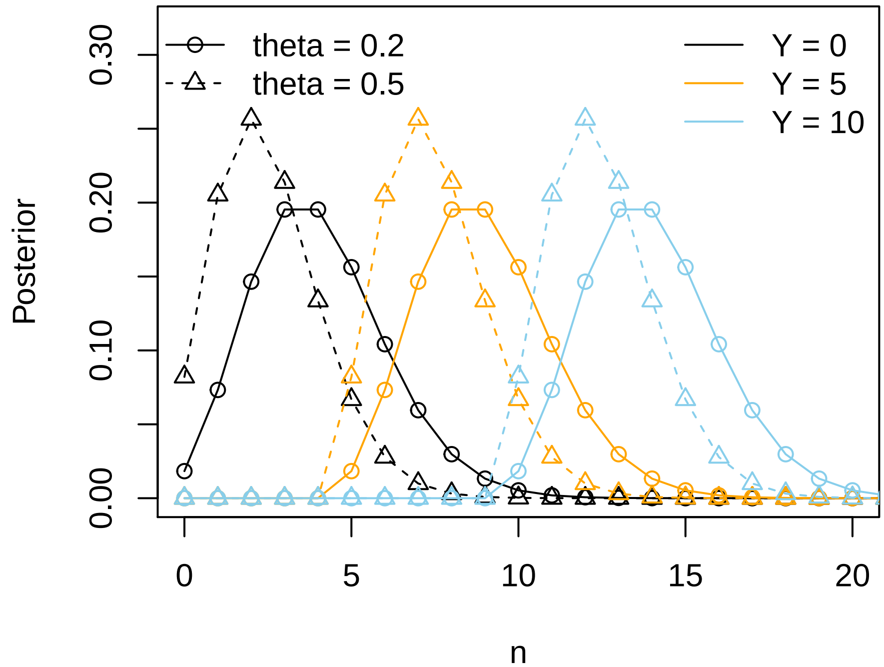
Homework 3 Solution

Math 315, Fall 2019

1. (1.14 from *BSM*)

The posterior increases with Y because more customers making a purchase suggests there are more total customers. For a given Y, larger θ suggests a smaller n as there are likely fewer missing customers (i.e., those that do not make a purchase).

```
y     <- c(0, 5, 10)
theta <- c(0.2, 0.5)
n     <- 0:100

# Empty plotting window
plot(NA, xlim = c(0, 20), ylim = c(0, 0.32),
     xlab = "n", ylab = "Posterior")

colors <- c("black", "orange", "skyblue")

for(i in 1:3){
  for(j in 1:2){
    post <- dbinom(y[i], n, theta[j]) * dpois(n, 5)
    post <- post / sum(post)
    lines(n, post, col = colors[i], lty = j)
    points(n, post, col = colors[i], pch = j)
  }
}

legend("topright",paste("Y =", y),lty=1, col = colors, bty="n")
legend("topleft",paste("theta =", theta), lty = 1:2, pch=1:2, bty="n")
```



2.

a.

Notice that the unnormalized posterior is given by

$$p(\theta|y_1, \dots, y_5) \propto \prod_{i=1}^{5} \frac{1}{1 + (y_i - \theta)^2}$$

In the below code, notice that I use a for loop to calculate this product (on the log scale and then I back transform).

