Exam 1 Practice Problem Solutions (Part 2)

Math 315, Fall 2019

BSM exercise 3.1 a-c

MAP: Fast but doesn't quantify uncertainty

Numerical integration: Accurate but fails in high dimensions

CLT: Fast but requires a large sample size

BSM exercise 3.3

a.

The conjugate prior is $\lambda \sim \text{Gamma}(a, b)$ and the posterior is

$$\begin{aligned} p(\lambda|\text{data}) &\propto \left[\prod_{i=1}^{n} f(Y_i|\lambda)\right] \pi(\lambda) \\ &\propto \left[\prod_{i=1}^{n} \lambda^{Y_i} e^{-N_i\lambda}\right] \cdot \lambda^{a-1} e^{-\lambda b} \\ &\propto \lambda^{(\sum y_i + a)} \exp\left[-\lambda\left(\sum N_i + b\right)\right] \end{aligned}$$

and so $\lambda|Y_1, \ldots, Y_n \sim \text{Gamma}(\sum Y_i + a, \sum N_i + b)$

b.

For $\lambda \in (0, 20)$ the posterior is

$$p(\lambda|\text{data}) \propto \prod_{i=1}^{n} f(Y_i|\lambda).$$

Therefore, the log posterior is a constant plus

$$\ell(\lambda) = \sum_{i=1}^{n} \left[-N_i\lambda + Y_i \log(\lambda)\right].$$

Taking the derivative and setting to sero gives

$$\ell(\lambda) = -\sum_{i=1}^{n} N_i + \frac{1}{\lambda} \cdot \sum_{i=1}^{n} Y_i = 0$$

and so the MAP estimate is $\widehat{\lambda} = \sum_{i=1}^{n} Y_i / \sum_{i=1}^{n} N_i$ (unless this is greater than 20, in which case the MAP is 20).

c.

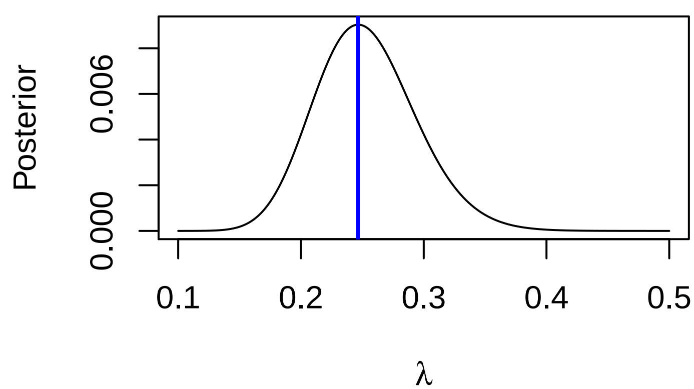
```r
lambda <- seq(0.1, 0.5, by = 0.001)

posterior <- dpois(x = 12, lambda = 50 * lambda) *
  dpois(x = 25, lambda = 100 * lambda)

MAP <- (12 + 25) / (50 + 100)
MAP
```

```
## [1] 0.2466667
```

```r
plot(lambda, posterior, type="l", xlab = expression(lambda),
     ylab = "Posterior")
abline(v = MAP, col = "blue", lwd = 2)
```



d.

Taking the second derivative gives

$$\frac{d^2}{d\lambda^2}\ell(\lambda) = -\frac{1}{\lambda^2}\sum_{i=1}^{n} Y_i$$

and so the approximate variance is $\widehat{\lambda}^2/\sum_{i=1}^n Y_i = \sum_{i=1}^n Y_i/[\sum_{i=1}^n N_i]^2$

```
lambda <- seq(0.1, 0.5, 0.001)
VAR    <- MAP * MAP / (12 + 25)
clt    <- dnorm(lambda, MAP, sqrt(VAR))
plot(lambda, posterior / sum(posterior), type = "l",
     xlab = expression(lambda), ylab = "Posterior")
lines(lambda, clt / sum(clt), col = "blue", lty = 2)
legend("topright", c("Exact","CLT"), lty=1:2, col = c("black", "blue"), bty = "n")
```