Understanding the Levitron Through the Analysis of a Symmetric Spinning Top in a Gravitational Field

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The spinning top's ability to sustain an implausible balance has perplexed physicists for some time. It is a phenomenon that reaches beyond the scope of classical mechanics to the realms of quantum atomic traps, helicopter stability, and childish wonder. With the tools learned in 8.223, the use of abstract coordinates, Lagrangian analysis, integrals of motion, stable points found through minimizing the potential, and the bounds of stable oscillation we are able to easily study this system, and obtain the criteria necessary to attain a stable motion.

We begin by modeling the spinning top as a massless rod of length l, attached at the top to a wheel of mass m and radius R. This model is plausible due to the symmetry of the top. We will assume the top's pivot point, at the bottom of its axis of symmetry, to be fixed to the ground, at the origin of our reference frame. This constraint reduces the body's degrees of freedom to three coordinates (Fig.1), when using the Euler angle coordinate system of (ϕ, θ, ψ). Thus the system can be completely described with these coordinates.

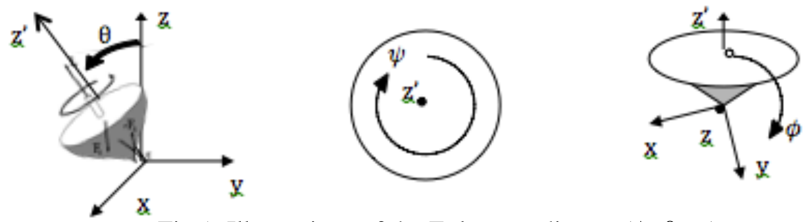

Fig 1. Illustrations of the Euler coordinates (ϕ, θ, ψ)

The Euler angle coordinate system is frequently used to represent the position and orientation of rigid bodies about a fixed origin. The notation used in this report will follow Landau's convention: where ψ represents the spin of the top about its own axis, z' (an axis passing from the contact point with the ground, to the upper peak of the spinning top), θ represents the tilt of the top with respect to the z-axis, and ϕ represents the rotation of the top's center of mass around the z-axis, precession. The angular frequencies of our top, (ω_x, ω_y, $\omega_{z'}$), are obtained from the Euler Coordinate Rotation Matrix. They represent the rotation of our spinning top around the x, y, and z'-axes respectively.

To study the system, we use a Lagrangian of kinetic and potential energies ($L = T + U$)

$$L = \frac{1}{2} I_s(\omega_{z'})^2 + \frac{1}{2} I_p\left(\sqrt{{\omega_x}^2 + {\omega_y}^2}\right)^2 = \frac{1}{2} I_s\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + \frac{1}{2} I_p\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mgl\cos\theta$$

Where the first term is the kinetic energy of rotation of the top about its own axis, the second term is the kinetic energy of the top's rotation about the x and y-axes, and the final term is the gravitational potential. Since the Lagrangian does not explicitly depend on the Euler angles ψ and ϕ, this eludes to the conservation of momenta in those coordinates over time: $\partial L/\partial\dot{\psi} = p_\psi$ and $\partial L/\partial\dot{\phi} = p_\phi$. This leaves the system with one crucial variable, θ.

A stable spin would occur at $\partial U/\partial\theta = mgl\dot{\theta}\sin\theta = 0$, which entails the top is either on its side on the ground ($\theta = \pi/2$), which is arbitrary, or $\dot{\theta} = 0$. With this information, (no nutation $\dot{\theta} = 0$), we will attempt to find explicit expressions for our coordinates at these stable points. We define the full energy of the system as ($E = T + U$), and can find an equation for $\dot{\phi}$ by solving $\partial L/\partial\dot{\psi} = p_\psi$ and $\partial L/\partial\dot{\phi} = p_\phi$, which will constitute our first equation of motion. We arrive at a final explicit equation for $\dot{\phi}$, after requiring $\partial E/\partial\theta = 0$. This is allowed since E is conserved during the motion, because it has no explicit dependence on time, and only relies on the initial conditions.

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_p\sin^2\theta} = \frac{p_\psi}{I_p\cos\theta}\left(1 \pm \sqrt{1 - \frac{4mgl\, I_p\cos\theta}{{p_\psi}^2}}\right)$$

It is clear that this equation results in two solutions for $\dot{\phi}$, which we will refer to as "fast" and "slow." In both cases we assume $p_\psi \gg \sqrt{4mgl\, I_p\, cos\theta}$, because the rotation of the wheel around *z'* is assumed to be large [1]. Now the expression for fast precession can be reduced to $\dot{\phi}_{fast} = p_\psi / I_s cos\theta$. And to find the slow precession we use the first order Taylor expansion of $\sqrt{1-x}$, which results in $\dot{\phi}_{slow} = mgh/p_\psi$.

