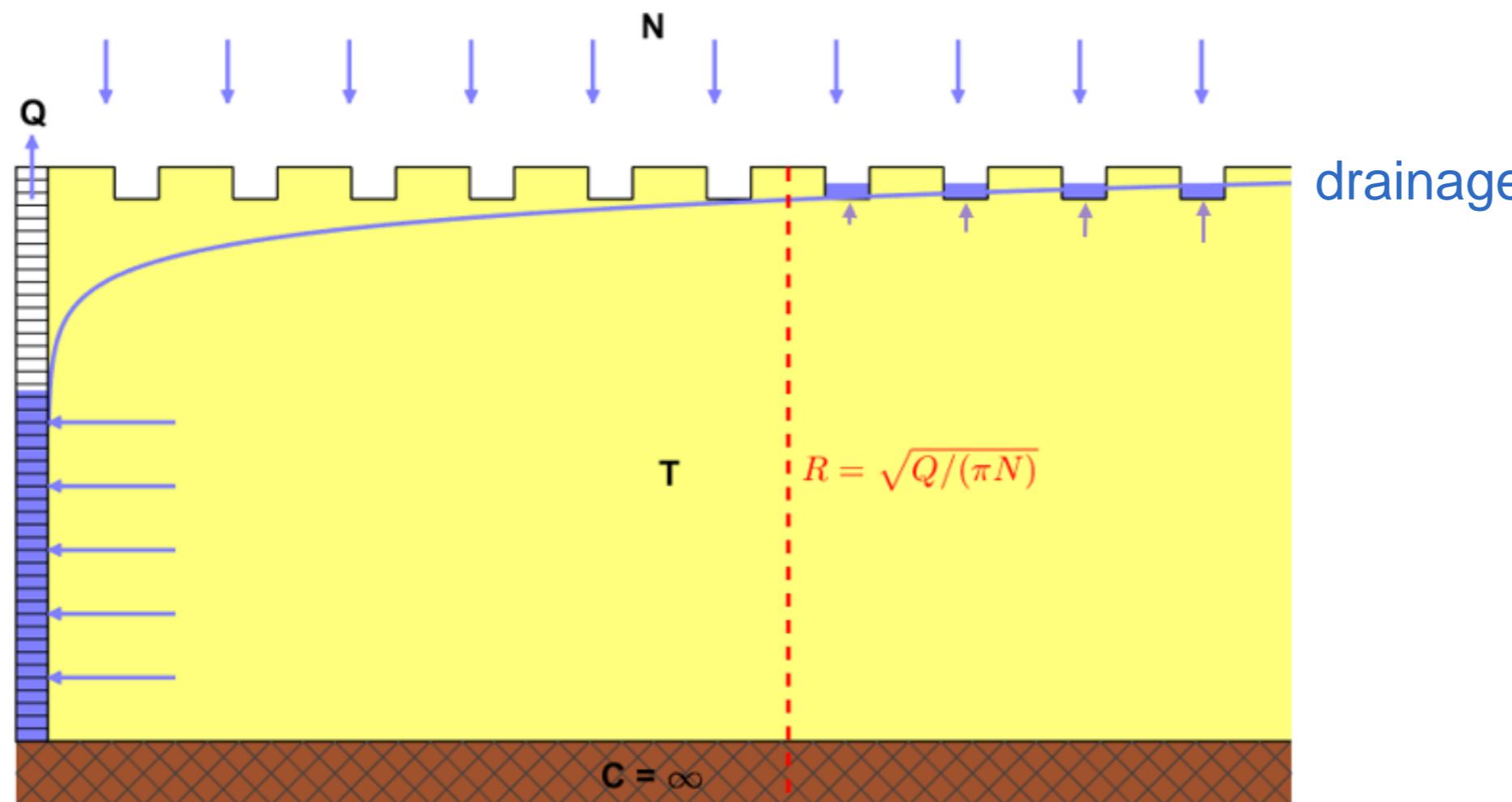
Hantush-Jacob \rightarrow de Glee



THE ERNST MODEL

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ANALYSIS OF GROUNDWATER FLOW TO DEEP WELLS IN AREAS WITH A NON-LINEAR FUNCTION FOR THE SUBSURFACE DRAINAGE

L. F. ERNST

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Abstract: The quantitative analysis of groundwater flow to deep wells in areas where the excess precipitation is discharged for a major part by surface drains, requires information about the system of this surface drainage. The non-linear relation between the discharge and the phreatic level (areal mean) can be explained mainly by the fact that the length of the drains containing water and giving discharge is varying in the same sense as the discharge and the phreatic level. Changes in evaporation by the plants are of less importance in this connection. As there is some evidence that the amplitude of the seasonal fluctuations of the phreatic surface will not be influenced very much when there is a constant pumping of water from deep wells, the change in that surface effected by pumping can be put equal to the drawdown during a steady state flow to the deep well. When the relation between hydraulic head and discharge by drains is linearized, i.e. represented in a graph as a broken straight line with for each part a specific value for the effective drainage resistance Y_e , the basic differential equation is reduced to a Bessel equation of zero order. The steady state solution either contains a combination of modified Bessel functions (for finite values of Y_e) or a logarithm (when the effective drainage resistance $Y_e = \infty$). The determination of the integration constants for several zones around the well is in principle not difficult. In the paper an explicit solution is only given for a rather simple case.

Introduction

Where in humid areas the ground surface has only relatively small differences in elevation and the transmissivity of the underground is not very small, the excess precipitation is mainly carried off by groundwater flow to a system of rather closely spaced surface drains of different size and level (Fig. 1). The depth of the groundwater table and the discharge by the drains are variable owing to seasonal fluctuations of the evaporation and irregular variations of the precipitation.

Deep well pumping of groundwater from thick phreatic aquifers or from semi-confined aquifers will cause a decline of the phreatic surface, especially in the case of phreatic aquifers. Primarily this involves a smaller discharge of water by the surface drains. In those cases that formerly during summer (period with main evaporation) the depth of the phreatic surface was rather

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Steady flow with recharge and nonlinear drainage (Ernst, 1971)

$$s(r) = \frac{Q}{2\pi K D} \ln \left(\frac{R}{r} \right) - \frac{N}{4 K D} (R^2 - r^2) \quad \text{if } c \rightarrow 0$$

ERNST: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Steady-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Constant pumping rate
 - Infinitesimal radius
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded
 - Uniform areal infiltration + drainage

ERNST: PROBLEM STATEMENT

Steady flow with infiltration:

$$T \left(\frac{d^2 h_1}{dr^2} + \frac{1}{r} \frac{dh_1}{dr} \right) = -N$$

infiltration

(1)

Inner boundary condition at zero:

$$Q = \lim_{r \rightarrow 0} \left(2\pi r T \frac{dh_1}{dr} \right)$$

(2)

Outer boundary condition at R :

$$h_1(R) = 0$$

(3)

R

Steady flow with infiltration & drainage:

$$T \left(\frac{d^2 h_2}{dr^2} + \frac{1}{r} \frac{dh_2}{dr} \right) = -N + \frac{h_2}{c}$$

infiltration drainage

(4)

Inner boundary condition at R :

$$h_2(R) = 0$$

(5)

Outer boundary condition at infinity:

$$h_2(\infty) = 0$$

(6)

ERNST: SOLUTION

General solution of (1):

$$h_1 = \frac{-Nr^2}{4T} + \alpha_1 \ln r + \beta_1 \quad (7)$$

Applying inner BC (2) and outer BC (3):

$$\begin{aligned} \alpha_1 &= \frac{Q}{2\pi T} \\ \beta_1 &= \frac{NR^2}{4T} - \frac{Q}{2\pi T} \ln R \end{aligned} \quad (8)$$

Introducing (8) in (7):

$$h_1 = \frac{N}{4T} (R^2 - r^2) + \frac{Q}{2\pi T} \ln \frac{r}{R} \quad (9)$$

zone 1: $r \leq R$

R

General solution of (4):

$$h_2 = \alpha_2 I_0 \left(\frac{r}{\lambda} \right) + \beta_2 K_0 \left(\frac{r}{\lambda} \right) + Nc \quad (10)$$

Applying outer BC (6) and inner BC (5):

$$\alpha_2 = 0$$

$$\beta_2 = \frac{-Nc}{K_0 \left(\frac{R}{\lambda} \right)} \quad (11)$$

Introducing (11) in (10):

$$h_2 = Nc \left[1 - \frac{K_0 \left(\frac{r}{\lambda} \right)}{K_0 \left(\frac{R}{\lambda} \right)} \right]$$

zone 2: $r \geq R$

ERNST: DETERMINING R

Continuity of flow at boundary R :

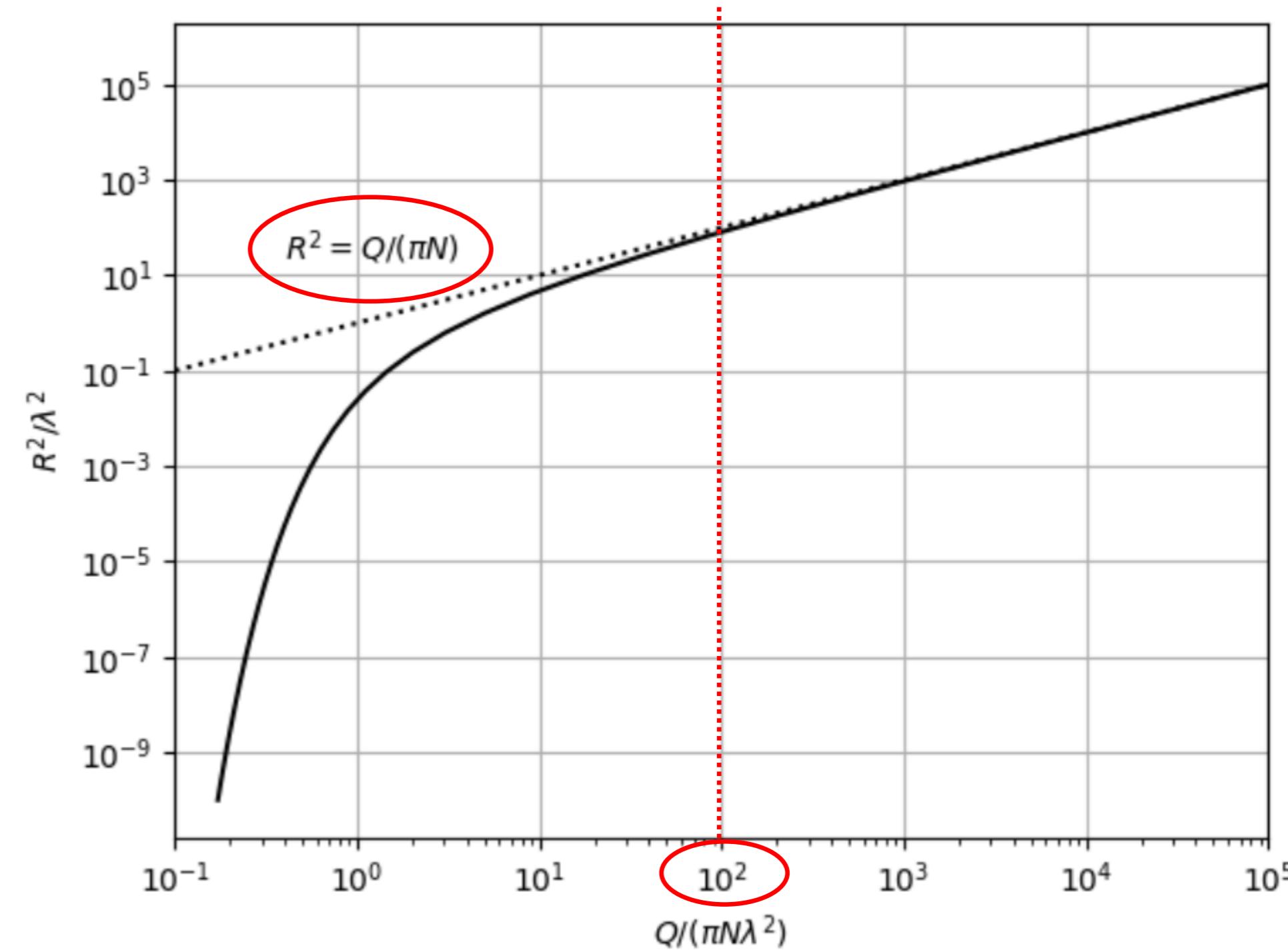
$$\frac{dh_1}{dr} = \frac{dh_2}{dr} \quad (r = R) \quad (13)$$

Solving (13) using (9) and (12) and rearranging:

$$R \left[\frac{R}{\lambda} + 2 \frac{K_1\left(\frac{R}{\lambda}\right)}{K_0\left(\frac{R}{\lambda}\right)} \right] - \frac{Q}{\pi N \lambda^2} = 0 \quad (14)$$

Expression (14) simplifies to:

$$Q = \pi R^2 N \quad \text{if } Q > 100\pi N \lambda^2$$



ERNST: ASYMPTOTIC SOLUTIONS

Low drainage resistance: $\lambda \rightarrow 0$

$$R = \sqrt{Q/(\pi N)}$$

Radius of influence!

(15)

$$h_1 = \frac{N}{4T} (R^2 - r^2) + \frac{Q}{2\pi T} \ln \frac{r}{R}$$

(16)

$$h_2 = 0$$

(17)

Initial head h_0 is equal to Nc

High drainage resistance: $\lambda \rightarrow \infty$

$$R = 0$$

(18)

$$\cancel{h_1 = -\infty}$$

(19)

$$h_2 = Nc - \frac{Q}{2\pi T} K_0 \left(\frac{r}{\lambda} \right)$$

(20)

$$\cancel{Q = 0}$$

Applying definition of drawdown s to (16):

$$s = -h_1 = \frac{Q}{2\pi T} \ln \frac{R}{r} - \frac{N}{4T} (R^2 - r^2)$$

Applying definition of drawdown s to (20):

$$s = Nc - h_2 = \frac{Q}{2\pi T} K_0 \left(\frac{r}{\lambda} \right)$$

ERNST VS THIEM VS DE GLEE

Relatively low resistance:

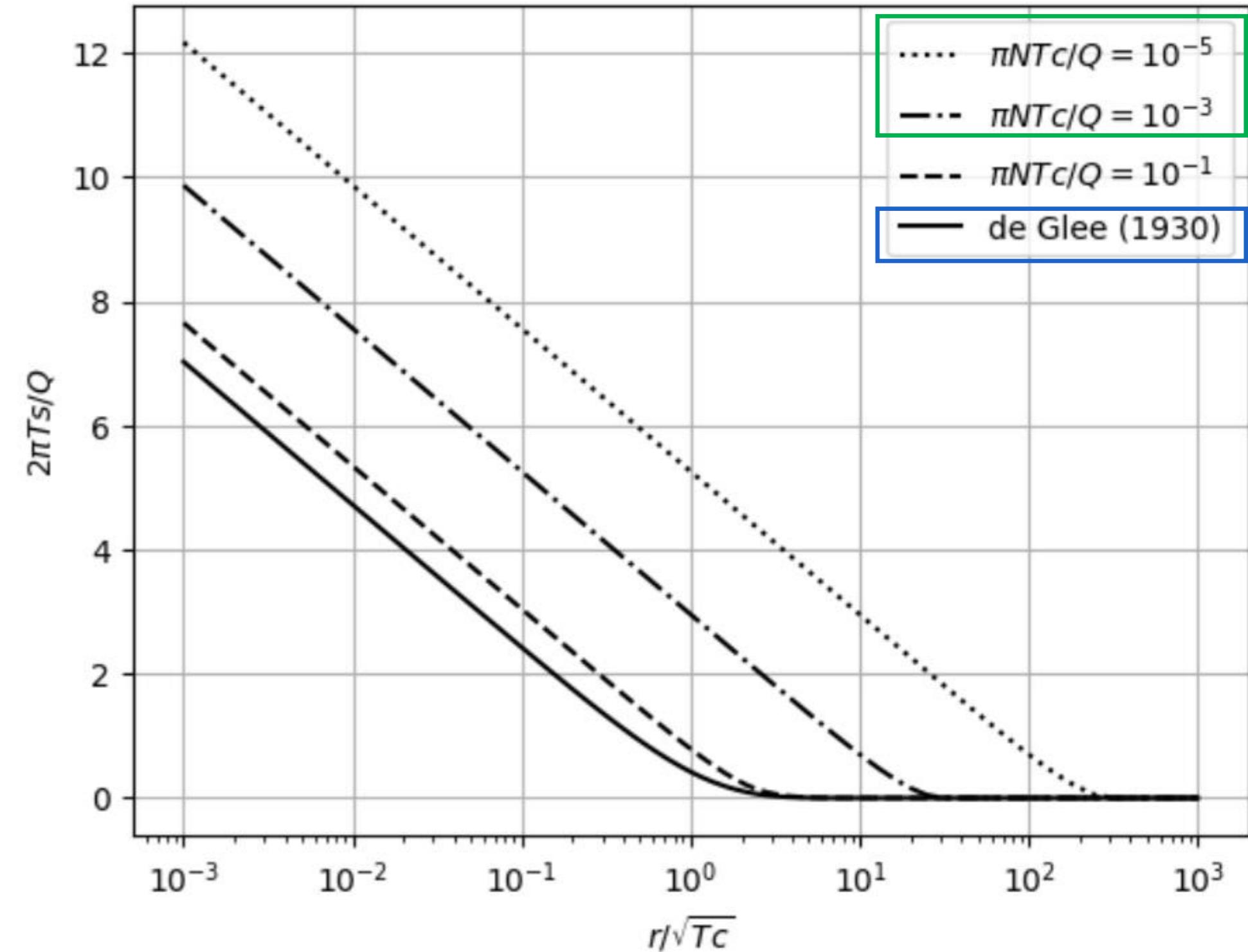
$$Q > 100\pi N \lambda^2$$

Ernst → Thiem + Infiltration pond

Relatively high resistance:

$$Q < \pi N \lambda^2$$

Ernst → de Glee



THE RADIUS OF INFLUENCE MYTH

THE RADIUS OF INFLUENCE

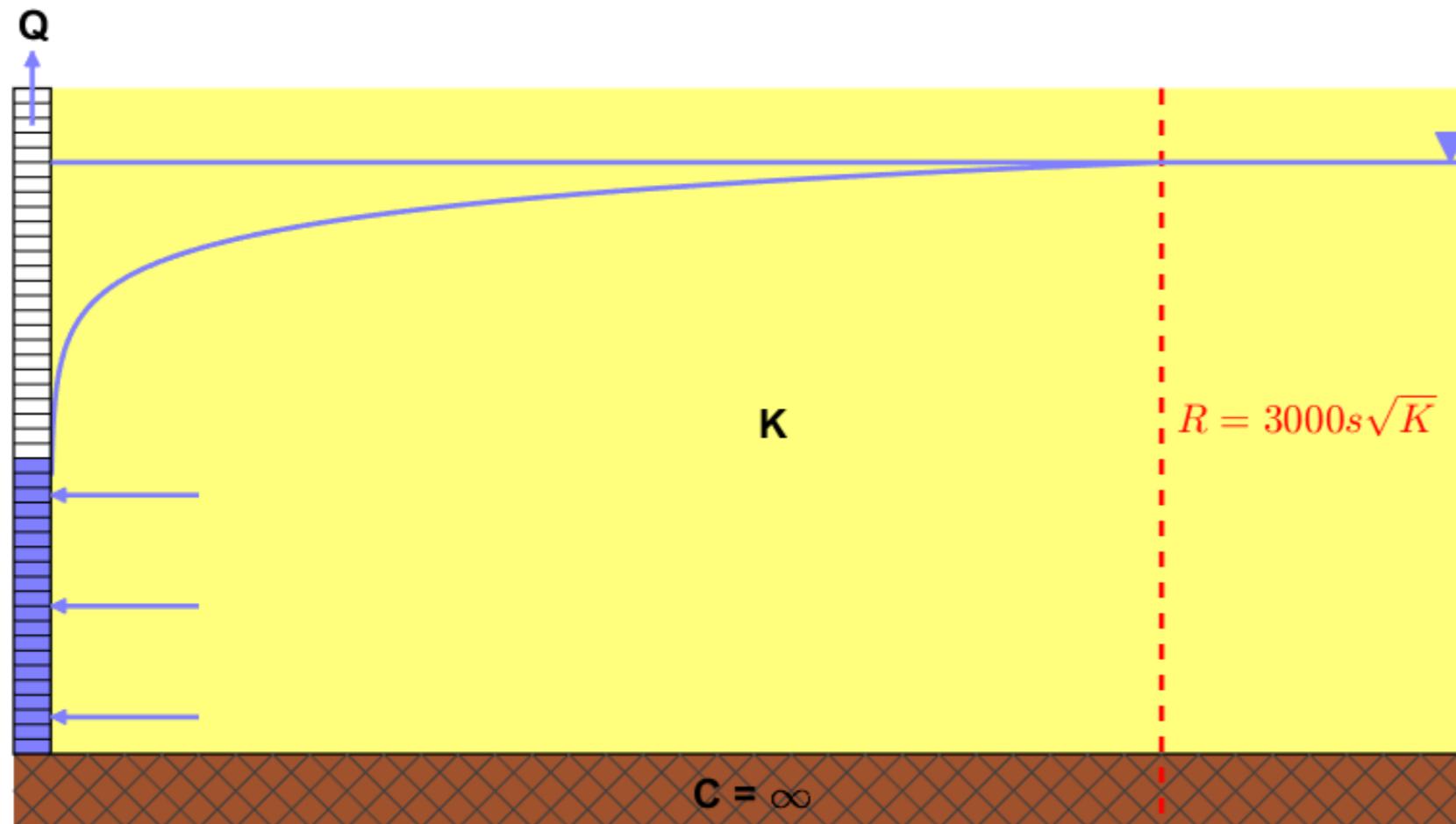
= *the radial distance from the pumping well where there is no lowering of the head or beyond which drawdown is negligible*

- Thiem-Dupuit equations: outer boundary
- How to determine?
 - empirical formulas
 - models defining more realistic boundary conditions

Author	Reference	Formula Influence	Radius of Influence
Lembke	(1886, 1887)	$R = h_o \times \sqrt{\frac{K}{2N}}$	
Weber	[Kyrieleis-Sichardt, 1930]	$R = 3 \times \sqrt{\frac{h_o \times K \times t}{n_e}}$	
Kusakin	Chertusov, 1949	$R = 575 \times s_w \times \sqrt{K \times h_o}$	
Kusakin	Aravin and Numerov, 1953	$R = 1,9 \times \sqrt{\frac{h_o \times K \times t}{n_e}}$	
Sichardt	[Kirieleis-Sichardt, 1930]	$R = 3000 \times s_w \times \sqrt{K}$	

<https://hatariwater.tumblr.com/post/138690184404/overview-of-the-radius-of-influence>

EMPIRICAL SICHARDT FORMULA



weite begnügen kann. Einen gewissen Anhalt für solche Schätzungen gibt eine von Sichardt empirisch gefundene Formel, die bisher noch nicht veröffentlicht worden ist und hier mitgeteilt sei. Sie gilt für den Beharrungszustand und lautet

$$R = 3000 s \sqrt{k}, \quad (26)$$

worin s = Absenkung in m .

Grundwasserabsenkung bei Fundierungsarbeiten

von
Dr.-Ing. Wilhelm Kyrieleis

In zweiter Auflage neubearbeitet
von
Dr.-Ing. Willy Sichardt

Mit 152 Abbildungen im Text
und 3 Tafeln



Berlin
Verlag von Julius Springer
1930

FORWARD AND INVERSE PROBLEMS

- **forward problem**

- simulate head h or drawdown s
- e.g. assessing the environmental impact of extractions

$$s = \frac{Q}{2\pi KD} \ln \left(\frac{R}{r} \right)$$

- **inverse problem type I**

- derive transmissivity KD
- e.g. pumping test interpretation

$$KD = \frac{Q}{2\pi} \frac{\ln r_2 - \ln r_1}{s_1 - s_2}$$

- **inverse problem type II**

- derive pumping rate Q
- e.g. construction dewatering

$$Q = 2\pi KD \frac{s_w}{\ln R - \ln r_w}$$

DIFFERENT PERSPECTIVES

- Well performance and efficiency
- Hydraulic characteristics of aquifers
- The groundwater **basin** as part of the hydrological system
- Groundwater **sustainability** which also considers water quality, ecological and socio-economic aspects.

THE RADIUS OF INFLUENCE MYTH

Applying the Sichardt formula:

- is inconsistent with fundamental hydrogeological principles
- may underestimate the extent of the cone of depression
- is not recommended to assess the impact of extractions



Article

The Radius of Influence Myth

by Andy Louwyck ^{1,*} Alexander Vandenbohede ² Dirk Libbrecht ³ ,
 Marc Van Camp ¹ and Kristine Walraevens ¹

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ALTERNATIVES

1D axisymmetric models consistent with fundamental principles:

- **de Glee (1930)**: steady well-flow in a leaky aquifer
- **Theis (1935)**: transient well-flow in a confined aquifer
- **Hantush-Jacob (1955)**: transient well-flow in a leaky aquifer
- **Ernst (1971)**: steady well-flow in a phreatic aquifer subject to uniform infiltration and drainage

Table 1. Summary of the analytical models discussed in the paper applied to simulate axisymmetric flow towards a fully penetrating well with infinitesimal radius and constant pumping rate in a homogeneous aquifer with impervious base. From the solutions of these models, equations and rules of thumb are derived to estimate the radius of influence R , with KD the transmissivity, c the resistance, S the storage coefficient, N the infiltration flux, Q the pumping rate, and t the time. See text for explanation and definitions.

Model	Flow Regime	Outer Boundary	Upper Boundary	Initial Flow	Super-Position	Radius of Influence R
Dupuit [76]	Steady	Finite	Water table	None	No ⁴	Outer boundary (=input parameter)
Thiem [70]	Steady	Finite	Impervious ¹	Steady	Yes	Outer boundary (=input parameter)
de Glee [71,86]	Steady	Infinite	Leaky ²	Steady	Yes	$R = 4\sqrt{cKD}$
Theis [72]	Transient	Infinite	Impervious ¹	Steady	Yes	$R = 1.5\sqrt{\frac{tKD}{S}}$
Hantush-Jacob [73]	Transient	Infinite	Leaky ²	Steady	Yes	$R = 1.5\sqrt{\frac{tKD}{S}} \text{ if } t < 0.01Sc$ $R = 4\sqrt{cKD} \text{ if } t > 10Sc$
Ernst [74]	Steady	Infinite	Drainage + Recharge	None ³	No ⁴	$R = \sqrt{\frac{Q}{\pi N}} \text{ if } \frac{Q}{\pi NKDc} > 100$ $R = 4\sqrt{cKD} \text{ if } \frac{Q}{\pi NKDc} < 1$
Transient Ernst (Appendix A)	Transient	Infinite	Drainage + Recharge	None ³	No ⁴	See Figure 5

¹ Or water table if drawdown is less than 10% of initial saturated thickness. ² Leakage through incompressible aquitard or linear surface water interaction (cfr. MODFLOW river). ³ Initial heads equal to Nc are relative to the steady drainage levels, which are set to zero for convenience. ⁴ Unless the solution may be approximated by its corresponding linear equation.

THE WATER BUDGET

MYTH AND

THE SUPERPOSITION

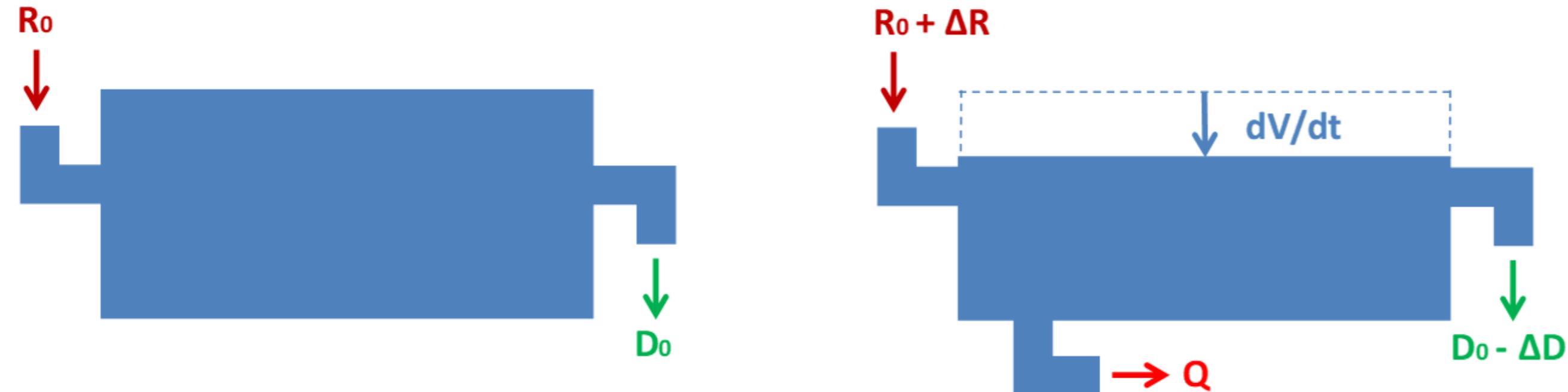
PRINCIPLE

THE WATER BUDGET MYTH

– Safe yield: $\cancel{Q = R_0}$

– Capture equation: $Q = \Delta R - \Delta D - \frac{dV}{dt}$

(Theis, 1940; Bredehoeft et al., 1982; Bredehoeft, 2002)



Michael E. Campana
(Read this paper before the
one by Bredehoeft et al.)
The Source of Water Derived from Wells
Essential Factors Controlling the Response of an Aquifer to Development
FROM A PAPER PRESENTED BEFORE THE ARIZONA SECTION
By CHARLES V. THEIS

Bredehoeft, J.D., S.S. Papadopoulos and H.H. Cooper, 1982. Groundwater: the Water-Budget Myth. In *Scientific Basis of Water-Resource Management*, Studies in Geophysics, Washington, DC: National Academy Press, pp. 51-57.

Groundwater:
The Water-Budget Myth



John D. Bredehoeft

JOHN D. BREDEHOEFT
U.S. Geological Survey

STEPHEN S. PAPADOPOLOS
S. S. Papadopoulos and Associates, Inc.

