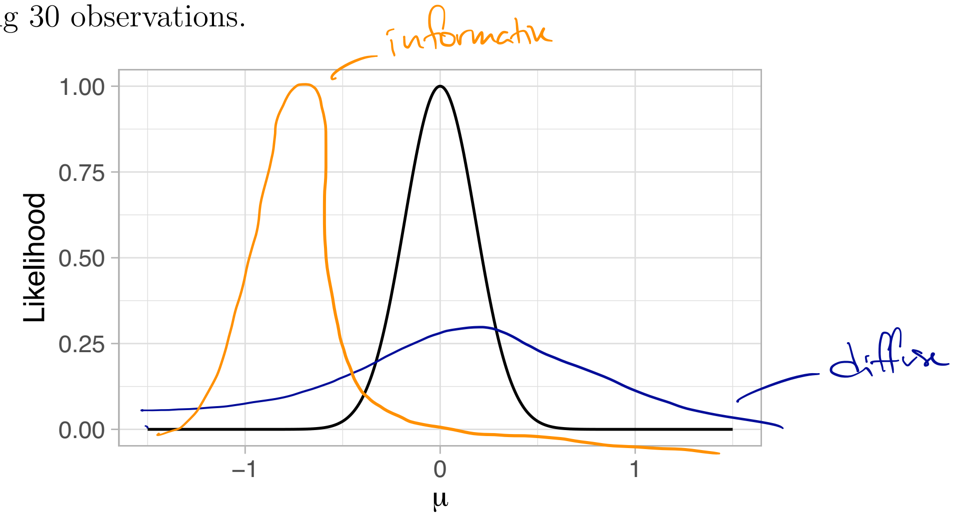
1. **Priors.** The plot below is the likelihood function for the mean of a $\mathcal{N}(\mu, 1)$ distribution evaluated using 30 observations.



(a) Sketch (and clearly label) a *diffuse* prior on the plot above. Write a short definition of a diffuse prior.

4 pts

A diffuse prior "lets the data speak" by placing roughly equal density across the parameter space. Above is an example of a diffuse Normal dsn, but it need not be normal.

(b) Sketch (and clearly label) an *informative* prior on the plot above. Write a short definition of an informative prior.

4 pts

An informative prior uses prior belief about the parameter to restrict (i.e. up- and/or down-weight) regions of the parameter space.

2. **Bayesian updating and prediction.** Suppose that $Y_1, \ldots, Y_n|\theta \sim \text{Gamma}(1, \theta)$ and that $\theta \sim \text{InvGamma}(a, b)$.

(a) Find the posterior distribution of θ. If it is a member of a named family of distributions, be sure to specify this, along with its parameter values.

6 pts

$$f(y_1, \ldots, y_n|\theta) = \prod_{i=1}^{n} \frac{\theta}{\Gamma(1)} e^{-\theta y_i} y_i^{1-1} = \theta^n e^{-\theta \sum y_i}$$

$$p(\theta|y_1, \ldots, y_n) \propto p(\theta) f(y_1, \ldots, y_n|\theta)$$

$$\propto e^{-b/\theta} \theta^{-a-1} \cdot \theta^n e^{-\theta \sum y_i}$$

$$= \theta^{(n-a)-1} e^{-\frac{b}{\theta} - \theta \sum y_i}$$

This is not a member of one of the named dsns attached to the exam.

(b) Is the inverse-gamma prior a conjugate family to the gamma likelihood?

2 pt.

No, since the posterior is not a member of the inverse-gamma family of dsns it is not a conjugate prior.

(c) Write down two integrals (but do not evaluate them) that could be solved to find the 97% percentile interval for θ.

6 pts

$$\int_0^{l} p(\theta|y_1, \ldots, y_n) d\theta = 0.015$$

$$\int_0^{u} p(\theta|y_1, \ldots, y_n) d\theta = 0.985$$

(d) Outline (describe) the steps necessary to draw a sample from the posterior predictive distribution of $\widetilde{Y}$, assuming that you have a sample $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(m)}$ from the posterior distribution of θ.

5 pts

We already have draws for θ, so to obtain a $\tilde{y}_i$ draw from $f(\tilde{y}|y)$ we draw from the likelihood evaluated at $\theta^{(i)}$, a Gamma$(1, \theta^{(i)})$. Do this for each $\theta^{(i)}$ drawn from the posterior.

4. **Sampling.**

6 pts

(a) Suppose that you are interested in a posterior distribution, $p(\theta|x_1, \ldots, x_n)$, with support on $(0, 1)$. Describe how you would use grid approximation to draw a random sample from $p(\theta|x_1, \ldots, x_n)$.

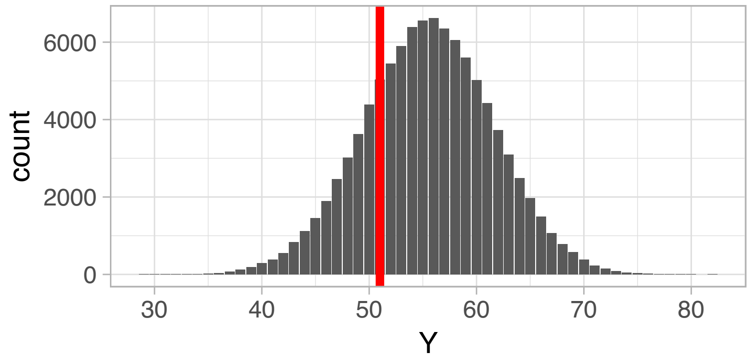
1) Create a grid on (0,1), say of length 1000. These are θ_is.
2) Evaluate the prior and likelihood at each point (θ_i) on the grid.
3) Approximate the posterior by standardizing prior x likelihood.
4) Draw a random sample from the grid of θ_is, with replacement, with sampling probabilities given by the posterior probabilities.

(b) Suppose that we have a sample of size m drawn from $p(\theta|x_1, \ldots, x_n)$—that is, we have $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(m)}$ in hand. How could we use this sample to evaluate the posterior probability that $\theta > 0.8$?

3 pts

Calculate the proportion of the $\theta^{(i)}$ draws that are greater than 0.8.

6. **Model checking.** Below is the posterior predictive distribution for the following model: $Y_i|p \sim \text{Binom}(n, p)$, where $p \sim \text{Beta}(0.5, 0.5)$. The observed value of Y is diplayed as a vertical line.



4 pts

What does this plot reveal about the model's fit?

There is general agreement between the observed data and the predictions made by the model. While the observed Y is not right at the mode, it is well within the dsn of the simulated data (i.e. it could have plausibly been generated by the model).

3. Suppose $y_1, \dots, y_n$ form a random sample from $\mathcal{N}(\mu, \sigma^2)$. The joint posterior distribution that results from the reference prior is

$$p(\mu, \sigma^2 | y_1, \dots, y_n) \propto \left(\sigma^2\right)^{-n/2-1} \exp\left\{\sum_{i=1}^{n} -\frac{1}{2\sigma^2}(y_i - \mu)^2\right\}$$

Find $p(\sigma^2|\mu, y_1, \dots, y_n)$, the conditional posterior of σ^2 given μ and the data. If it is a member of a named family of distributions, be sure to specify this, along with its parameter values.

$$p(\sigma^2 | \mu, \vec{y}) \propto (\sigma^2)^{-n/2-1} \exp\left\{ -\frac{1}{2} \sum (y_i - \mu)^2 / \sigma^2 \right\}$$

This is the kernel of the InvGamma $\left(\frac{n}{2}, \frac{1}{2} \sum (y_i - \mu)^2 \right)$