We are now able to find an expression for the tilt of the top θ at equilibrium ($\dot{\phi} = \dot{\phi}_{slow}$) by using the expression for $\dot{\phi}_{slow}$ in the original expression for $\dot{\phi}$. One value of $|cos\theta|$ is neglected since it is greater than one, but by applying the first order Taylor approximation on the remaining root, we are left with the final expression $cos\theta_{stable}(\dot{\phi} = \dot{\phi}_{slow}) = p_\theta / p_\psi$.

Now that we have analyzed the system at its stable points, we will study how it evolves around those points. We will discover that the motion may be bounded between two maximum angles of tilt ($\theta_{min} < \theta_{stable} < \theta_{max}$), for which the system will oscillate between. Using our equation for total energy, we may define a potential function, call it the restoring potential in this case, and state it as follows:

$$U_{res}(\theta) = E - \frac{I_p}{2}\dot{\theta}^2 = \frac{p_\psi^2}{2I_s} + \frac{(p_\phi - p_\psi cos\theta)^2}{2I_p sin^2\theta}$$

By plotting $U_{res}(\theta)$, and keeping in mind that E is constant, it is easy to see that the spinning top will exhibit bounded oscillatory motion depending on θ and initial conditions (Fig 2.). We find the range of oscillation by equating E and $U_{res}(\theta)$, and solving for $cos\theta$.

We have now found all the stable points of the system, and the range for stable oscillations. All that is left is to state the equations of motion, which are directly derived from the Lagrangian, by implementing the Euler-Lagrange equation. This results in a system of first-order non-linear differential equations that can be solved numerically for $\theta(t)$, and from thereon, $\psi(t)$ and $\phi(t)$ are solved analytically with constants based on the initial conditions.

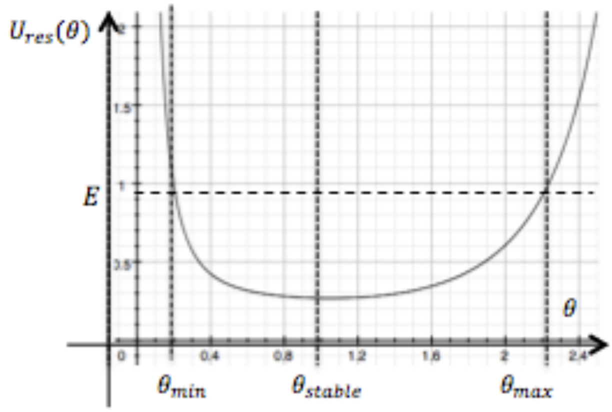

Fig 2. Restoring Potential

Based on Elliott's numerical simulations [2], there are three main motions for the apex of the top (three loci of the top's apex on a sphere of radius *l*), dependent on the sign of $\dot{\phi}$ at $\theta = \theta_{min}$ and $\theta = \theta_{max}$. If $\dot{\phi} > 0$ at $\theta = \theta_{min}$ and $\theta = \theta_{max}$, the motion will resemble that of Fig 2. (a) We attain this by giving the system a positive initial precession rate. This will cause continuous unidirectional motion. If $\dot{\phi} > 0$ at $\theta = \theta_{min}$ but $\dot{\phi} < 0$ at $\theta = \theta_{max}$, the motion will resemble that of Fig 2. (b) We attain this by giving a negative initial precession rate. This results in looping precession. If $\dot{\phi} > 0$ at $\theta = \theta_{min}$ but $\dot{\phi} = 0$ at $\theta = \theta_{max}$, the motion will resemble that of Fig 2. (c) Where given an angle θ_0, the top will begin to fall due to gravity, so the nutation angle increases. To keep p_ϕ constant, while the top is falling, the precession rate must change direction and increase with the direction of precession following the direction of ω_z.

With our new understanding of the parameters required to attain a stable spin with a mechanical top, we may expand our analysis to understand the stability of a Levitron. Since the tools to do so are beyond the scope of this paper, our discussion will be based on Berry's results, aiming to explain the analogy between both systems [3].