H. H. COOPER, JR.
U.S. Geological Survey

DIFFERENT PERSPECTIVES

- **Sustainable pumping:**
 - = the well does not go dry
 - only considers well performance
- **Sustainability:**
 - = much broader concept
 - also considers water quality, socio-economic and ecological aspects



LINEAR VS NONLINEAR MODELS

- Linear model: $Q = \Delta R - \Delta D - \frac{dV}{dt}$
 - superposition: recharge is canceled out
 - implicit assumption of infinite sources of water
 - e.g.: Thiem, de Glee, Theis, Hantush-Jacob
- Nonlinear model: $Q = [R_t - R_0] - [D_t - D_0] - \frac{dV}{dt}$
 - initial conditions are relevant
 - so is recharge!
- Ernst model: $Q = [D_t - D_0]$ $= \pi R^2 N$



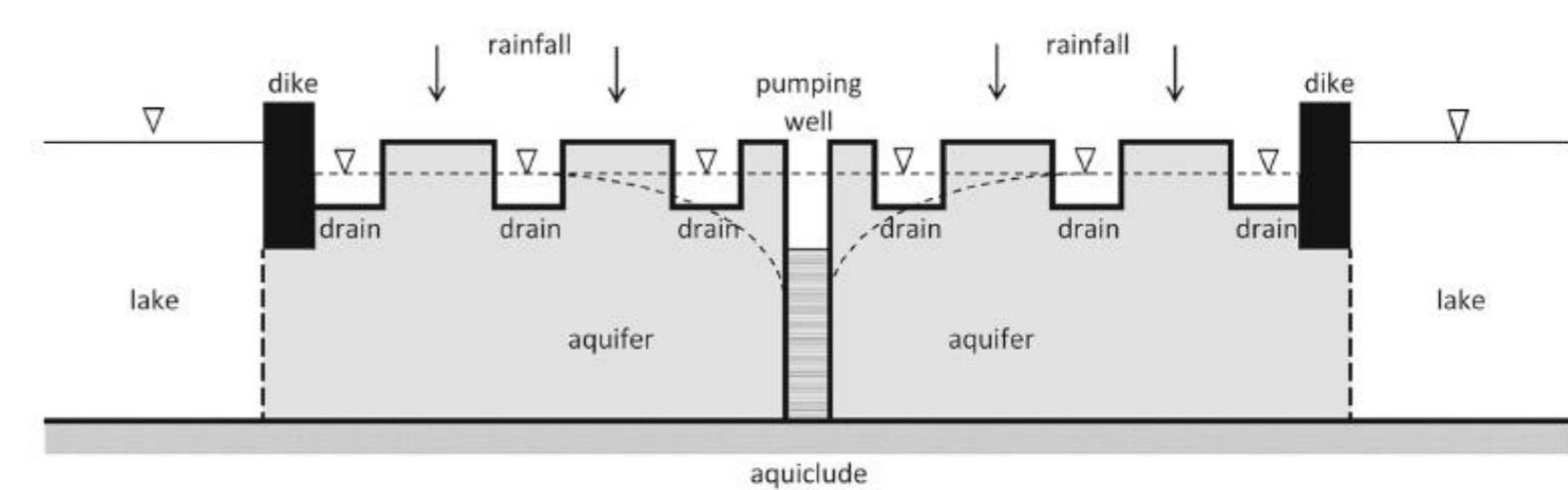
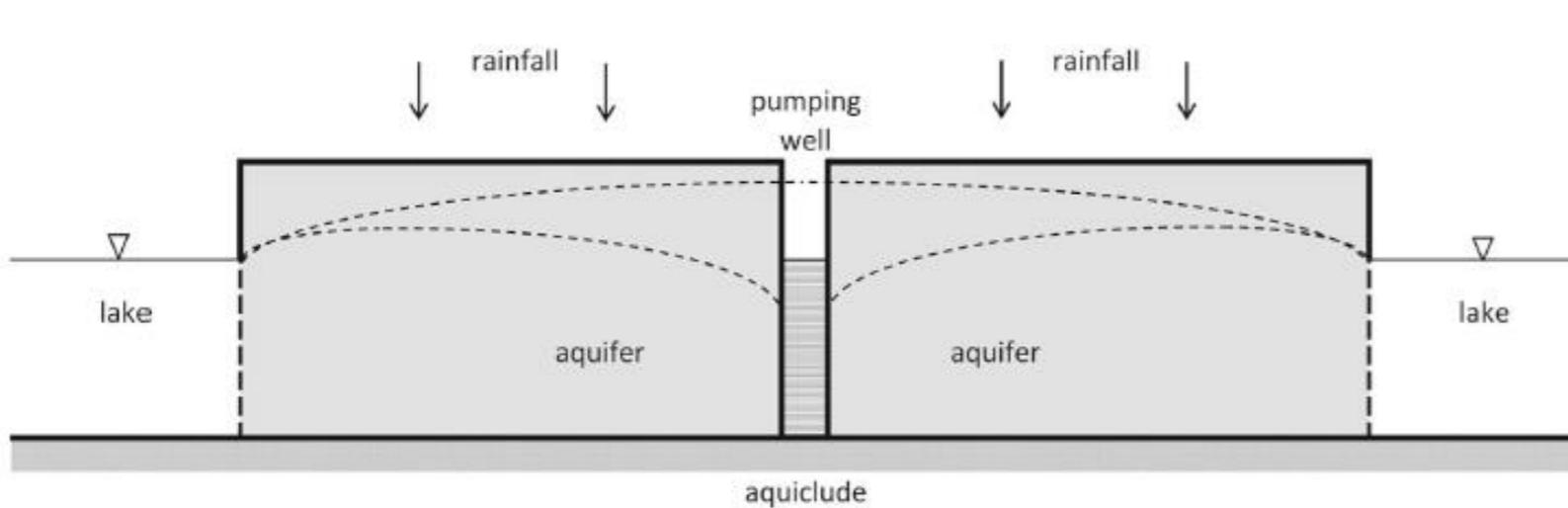
Research Paper/

The Water Budget Myth and Its Recharge Controversy: Linear vs.
Nonlinear Models

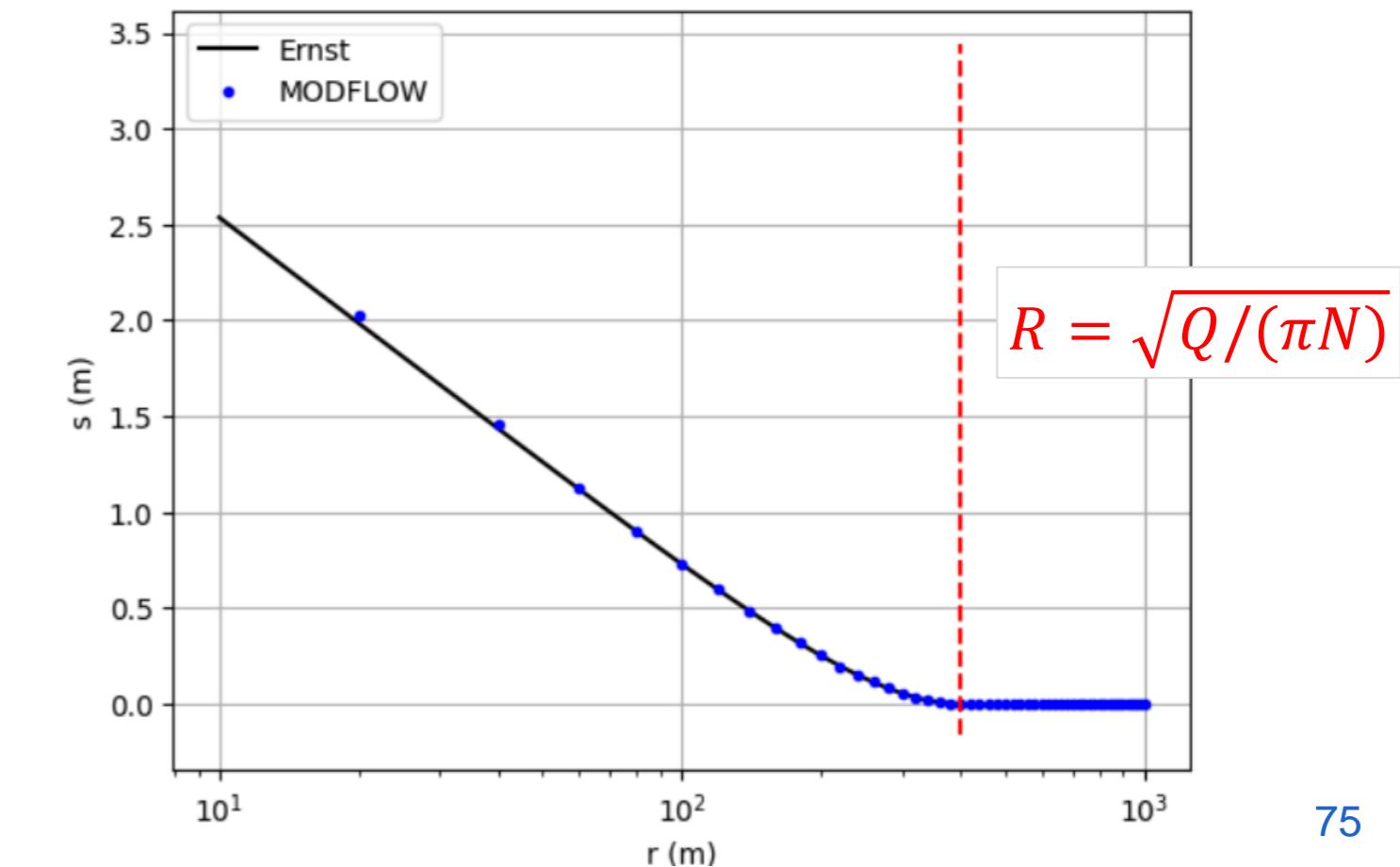
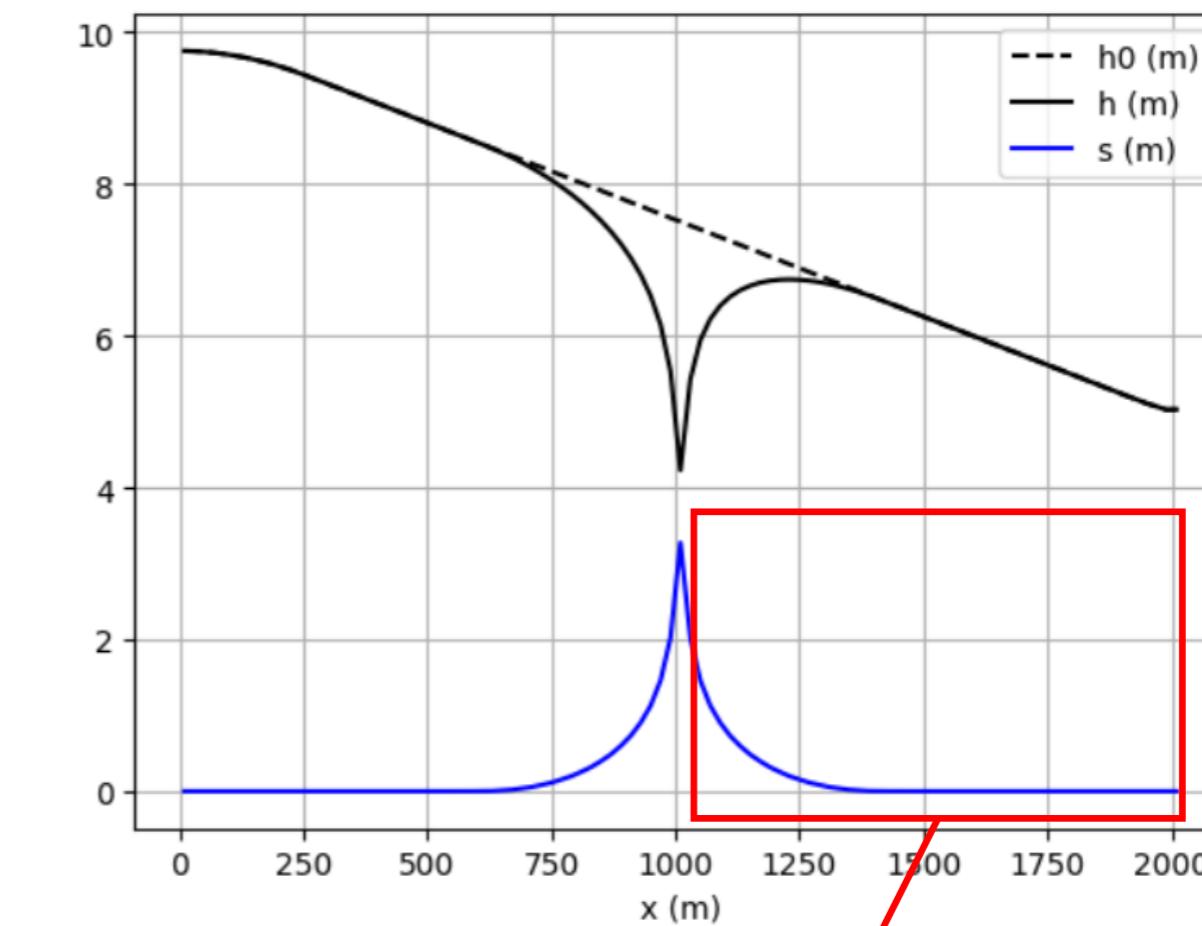
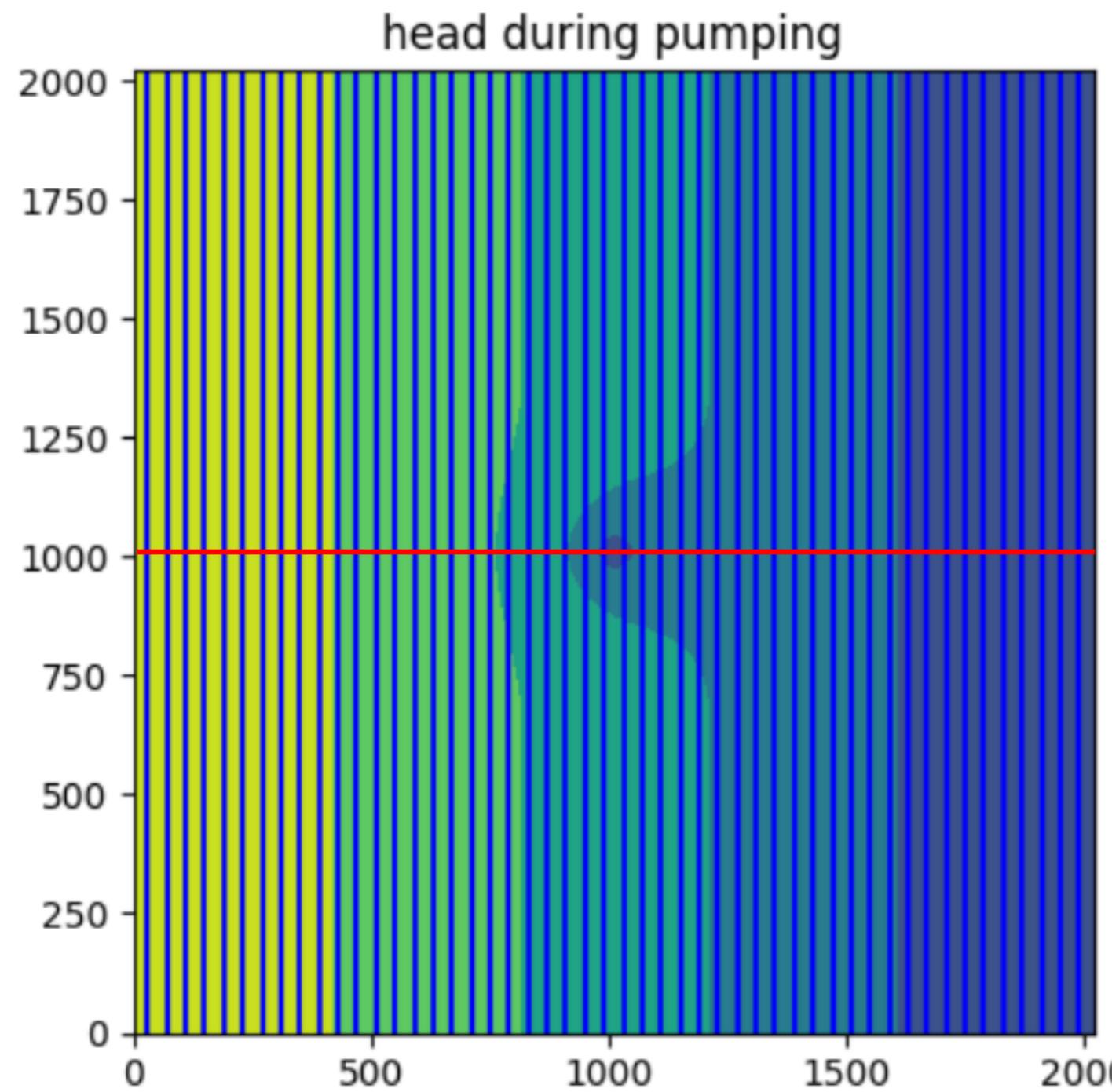
Andy Louwyck✉, Alexander Vandenbohede, Griet Heuvelmans, Marc Van Camp, Kristine Walraevens

First published: 15 August 2022 | <https://doi.org/10.1111/gwat.13245> | Citations: 1

BREDEHOEFT'S VS POLDER ISLAND



MODFLOW VS ERNST



THE SUPERPOSITION PRINCIPLE

- Property of linear models:

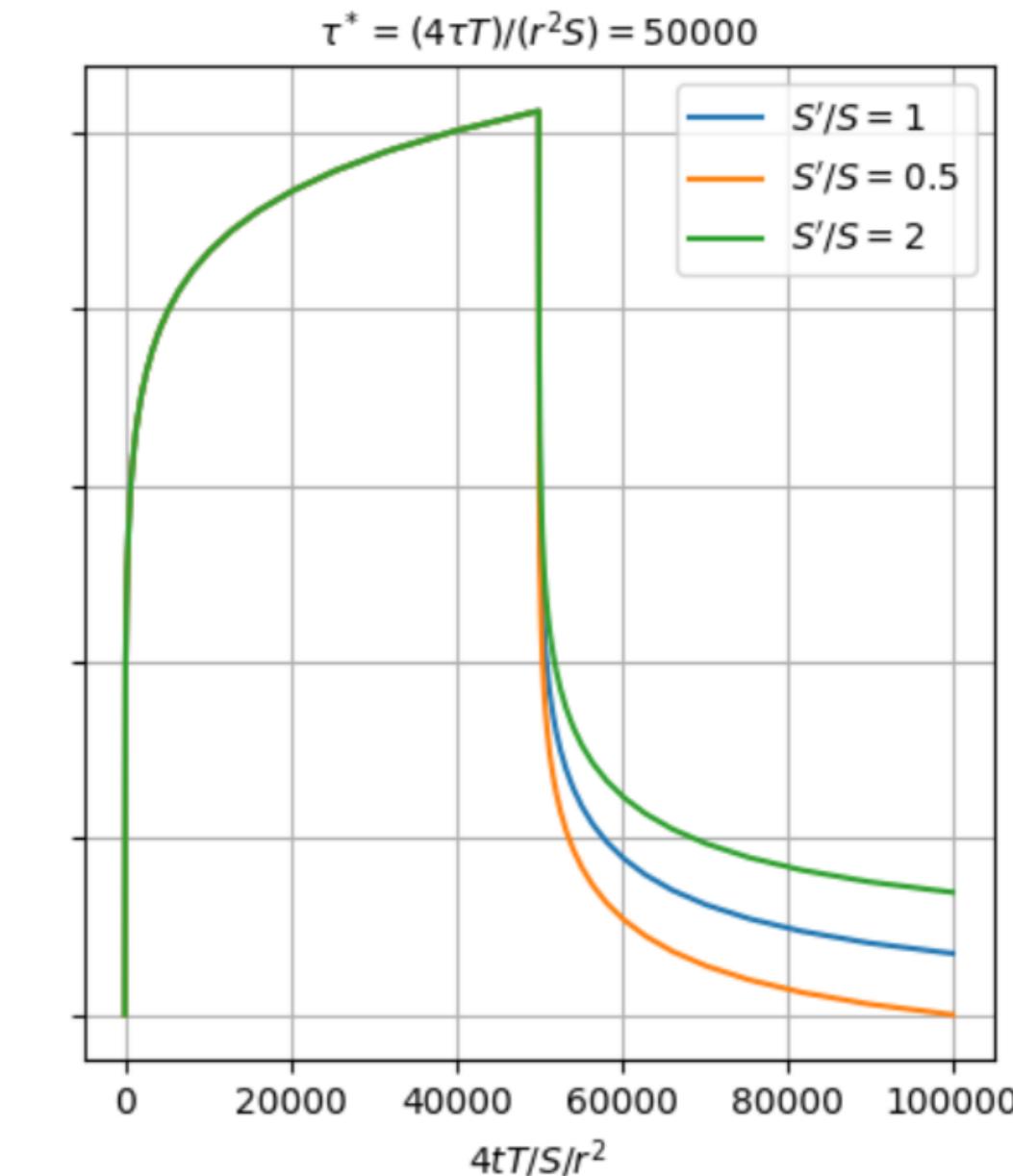
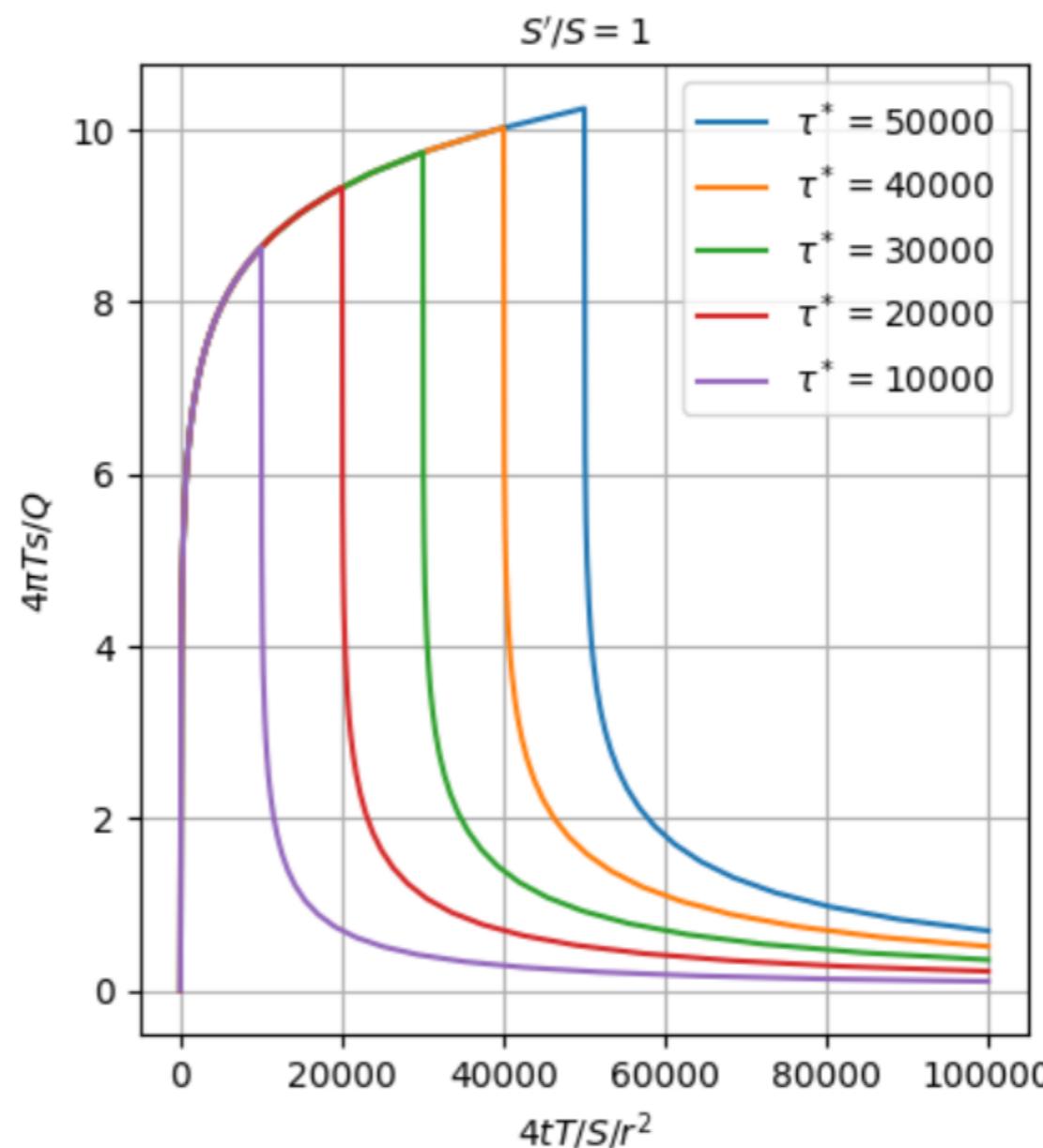
$$s = \sum_i Q_i \sigma_i \quad \text{with } \sigma_i \text{ drawdown according to unit discharge}$$

- Linear model:
 - linear differential equation
 - linear boundary conditions
 - **homogeneous** differential equation (mostly)
 1. model before pumping: $\nabla^2 h_0 = -N$
 2. model during pumping: $\nabla^2 h = -N$
 3. drawdown model: $\nabla^2 s = \nabla^2 h_0 - \nabla^2 h = 0$

SUPERPOSITION IN TIME: EXAMPLE

residual drawdown s'
during recovery (Theis, 1935):

- pump shuts down at time τ
= start of injection $-Q$
- storativity S' during recovery



$$s'(r, t) = \frac{Q}{4\pi T} \left[W\left(\frac{r^2 S}{4tT}\right) - W\left(\frac{r^2 S'}{4\Delta t T}\right) \right]$$

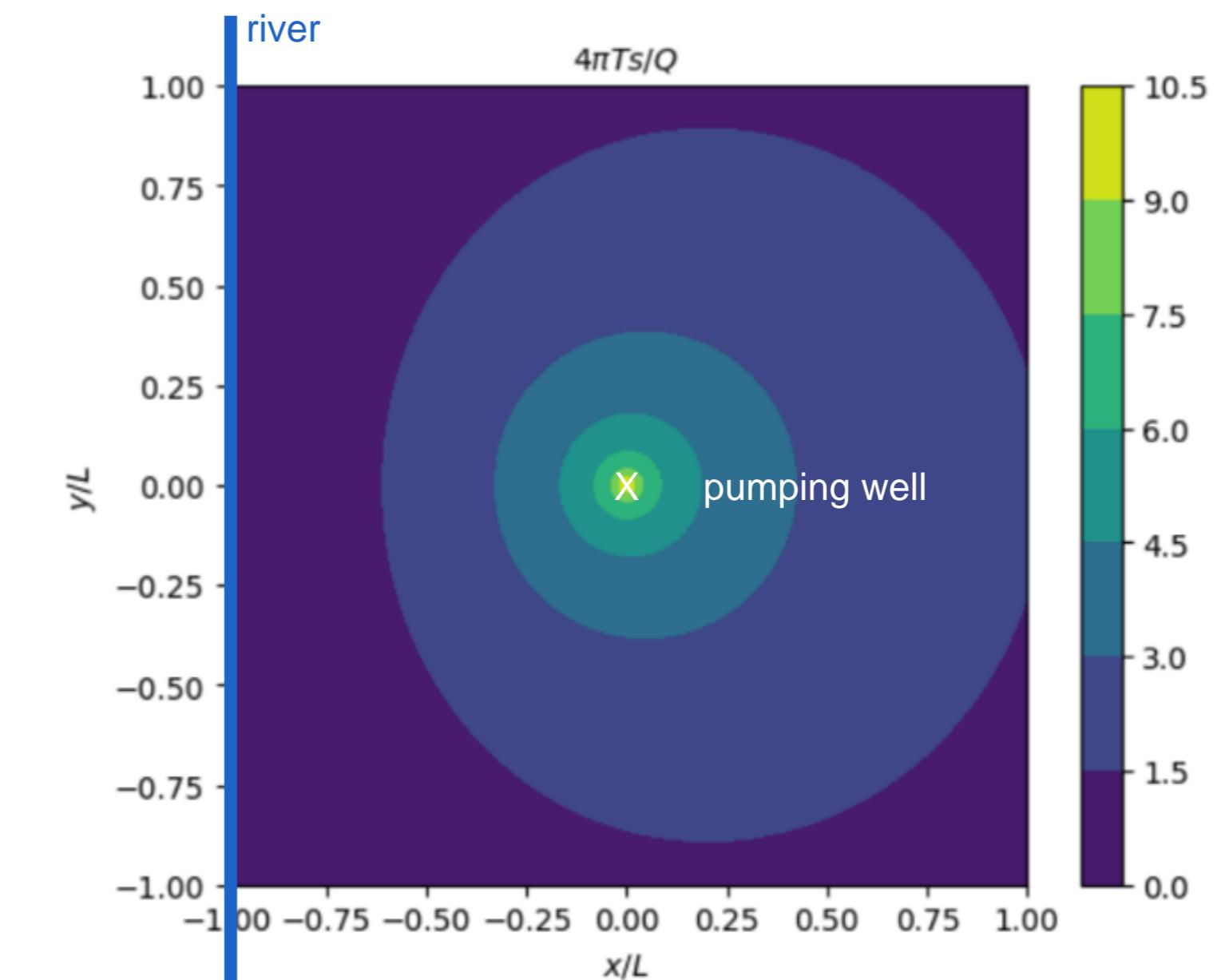
pumping + injection

with $\Delta t = t - \tau$

SUPERPOSITION IN SPACE: EXAMPLE

pumping well near straight
constant-level river (Theis, 1941):

- well at position (0,0)
- straight river: $x = -L$
= constant-head boundary
- **method of images:**
add injection well at position $(-2L, 0)$

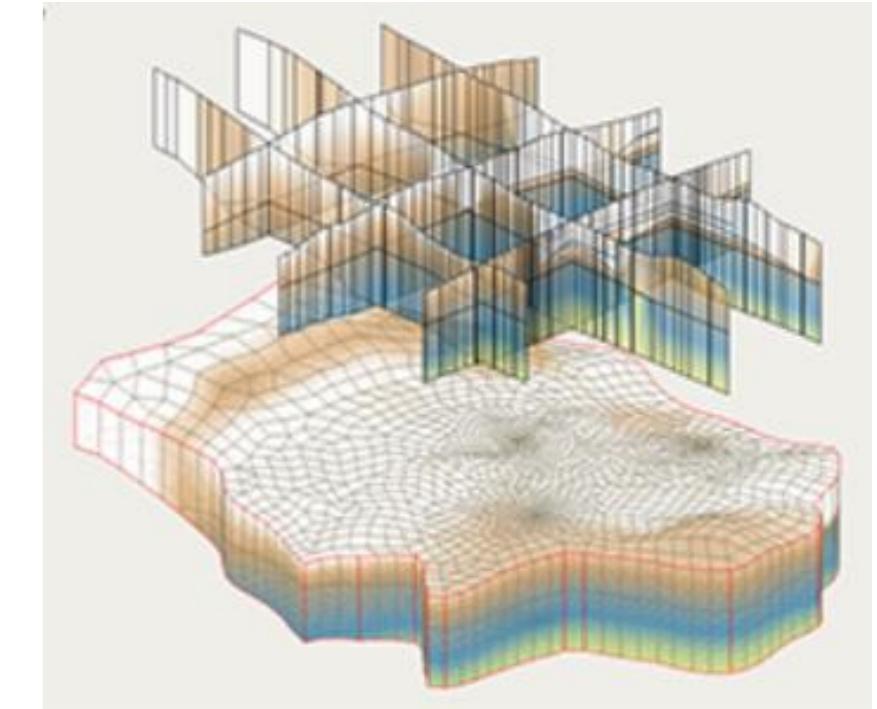


$$s(x, y, t) = \frac{Q}{4\pi T} \left[W\left(\frac{[x^2 + y^2]S}{4tT}\right) - W\left(\frac{[(x + 2L)^2 + y^2]S}{4tT}\right) \right]$$

pumping well + imaginary injection well

CONCLUSIONS

- assessing sustainability and impact of extractions requires **advanced numerical modeling**
- **analytical models** may be useful:
 - time and budget constraints
 - lack of data
 - *they offer insight!*



The Role of Hand Calculations in Ground Water Flow Modeling

Henk Haitjema

First published: 08 March 2006 | <https://doi.org/10.1111/j.1745-6584.2006.00189.x> | Citations: 66

MORE ADVANCED AXISYMMETRIC MODELS

EVOLUTION OF AXISYMMETRIC MODELS

- 1 layer
- incompressible aquitards
- well:
 - fully penetrating (mostly)
 - infinitesimal radius (mostly)

1856	Darcy
1857 & 1863	Dupuit
1870	A. Thiem
1906	G. Thiem
1914	Kooper
1930	de Glee
1935	Theis
1946	Jacob
1955	Hantush & Jacob

EVOLUTION OF AXISYMMETRIC MODELS

- 1, 2 or 3 layers
- compressible aquitards
- anisotropy
- well:
 - partially penetrating
 - multi-aquifer
 - finite diameter (wellbore storage)
 - instantaneous head change (slug test)
 - finite-thickness skin
- water table conditions:
 - delayed yield
 - infiltration and drainage
 - confined-unconfined flow

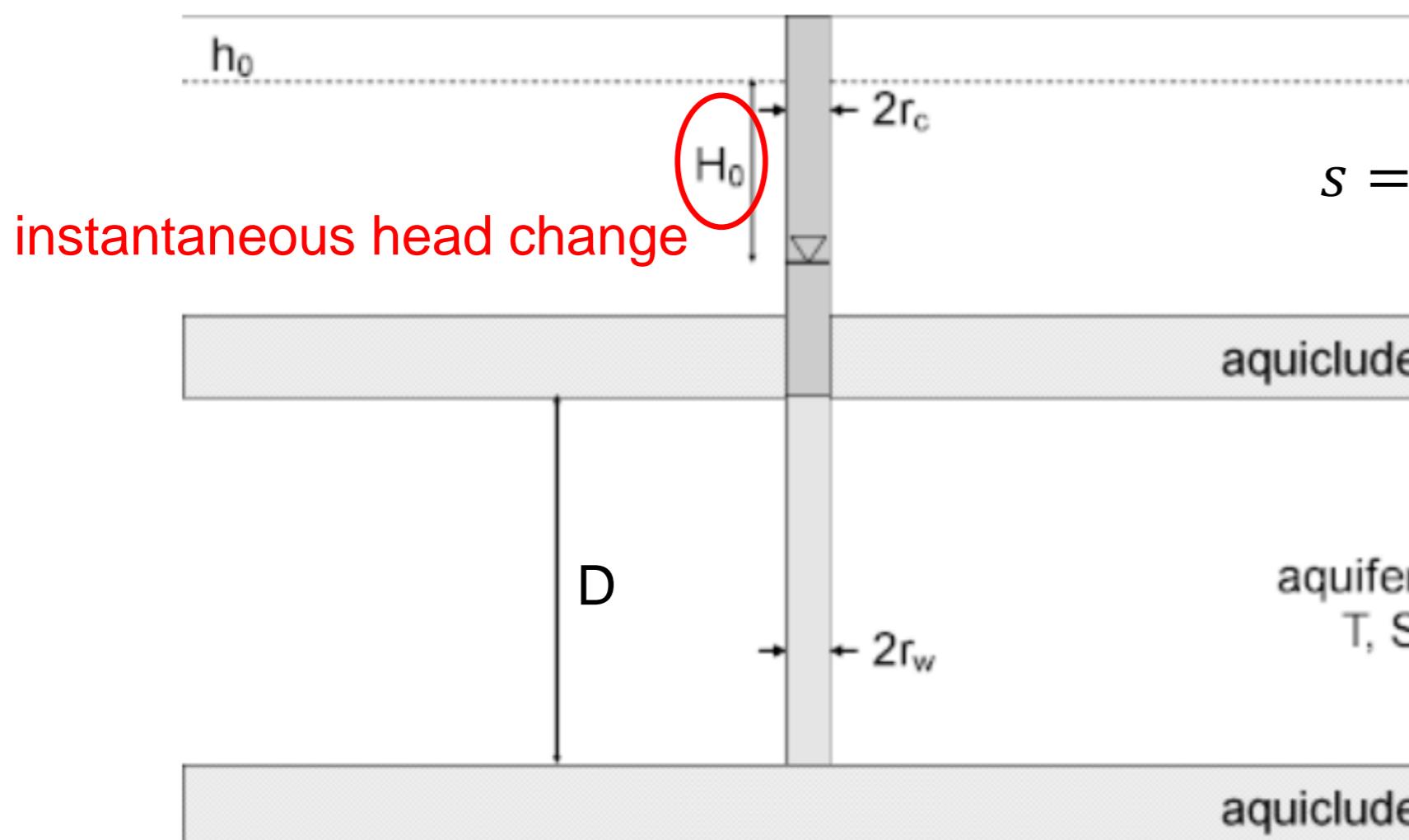
1951	Huisman & Kemperman
1954 & 1963	Boulton
1964 & 1967	Hantush
1966	Papadopoulos
1967	Papadopoulos & Cooper
1967	Cooper et al.
1969	Neuman & Witherspoon
1971	Ernst
1972	Moench & Prickett
1972	Bruggeman
1972 & 1974	Neuman
1983	Javandel & Witherspoon
1984	Moench
1984	Wikramaratna
1988	Butler
1994	Hyder et al.
1995 & 1996	Moench
2012	Mishra et al.
2022	Louwyck et al.

TWO EXAMPLES

Extensions/modifications of the Theis (1935) model:

- **Cooper et al. (1967):**
Instantaneous head change inside the well
(= slug test)
- **Butler (1988):**
Pumping well with finite-thickness skin

COOPER ET AL. MODEL



Slug test in confined aquifer (Cooper et al., 1967)

$$\bar{s}(r, p) = \frac{H_0 r_w S K_0(r\omega)}{T\omega[r_w \omega K_0(r_w \omega) + 2\alpha K_1(r_w \omega)]}$$

$$\text{with } \alpha = \frac{r_w^2}{r_c^2} S$$

Response of a Finite-Diameter Well to an Instantaneous Charge of Water¹

HILTON H. COOPER, JR., JOHN D. BREDEHOEFT, AND
ISTAVROS S. PAPADOPULOS

Water Resources Division, U. S. Geological Survey, Washington, D. C.

Abstract. A solution is presented for the change in water level in a well of finite diameter after a known volume of water is suddenly injected or withdrawn. A set of type curves computed from this solution permits a determination of the transmissibility of the aquifer. (Key words: Aquifer tests; groundwater; hydraulics; permeability.)

INTRODUCTION

Ferris and Knowles [1954] introduced a method for determining the transmissibility of an aquifer from observations of the water level in a well after a known volume of water is suddenly injected into the well. (See also Ferris et al. [1962].) They reasoned that for practical purposes the well may be approximated by an instantaneous line source in the infinite region, for which the residual head differences due to the injection are described by

$$h/H_0 = (r_e^2/4Tt)e^{-r^2S/4Tr} \quad (1)$$

where

h = change in head at distance r and time t due to the injection;
 r = distance from the line source or center of well;
 t = time since instantaneous injection;
 V = volume of water injected;
 T = transmissibility of aquifer;
 S = coefficient of storage of aquifer.

They reasoned further that the head H in the injected well would be described closely by (1) when r is set equal to the effective radius r_e [Jacob, 1947, p. 1049] of the screen or open hole. Then, since r_e is small, the exponential approaches unity quickly, so that the equation approaches $H = V/4\pi Tt$, which can be written

$$T = V(1/t)/4\pi H \quad (2)$$

To the extent that the equation is valid for a

¹Publication authorized by the Director, U. S. Geological Survey.

well of finite diameter, a determination of the transmissibility can be obtained from the slope of a plot of head H versus the reciprocal of time ($1/t$).

Since the volume of water injected into the well is $\pi r_e^2 H_0$, where r_e is the radius of the easing in the interval over which the water level fluctuates and H_0 is the initial head increase in the well, equation 1 can be written

$$h/H_0 = (r_e^2/4Tt)e^{-r^2S/4Tr} \quad (3)$$

and equation 2 can be written

$$H/H_0 = r_e^2/4Tt \quad (4)$$

Recently Bredehoeft et al. [1966] demonstrated by means of an electrical analog model of a well-aquifer system that equation 3 gives a satisfactory approximation of the head in an injected well only after the time t is large enough for the ratio H/H_0 to be very small (see Figure 1). The observed discrepancy appears to arise from the assumption that the injected well can be approximated by a line source.