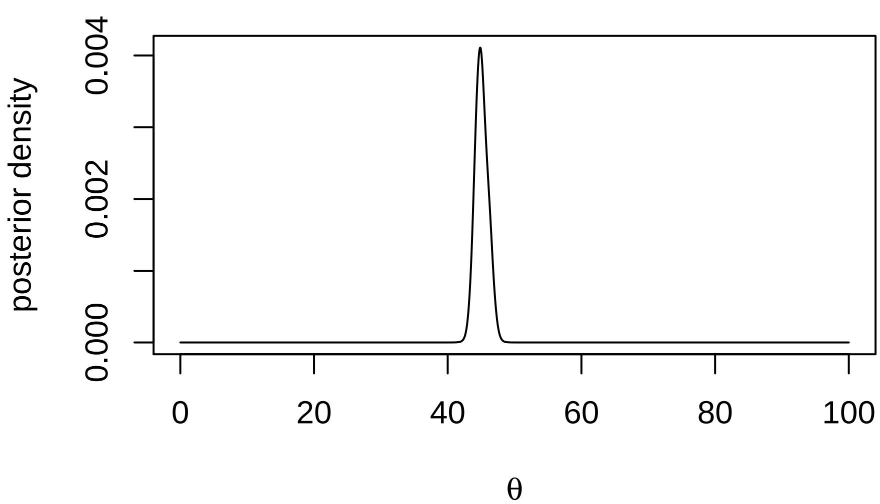
```
y <- c(43, 44, 45, 46.5, 47.5)  # obs data
theta <- seq(0, 100, by = 0.01) # grid for theta

# Prior for theta
prior <- dunif(theta, 0, 100)

# Calculate the likelihood of the data | theta
log_likelihood <- 0
for(i in seq_along(y)) {
  log_likelihood <- log_likelihood + dcauchy(y[i], location = theta, scale = 1, log = TRUE)
}

likelihood <- exp(log_likelihood)
```

With the prior and likelihood calculated, we can calculate the unnormalized posterior (shown above) on the grid, and normalize by dividing by the sum:

```
unnorm_posterior <- prior * likelihood
posterior <- unnorm_posterior / sum(unnorm_posterior)
```

Now, we can plot the posterior

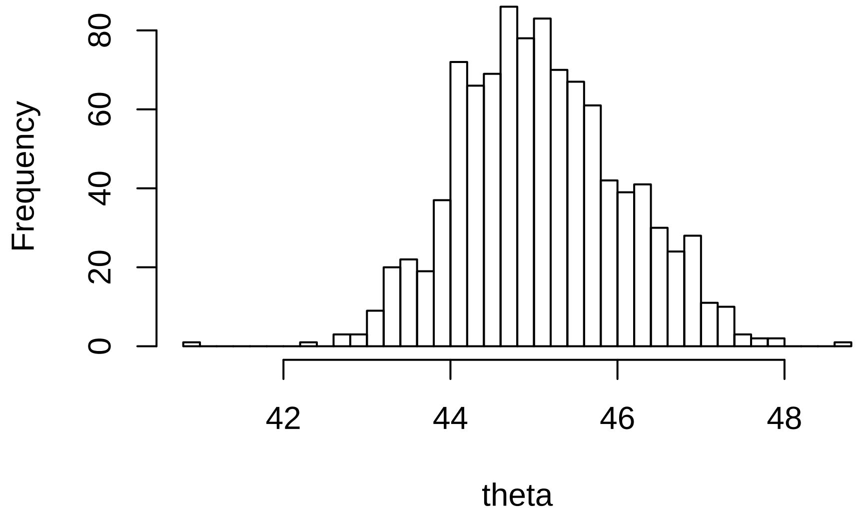
```
plot(x = theta, y = posterior, type = "l", xlab = expression(theta), ylab = "posterior density")
```



b.

Here, we draw a Monte Carlo sample of size $S = 1000$ from the posterior and plot the results. Here, 30 bins is reasonable, but answers will vary due to the high variability of the posterior.

```
theta_draws <- sample(theta, size = 1000, replace = TRUE, prob = posterior)
hist(theta_draws, xlab = "theta", main = NULL, nclass = 30)
```



c.

Here, we use the 1000 $\theta^{(i)}$s to simulate 1000 Y_6s (one for each θ). Notice that your histogram likely looks ridiculous. This is due to the high variability of the Cauchy likelihood. (In fact, the variance of a Cauchy is ∞!) This creates some severe outliers. An alternative visualization is a density plot (shown on the left).

```
y6 <- rcauchy(1000, location = theta_draws, scale = 1)
par(mfrow = c(1, 2)) # 1 row, 2 cols
hist(y6, xlab = "predicted Y", nclass = 100, main = "Posterior predictive density")
plot(density(y6), main = "Posterior predictive density")
```