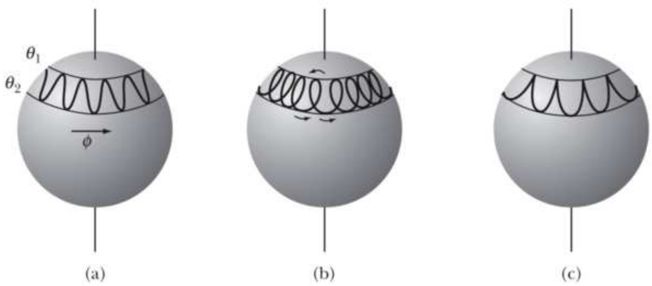

Fig 3. Types of Motion [2]

Although the Levitron presents a system similar to our mechanical top, an adiabatic condition must be satisfied in the Levitron for spin-stabilized levitation. The gyroscopic motion should prevent the top from toppling over like the case in our mechanical top, but the gyroscopic precession must be fast enough to continuously align the top's precession axis to $\hat{B}$, defined as the direction of the magnetic field as seen by the moving top, keeping the adiabatic invariant *approximately** conserved $(\vec{L} \cdot \hat{B})$.

Since both gravitational and magnetic fields are harmonic potentials, by the theorem of Earnshaw, critical points must be saddles. Therefore, for a minimum of potential to be possible, there must be small differences between $|\vec{B}|$ and B_z. As explained by Berry, this is because once the precession is much greater than $d(\hat{B})/dt$, the precession axis and dipole moment become slaved to $\hat{B}$ and this alignment allows for a minimum within given conditions. [3]

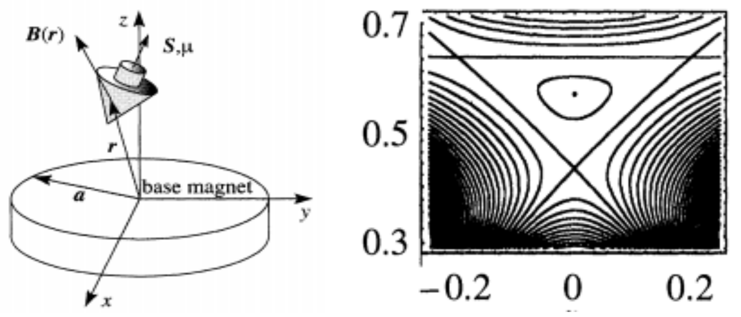

Fig 4. (a) Levitron commercial set-up (b) contour of axial section for scaled potential energy [3]

First of all, the gradient of the potential energy must equal zero for equilibrium. Horizontal equilibrium is guaranteed by the symmetry of this system; for vertical equilibrium, the gravitational force must be balanced by an upward magnetic force determined by the gradient of the magnitude of magnetic field. Second, $\partial^2 U/\partial z > 0$ must be satisfied for vertical stability. So, for a uniformly magnetized disc, the first derivative and second derivative with respect to z of magnetic potential are negative and positive, respectfully. This condition always holds for this commercial set-up of the Levitron, which allows for gravity to cancel out the magnetic repulsion by choosing mass accordingly. Also, a lower limit for the stable height of the Levitron is derived from this condition. Finally, $\partial^2 U/\partial x$ and $\partial^2 U/\partial y$ must be positive for horizontal stability and this is satisfied under an upper limit for the stable height of the Levitron. These conditions create a small range of mass within a small range of height for a stable minimum that is ultimately created from an axial saddle that, by local bifurcation, generates a ring of saddles that recedes from the axis, as illustrated in Fig. 4 (b). [3]

Now, for the Levitron with the correct mass and height, a slower $\dot{\psi}$ will allow the spinning top to fall by our analysis of a mechanical spinning top. However, a faster $\dot{\psi}$ will break the conservation of the adiabatic invariant and will also end up falling because $\dot{\phi}$ will be[1] smaller and will no longer be able to continuously align the axis of precession with $\hat{B}$. [3]

It is very exciting to realize that the childish wonder of a spinning top has such astonishing physics and beautiful mathematics behind it. The principles underlying stable gyroscopic motion in conservative fields are key to various aspects of modern technology. Beginning with classical mechanics, understanding this phenomenon has contributed towards discoveries that extend to astrology, electromagnetism, atomic physics, and quantum mechanics.

References:

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[2] Elliott, Chloe. The Spinning Top. Durham University. Durham, UK, 2009.
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* The correction to this approximation is based on geometric magnetism impacting a velocity-dependent force on the magnetic dipole that is similar to the Lorentz force [3]