We present here an exact solution for the head in and around a well of finite diameter after the well is instantaneously charged with a known volume of water.

ANALYSIS

Consider a nonflowing well cased to the top of a homogeneous isotropic artesian aquifer of uniform thickness, and screened (or open) throughout the thickness of the aquifer (Figure 2). Suppose that the well is instantaneously charged with a volume V of water. (We will consider

COOPER ET AL.: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Transient-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Instantaneous initial head change
 - Finite radius → wellbore storage!
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded

COOPER ET AL.: PROBLEM STATEMENT

Continuity of transient 1D confined flow:

$$T \left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right) = \boxed{S \frac{\partial s}{\partial t}}$$

storage change

(1)



Partial differential equation (PDE):

- drawdown s is function of r and t
- apply Laplace transform w.r.t. t

Inner boundary condition at zero:

$$2\pi r T \frac{\partial s}{\partial r} = \boxed{\pi r_c^2 \frac{dH}{dt}} \quad \text{wellbore storage}$$

$(r = r_w)$

with H the head change in the well

(2)

Initial condition at $t = 0$:

Outer boundary condition at infinity:

$$s(\infty, t) = 0$$

(3)

$$s(r, 0) = \begin{cases} 0 & (r > r_w) \\ \boxed{H_0} & (r = r_w) \end{cases}$$

instantaneous head
change in the well

(4)

COOPER ET AL.: LAPLACE TRANSFORM

Laplace transform of PDE (1):

$$\frac{d^2\bar{s}}{dr^2} + \frac{1}{r} \frac{d\bar{s}}{dr} = \frac{S}{T} p \bar{s}$$

(5)



Modified Bessel differential equation:

$$\frac{d^2\bar{s}}{dr^2} + \frac{1}{r} \frac{d\bar{s}}{dr} = a \bar{s}$$

Laplace transform of inner BC (2):

$$2\pi r T \frac{d\bar{s}}{dr} = \pi r_c^2 (p \bar{H} - H_0) \quad \text{initial condition!}$$

(6)

Laplace transform of outer BC (3):

$$\bar{s}(\infty, p) = 0 \quad (7)$$

General solution:

$$\bar{s} = \alpha I_0(r\sqrt{a}) + \beta K_0(r\sqrt{a})$$

with:

- I_0, K_0 : the zero order modified Bessel functions of the first and second kind, resp.
- α, β : integration constants

COOPER ET AL.: LAPLACE SOLUTION

General solution of (5) after applying BC (7) which yields $\alpha = 0$:

$$\bar{s}(r, p) = \beta K_0(r\omega)$$

$$\text{with } \omega = \sqrt{Sp/T}$$

(8)

From (9) it follows that:

$$\beta = \frac{r_c^2 H_0}{r_c^2 p K_0(r_w \omega) + 2 r_w \omega T K_1(r_w \omega)} \quad (10)$$

Introducing (10) in (8) and rearranging:

$$\bar{s}(r, p) = \frac{H_0 r_w S K_0(r\omega)}{T \omega [r_w \omega K_0(r_w \omega) + 2 \alpha K_1(r_w \omega)]} \quad (11)$$

with $\alpha = \frac{r_w^2}{r_c^2} S$

Applying BC (6) with $\bar{H}(p) = \bar{s}(r_w, p)$

$$-2\pi r_w \omega T \beta K_1(r_w \omega) =$$

$$\pi r_c^2 [p \beta K_0(r_w \omega) - H_0]$$

(9)

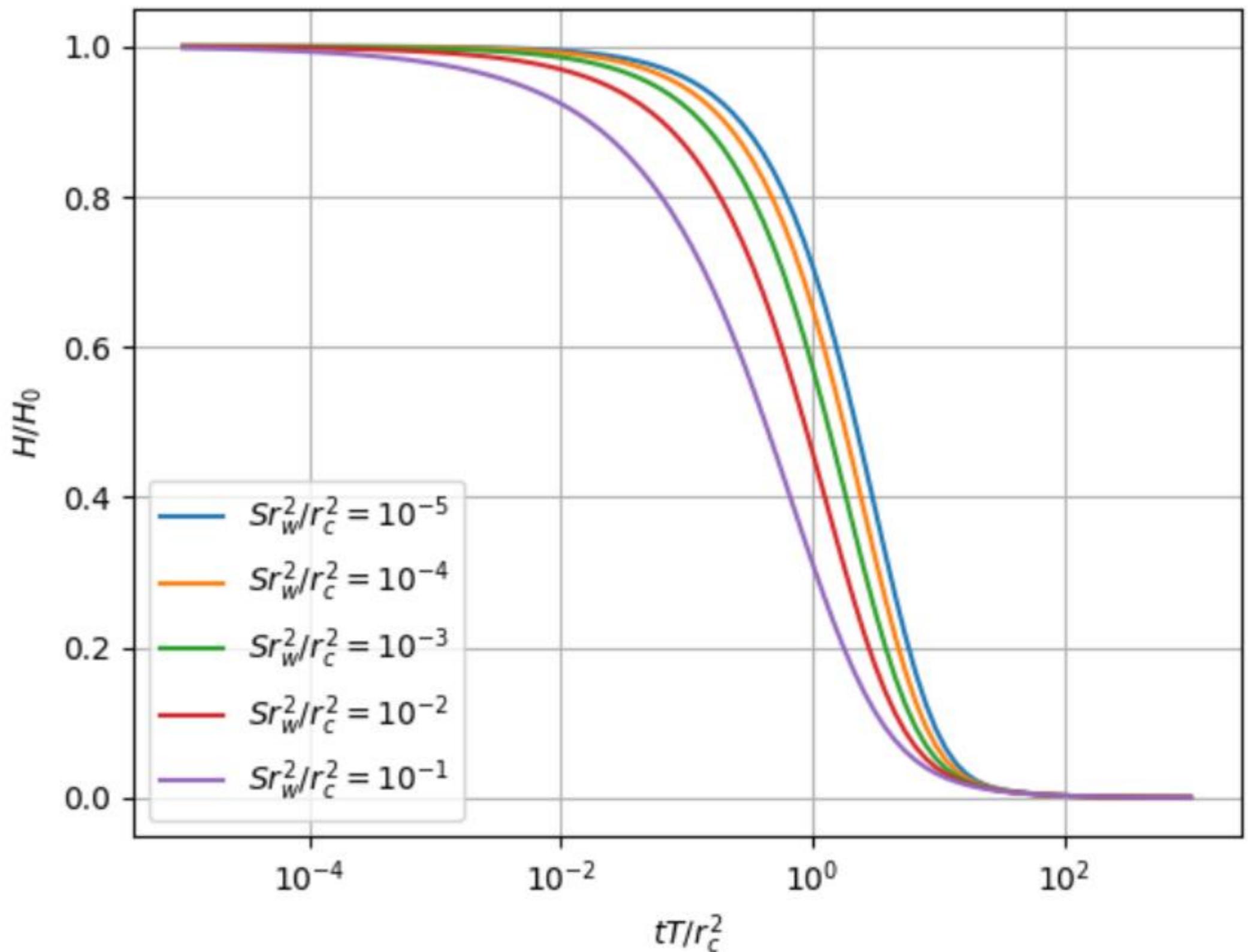
COOPER ET AL.: NUMERICAL INVERSION

Applying the Stehfest algorithm

to Laplace solution (11)

with $r = r_w$

as $H(t) = s(r_w, t)$



BUTLER MODEL

[2]

PUMPING TESTS IN NONUNIFORM AQUIFERS — THE RADIALLY SYMMETRIC CASE

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(Received May 22, 1987; revised and accepted December 12, 1987)

ABSTRACT

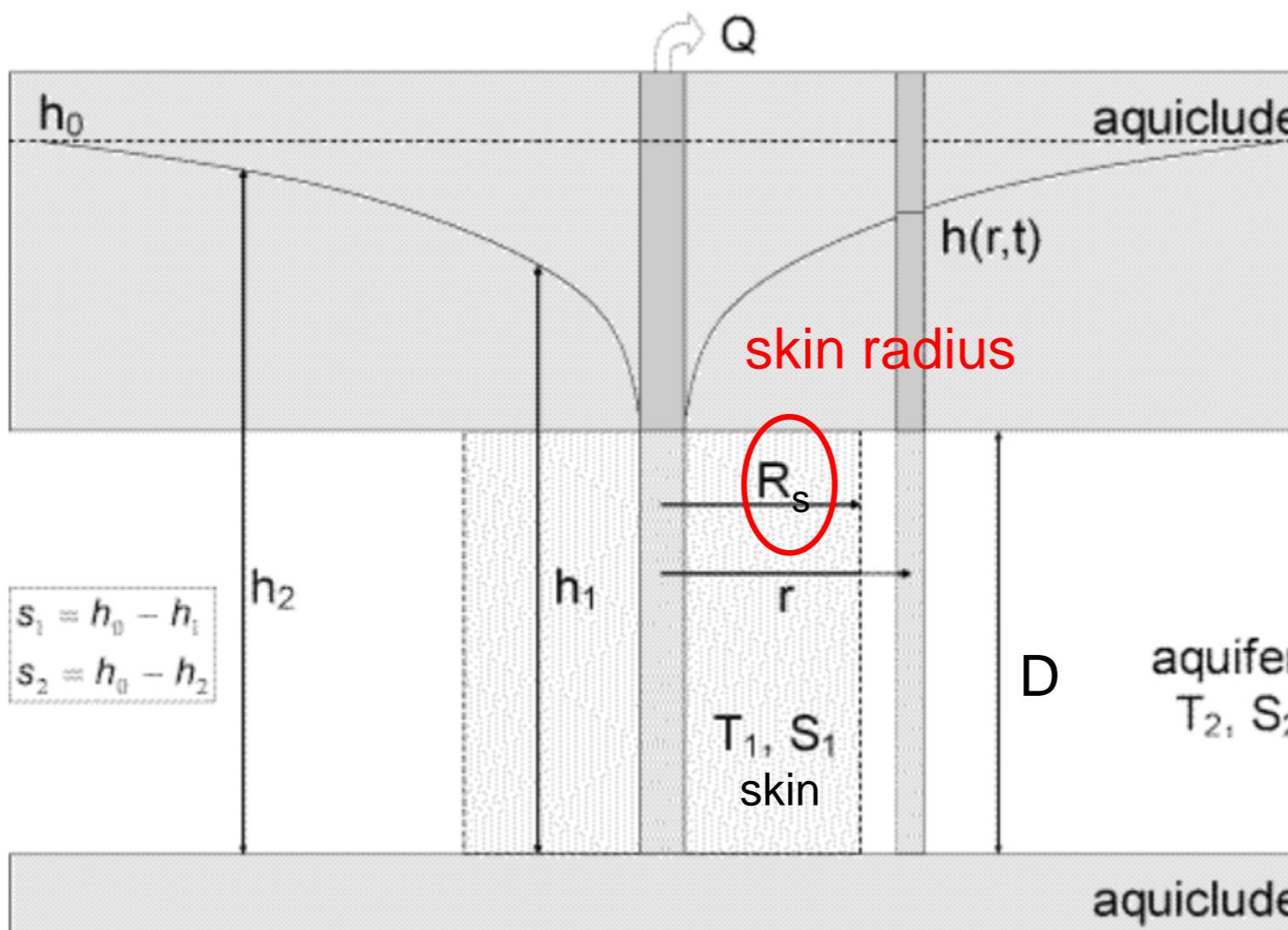
Butler, J.J., Jr., 1988. Pumping tests in nonuniform aquifers—The radially symmetric case. *J. Hydrol.*, 101: 15–30.

Traditionally, pumping-test-analysis methodology has been limited to applications involving aquifers whose properties are assumed uniform in space. This work attempts to assess the applicability of analytical methodology to a broader class of units with spatially varying properties. An examination of flow behavior in a simple configuration consisting of pumping from the center of a circular disk embedded in a matrix of differing properties is the basis for this investigation. A solution describing flow in this configuration is obtained through Laplace-transform techniques using analytical and numerical inversion schemes. Approaches for the calculation of flow properties in conditions that can be roughly represented by this simple configuration are proposed. Possible applications include a wide variety of geologic structures, as well as the case of a well skin resulting from drilling or development. Of more importance than the specifics of these techniques for analysis of water-level responses is the insight into flow behavior during a pumping test that is provided by the large-time form of the derived solution. The solution reveals that drawdown during a pumping test can be considered to consist of two components that are dependent and independent of near-well properties, respectively. Such an interpretation of pumping-test drawdown allows some general conclusions to be drawn concerning the relationship between parameters calculated using analytical approaches based on curve-matching and those calculated using approaches based on the slope of a semilog straight line plot. The infinite-series truncation that underlies the semilog analytical approaches is shown to remove further contributions of near-well material to total drawdown. In addition, the semilog distance-drawdown approach is shown to yield an expression that is equivalent to the Thiem equation. These results allow some general recommendations to be made concerning observation-well placement for pumping tests in nonuniform aquifers. The relative diffusivity of material on either side of a discontinuity is shown to be the major factor in controlling flow behavior during the period in which the front of the cone of depression is moving across the discontinuity. Though resulting from an analysis of flow in an idealized configuration, the insights of this work into flow behavior during a pumping test are applicable to a wide class of nonuniform units.

INTRODUCTION

The pumping test has traditionally been the standard method used to evaluate the transmissive and storage properties of subsurface material for

0022-1694/88/\$03.50 © 1988 Elsevier Science Publishers B.V.



Pumping well with finite-thickness skin in confined aquifer (Butler, 1988)

$$s(r, t) \approx \begin{cases} \frac{Q}{2\pi T_2} \ln \frac{R}{R_s} + \frac{Q}{2\pi T_1} \ln \frac{R_s}{r} & (r \leq R_s) \\ \frac{Q}{2\pi T_2} \ln \frac{R}{r} & (r \geq R_s) \end{cases}$$

with $R = \sqrt{\frac{4tT_2}{e^{\gamma} S_2}}$ and $t \rightarrow \infty$



James J. Butler, Jr.

BUTLER (1988) MODEL: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Transient-state
 - Strictly horizontal
- Well:
 - Fully penetrating
 - Constant pumping rate
 - Infinitesimal radius
 - **Finite-thickness skin**
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded

BUTLER: PROBLEM STATEMENT

Transient flow in the skin zone:

$$T_1 \left(\frac{\partial^2 s_1}{\partial r^2} + \frac{1}{r} \frac{\partial s_1}{\partial r} \right) = S_1 \frac{\partial s_1}{\partial t} \quad (1)$$

Inner boundary condition at zero:

$$Q = - \lim_{r \rightarrow 0} \left(2\pi r T_1 \frac{\partial s_1}{\partial r} \right) \quad (2)$$

Proximal zone 1: $r \leq R_s \rightarrow$ skin: T_1, S_1

R_s

Transient flow in the aquifer:

$$T_2 \left(\frac{\partial^2 s_2}{\partial r^2} + \frac{1}{r} \frac{\partial s_2}{\partial r} \right) = S_2 \frac{\partial s_2}{\partial t} \quad (3)$$

Outer boundary condition at infinity:

$$s_2(\infty) = 0 \quad (4)$$

Distal zone 2: $r \geq R_s \rightarrow$ aquifer: T_2, S_2

Continuity of flow at the common boundary:

$$s_1(R_s, t) = s_2(R_s, t) \quad (5)$$

$$2\pi R_s T_1 \frac{\partial s_1}{\partial r} = 2\pi R_s T_2 \frac{\partial s_2}{\partial r} \quad (6)$$

BUTLER: LARGE-TIME APPROXIMATION

Pseudo-steady flow in the skin zone:

$$\frac{d^2 s_1}{dr^2} + \frac{1}{r} \frac{ds_1}{dr} \approx 0 \quad (7)$$

Condition (2) is true for all distances r :

$$Q \approx -2\pi r T_1 \frac{ds_1}{dr} \quad (8)$$

Proximal zone 1: $r \leq R_s \rightarrow$ skin: T_1

R_s

Laplace transform of PDE (3):

$$\frac{d^2 \bar{s}_2}{dr^2} + \frac{1}{r} \frac{d\bar{s}_2}{dr} = \frac{S_2}{T_2} p \bar{s}_2 \quad (9)$$

Laplace transform of BC (4):

$$\bar{s}_2(\infty) = 0 \quad (10)$$

Distal zone 2: $r \geq R_s \rightarrow$ aquifer: T_2, S_2

Laplace transform of conditions (5) and (6):

$$\bar{s}_1(R_s, p) = \bar{s}_2(R_s, p) \quad (11)$$

$$2\pi R_s T_2 \frac{d\bar{s}_2}{dr} \approx \frac{-Q}{p} \quad (12)$$

BUTLER: LARGE-TIME APPROXIMATION

Solution of (7) subject to (8) and (5):

$$s_1(r, t) \approx s_2(R_s, t) + \frac{Q}{2\pi T_1} \ln \frac{R_s}{r} \quad (13)$$

General solution of (9) after applying BC (10) which yields $\alpha = 0$:

$$\bar{s}_2(r, p) = \beta K_0(r\omega)$$

$$\text{with } \omega = \sqrt{S_2 p / T_2} \quad (14)$$

Applying BC (12):

$$2\pi R_s T_2 \beta \omega K_1(R_s \omega) \approx \frac{Q}{p} \quad (15)$$

From (14) and (15):

$$\bar{s}_2(r, p) \approx \frac{Q}{2\pi T_2 p} \frac{K_0(r\omega)}{R_s \omega K_1(R_s \omega)} \quad (16)$$

If $t \rightarrow \infty$, then $p \rightarrow 0$ and $R_s \omega K_1(R_s \omega) \rightarrow 1$:

$$\bar{s}_2(r, p) \approx \frac{Q}{2\pi T_2 p} K_0(r\omega) \quad (17)$$

Analytically inverting (17):

$$s_2(r, t) \approx \frac{Q}{4\pi T_2} W\left(\frac{r^2 S_2}{4tT_2}\right) \quad (18)$$

If $t \rightarrow \infty$, then (18) is approximated as:

$$s_2(r, t) \approx \frac{Q}{2\pi T_2} \ln\left(\frac{1}{r} \sqrt{\frac{4tT_2}{e^\gamma S_2}}\right) \quad (19)$$

BUTLER: LARGE-TIME SOLUTION

Combining (13) and (19):

$$s_1(r, t) \approx \frac{Q}{2\pi T_2} \ln \frac{R}{R_s} + \frac{Q}{2\pi T_1} \ln \frac{R_s}{r}$$

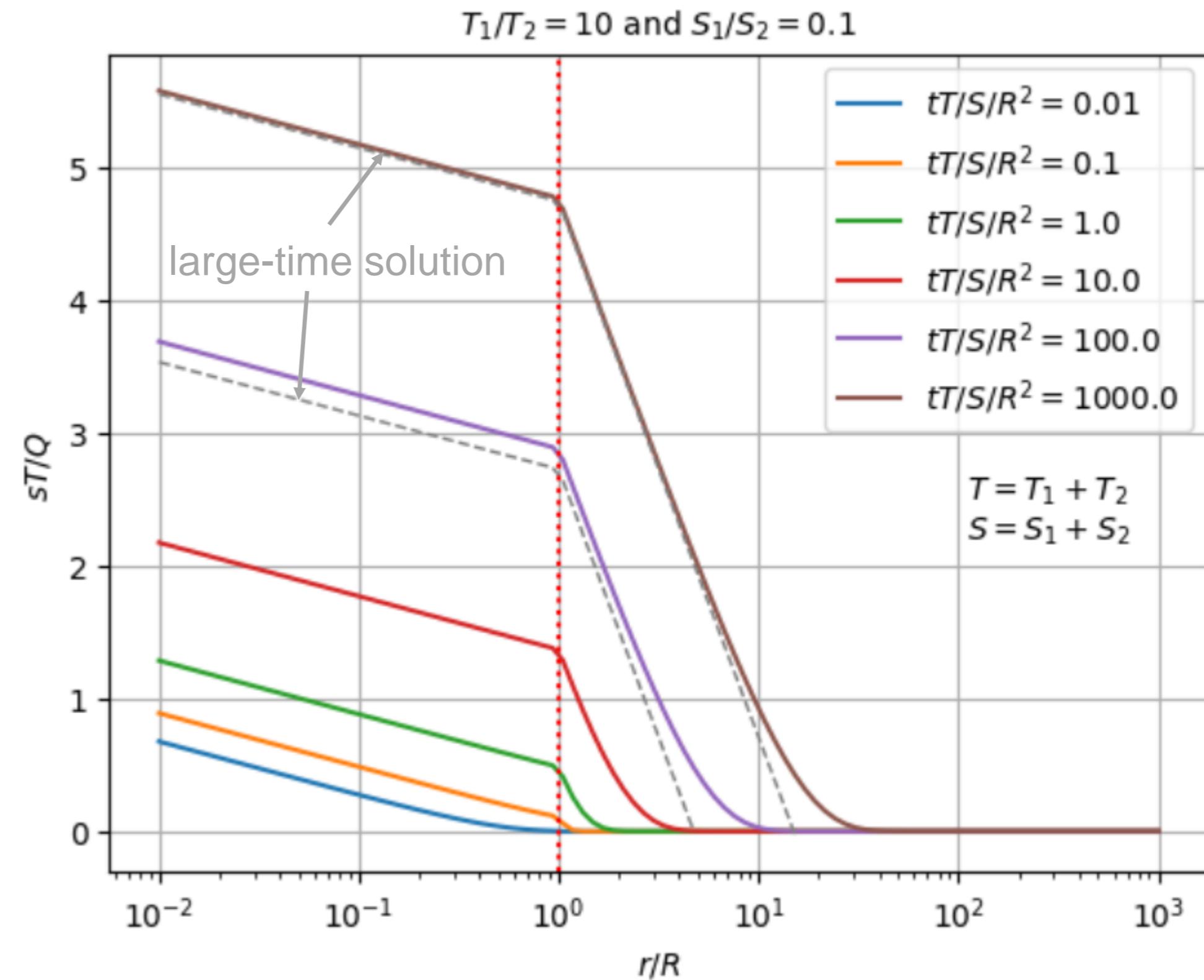
Cooper-Jacob Thiem

$$s_2(r, t) \approx \frac{Q}{2\pi T_2} \ln \frac{R}{r}$$

with $R = \sqrt{\frac{4tT_2}{e^\gamma S_2}}$

radius of influence

(20)



SKIN FACTOR

Definition of dimensionless skin factor F :

$$F = \frac{T_2}{T_1} \ln \frac{R_s}{r_w} \quad (21)$$

Drawdown s_w in pumping well:

$$s_w(t) = s(R_s, t) + \frac{Q}{2\pi T_2} F \quad (22)$$

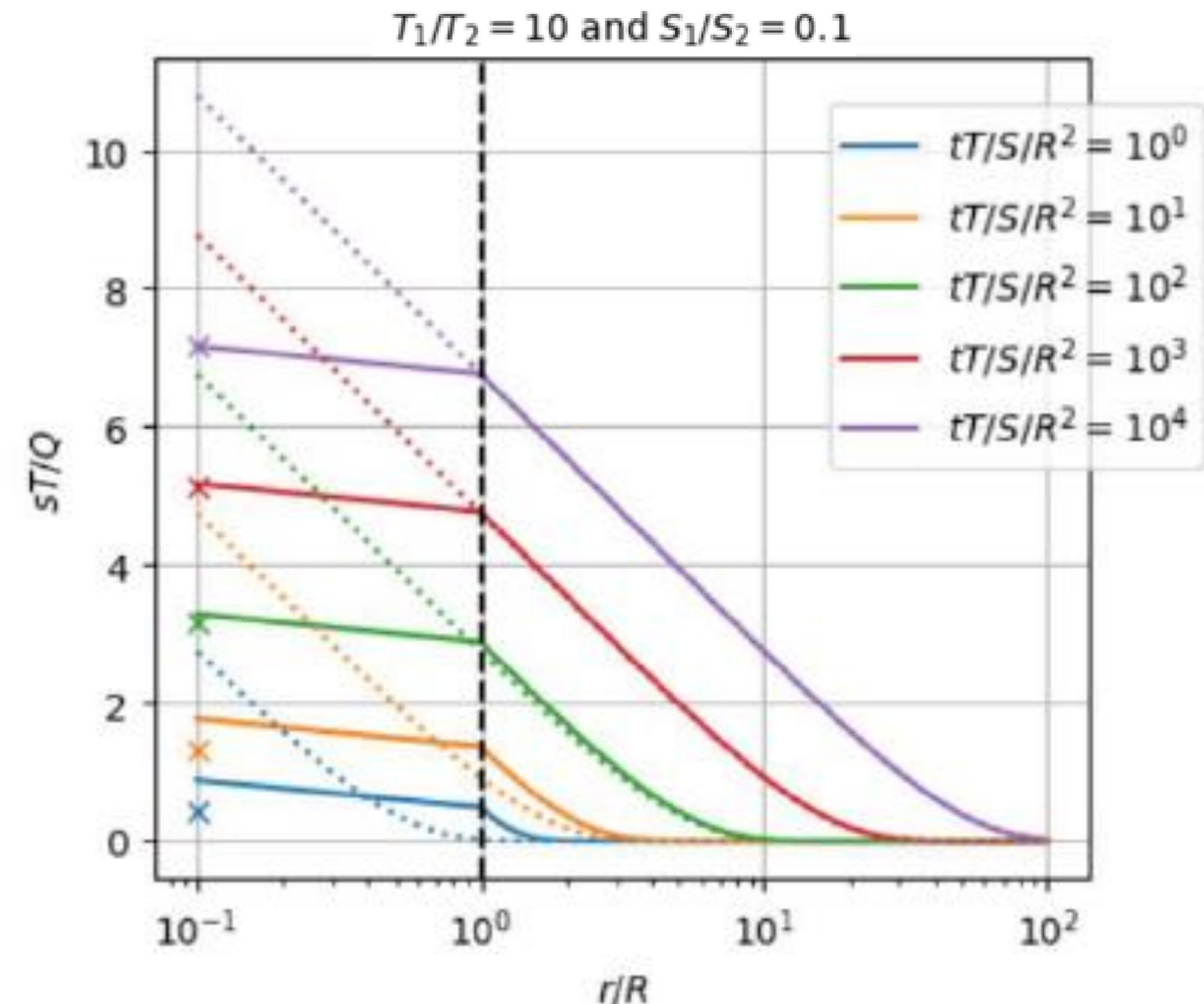
(22) corresponds to (20) as:

$$s_w(t) = s_1(r_w, t)$$

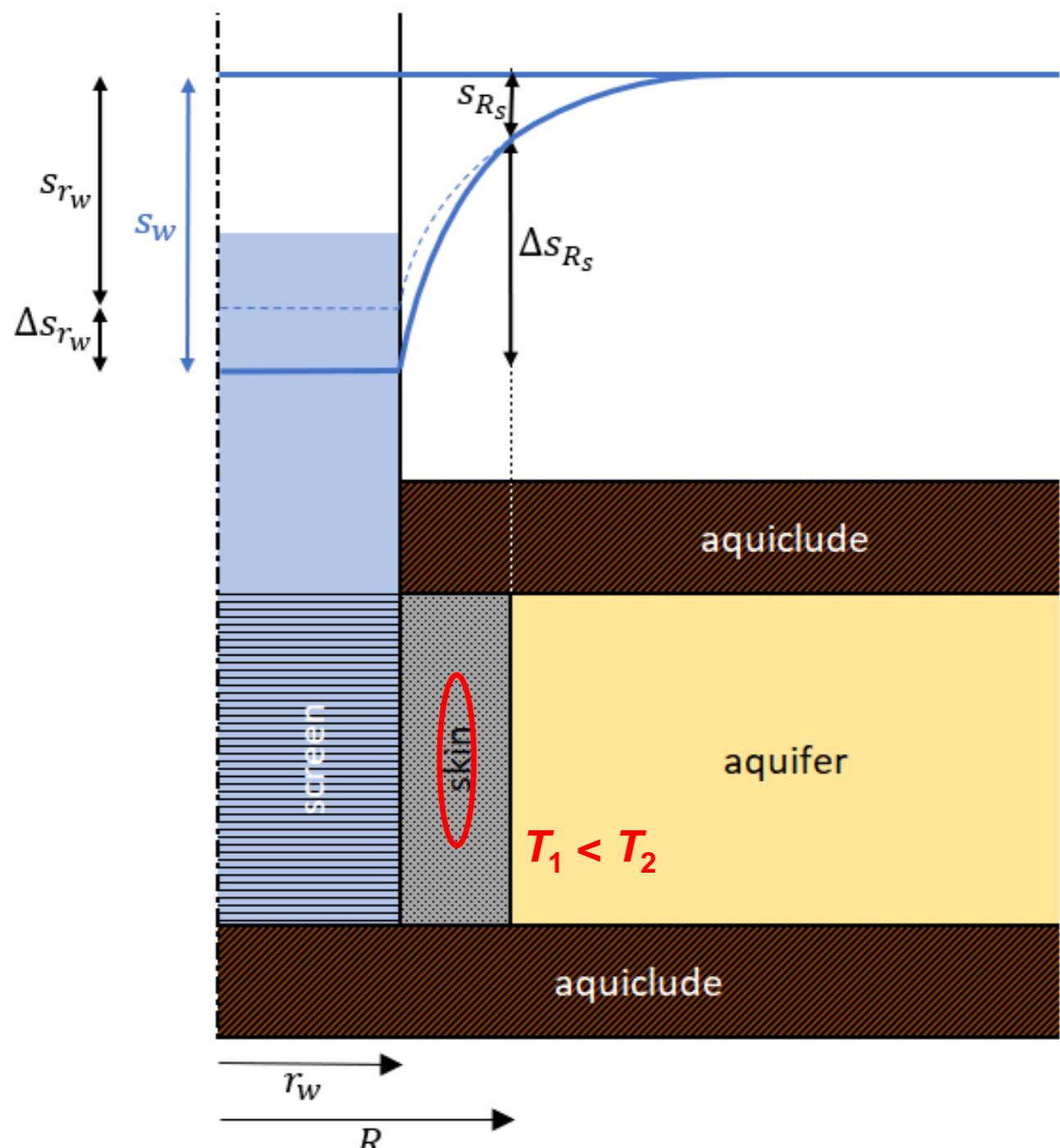
$$s(R_s, t) = \frac{Q}{2\pi T_2} \ln \frac{R}{R_s} \quad \text{Cooper-Jacob}$$

$$\frac{Q}{2\pi T_2} F = \frac{Q}{2\pi T_1} \ln \frac{R_s}{r_w} \quad \text{Thiem}$$

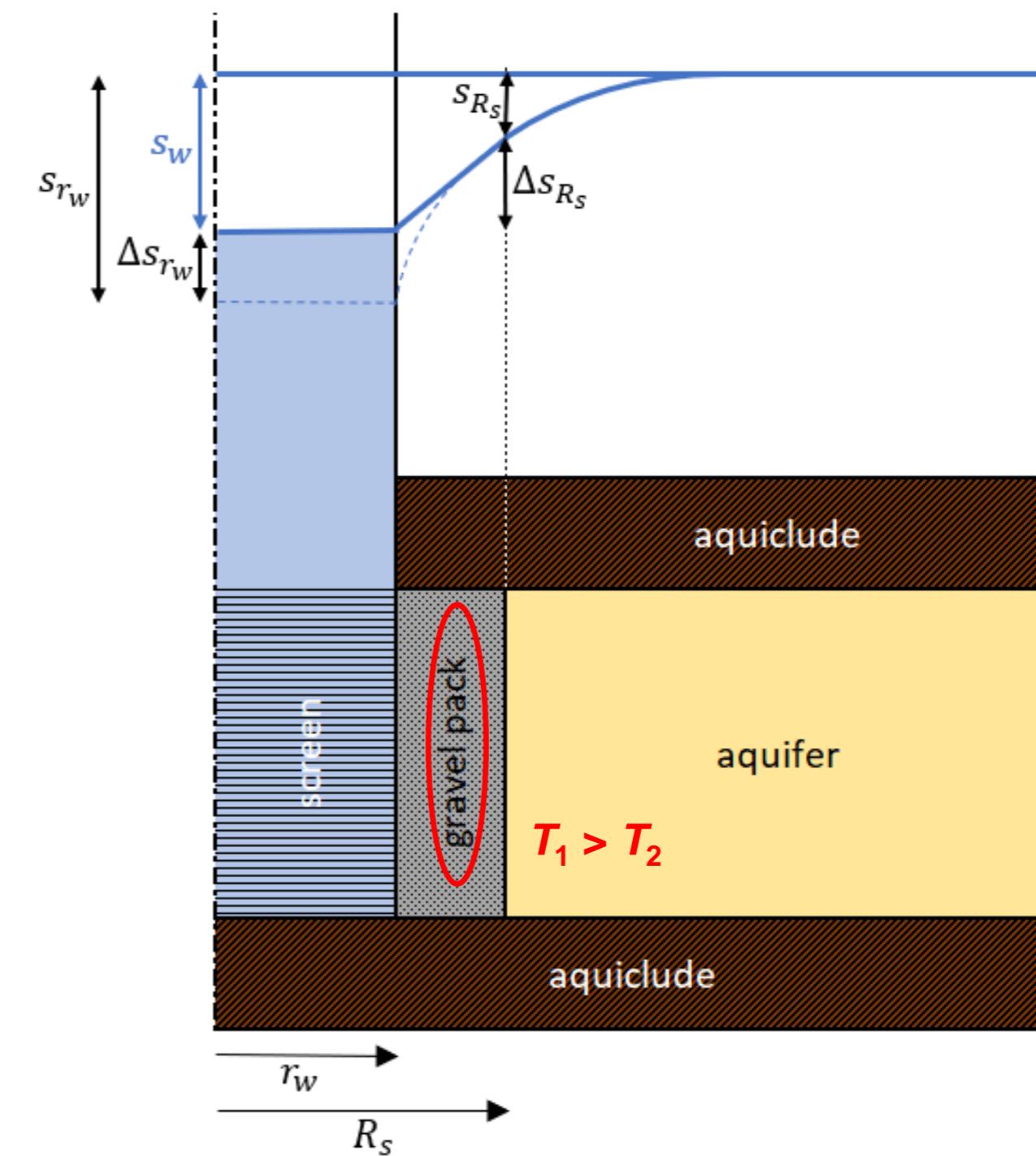
- solid lines: Butler (1988)
- dotted lines: Theis (1935)
- crosses: Theis + skin factor



