Bayesian Updating: Additional Examples

Math 315, Fall 2018

Example: Poisson-Gamma model

Let $X_1, X_2, \ldots, X_n$ be a random sample from the Poisson distribution with PMF

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!},\ x = 0, 1, 2, \ldots$$

1. Write down the likelihood function of λ.

$$f(x_1, \ldots, x_n \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \left[\prod_{i=1}^{n} \frac{1}{x_i!}\right] \lambda^{\sum x_i} e^{-n\lambda}$$

2. Suppose that you decide to use a Gamma(a,b) prior distribution for λ with PDF

$$p(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda},\ \lambda > 0.$$

Find the posterior density of λ.

$$p(\lambda \mid x_1, \ldots, x_n) \propto p(\lambda) f(x_1, \ldots, x_n \mid \lambda)$$
$$\propto \lambda^{a-1} e^{-b\lambda} \lambda^{\sum x_i} e^{-n\lambda}$$
$$= \lambda^{(a + \sum x_i) - 1} e^{-\lambda(b+n)}$$

This is the kernel (i.e. up to a constant) the Gamma$(a + \sum x_i, b+n)$.

3. Is the gamma prior a conjugate family to the Poisson likelihood?

Both the prior ad posterior belong to the same dsn, so the gamma prior is a conjugat family to the Poisson likelihood.

Example: Exponential-Gamma model

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the Exponential distribution with PDF

$$f(y|\lambda) = \lambda e^{-\lambda y},\ y \geq 0,\ \lambda > 0$$

1. Write down the likelihood function of μ.

$$f(y_1, \ldots, y_n | \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda y_i} = \lambda^n e^{-\lambda \sum y_i}$$

2. Suppose that you decide to use a Gamma(a,b) prior distribution for λ with PDF

$$p(\theta) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda},\ \lambda > 0.$$

Find the posterior density of λ.

$$\begin{aligned} p(\lambda | x_1, \ldots, x_n) &\propto p(\lambda) f(y_1, \ldots, y_n | \lambda) \\ &\propto \lambda^n e^{-\lambda \sum y_i} \lambda^{a-1} e^{-b\lambda} \\ &= \lambda^{(n+a)-1} e^{-\lambda(\sum y_i + b)} \end{aligned}$$

This is the Kernel (i.e. up to a constant) the

Gamma$(n+a , \sum y_i + b)$.

3. Is the gamma prior a conjugate family to the exponential likelihood?

Both the prior ad posterior belong to the same dsn, so the gamma prior is a conjugat family to the exponential likelihood.