SKIN EFFECT



positive skin effect



negative skin effect

AXISYMMETRIC FLOW

IN MULTILAYER

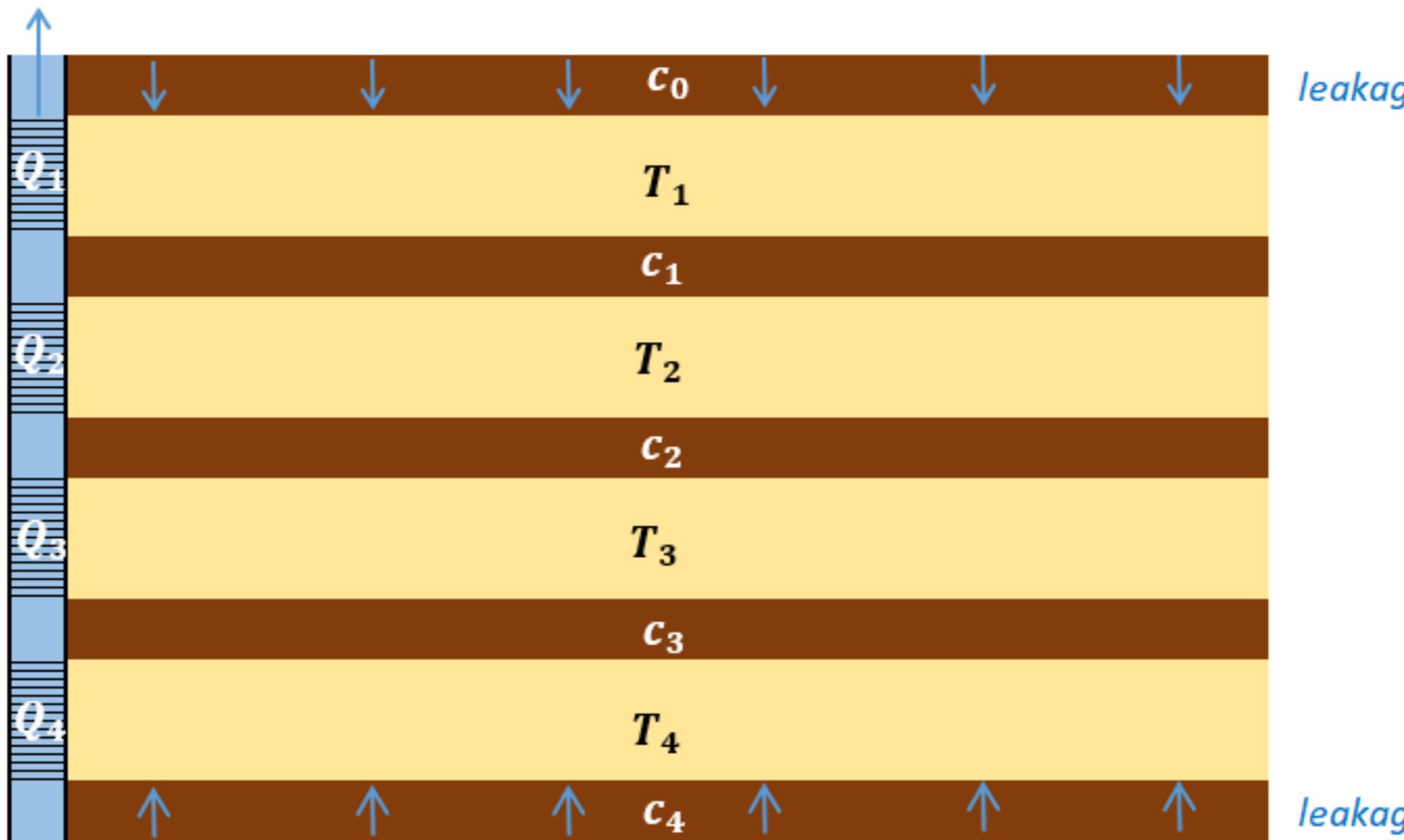
AQUIFER SYSTEMS

EVOLUTION OF AXISYMMETRIC MODELS

- N layers
- compressible aquitards
- anisotropy
- well:
 - partially penetrating
 - multi-aquifer
 - finite diameter (wellbore storage)
 - instantaneous head change (slug test)
 - finite-thickness skin
- water table conditions:
 - delayed yield
 - infiltration and drainage
 - confined-unconfined flow

1984 & 1985	Hemker
1985 & 1986	Hunt
1986 & 1987	Maas
1987	Hemker & Maas
1987	Yu
1993	Cheng & Morohunfola
1999	Hemker
2001	Bakker
2002 & 2004	Bakker & Hemker
2003	Bakker & Strack
2004	Meesters et al.
2006	Bakker & Hemker
2007	Hunt & Scott
2009	Veling & Maas
2023	Louwyck

HEMKER: STEADY MULTI-AQUIFER FLOW



Journal of Hydrology

Volume 72, Issues 3–4, 15 June 1984, Pages 355-374



Research paper

Steady groundwater flow in leaky multiple-aquifer systems

C.J. Hemker^{1,2}

Show more ▾

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[https://doi.org/10.1016/0022-1694\(84\)90089-1](https://doi.org/10.1016/0022-1694(84)90089-1)

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Steady flow to a well in a leaky multi-aquifer system (Hemker, 1984)

$$s = V^{-1} KVQ$$

HEMKER (1984) MODEL: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - **Steady-state**
 - Aquifers: strictly horizontal
 - Aquitards: strictly vertical
- Well:
 - Fully penetrating screens
 - Screens are not connected
 - Constant pumping rates
 - Infinitesimal radius
- Aquifer system:
 - Homogeneous aquifers and aquitards
 - Aquifers have constant saturated thickness
 - Incompressible aquitards → zero-thickness resistance layers
 - Laterally unbounded
 - Leaky top and bottom (**both top and bottom impervious is not possible!**)

HEMKER (1984): PROBLEM STATEMENT

Steady flow in each layer i :

$$T_i \left(\frac{d^2 s_i}{dr^2} + \frac{1}{r} \frac{ds_i}{dr} \right) = \frac{s_i - s_{i-1}}{c_{i-1}} + \frac{s_i - s_{i+1}}{c_i}$$

leakage from adjacent layers

(1)

Matrix notation:

$$\frac{d^2 \mathbf{s}}{dr^2} + \frac{1}{r} \frac{d\mathbf{s}}{dr} = \mathbf{A}\mathbf{s}$$

(4)

Inner boundary condition at zero:

$$Q_i = - \lim_{r \rightarrow 0} \left(2\pi r T_i \frac{ds_i}{dr} \right)$$

(2)

$$Q = - \lim_{r \rightarrow 0} \left(r \frac{d\mathbf{s}}{dr} \right)$$

(5)

Outer boundary condition at infinity:

$$s_i(\infty) = 0$$

(3)

$$s(\infty) = 0$$

(6)

HEMKER (1984): MATRICES

Vector \mathbf{s} :

$$\mathbf{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_{n_l} \end{bmatrix}$$

Vector \mathbf{Q} :

$$\mathbf{Q} = \frac{1}{2\pi} \begin{bmatrix} Q_1/T_1 \\ \vdots \\ Q_{n_l}/T_{n_l} \end{bmatrix}$$

with n_l the number of layers

System matrix \mathbf{A} :

$$\mathbf{A} = \begin{bmatrix} a_1 + b_1 & -b_1 & 0 & \cdots \\ -a_2 & a_2 + b_2 & -b_2 & \\ 0 & & \ddots & \\ \vdots & & & \\ \cdots & -a_{n_l} & a_{n_l} + b_{n_l} & \end{bmatrix}$$

with

$$a_i = \frac{1}{c_{i-1} T_i}$$

$$b_i = \frac{1}{c_i T_i}$$

HEMKER (1984): EIGENDECOMPOSITION

Eigendecomposition of matrix A :

$$A = V^{-1}DV \quad (7)$$

Applying (7) to ODE (4) with $g = Vs$:

$$\frac{d^2g}{dr^2} + \frac{1}{r} \frac{dg}{dr} = Dg \quad (8)$$

Multiplying each side of (5) and (6) by V :

$$q = VQ = -\lim_{r \rightarrow 0} \left(r \frac{dg}{dr} \right) \quad (9)$$

$$g(\infty) = 0 \quad (10)$$

General solution of (8):

$$g = I\alpha + K\beta \quad (11)$$

Diagonal matrices I and K :

$$I_{ii} = I_0(r\sqrt{d_i}) \text{ and } K_{ii} = K_0(r\sqrt{d_i})$$

with d_i the i -th eigenvalue of A

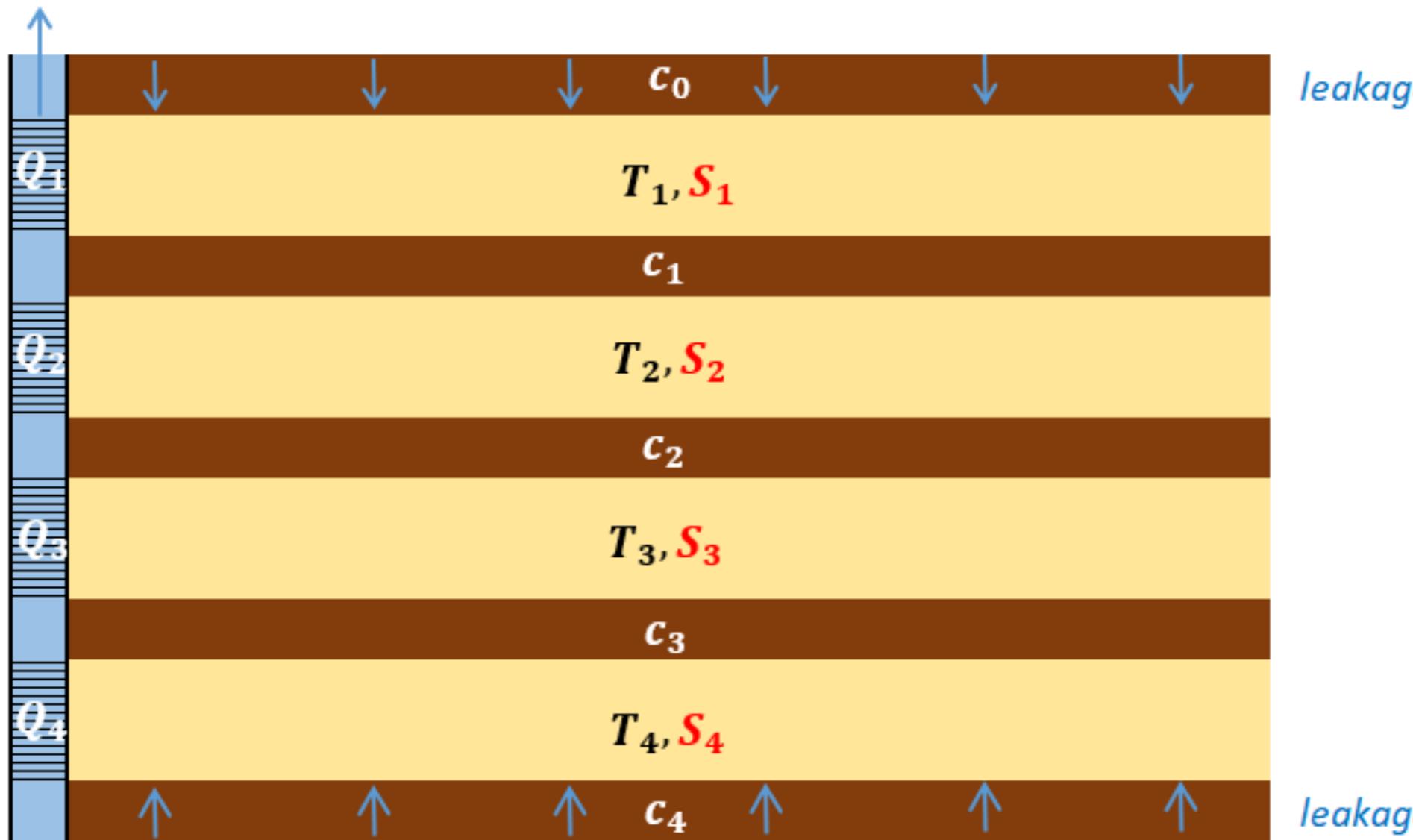
From (10): $\alpha = 0$

From (9): $\beta = q$

Introducing α and β in (11):

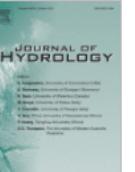
$$s = V^{-1}KVQ$$

HEMKER: TRANSIENT MULTI-AQUIFER FLOW



Journal of Hydrology

Volume 81, Issues 1–2, 30 October 1985, Pages 111-126



Research paper

Transient well flow in leaky multiple-aquifer systems

C.J. Hemker^{1,b}

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[https://doi.org/10.1016/0022-1694\(85\)90170-2](https://doi.org/10.1016/0022-1694(85)90170-2)

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Transient flow to a well in a leaky multi-aquifer system (Hemker, 1985)

$$\bar{\mathbf{s}} = V^{-1} \mathbf{K} V \mathbf{Q}$$

= Laplace solution which is numerically inverted

HEMKER (1985) MODEL: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - **Transient-state**
 - Aquifers: strictly horizontal
 - Aquitards: strictly vertical
- Well:
 - Fully penetrating screens
 - Screens are not connected
 - Constant pumping rates
 - Infinitesimal radius
- Aquifer system:
 - Homogeneous aquifers and aquitards
 - Aquifers have constant saturated thickness
 - Incompressible aquitards → zero-thickness resistance layers
 - Laterally unbounded
 - Leaky top and bottom (**both top and bottom impervious is possible!**)

HEMKER (1985): PROBLEM STATEMENT

Transient flow in each layer i :

$$T_i \left(\frac{\partial^2 s_i}{\partial r^2} + \frac{1}{r} \frac{\partial s_i}{\partial r} \right) = \boxed{S_i \frac{\partial s_i}{\partial r}} + \boxed{\frac{s_i - s_{i-1}}{c_{i-1}} + \frac{s_i - s_{i+1}}{c_i}} \quad (1)$$

storage change leakage

Matrix notation + Laplace transform:

$$\frac{d^2 \bar{s}}{dr^2} + \frac{1}{r} \frac{d\bar{s}}{dr} = A \bar{s} \quad (4)$$

Inner boundary condition at zero:

$$Q_i = - \lim_{r \rightarrow 0} \left(2\pi r T_i \frac{\partial s_i}{\partial r} \right) \quad (2)$$

$$Q = - \lim_{r \rightarrow 0} \left(r \frac{d\bar{s}}{dr} \right) \quad (5)$$

Outer boundary condition at infinity:

$$s_i(\infty) = 0 \quad (3)$$

$$\bar{s}(\infty) = 0 \quad (6)$$

HEMKER (1985): MATRICES

Vector \bar{s} :

$$\bar{s} = \begin{bmatrix} \bar{s}_1 \\ \vdots \\ \bar{s}_{n_l} \end{bmatrix}$$

Vector Q :

$$Q = \frac{1}{2\pi p} \begin{bmatrix} Q_1/T_1 \\ \vdots \\ Q_{n_l}/T_{n_l} \end{bmatrix}$$

with n_l the number of layers

System matrix A :

$$A = \begin{bmatrix} a_1 + b_1 + \omega_1^2 & -b_1 & 0 & \cdots \\ -a_2 & a_2 + b_2 + \omega_2^2 & -b_2 & \\ 0 & \ddots & & \\ \vdots & & & \\ \cdots & -a_{n_l} & a_{n_l} + b_{n_l} + \omega_{n_l}^2 & \end{bmatrix}$$

with

$$a_i = \frac{1}{c_{i-1} T_i}$$

$$b_i = \frac{1}{c_i T_i}$$

$$\omega_i = \sqrt{p S_i / T_i}$$

HEMKER (1985): EIGENDECOMPOSITION

Eigendecomposition of matrix A :

$$A = V^{-1}DV \quad (7)$$

Applying (7) to ODE (4) with $g = V\bar{s}$:

$$\frac{d^2g}{dr^2} + \frac{1}{r} \frac{dg}{dr} = Dg \quad (8)$$

Multiplying each side of (5) and (6) by V :

$$q = VQ = -\lim_{r \rightarrow 0} \left(r \frac{dg}{dr} \right) \quad (9)$$

$$g(\infty) = 0 \quad (10)$$

General solution of (8):

$$g = I\alpha + K\beta \quad (11)$$

Diagonal matrices I and K :

$$I_{ii} = I_0(r\sqrt{d_i}) \text{ and } K_{ii} = K_0(r\sqrt{d_i})$$

with d_i the i -th eigenvalue of A

From (10): $\alpha = 0$

From (9): $\beta = q$

Introducing α and β in (11):

$\bar{s} = V^{-1}KVQ$

HEMKER: STEADY VS TRANSIENT

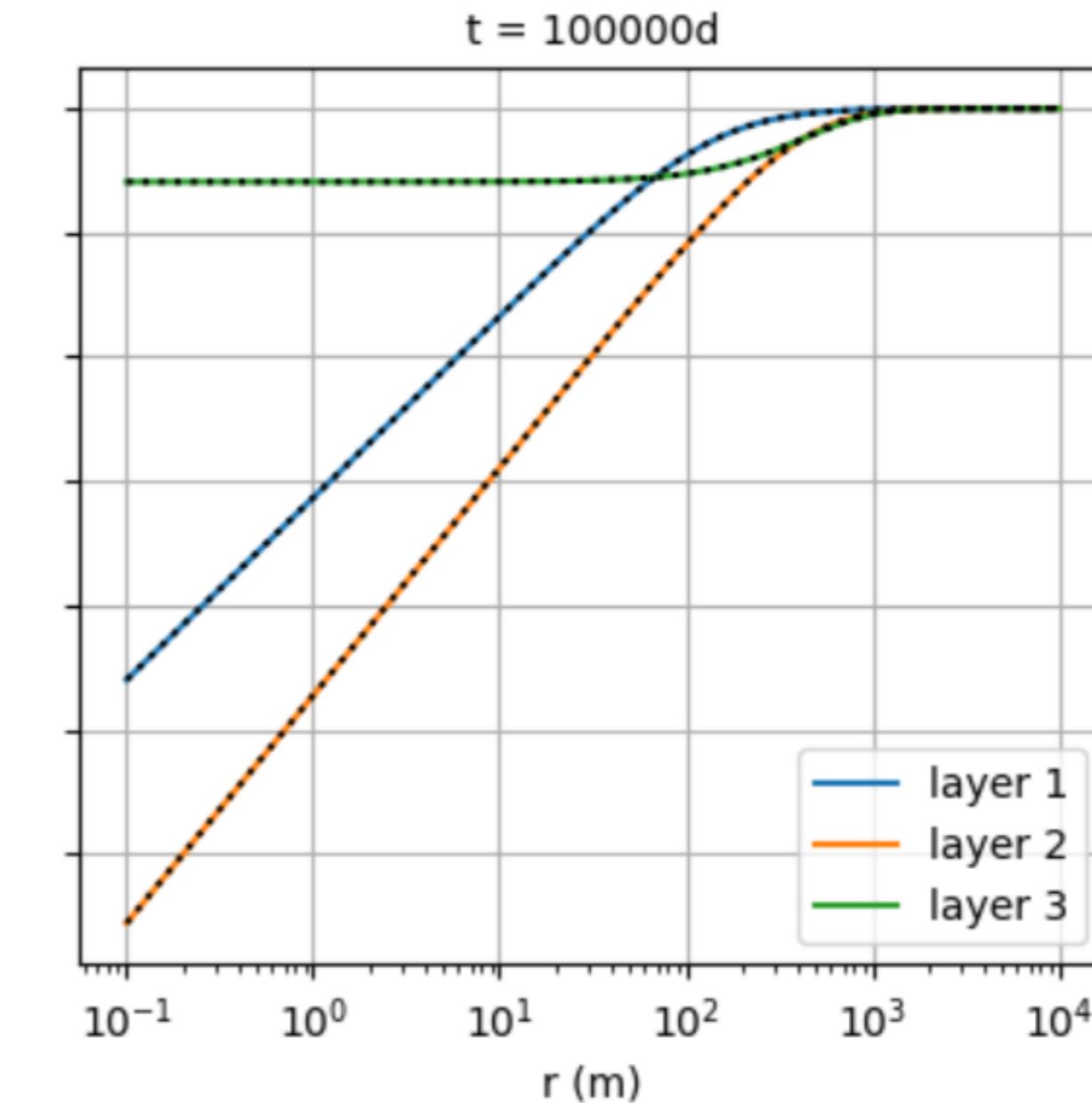
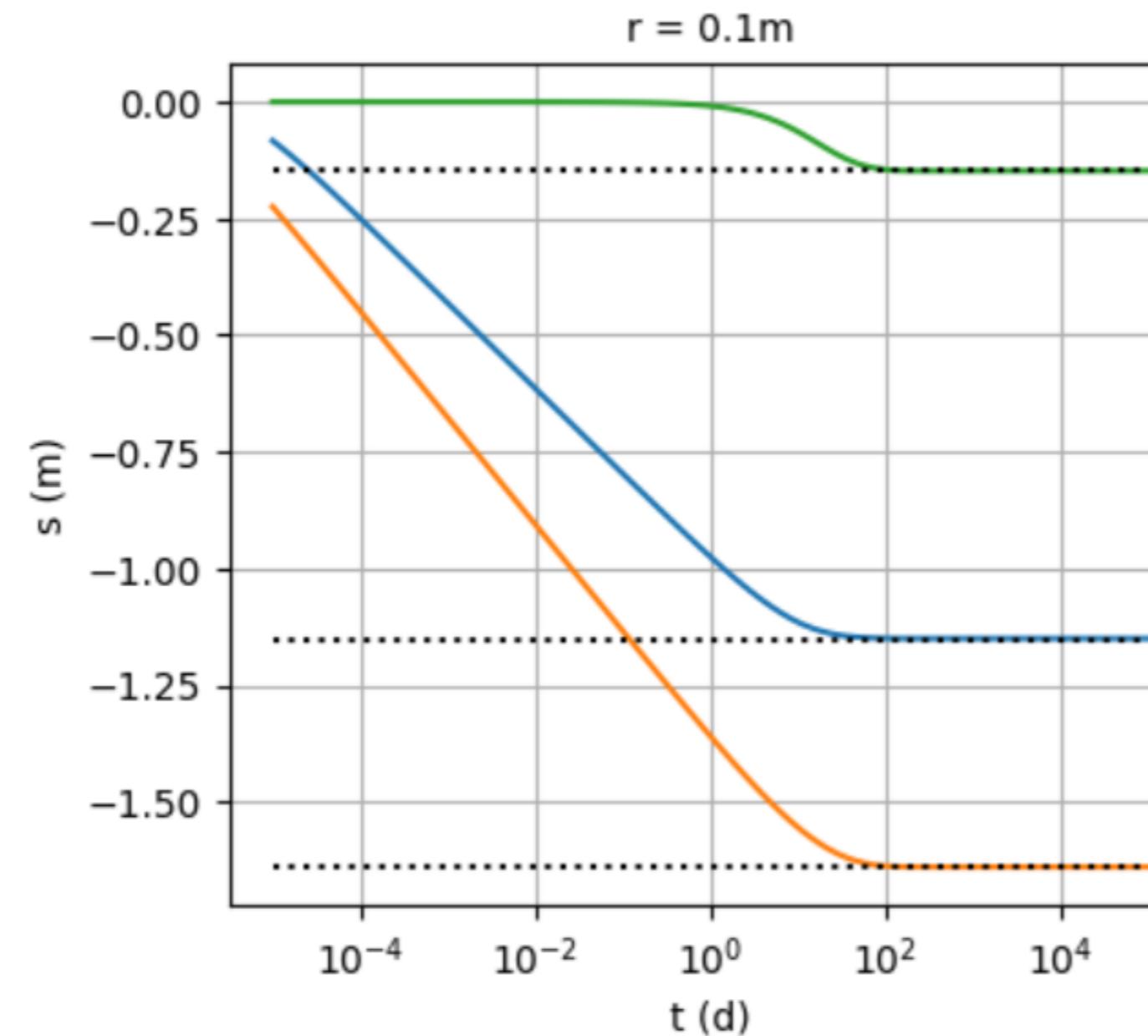


alouwyck / maxsypy

```
T = [100, 200, 50]          # aquifer transmissivities (m²/d)
S = [0.1, 0.05, 0.01]        # aquifer storativities (-)
c = [100, 500, 1000, np.inf] # aquitard resistances (d)
Q = [-100, -250, 0]          # pumping rates (m³/d)

model1 = Steady(T=T, Q=Q, c_top=c[0], c=c[1:-1], c_bot=c[-1])
model2 = Transient(T=T, S=S, Q=Q, c_top=c[0], c=c[1:-1], c_bot=c[-1])
```

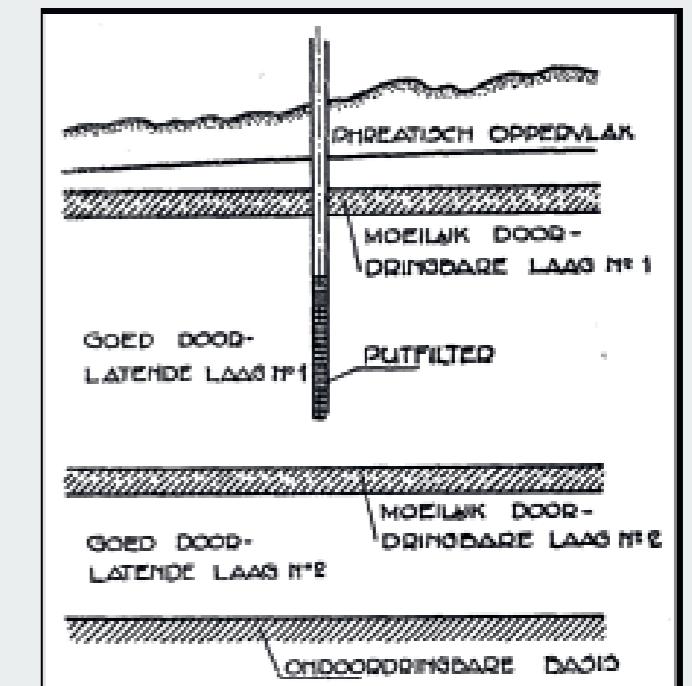
- solid lines: transient-state
- dotted lines: steady-state





- **Semi-analytical method:**
 - Generalized semi-analytical solution for multilayer flow
 - Extension to multilayer-multizone flow
- **Finite-difference approach:**
 - Matlab tool MAxSym for multilayer flow
 - MODFLOW procedure for axisymmetric flow
 - Extension to multi-node wells
- **Comparing both solution methods**

Axisymmetric Flow in
Multilayer Aquifer Systems:
Solutions and Theoretical Considerations



GENERALIZED SEMI-ANALYTICAL SOLUTION

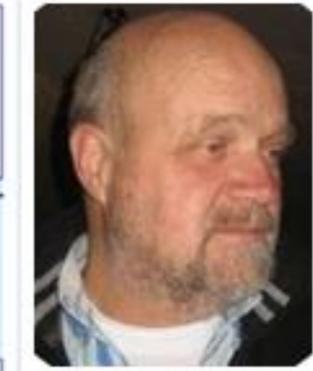
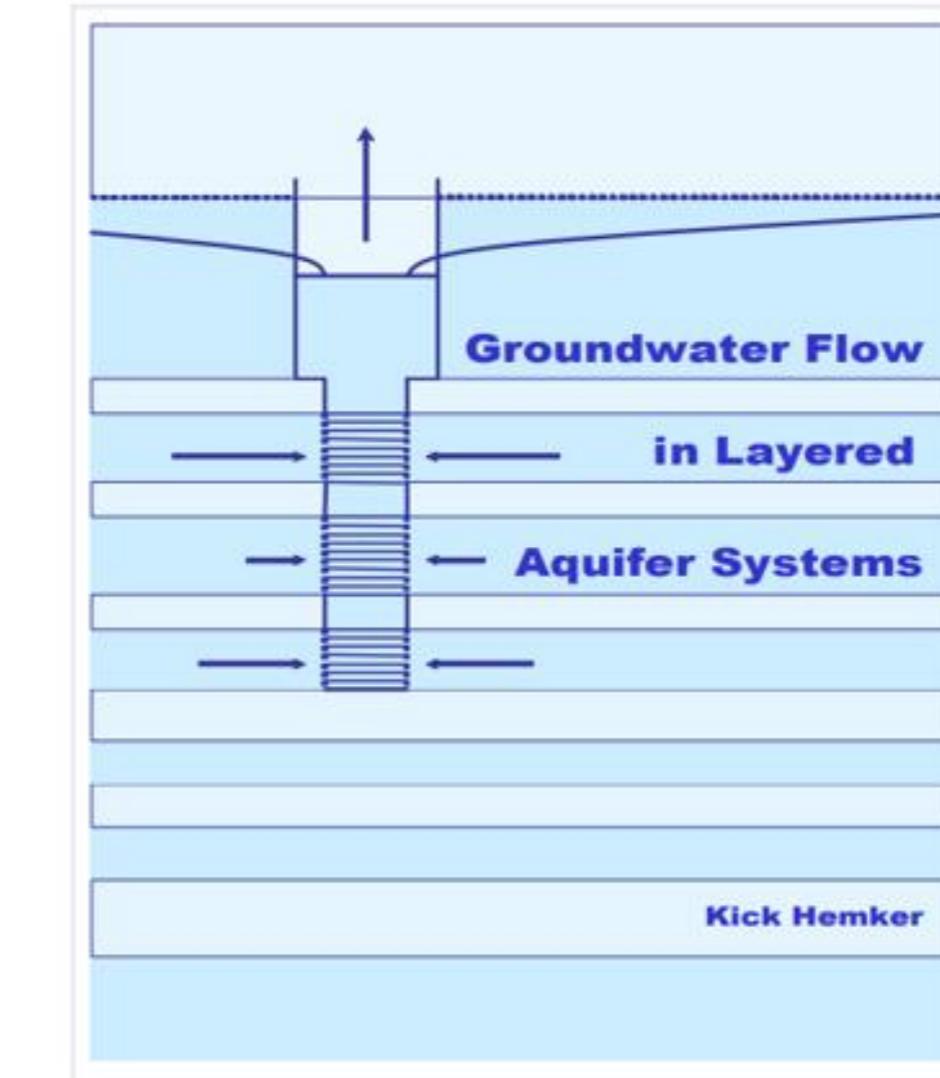
- Python code:
 - axisymmetric or parallel flow
 - steady or transient state
 - specified discharge or head
 - laterally bounded or unbounded
 - confined or leaky + recharge
 - superposition in space and time



```
model = Transient(T=[100, 200, 50],      # transmissivities (m²/d)
                  S=[0.1, 0.05, 0.01],    # storativities (-)
                  Q=[-100, -250, 0],     # pumping rates (m³/d)
                  c=[500, 1000],          # resistances (d)
                  c_top=100)             # top resistance (d)

t = np.logspace(-5, 5, 100) # simulation times (d)
r = 0.1 # well-radius (m)
s = model.h(r, t) # drawdown s (m)
```

- based on earlier work
 - Hemker (1984, 1985, 1999, 2000)
 - Bakker & Strack (2003)
 - MLU app (Hemker & Post)
 - Python packages TimML and TTIm (Bakker)

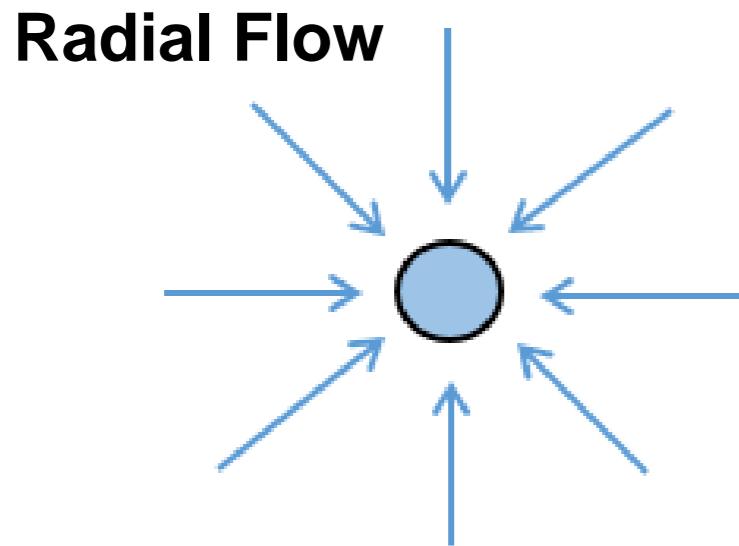


Kick Hemker

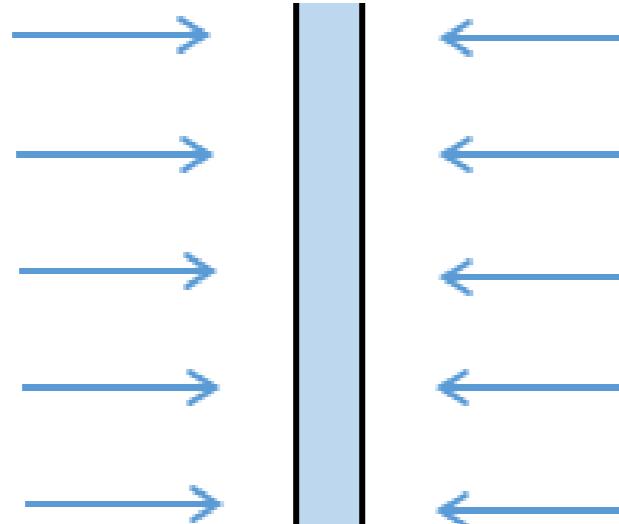


Mark Bakker

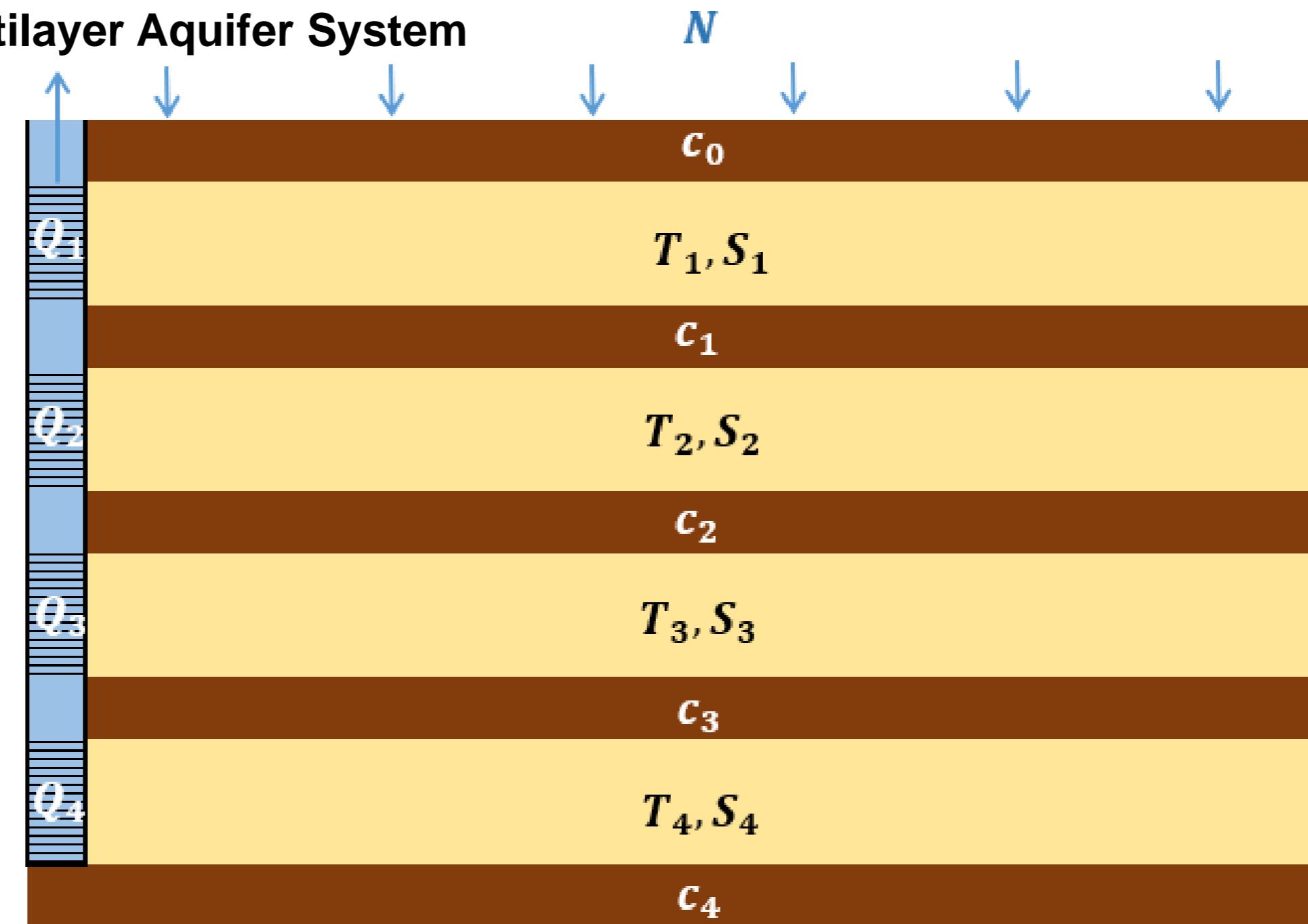
2D MULTILAYER FLOW



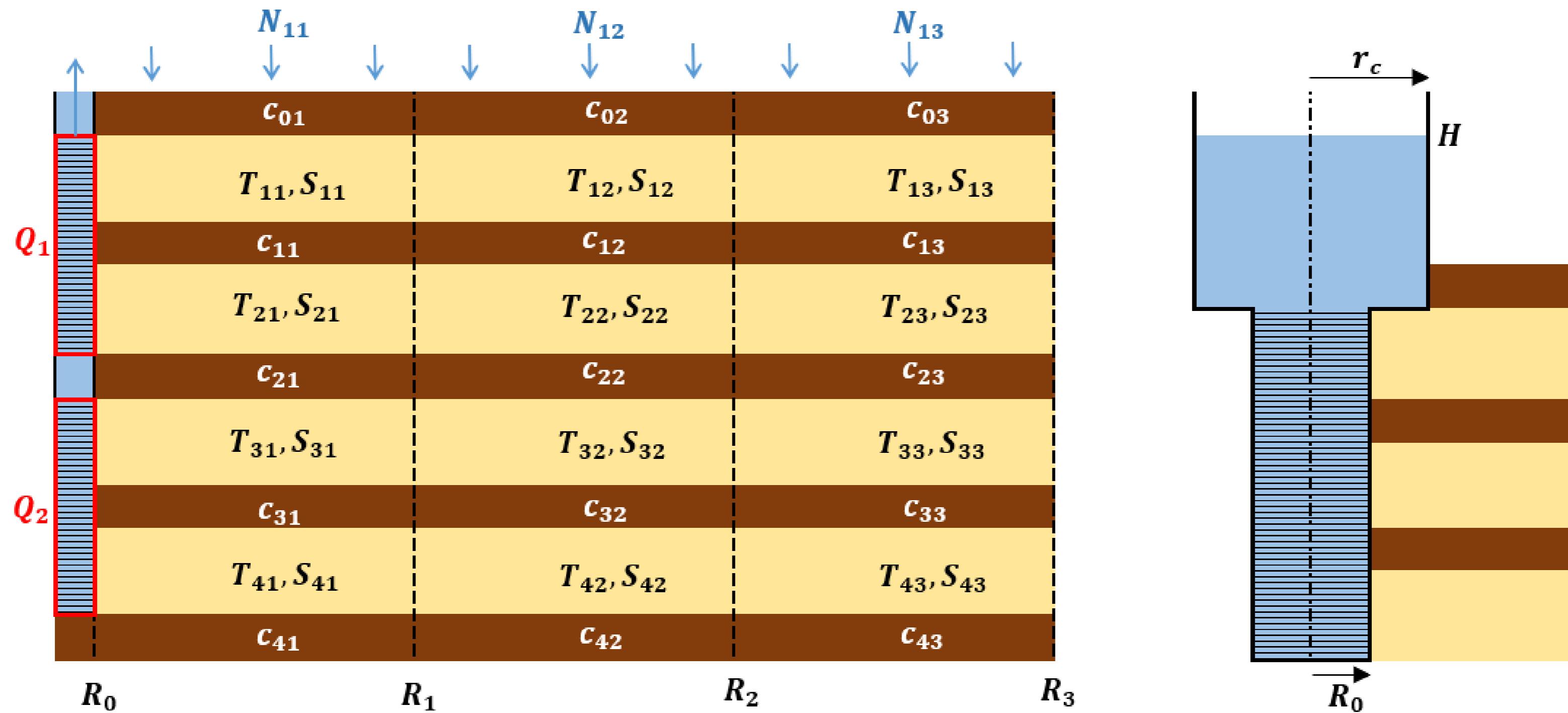
Parallel Flow



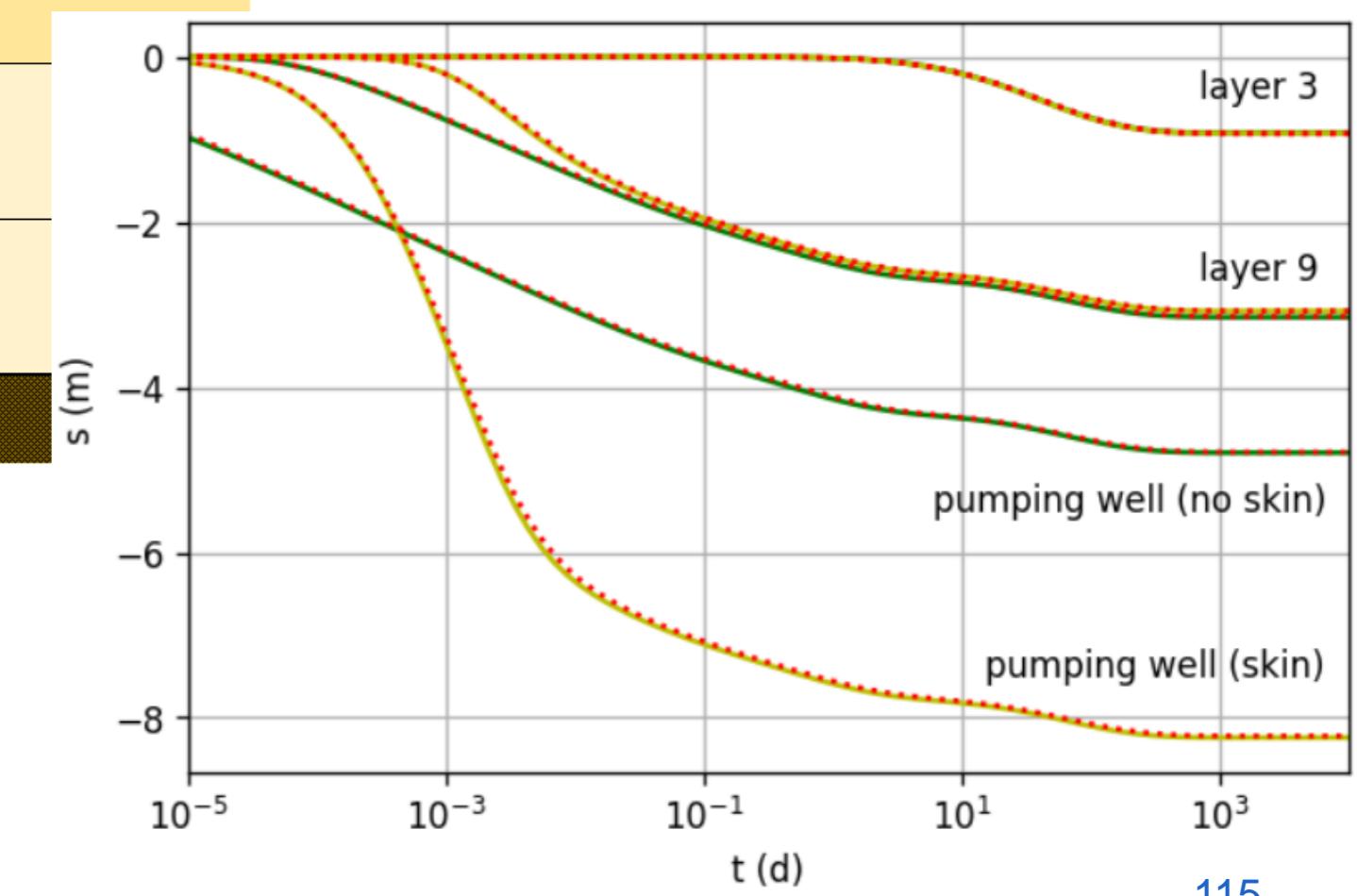
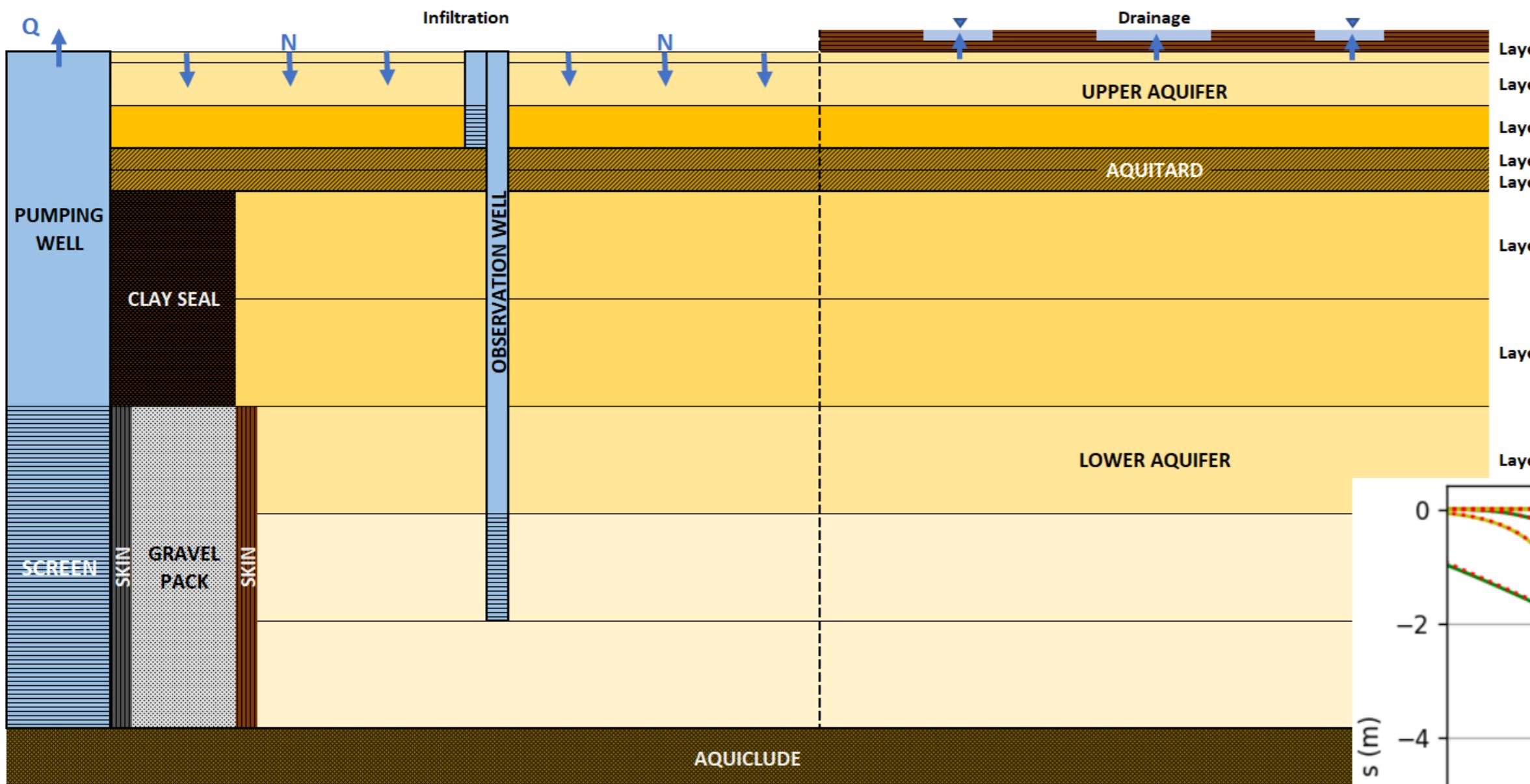
Multilayer Aquifer System



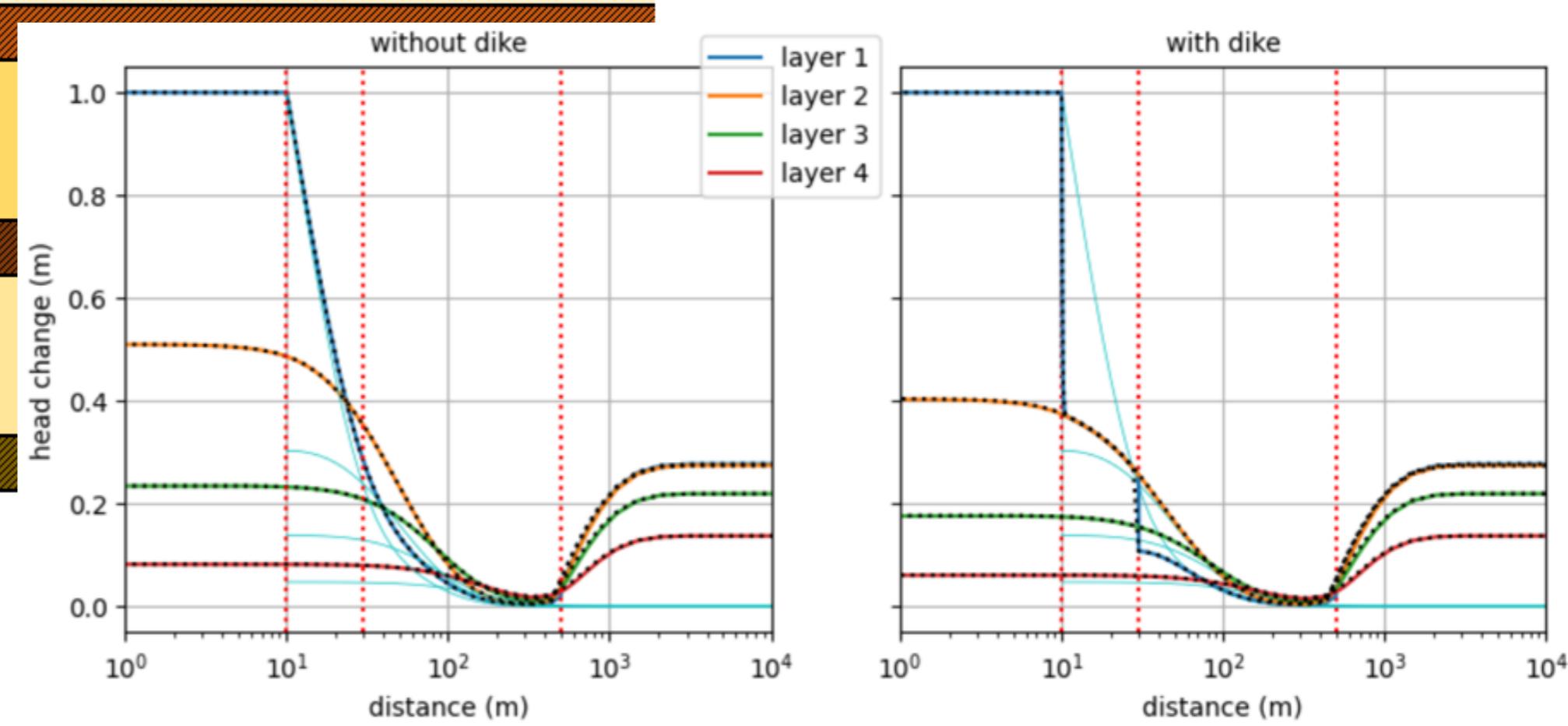
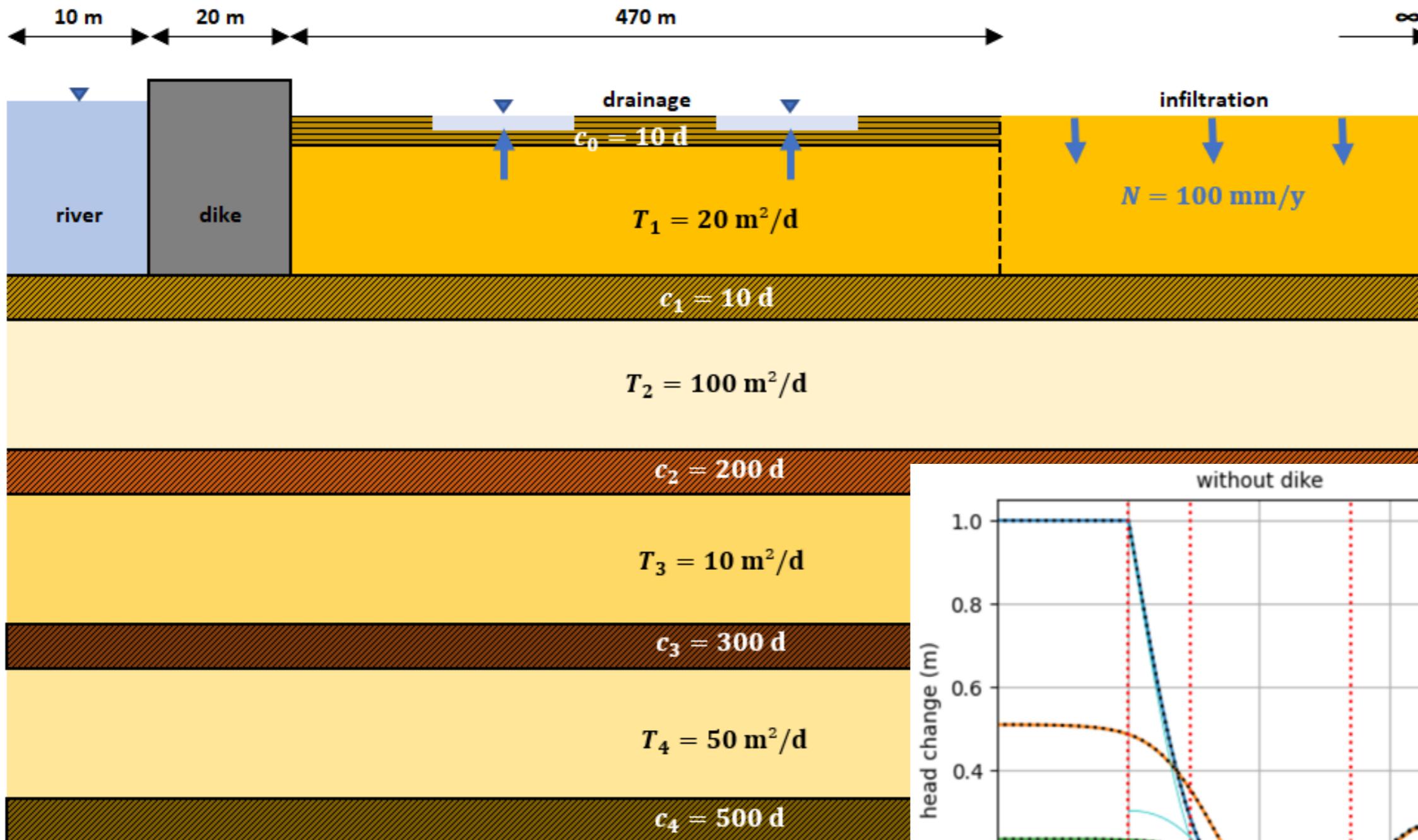
EXTENSION: MULTILAYER-MULTIZONE FLOW



EXAMPLE: MULTILAYER WELL



EXAMPLE: EMBANKED RIVER



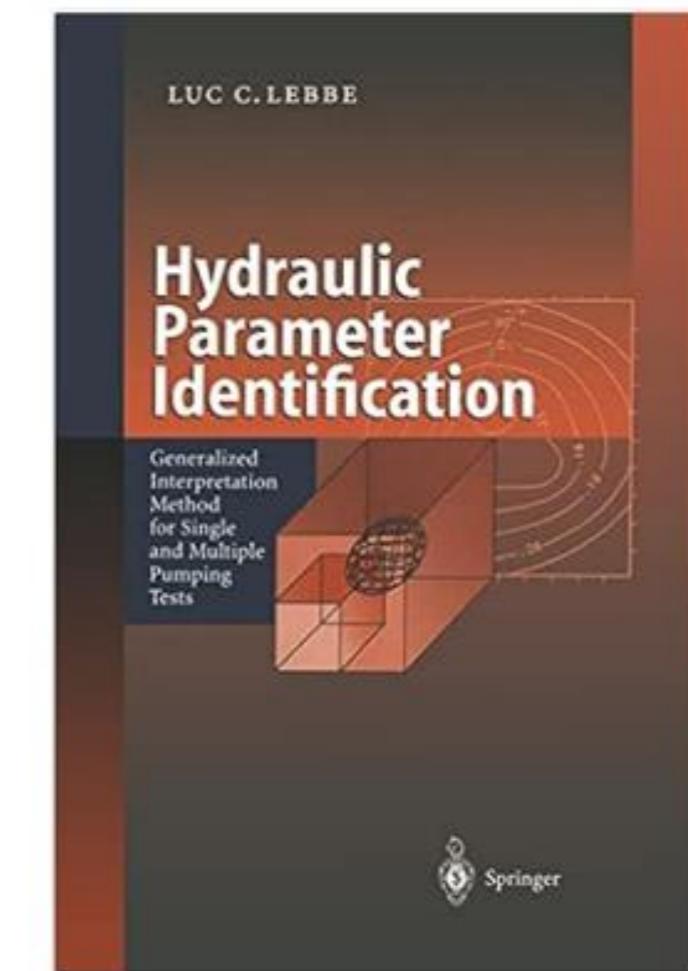


– software implementations

- AS2D Matlab wrapper
- OGMA-RF (Louwyck et al., 2007, 2010; Vandenbohede et al., 2008, 2009)
- MAxSym (Louwyck, 2011, 2015; Louwyck et al., 2012)
- MODFLOW procedure (Louwyck et al., 2012, 2014)
- Python version

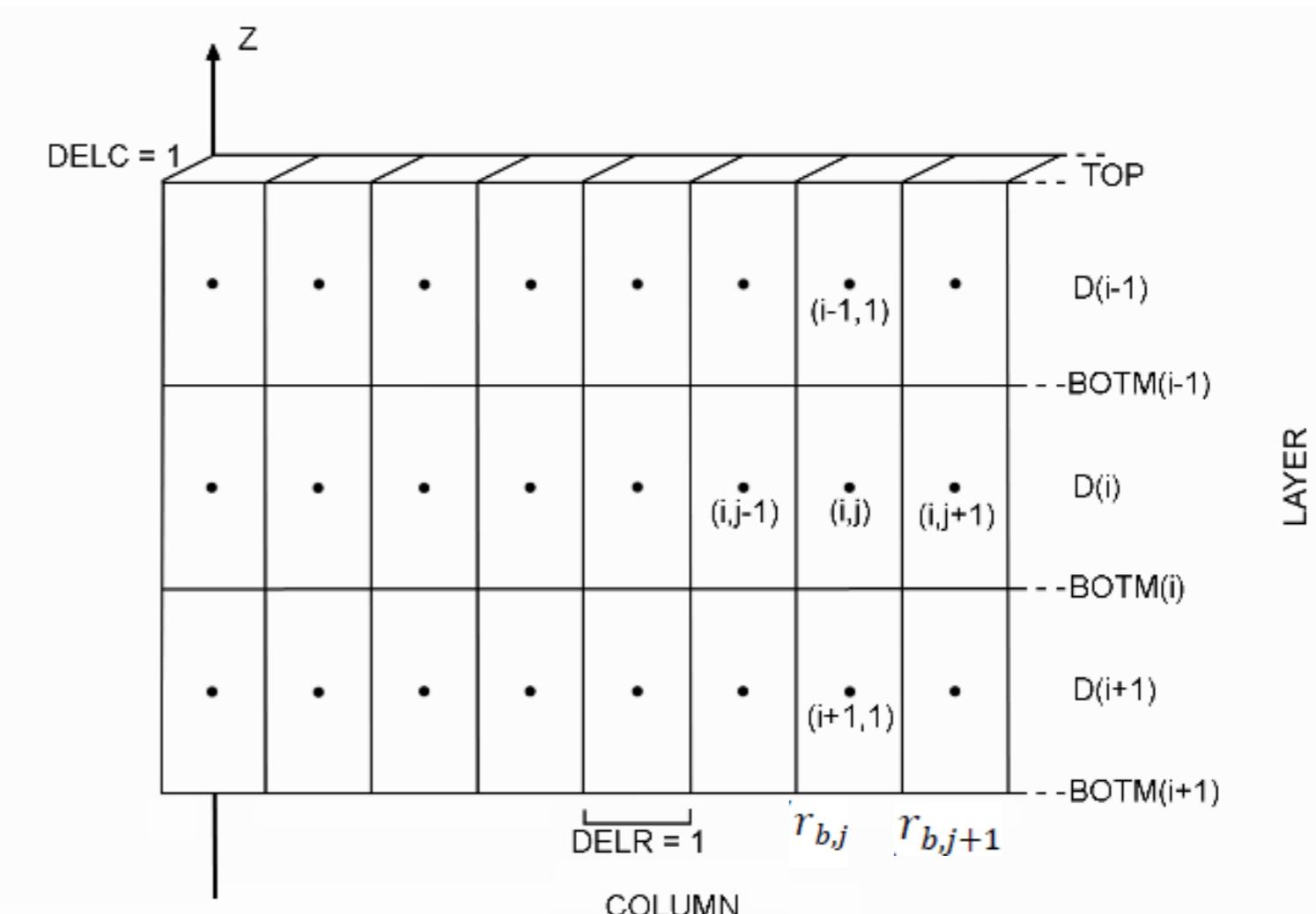
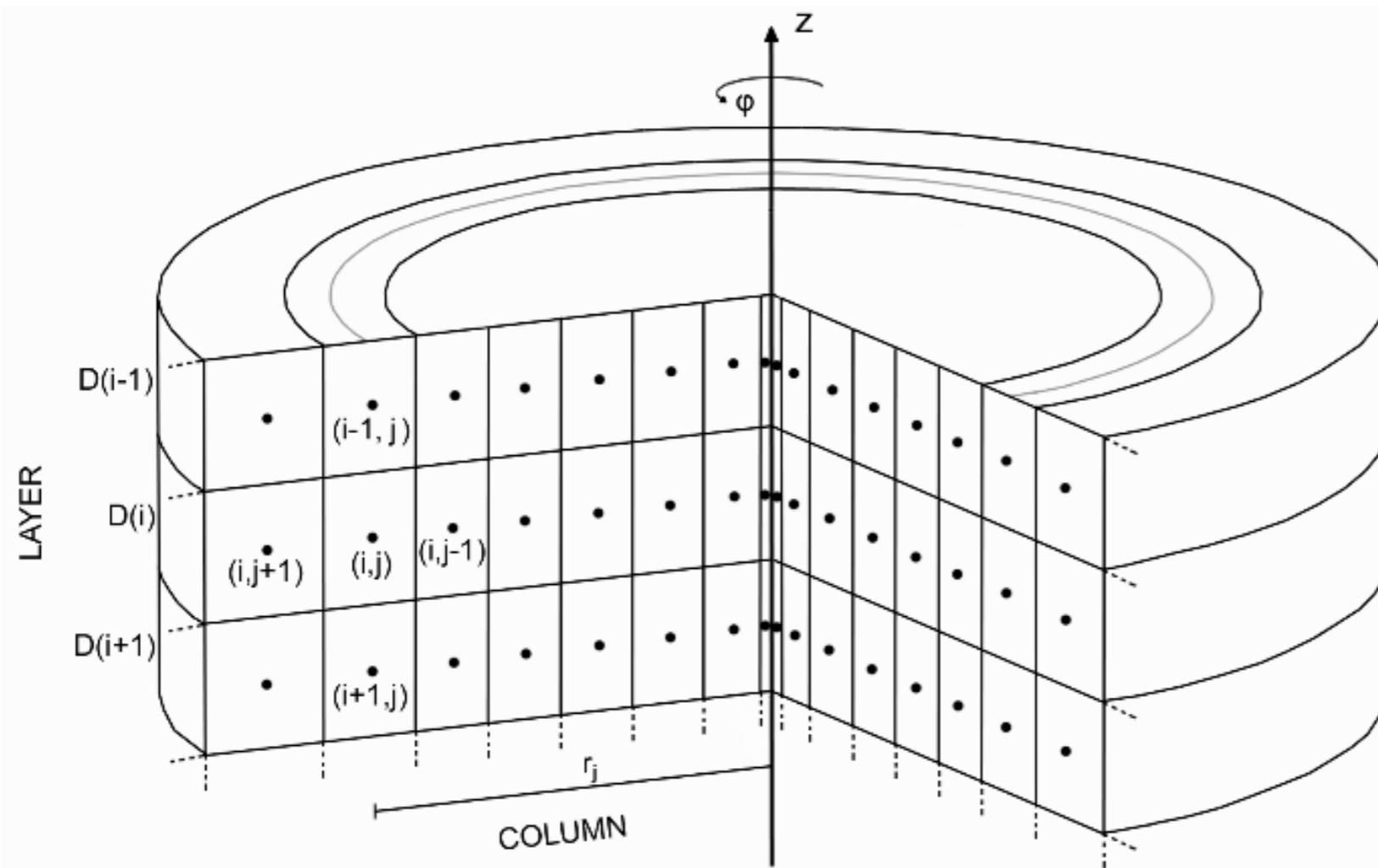
– based on earlier work

- AS2D (Lebbe, 1983, 1988, 1999)
- MODFLOW (1984, 1988, 1996, 2000, 2005)
- MODFLOW procedure (Langevin, 2008)



Luc Lebbe

MODFLOW PROCEDURE



Hydrogeology Journal (2014) 22: 1217–1226
DOI 10.1007/s10040-014-1150-0



MODFLOW procedure to simulate axisymmetric flow in radially heterogeneous and layered aquifer systems

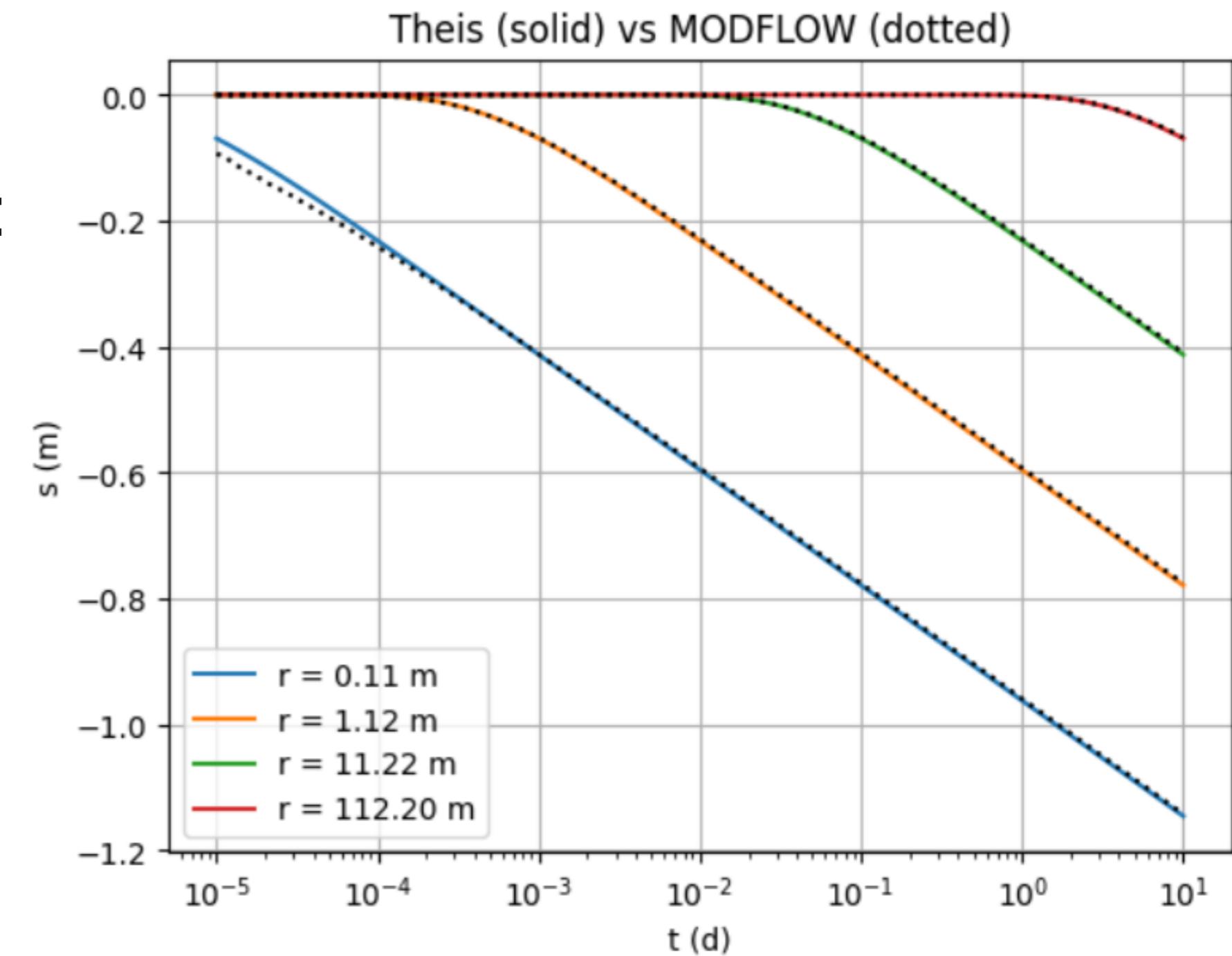
PARAMETER CONVERSIONS

Aquifer parameters are modified to take into account the axisymmetric grid geometry:

$$HK_{ij} = \frac{2\pi K_{ij}^h}{\ln(r_{b,j+1}/r_{b,j})}$$

$$VKA_{ij} = \pi(r_{b,j+1}^2 - r_{b,j}^2)K_{ij}^v$$

$$SS_{ij} = \pi(r_{b,j+1}^2 - r_{b,j}^2)S_{ij}^s$$



CONCLUSIONS

Semi-Analytical (SA) vs Finite-Difference (FD):

- both very accurate and fast
- FD easier to implement in case of
 - heterogeneities
 - nonlinearities
- SA offers insight!

AQUIFER TESTS

AQUIFER TESTING

- Used to characterize aquifer systems
 - E.g., pumping test, step-drawdown test, slug test, recovery test, ...
- Test conducted in the field:
 - Stimulate the aquifer by stressing a well
 - Measure drawdown in observation wells
- Interpretation of the test:
 - Fit the observed data
 - Derive hydraulic parameters
 - Inverse problem type I



TRADITIONAL MANUAL CURVE FITTING

- Analytical solution produces type curves
- Test data are visually fit to these curves
- See Kruseman & de Ridder (2000)
E.g., the Theis curve-fitting method

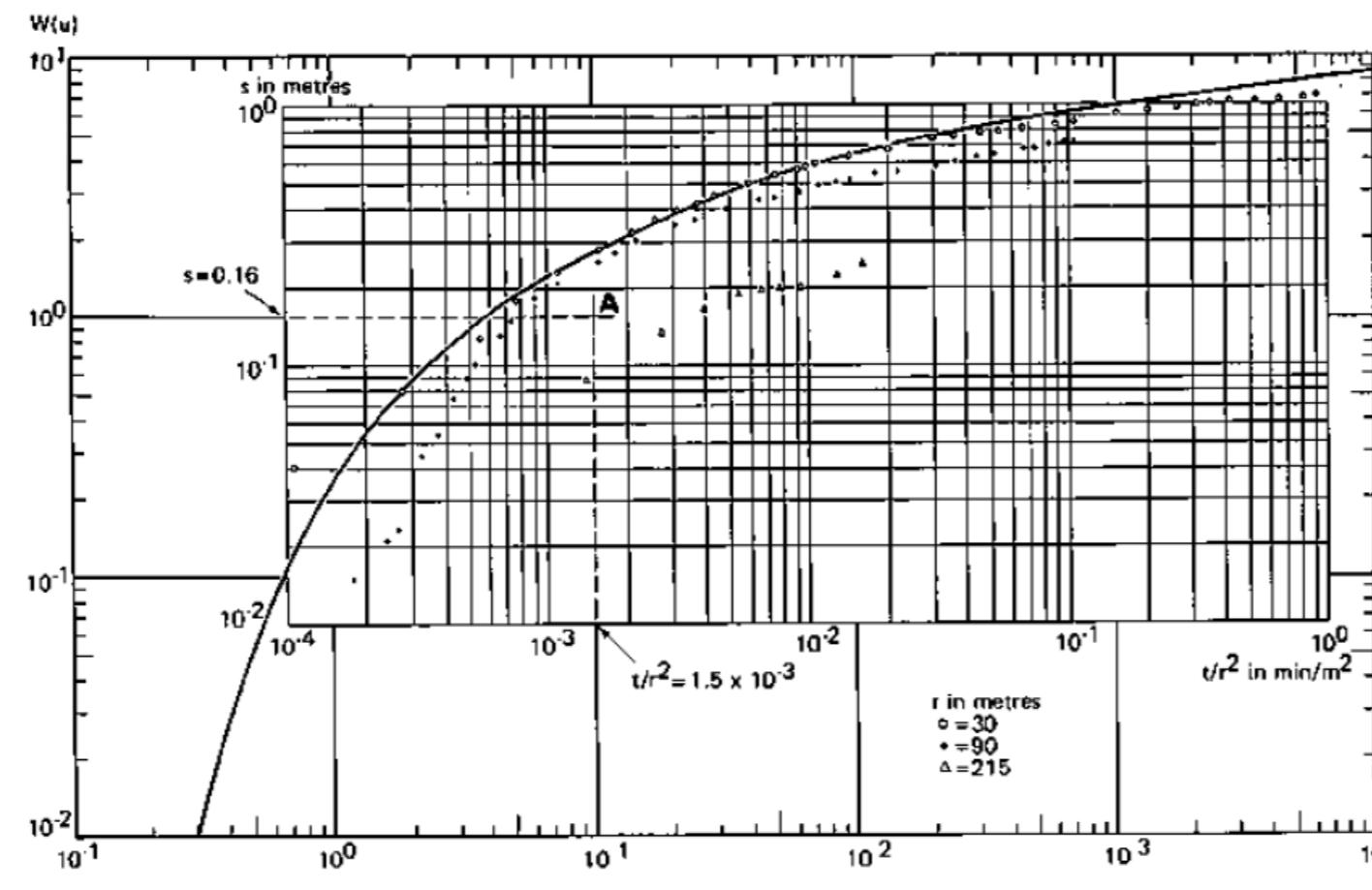
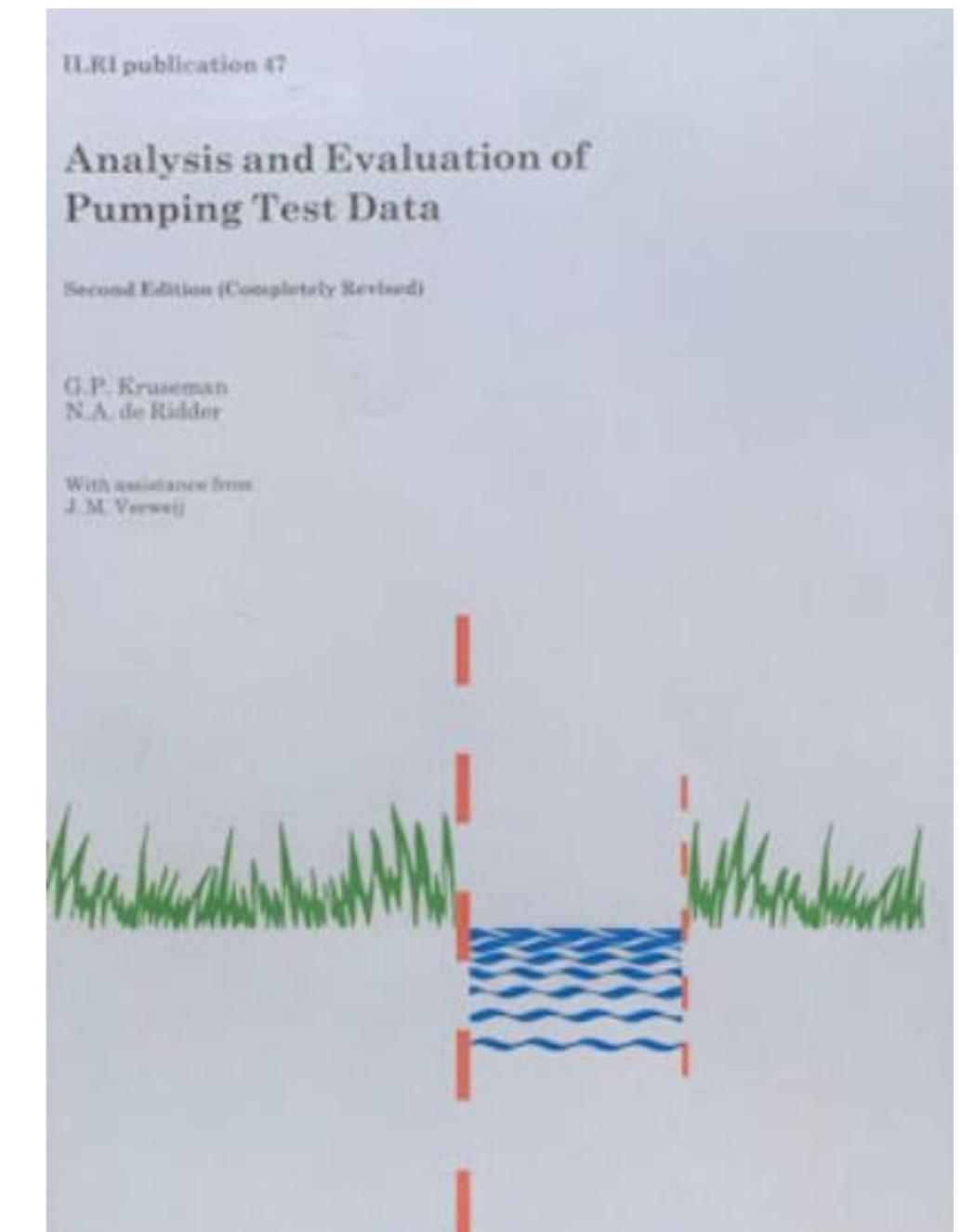
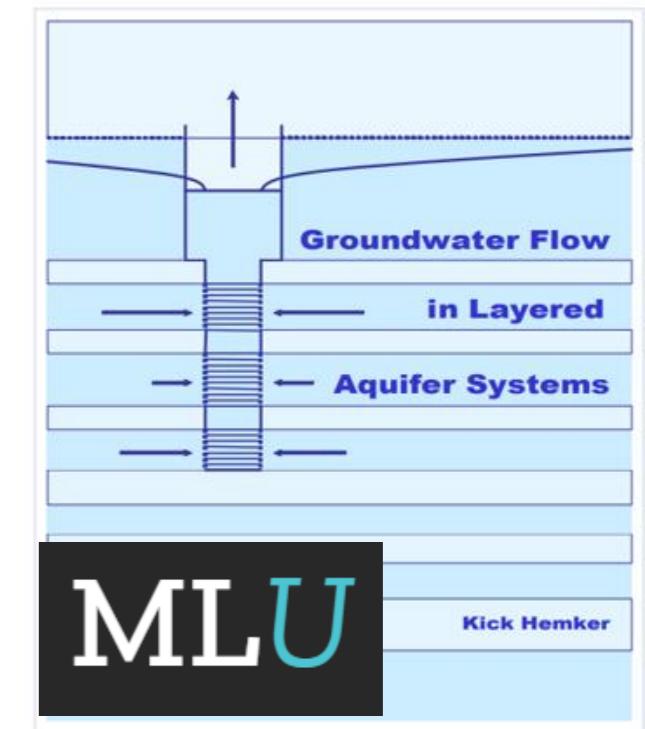
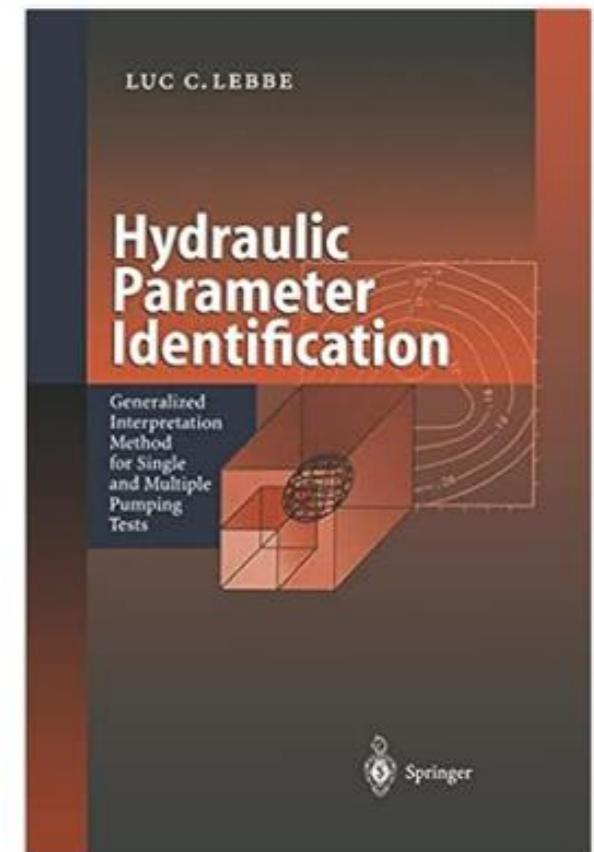
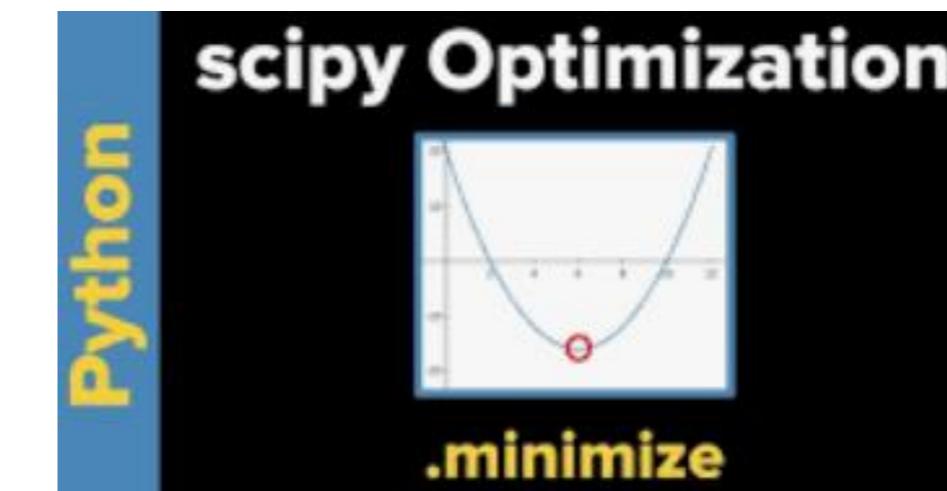
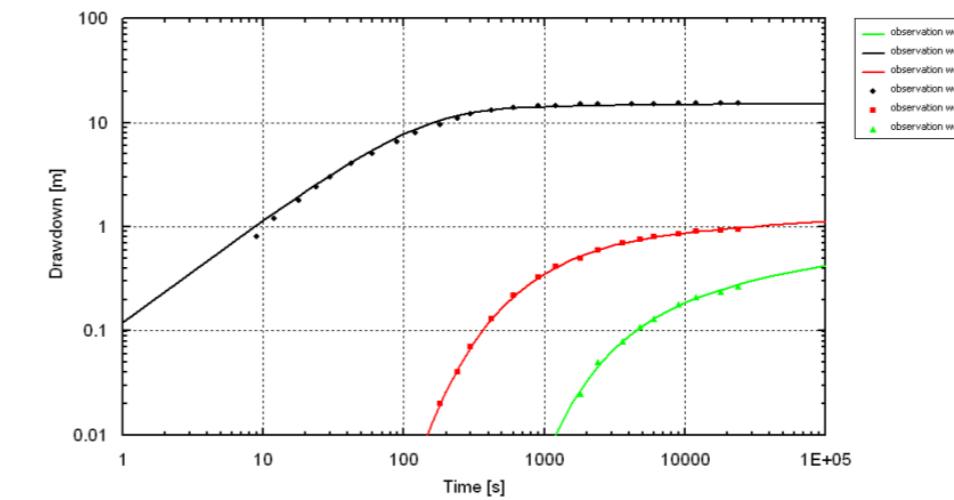


Figure 3.6 Analysis of data from pumping test 'Oude Korendijk' with the Theis method, Procedure 3.3



NONLINEAR REGRESSION

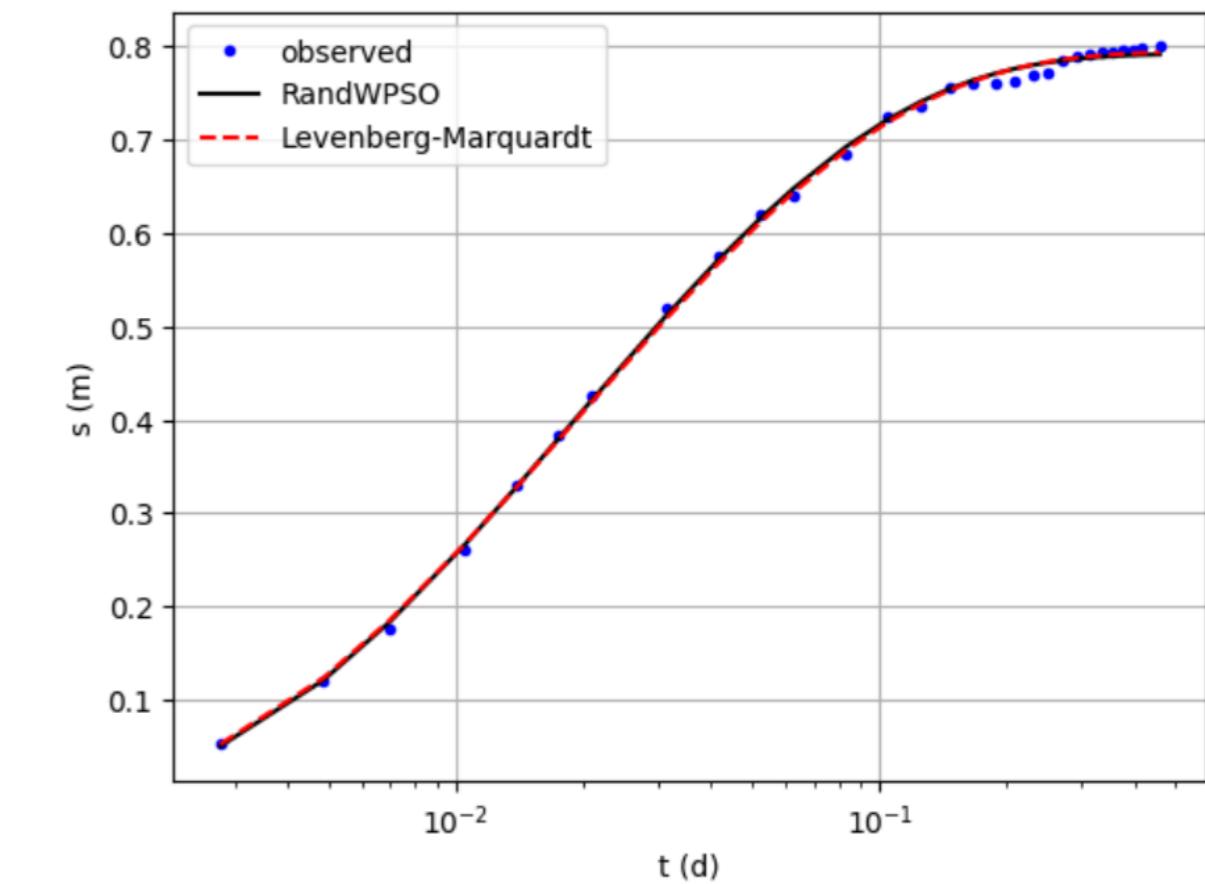
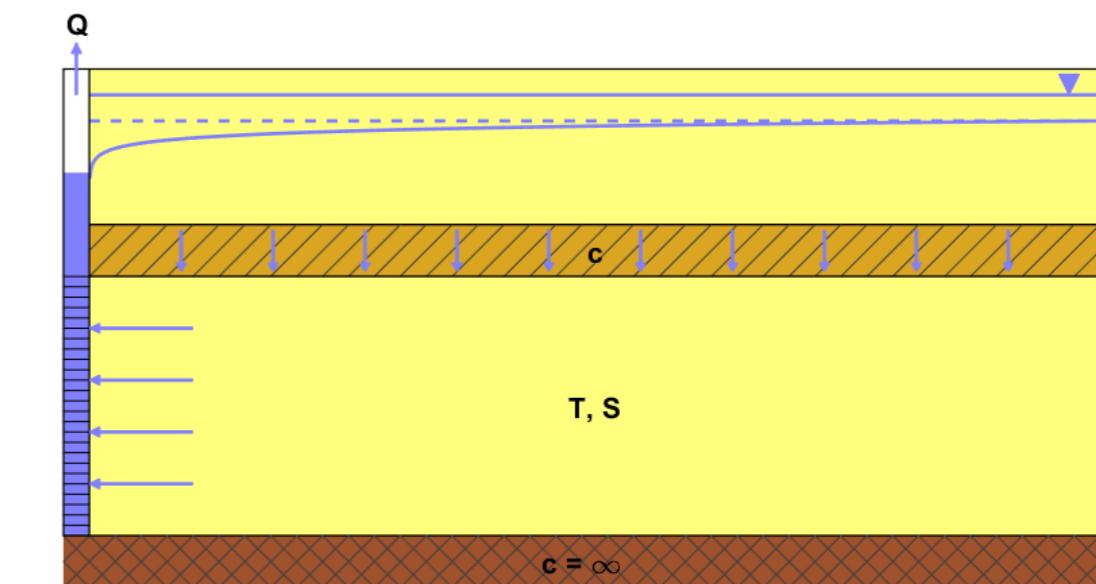
- Test data are fit through nonlinear regression
 - E.g., Gauss-Newton, Levenberg-Marquardt
- Pros:
 - Automated curve-fitting
 - Reproducible results
- Software:
 - HYPARIDEN (Lebbe, 1999)
 - MLU: <https://mlu.app/>
 - Python: SciPy



PUMPING TEST IN LEAKY AQUIFER

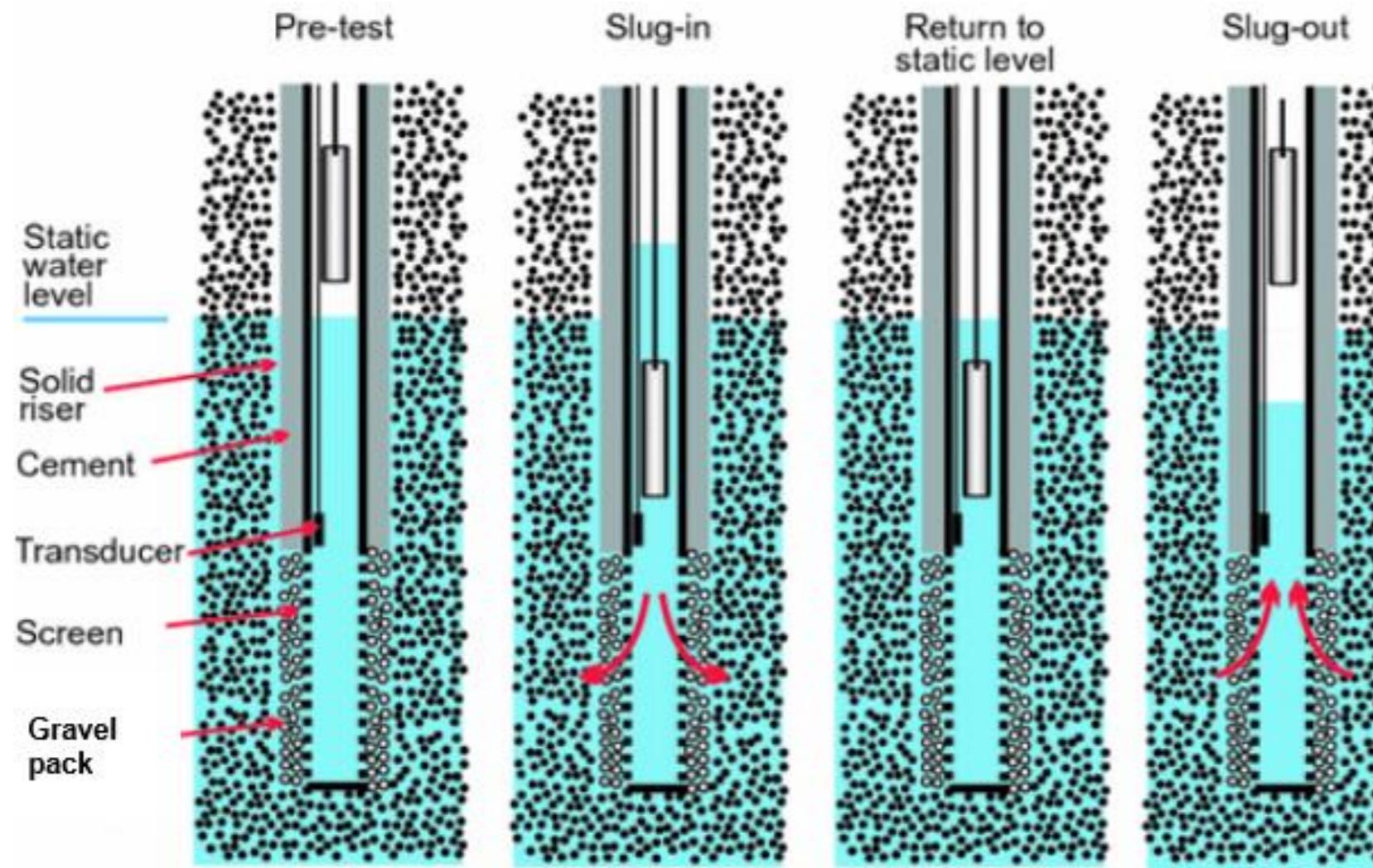
- Pumping test (Jiang Hui et al., 2009):
 - Leaky aquifer consisting of gravel
 - Fully penetrating pumping well: $Q = 69.1 \text{ m}^3/\text{h}$
 - Observation well: $r = 197 \text{ m}$
 - 30 drawdown observations
- Interpretation:
 - Forward model: Hantush-Jacob (MAxSyPy)
 - Curve fitting: Levenberg-Marquardt (SciPy)
 - Optimal parameter values:

$$\begin{aligned} T &= 418.38884431293474 \\ S &= 0.0001391052948927095 \\ c &= 804.0082714958618 \end{aligned}$$



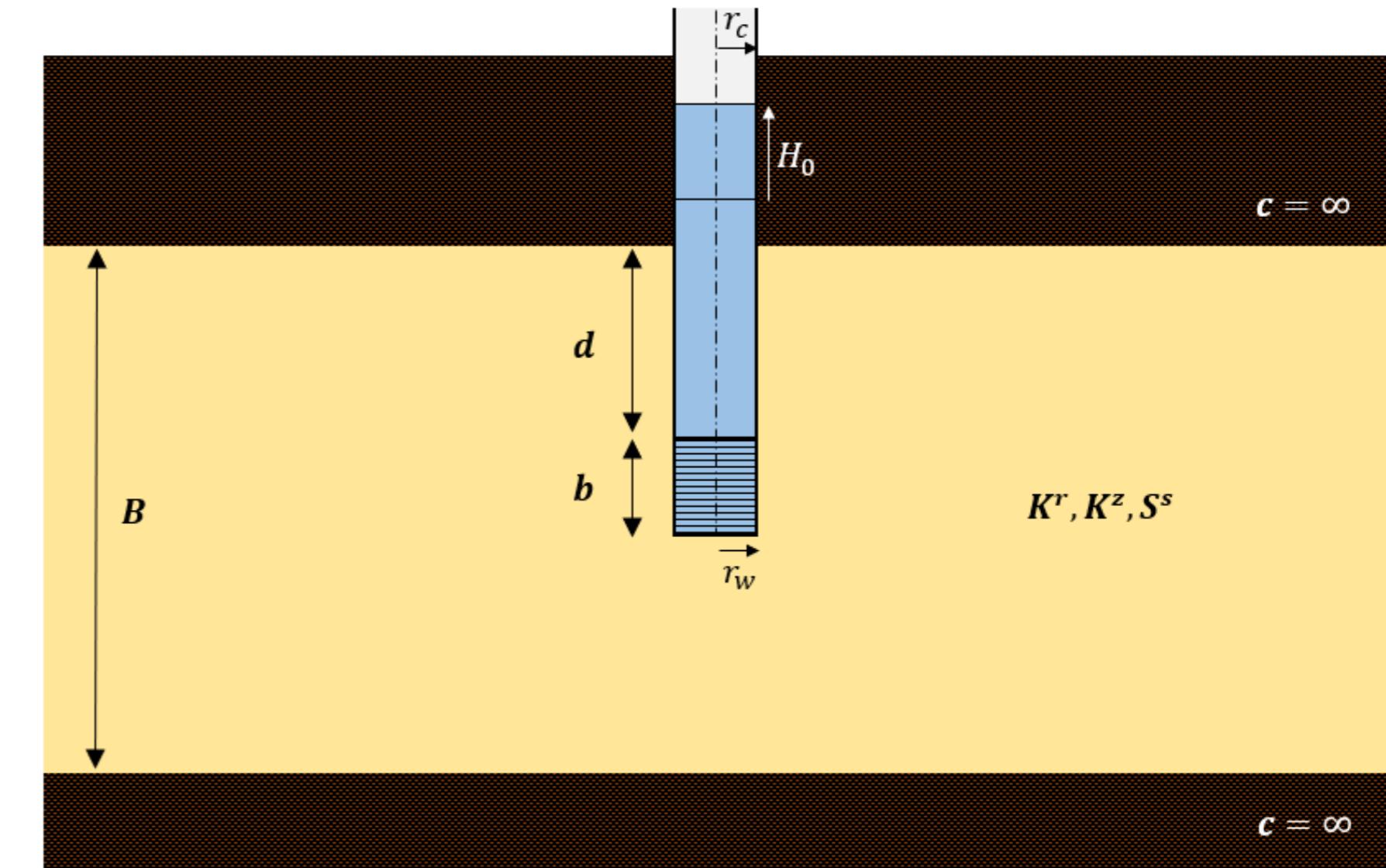
SLUG TEST IN PARTIALLY PENETRATING WELL

- Add/remove water (slug) to a well
- Monitor head change in well



KGS MODEL

- Simulation of a slug test in a partially penetrating well
- Semi-analytical solution applying integral transforms
- Developed at Kansas Geological Survey (KGS)
- Originally written in Fortran
- Python implementation available with MAxSyPy



```
from maxsypy.kgs import KGS
```

Water Resources Research[®]

Subsurface Hydrology

Slug tests in partially penetrating wells

Zafar Hyder, James J. Butler Jr., Carl D. McElwee, Wenzhi Liu

First published: November 1994 | <https://doi.org/10.1029/94WR01670> | Citations: 139

KGS MODEL: ASSUMPTIONS

- Flow:
 - Axisymmetric
 - Transient-state
 - Horizontal + **vertical**
- Well:
 - **Partially penetrating**
 - Instantaneous initial head change
 - Finite radius → wellbore storage!
 - Finite-thickness skin
- Aquifer:
 - Homogeneous
 - Constant saturated thickness
 - Laterally unbounded
 - Confined or leaky (= static water table)

SLUG TESTS IN DAMME

- Slug test campaign conducted on wells of the groundwater monitoring network
- Multi-level well **3-0523** with 3 screens in different geological formations
- Number of slug test performed:
 - Screen 1: 6 tests
 - Screen 2: 6 tests
 - Screen 3: 4 tests

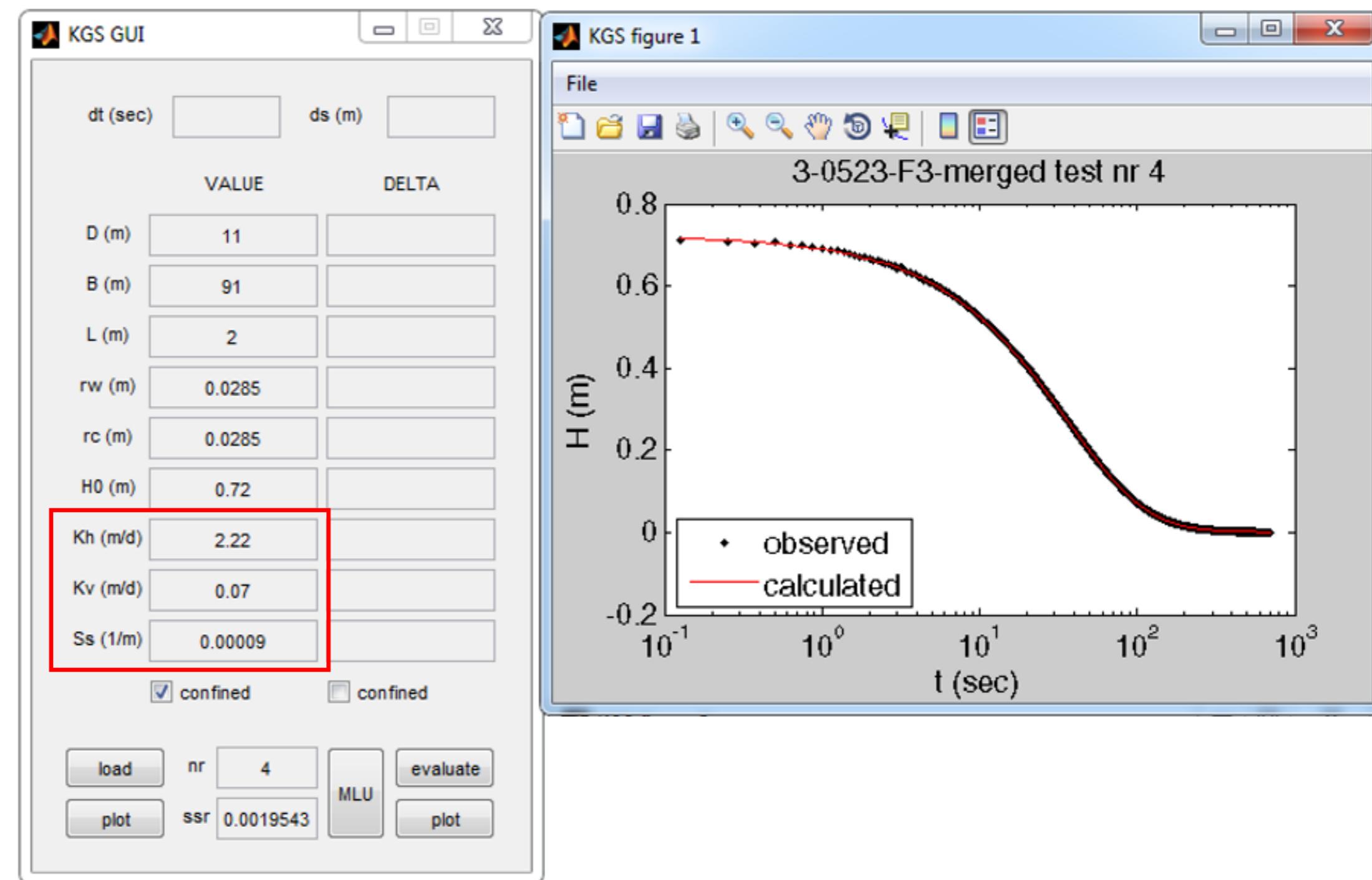


Depth (m)	Lithology	Well
0,0	fine sand	1
13,5		
15,5		
20,0		
20,0	very fine sand	2
40,0		
42,0		
58,0		
58,0	fine to medium sand	3
80,0		
80,0	clay	
87,0		
87,0	clayey sand	3
91,0		
93,0		
98,0		
98,0	heavy clay	
200?		



INTERPRETATION OF TEST 4 IN SCREEN 3

Manual curve-fitting
using KGS model:



INTERPRETATION OF TEST 4 IN SCREEN 3

KGS model +
Nelder-Mead method
to minimize MSE:

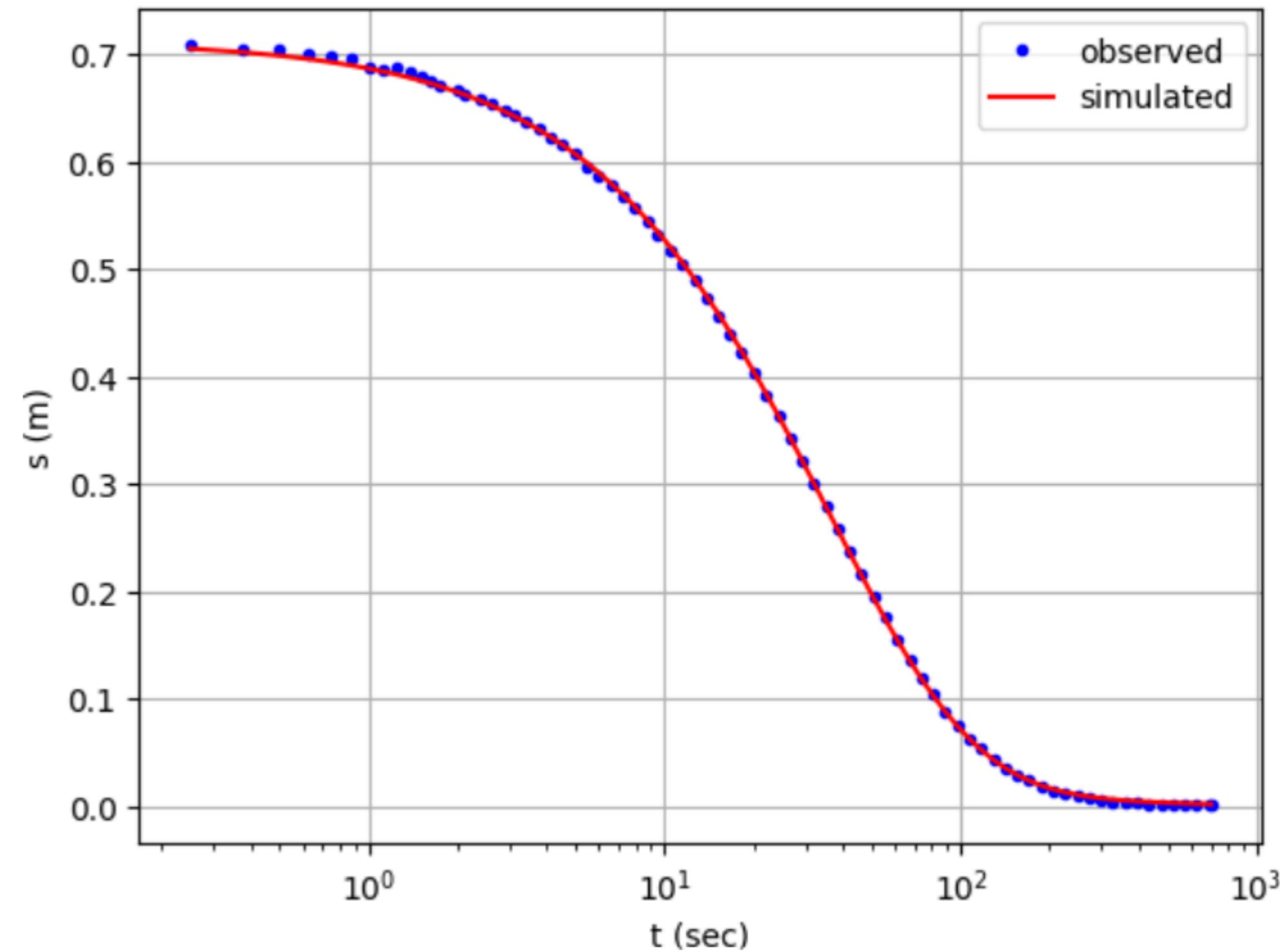
```
from scipy.optimize import fmin

x0 = np.array([0, 0, -5]) # starting values for logP
x = fmin(func=MSE, x0=x0)
print(f'\tMSE = {MSE(x)}')

Kr, Kz, Ss = tuple(10**x)
print('\nOptimal values:')
print(f'\tKr = {Kr} m/d')
print(f'\tKz = {Kz} m/d')
print(f'\tSs = {Ss} m^-1')

Optimization terminated successfully.
    Current function value: 0.000005
    Iterations: 232
    Function evaluations: 412
    MSE = 4.58556435401376e-06

Optimal values:
    Kr = 3.2437663500170486 m/d
    Kz = 0.0001843799706562987 m/d
    Ss = 1.718446610208439e-06 m^-1
```



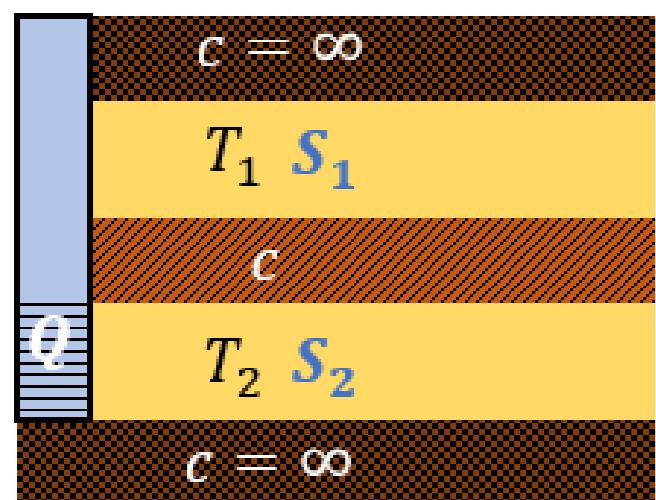
A THEORETICAL CASE

STUDY

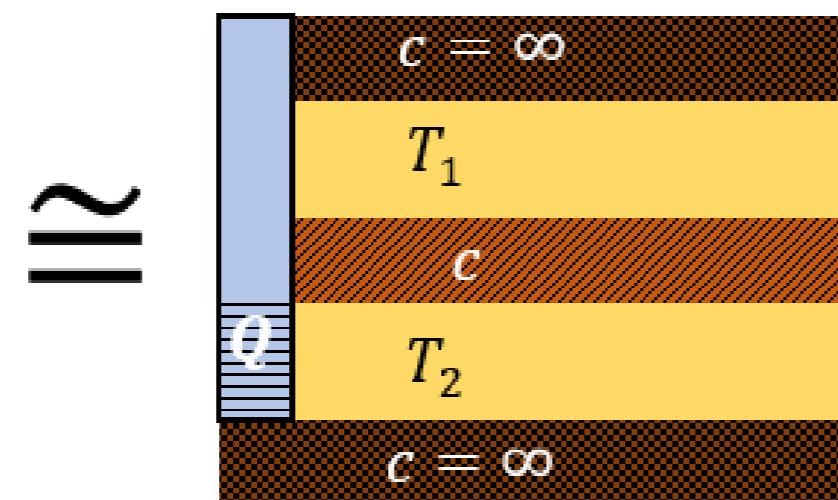
LARGE TIME APPROXIMATION

for confined multilayer well-flow:

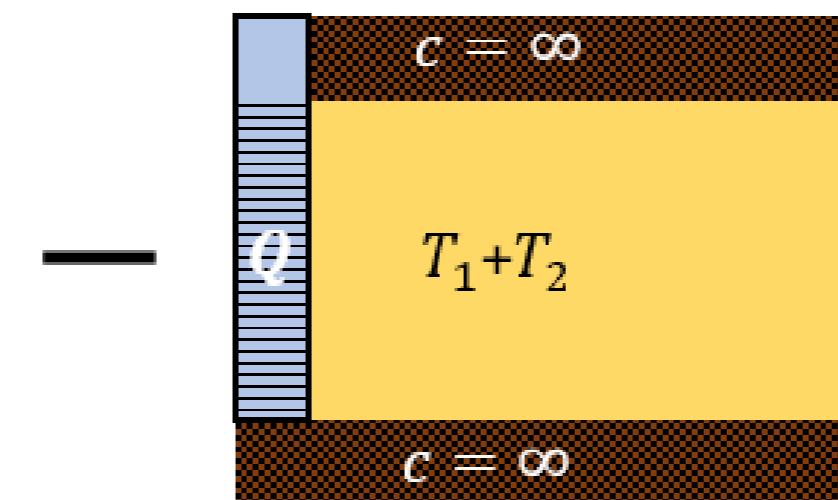
$$s(i, r, t) \sim s_{steady}(i, r) - s_{thiem}(r) + s_{theis}(r, t) \quad (t \rightarrow \infty)$$



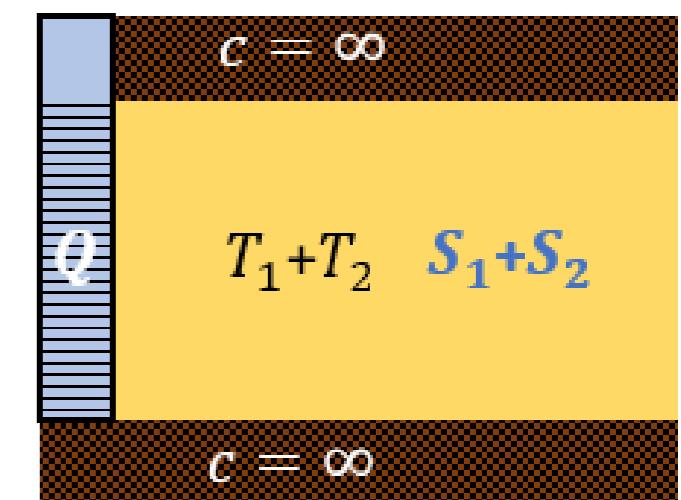
transient-state solution



steady-state solution



Thiem solution



Theis solution

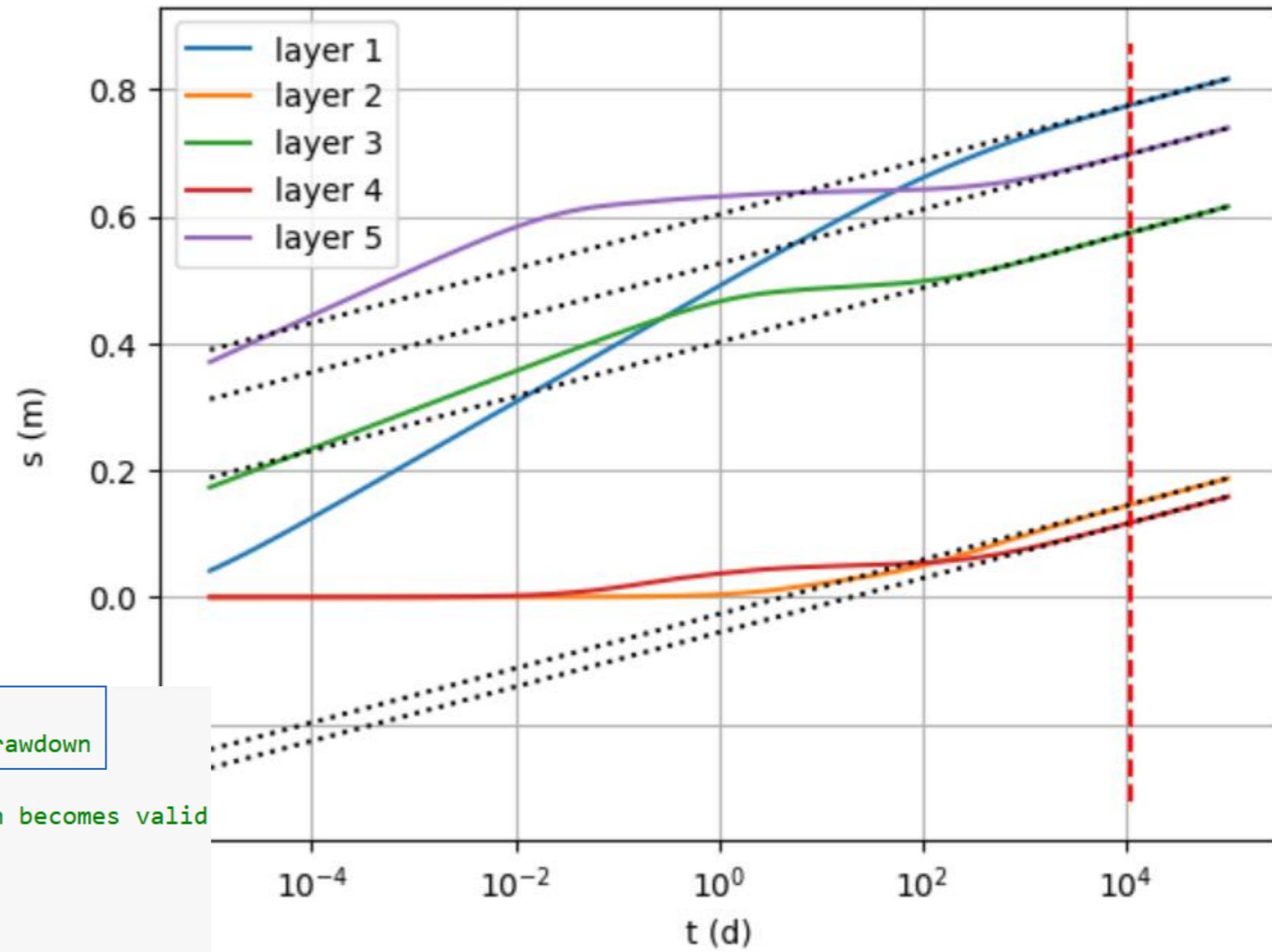
EXAMPLE

$$s_{trans} \sim s_{steady} - s_{thiem} + s_{theis}$$

```
plt.semilogx(t, s_trans); # exact drawdown
plt.semilogx(t, s_steady - s_thiem + s_theis, 'k:'); # approximate drawdown

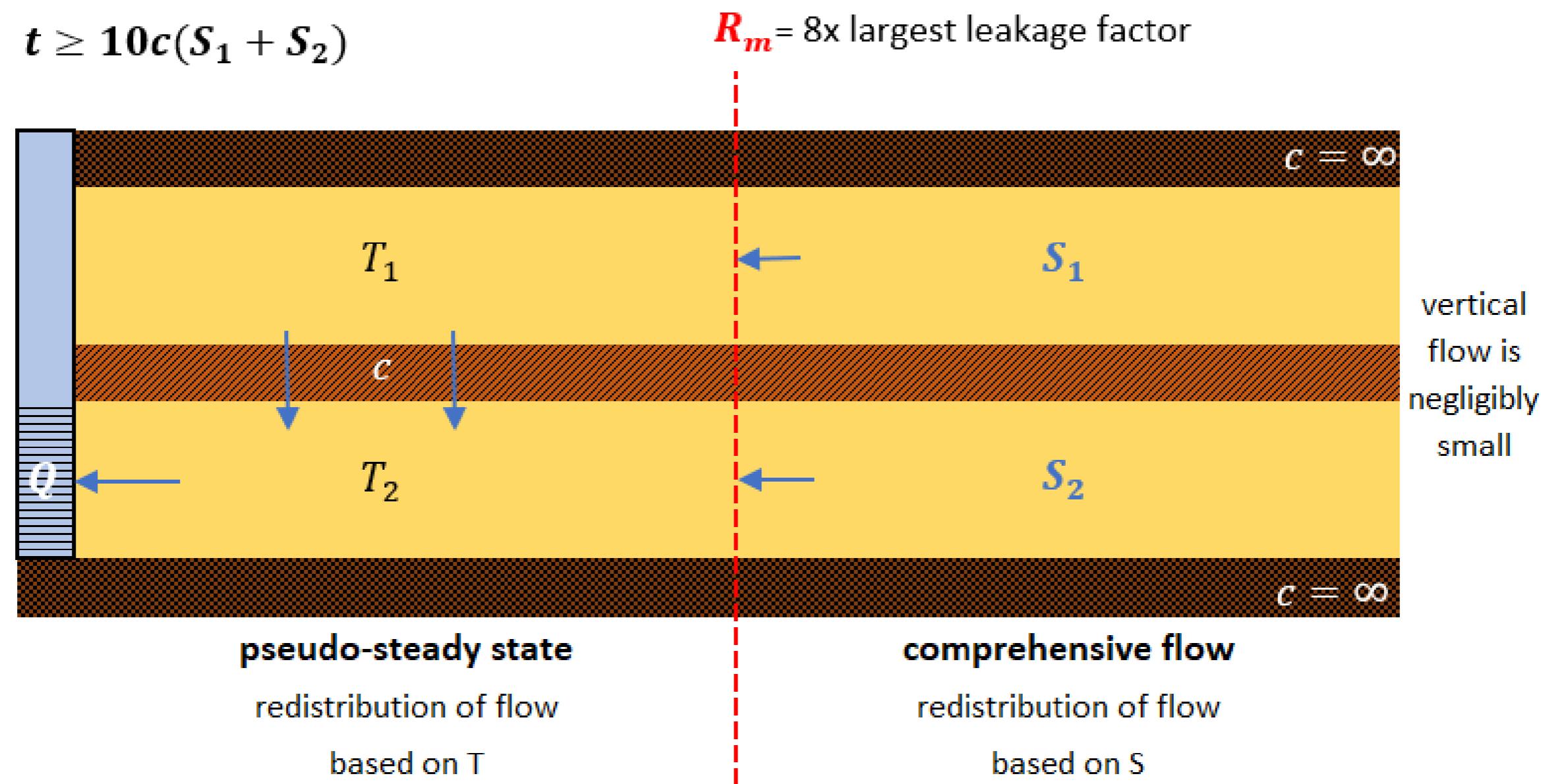
t_approx = 10 * S.sum() * c.sum() # time from which the approximation becomes valid
plt.semilogx([t_approx, t_approx], plt.ylim(), 'r--');

plt.legend([f'layer {i+1}' for i in range(transient.nl)]);
plt.grid();
plt.xlabel('t (d)');
plt.ylabel('s (m)');
```

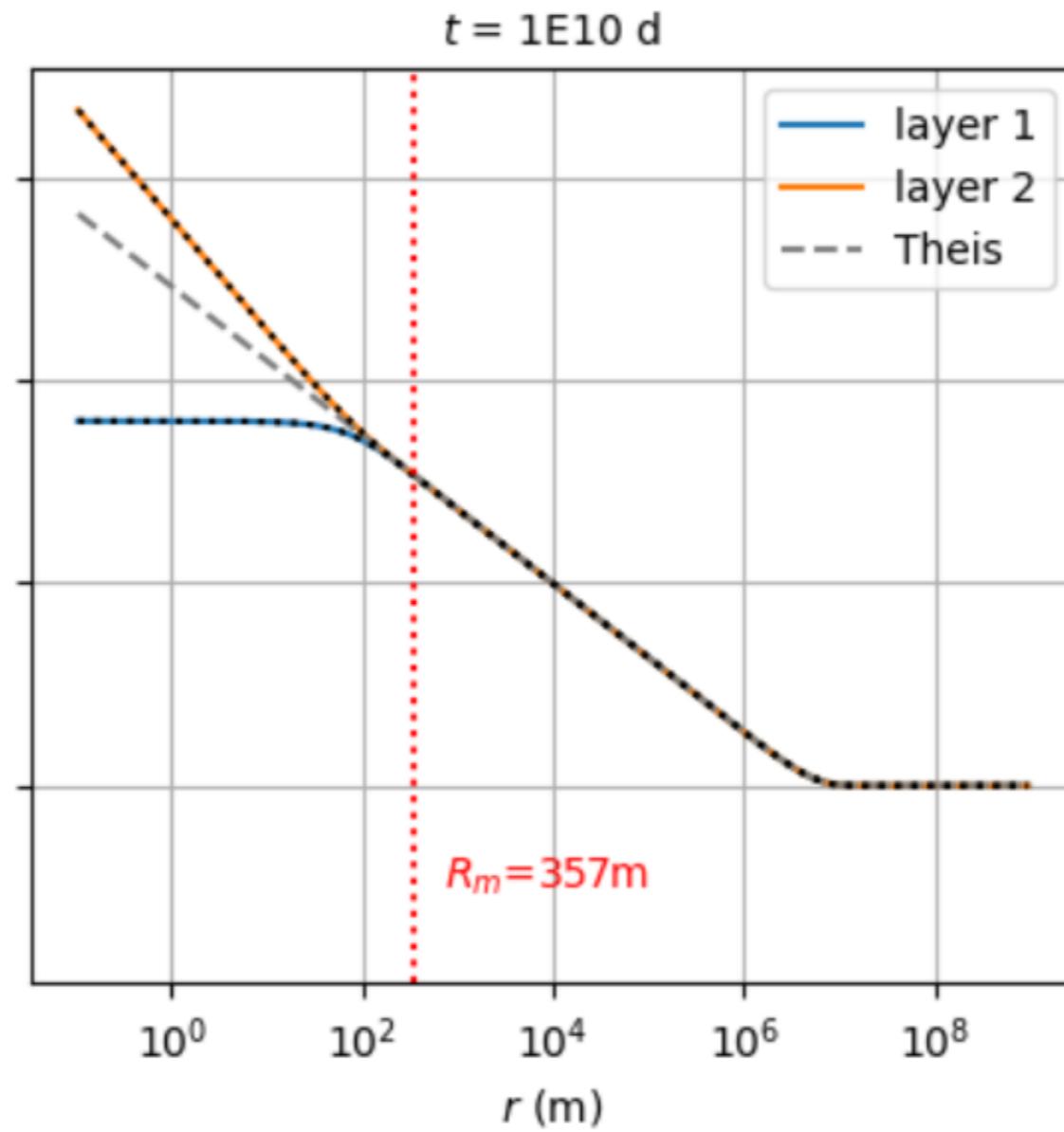
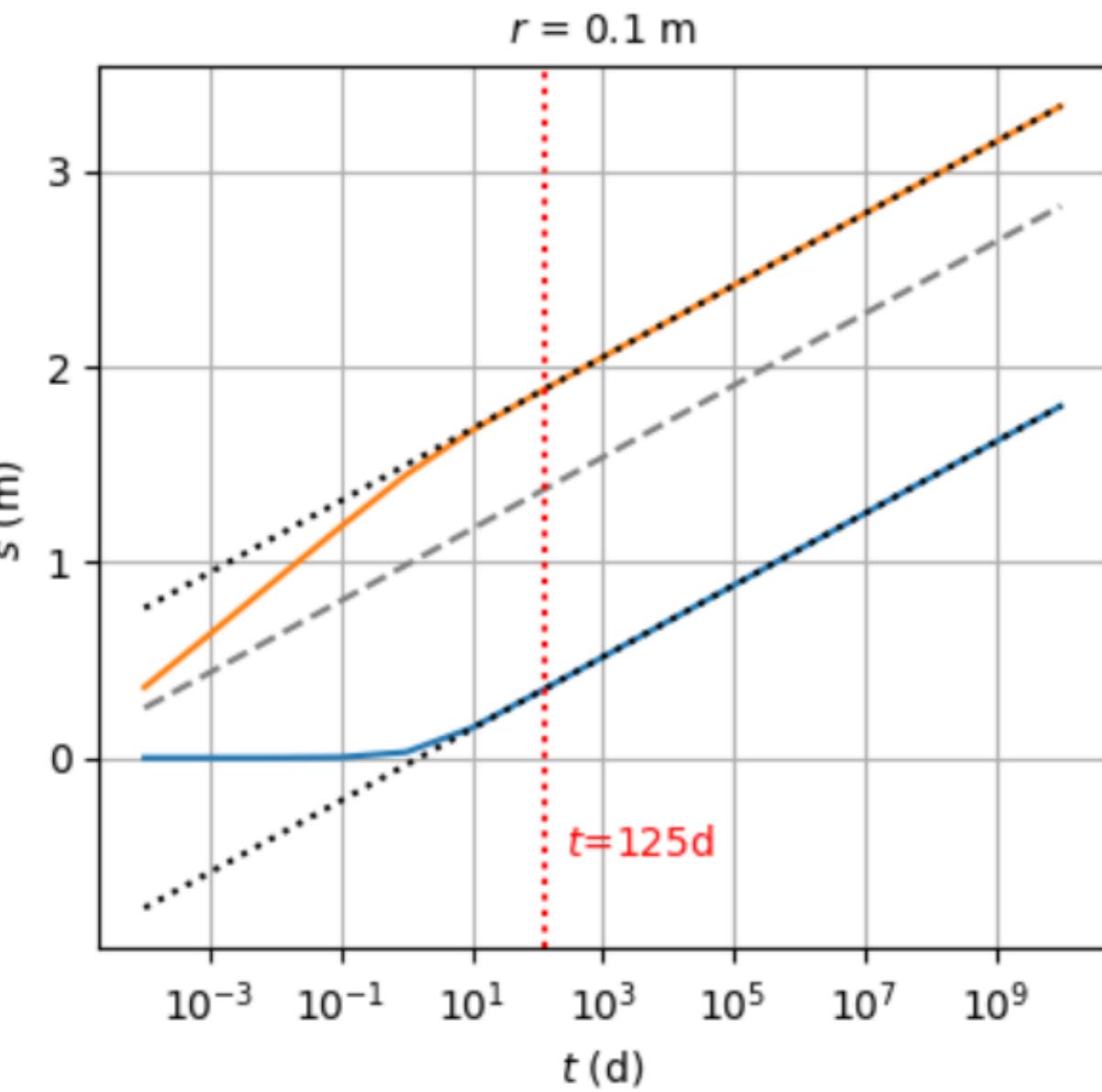


PSEUDO-STEADY STATE

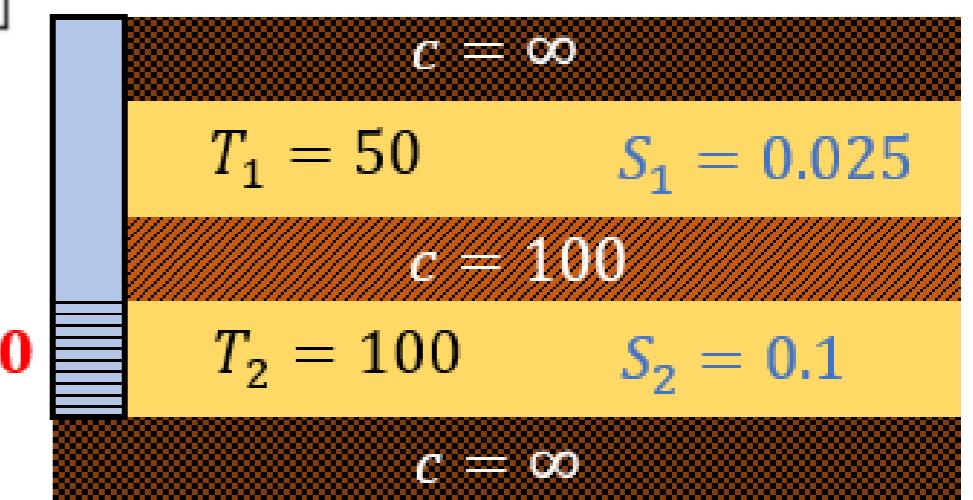
also called *steady shape* (Bohling et al., 2002)



EXAMPLE

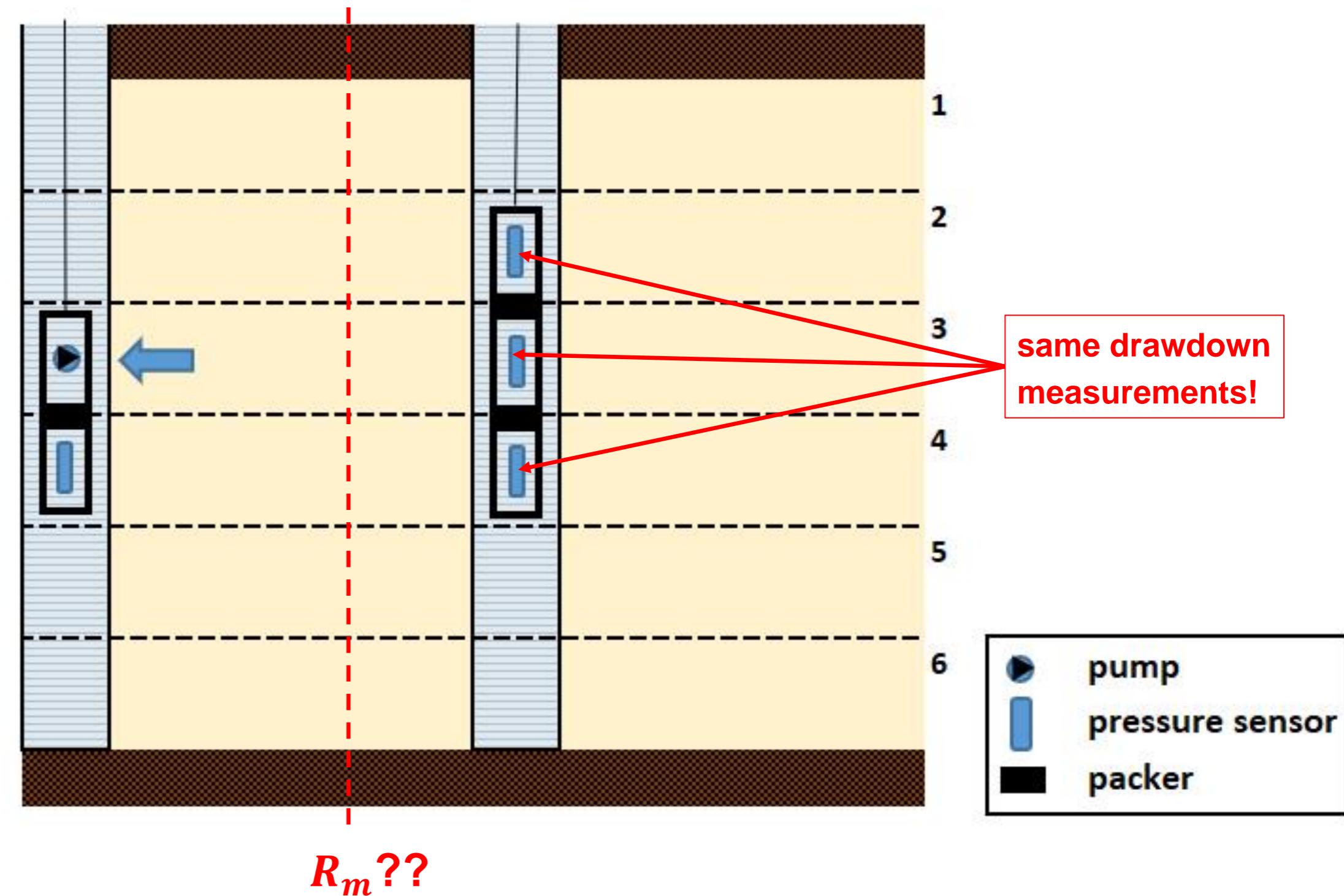
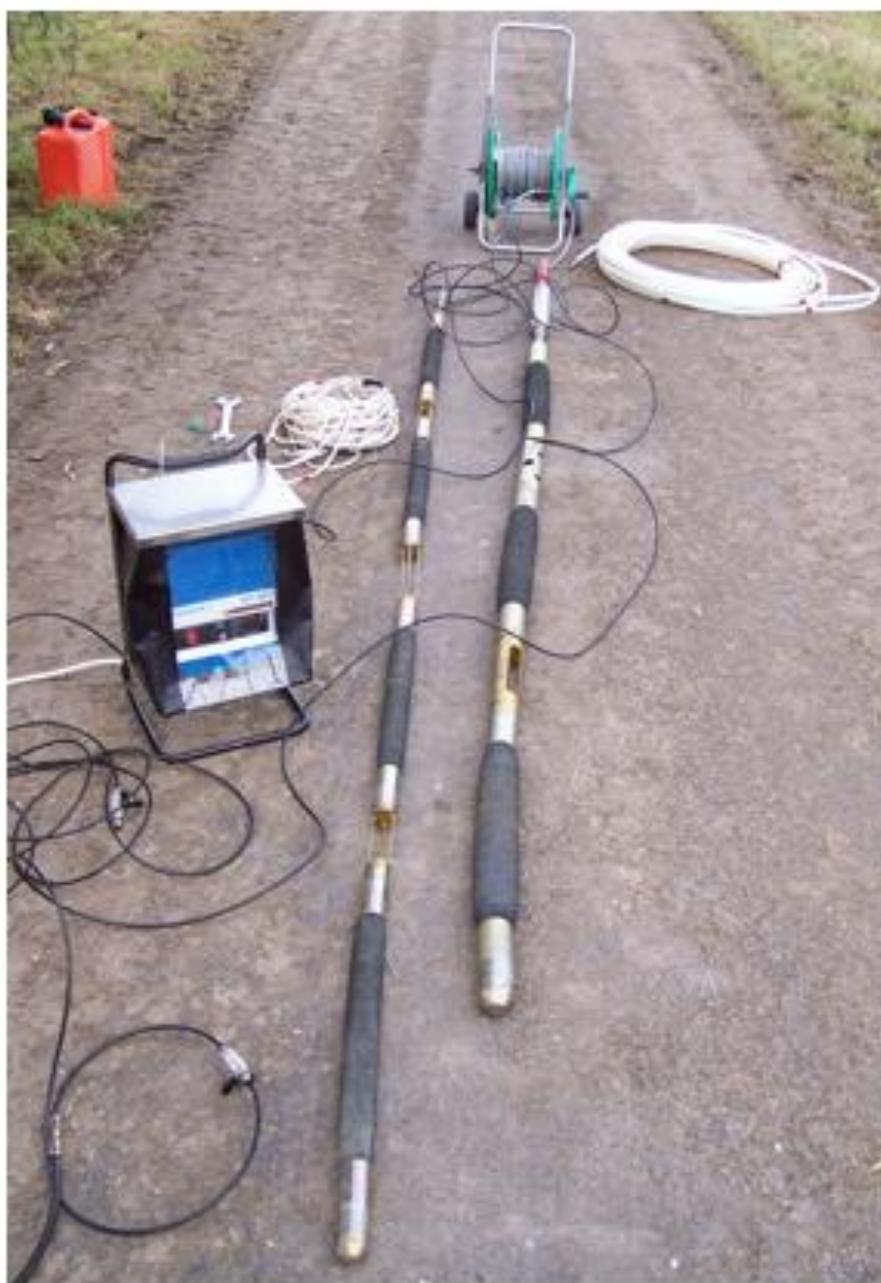


- solid lines: exact
- dotted lines: approximate



$$Q = 150$$

HYDRAULIC TOMOGRAPHY



= sequence of small-scale short-term pumping tests

CONCLUSIONS

- Theis curves (again) at large values of time
- inversion of flow in the distal zone possible
- spatial averaging in drawdown measurements
- multilevel pumping tests have inherent limitations



Inherent Limitations of Hydraulic Tomography

Geoffrey C. Bohling ✉, James J. Butler Jr.

First published: 22 September 2010 | <https://doi.org/10.1111/j.1745-6584.2010.00757.x> | Citations: 63

A PRACTICAL CASE

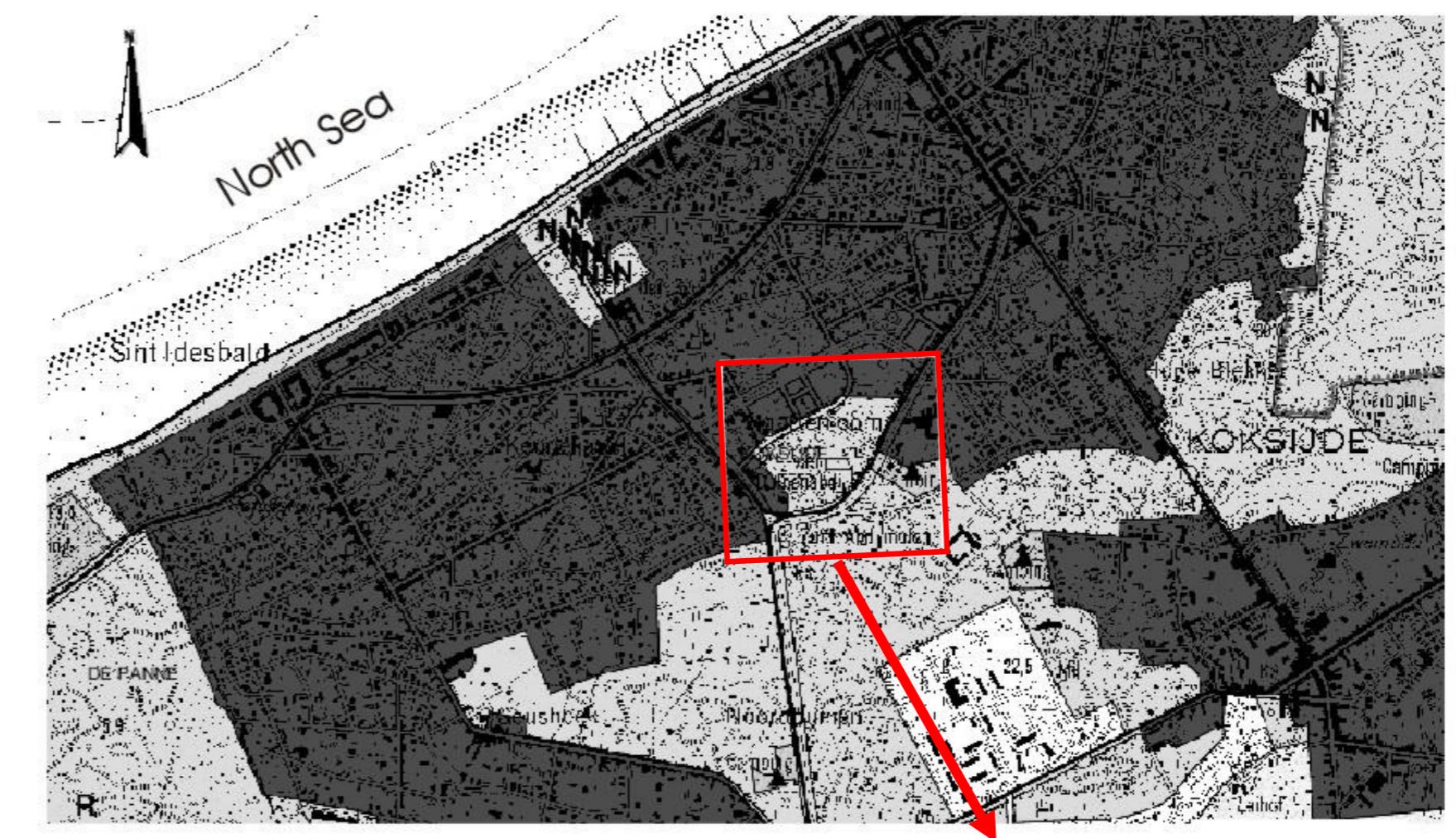
STUDY

OPTIMIZING A DRAINAGE SYSTEM

Excavation site “Duinenabdij”

(Koksijde, Belgium)

- Valuable dune area in the Belgian coastal plane
- Multilayer aquifer system
- Shallow semi-pervious layer caused flooding during the winter

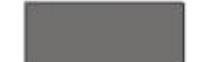


LEGEND

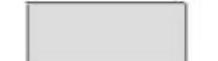
military area



urbanised area

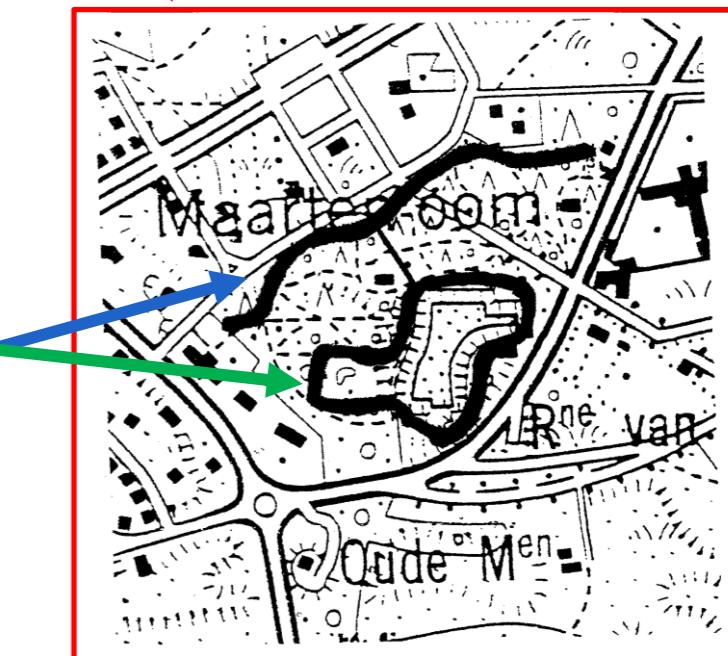


preserved dune area



0 300m

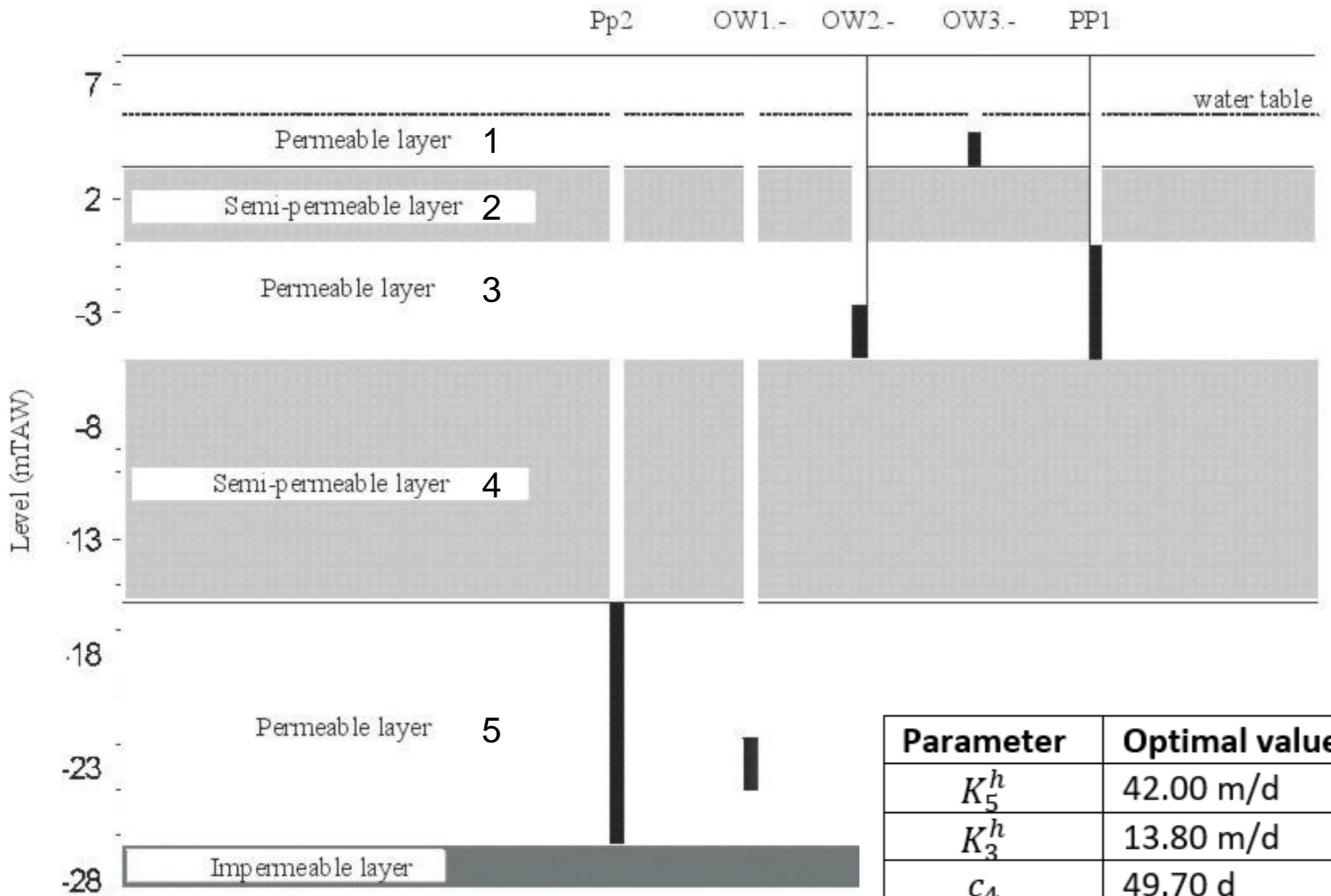
- Combined system of pumping and injection wells
 - Pumping to drain excess groundwater
 - Re-injecting the extracted groundwater
 - Re-injecting to protect the dunes in the north



KOKSIJDE



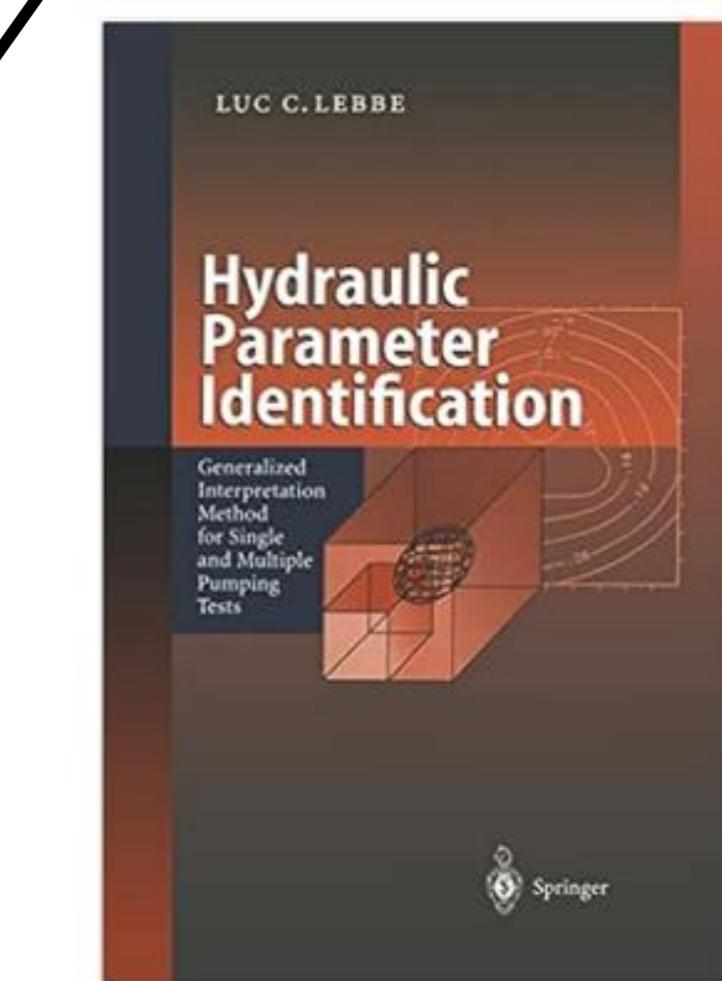
THE MULTILAYER AQUIFER SYSTEM



Parameter	Optimal value
K_5^h	42.00 m/d
K_3^h	13.80 m/d
c_4	49.70 d
S_5^s	$7.12 \times 10^{-5} \text{ m}^{-1}$
$S_2^s = S_3^s$	$7.80 \times 10^{-5} \text{ m}^{-1}$
S_4^s	$2.09 \times 10^{-5} \text{ m}^{-1}$
c_2	735.00 d

Double pumping test:

- Pumping test on layer 3
- Pumping test on layer 5
- Observations in all layers
- Simultaneous interpretation



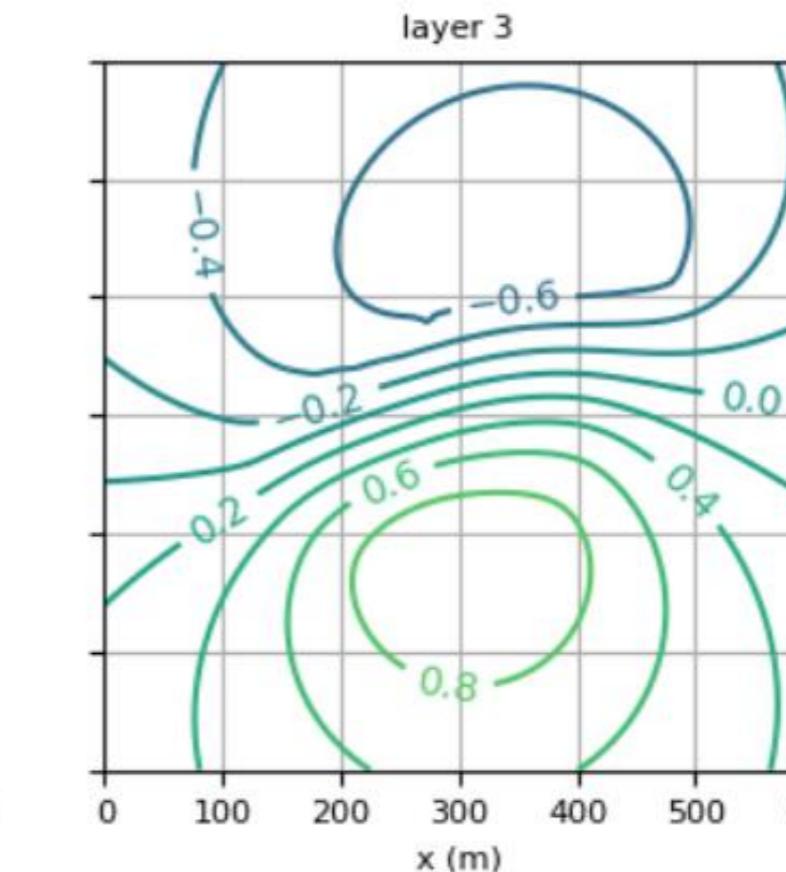
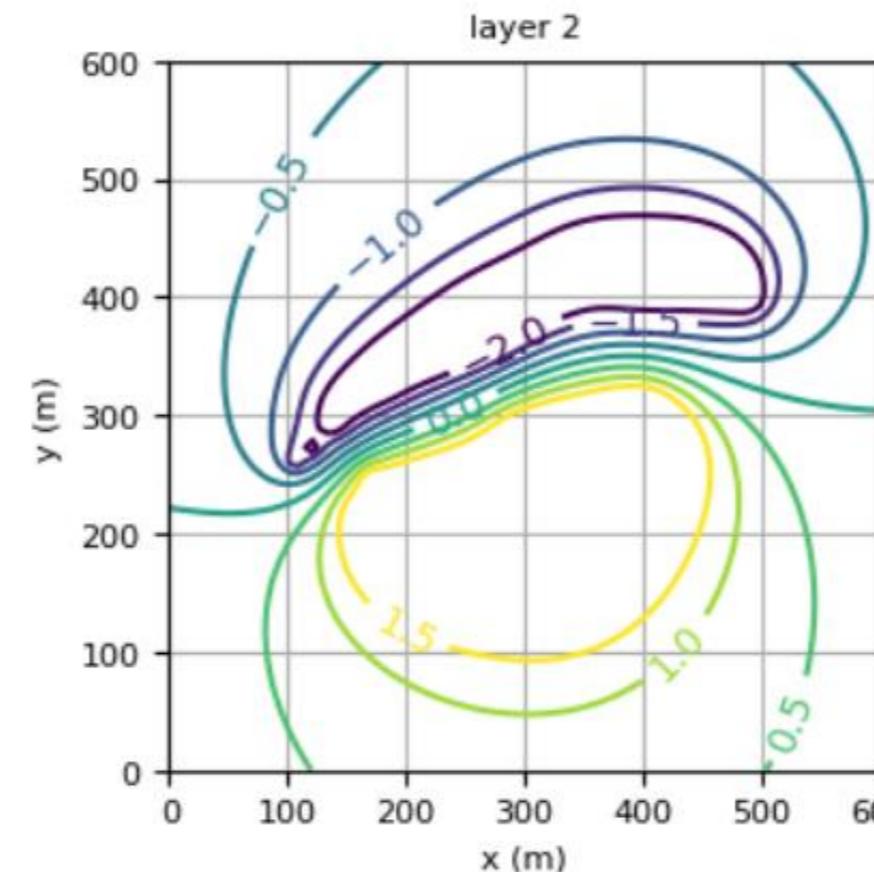
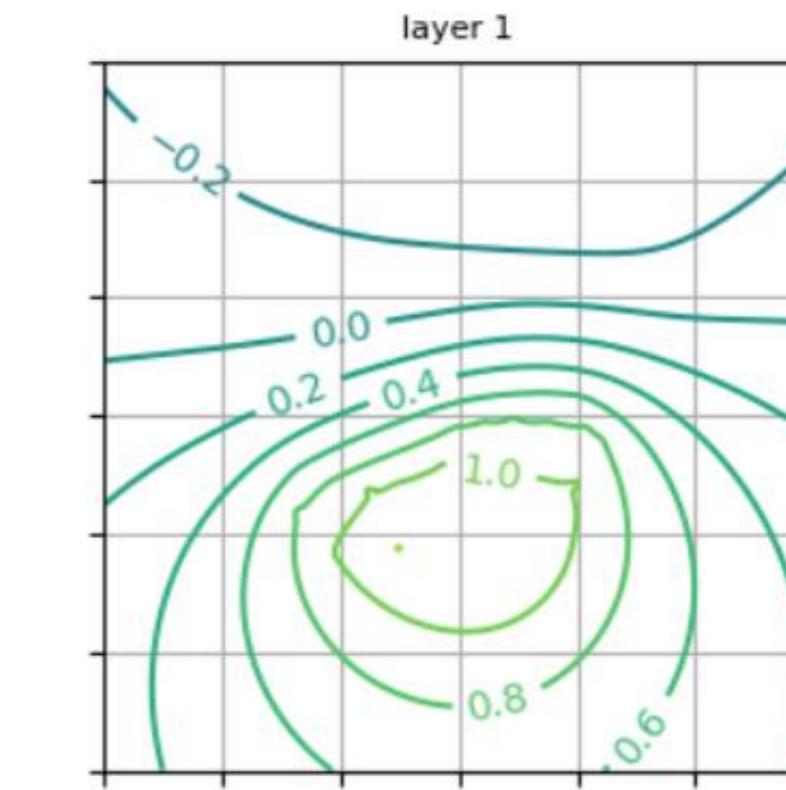
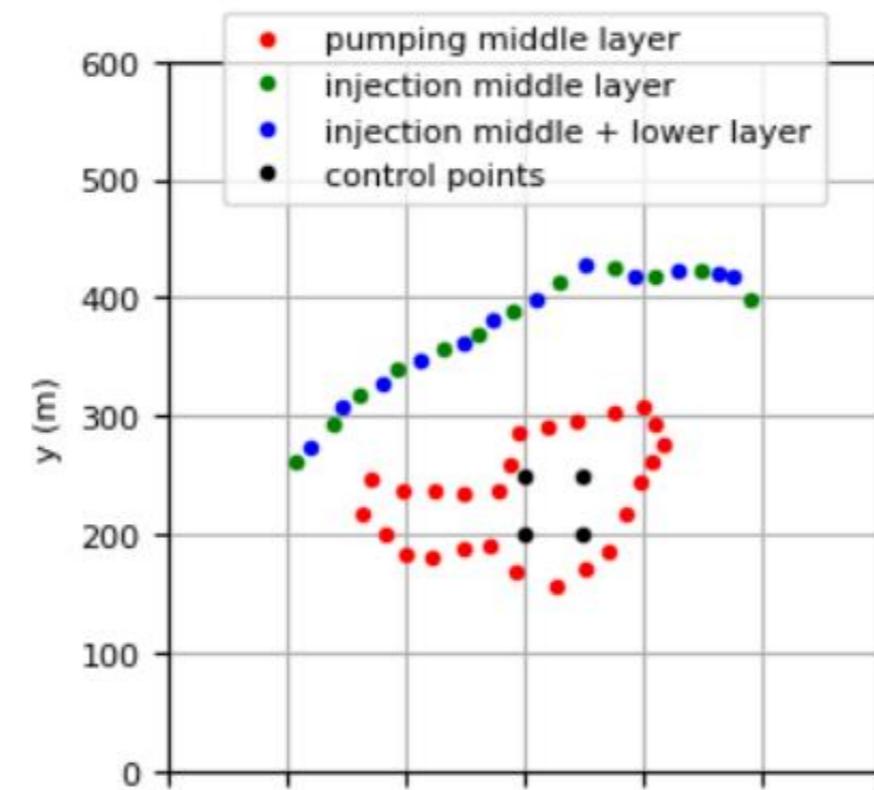
SIMULATING THE DRAINAGE SYSTEM



- Analytical solution:
 - steady-state
 - axisymmetric flow
 - confined system
 - 3 homogeneous layers
 - 2 separating resistances
- Superposition in space

$$s_i(x, y, t) = \sum_{p=1}^{n_w} \frac{Q_p}{Q_{ref}} s_i(r_p, t)$$

with $r_p = \sqrt{(x_p - x)^2 + (y_p - y)^2}$





= Minimizing the total pumping rate:

- Linear programming
- Python package PuLP

```
!pip install pulp  
from pulp import *
```

- Constraints:
 - 4 control points
 - drawdown min 1m

$$s \geq 1$$

```
# instantiating LpProblem object (linear programming problem)  
prob = LpProblem("Scenario_1", LpMinimize) # it's a minimization problem  
  
# defining the variables  
Q_pump1 = LpVariable("Q_pump", lowBound=0) # pumping rate middle layer (Q > 0)  
Q_inj1 = LpVariable("Q_inj", upBound=0) # injection rate middle layer (Q < 0)  
  
# defining the objective function  
prob += Q_pump1, "minimize total pumping rate"  
  
# adding constraint Q_out == Q_in  
prob += npw * (Q_grav + Q_pump1) + niw * Q_inj1 + niw / 2 * Q_deep == 0  
  
# add constraint at the 4 control points  
for i in range(len(xc)):  
    # drawdown in top layer must be at least 1 m: P1*s1 + P2*s2 + P3*s3 + P4*s4 >= 1  
    prob += Q_grav * s1[i] + Q_pump1 * s2[i] + Q_inj1 * s3[i] + Q_deep * s4[i] >= 1.0  
  
# solving the problem  
print(prob.solve())  
print(LpStatus[prob.status]) # checking the status of the solution  
print(Q_pump1.value(), Q_inj1.value(), Q_deep) # checking the optimized variables
```

CONCLUSIONS

- The **combined system of pumping and injection wells** is effective in creating local drawdown and protecting the surrounding dunes
- A **hydrogeological study including field tests** was necessary to reliably characterize the hydraulic properties of the aquifer system
- The **analytical multilayer solution** can be applied to efficiently solve a real-world problem without having to build a computationally expensive model
- **Linear programming** is an effective way of minimizing pumping rates subject to given drawdown constraints
- It is straightforward to implement both the analytical solution and the linear programming using **Python's packages for scientific computing**.

REFERENCES

- Alley, W.M., Reilly, T.E., & Franke, O.L., 1999. Sustainability of Ground-Water Resources. U.S. Geological Survey Circular 1186.
- Bakker, M., & Strack, O. D. L., 2003. Analytic elements for multiaquifer flow. *Journal of Hydrology*, 271(1–4).
- Bohling, G. C., & Butler, J. J. (2010). Inherent Limitations of Hydraulic Tomography. *Ground Water*, 48(6), 809–824.
- Bohling, G. C., Zhan, X., Butler, J. J., & Zheng, L. (2002). Steady shape analysis of tomographic pumping tests for characterization of aquifer heterogeneities. *Water Resources Research*, 38(12), 60_1-60_15.
- Bredehoeft, J. D., Papadopoulos, S. S., & Cooper, H. H., 1982. The water budget myth. In *Scientific Basis of Water Resources management Studies in Geophysics* (pp. 51–57). National Academy Press.
- Bredehoeft, J. D., 2002. The Water Budget Myth Revisited: Why Hydrogeologists Model. *Ground Water*, 40(4).
- Cooper, H. H., Bredehoeft, J. D., & Papadopoulos, I. S., 1967. Response of a finite-diameter well to an instantaneous charge of water. *Water Resources Research*, 3(1), 263–269.
- Butler, J. J., 1988. Pumping tests in nonuniform aquifers — The radially symmetric case. *Journal of Hydrology*, 101(1–4), 15–30.
- Cooper, H. H., & Jacob, C. E., 1946. A generalized graphical method for evaluating formation constants and summarizing well-field history. *Transactions, American Geophysical Union*, 27(4), 526–534.
- de Glee, G. J., 1930. Over grondwaterstroomingen bij wateronttrekking door middel van putten (in Dutch). PhD thesis, Technische Hoogeschool Delft, Drukkerij J. Waltman. Jr., Delft.
- de Hoog, F. R., Knight, J. H., & Stokes, A. N., 1982. An Improved Method for Numerical Inversion of Laplace Transforms. *SIAM Journal on Scientific and Statistical Computing*, 3(3), 357–366.
- Dupuit, J., 1857. Mouvement de l'eau à travers les terrains perméables (in French). *Comptes Rendus de l'Académie Des Sciences*, 45, 92–96.

REFERENCES

- Dupuit, J., 1863. Etude Théoriques et Pratiques Sur le Mouvement Des Eaux Dans Les Canaux Découverts et à Travers Les Terrains Perméables (in French). Paris: Dunod.
- Ernst, L. F., 1971. Analysis of groundwater flow to deep wells in areas with a non-linear function for the subsurface drainage. *Journal of Hydrology*, 14(2).
- Gaver, D. P., 1966. Observing Stochastic Processes, and Approximate Transform Inversion. *Operations Research*, 14(3), 444–459.
- Haitjema, H., 2006. The Role of Hand Calculations in Ground Water Flow Modeling. *Ground Water*, 44(6).
- Hantush, M. S., 1964. Hydraulics of wells. In V. T. Chow (Ed.), *Advances in Hydroscience* (Vol. I, pp. 281–432). New York and London: Academic Press.
- Hantush, M. S., & Jacob, C. E., 1955. Non-steady radial flow in an infinite leaky aquifer. *Transactions, American Geophysical Union*, 36(1), 95–100.
- Hemker, C. J., 1984. Steady groundwater flow in leaky multiple-aquifer systems. *Journal of Hydrology*, 72(3–4), 355–374.
- Hemker, C. J., 1985. Transient well flow in leaky multiple-aquifer systems. *Journal of Hydrology*, 81(1–2), 111–126.
- Hemker, C. J., 1999. Transient well flow in vertically heterogeneous aquifers. *Journal of Hydrology*, 225(1–2), 1–18.
- Hemker, C. J., 1999. Transient well flow in layered aquifer systems: the uniform well-face drawdown solution. *Journal of Hydrology*, 225(1–2), 19–44.
- Hemker, C. J., 2000. Groundwater flow in layered aquifer systems. PhD thesis, VU, Amsterdam.
- Hyder, Z., Butler, J. J., McElwee, C. D., & Liu, W. (1994). Slug tests in partially penetrating wells. *Water Resources Research*, 30(11), 2945–2957.
- Jacob, C. E., 1946. Radial flow in a leaky artesian aquifer. *Transactions, American Geophysical Union*, 27(2).
- Jiang Hui, Zeng Buo, Pan Hongyu, 2009. *Groundwater Dynamics*. Beijing, Geological Publishing House.
- Kooper, J., 1914. Beweging van het water in den bodem bij onttrekking door bronnen (in Dutch). *Ingenieur*, 29, 697–716.
- Kresic, N., 1997. Quantitative solutions in hydrogeology and groundwater modeling. Lewis publisher, New York.
- Kyrieleis, W., & Sichardt, W., 1930. *Grundwasserabsenkung bei Fundierungsarbeiten* (in German). Berlin: Springer.

REFERENCES

- Langevin, C. D., 2008. Modeling Axisymmetric Flow and Transport. *Ground Water*, 46(4), 579–590.
- Lebbe, L. C., 1983. Een matematisch model van de niet-permanente grondwaterstroming naar een pompput in een veellagig grondwaterreservoir en enkele beschouwingen over de stroomtijd (in Dutch). *Tijdschrift BECEWA*, 70, 35–48.
- Lebbe, L. C., 1988. Uitvoering van pompproeven en interpretatie door middel van een invers model (in Dutch). Ghent University, Ghent, Belgium.
- Lebbe, L. C., 1999. Hydraulic Parameter Identification. Generalized Interpretation Method for Single and Multiple Pumping Tests. Berlin Heidelberg: Springer-Verlag.
- Louwyck, A., 2011. MAxSym - A MATLAB Tool to Simulate Two-Dimensional Axi-Symmetric Groundwater Flow. Research Unit Groundwater Modeling, Ghent University. <https://github.com/alouwyck/MAxSym>
- Louwyck, A., 2015. Module Projectwerk programmeren B5 - Ontwikkeling MAxSymMer (in Dutch). Associate Degree Programming, CVO Lethas, Brussels.
- Louwyck, A., 2023. Axisymmetric Flow in Multilayer Aquifer Systems: Solutions and Theoretical Considerations. PhD thesis, Laboratory for Applied Geology and Hydrogeology, Department of Geology, Ghent University, Belgium.
- Louwyck, A., Vandenbohede, A., & Lebbe, L. C., 2005. The role of hydrogeological research in the realization of a combined pumping and deep infiltration system at the excavation “Duinenabdij.” In J.-L. Herrier, J. Mees, A. Salman, J. Seys, H. Van Nieuwenhuyse, & I. Dobbelaere (Eds.), Proceedings “Dunes and Estuaries 2005”: International Conference on nature restoration practices in European coastal habitats, Koksijde, Belgium 19-23 September 2005. (pp. 317–326). VLIZ Special Publication 19.
- Louwyck, A., Vandenbohede, A., & Lebbe, L. C., 2007. OGMA-RF: a user-friendly, modular program package to simulate and analyse radial flow. ModelCARE 2007, Sixth International Conference on Calibration and Reliability in Groundwater Modelling: Credibility in Modelling, Copenhagen, Denmark, 9-13 September 2007, 88–93.
- Louwyck, A., Vandenbohede, A., & Lebbe, L. C., 2010. Numerical analysis of step-drawdown tests: Parameter identification and uncertainty. *Journal of Hydrology*, 380(1–2), 165–179.

REFERENCES

- Louwyck, A., Vandenbohede, A., Bakker, M., & Lebbe, L. C., 2012. Simulation of axi-symmetric flow towards wells: A finite-difference approach. *Computers & Geosciences*, 44, 136–145.
- Louwyck, A., Vandenbohede, A., Bakker, M., & Lebbe, L. C., 2014. MODFLOW procedure to simulate axisymmetric flow in radially heterogeneous and layered aquifer systems. *Hydrogeology Journal*, 22(5), 1217–1226.
- Louwyck, A., Vandenbohede, A., Libbrecht, D., Van Camp, M., & Walraevens, K., 2022. The Radius of Influence Myth. *Water*, 14(2), 149.
- Louwyck, A., Vandenbohede, A., Heuvelmans, G., Van Camp, M., & Walraevens, K., 2023. The Water Budget Myth and Its Recharge Controversy: Linear vs. Nonlinear Models. *Groundwater*, 61(1), 100–110.
- Stehfest, H., 1970. Algorithm 368: Numerical inversion of Laplace transforms [D5]. *Communications of the ACM*, 13(1), 47–49.
- Theis, C. V., 1935. The relation between the lowering of the Piezometric surface and the rate and duration of discharge of a well using ground-water storage. *Transactions, American Geophysical Union*, 16(2), 519–524.
- Theis, C. V., 1940. The source of water derived from wells: essential factors controlling the response of an aquifer to development. *Civil Engineering*, 10, 277–280.
- Theis, C. V., 1941. The effect of a well on the flow of a nearby stream. *Transactions, American Geophysical Union*, 22(3), 734.
- Thiem, A., 1870. Die Ergiebigkeit artesischer Bohrlöcher, Schachtbrunnen und Filtergalerien (in German). *Journal Für Gasbeleuchtung Und Wasserversorgung*, 13, 450–467.
- Thiem, G., 1906. Hydrologische Methoden (in German). Leipzig: Gebhardt.
- Vandenbohede, A., Louwyck, A., & Lebbe, L. C., 2008. Identification and reliability of microbial aerobic respiration and denitrification kinetics using a single-well push–pull field test. *Journal of Contaminant Hydrology*, 95(1–2), 42–56.
- Vandenbohede, A., Louwyck, A., & Lebbe, L. C., 2009. Conservative Solute Versus Heat Transport in Porous Media During Push-pull Tests. *Transport in Porous Media*, 76(2), 265–287.
- Veling, E. J. M., & Maas, C., 2010. Hantush Well Function revisited. *Journal of Hydrology*, 393(3–4).

IMAGE SOURCES

- <https://www.azquotes.com/quote/531521>
- <https://www.eurocanals.com/Waterways/belgium.html>
- <https://www.weflycheap.be/blog/wat-te-doen-in-gent/>
- <https://www.sigmaplan.be/nl/nieuws/sigmaplan-maakt-omgeving-tussen-gent-en-wetteren-veiliger-natuurlijker-en-aangenamer>
- <https://www.ecopedia.be/ecohydrologie/hydrogeologie>
- <https://nl.wikipedia.org/wiki/Cilinderco%C3%B6rdinaten>
- <https://youtu.be/e6kf6DDQVYA?si=RKZVb50wTVVBmokS>
- <https://library.seq.org/doi/10.1190/geo2000-0001.1>
- https://en.wikipedia.org/wiki/Jules_Dupuit
- <https://gallica.bnf.fr/ark:/12148/bpt6k62096061.image>
- <https://hess.copernicus.org/articles/26/4055/2022/>
- <https://pubs.usgs.gov/circ/circ1186/pdf/circ1186.pdf>
- <https://hatariwater.tumblr.com/post/138690184404/overview-of-the-radius-of-influence>
- <https://www.amazon.de/-/en/Wilhelm-Kyrieleis/dp/3662235927>
- https://nl.wikipedia.org/wiki/Johan_Kooper
- <http://www.aqtesolv.com/cooper-jacob.htm>
- https://en.wikipedia.org/wiki/Bessel_function
- <https://timecapsule.iah.org/person/charles-vernon-theis/>
- <https://www.nmt.edu/research/deju/history/hantush.php>
- <https://honors.agu.org/winners/john-d-bredehoeft/>
- <https://link.springer.com/book/10.1007/978-3-031-13516-3>
- <https://www.natuurpunt.be/nieuws/samenwerken-aan-meer-ruimte-voor-natuur-%C3%A9n-water-het-viersels-gebroekt-20230208>
- <https://www.istockphoto.com/nl/foto/luchtfoto-van-het-typisch-nederlandse-landschap-met-grachten-polder-water-groene-gm1181771327-331581491>
- <https://peakvisor.com/adm/utah.html>
- <https://www.aquaveo.com/software/gms-groundwater-modeling-system-introduction>
- <http://www.aqtesolv.com/cooper-bredehoeft-papadopoulos.htm>
- <http://www.aqtesolv.com/butlernu.htm>
- <https://www.hydrology.nl/phd-theses.html>
- <https://www.linkedin.com/in/kick-hemker-20023316/?originalSubdomain=nl>
- <https://www.tudelft.nl/staff/mark.bakker/?cHash=5a0aa271d46cc02467568433bf0632d6>
- <https://link.springer.com/book/10.1007/978-3-642-60117-0>
- <https://www.researchgate.net/profile/Luc-Lebbe>
- <https://weisengineering.com/borehole-pump-testing-in-uganda/>
- https://link.springer.com/chapter/10.1007/978-3-319-32137-0_8
- <https://www.koksijde.be/nl/abdijmuseum-ten-duinen-koksijde>
- <https://www.tenduinen.be/en/node/435>
- <https://www.wereldreis.net/landeninfo/europa/belgie/wat-te-doen-in-koksijde/>
- <https://www.routezoeker.com/wandelroutes/duinenwandeling-koksijde-oostduinkerke>
- <https://www.dekust.be/wandelen/koksijde-oostduinkerke>

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