Math 315 - HW 1 - Fall 2019

2) $f(x) = \frac{1}{b-a}$ for $x \in [a,b]$

(a) i) $f(x) \geq 0$ for all $x \in [a,b]$ since $b > a$, and 0 otherwise.
[i.e. $f(x)$ is non-negative]

ii) $\int_a^b f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[x\Big|_{x=a}^{b}\right] = \frac{1}{b-a}\left[b-a\right] = 1$

(b) $E(X) = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx = \frac{1}{b-a}\left[\frac{1}{2}x^2\Big|_{x=a}^{b}\right]$

$= \frac{1}{2}\cdot\frac{1}{b-a}\left[b^2 - a^2\right] = \frac{1}{2}\cdot\frac{1}{b-a}\left[(a+b)(a-b)\right] = \frac{a+b}{2}$

$E(X^2) = \int_a^b x^2 \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{1}{3}x^3\Big|_{x=a}^{b}\right] = \frac{1}{3}\cdot\frac{1}{b-a}\left[b^3 - a^3\right]$

$= \frac{1}{3}\cdot\frac{1}{b-a}\left[(a-b)(a^2+ab+b^2)\right] = \frac{1}{3}(a^2+ab+b^2)$

$Var(X) = E(X^2) - [EX] = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{(a^2-b^2)}{12}$

4)

(a)

x_1	0	1	2
$P(X_1=x_2)$	0.3	.35	.35

(b)

x_2	0	1
$P(X_2=x_2)$	.45	.55

(c) $P(X_1=x_1 \mid X_2=x_2) = P(X_1=x_1, X_2=x_2) / P(X_2=x_2)$

x_1	0	1	2
$P(X_1=x_1 \mid X_2=0)$	$.15/.45$	$.15/.45$	$.15/.45$

x_1	0	1	2
$P(X_1=x_1 \mid X_2=1)$	$.15/.55$	$.2/.55$	$.2/.55$

(d) Similarly

x_2	0	1
$P(X_2=x_2 \mid X_1=0)$	$.15/.3$	$.15/.3$

x_2	0	1
$P(X_2=x_2 \mid X_1=1)$	$.15/.35$	$.2/.35$

x_2	0	1
$P(X_2=x_2 \mid X_1=2)$	$.15/.35$	$.2/.35$

(e) No, X_1 and X_2 are not independent because the conditional dsns are not equal to the marginal dsns.

8) Let Y = commut time ; let C denote a convention

Given: $Y \mid C^c \sim \text{Unif}(15, 16, \dots, 20)$

$Y \mid C \sim \text{Unif}(15, 16, \dots, 30)$

$P(C) = \frac{1}{4}$

*I used Discrete unif b/c asked about $Y = 18$ and 28.

(a) $$P(C \mid Y=18) = \frac{P(Y=18, C)}{P(Y=18)}$$

$$= \frac{P(Y=18 \mid C)\, P(C)}{P(Y=18 \mid C)\, P(C) + P(Y=18 \mid C^c)\, P(C^c)}$$

$$= \frac{\frac{1}{16}\left(\frac{1}{4}\right)}{\frac{1}{16}\left(\frac{1}{4}\right) + \frac{1}{6}\left(\frac{3}{4}\right)}$$

$$= \frac{1}{9}$$

(b) $$P(C \mid Y=28) = \frac{P(Y=28 \mid C)\, P(C)}{P(Y=28 \mid C)\, P(C) + P(Y=28 \mid C^c)\, P(C^c)}$$

$$= \frac{\frac{1}{16}\left(\frac{1}{4}\right)}{\frac{1}{16}\left(\frac{1}{4}\right) + 0\left(\frac{3}{4}\right)} = \underline{1}$$

* You should also be able to jump to the final solution using logic.

tree species occupies a forest

10) $X_1 | \theta \sim \text{Bernoulli}(\theta)$, $\theta \in [0,1]$

$X_2 | X_1, \theta \sim \text{Binomial}(n, \lambda X_1)$ $\lambda \in [0,1]$

of the species obs in sample of n trees.

(a) $P(X_1 = 0, X_2 = 0) = P(X_2 = 0 | X_1 = 0)\, P(X_1 = 0)$
$= 1 \cdot (1-\theta)$
$= 1-\theta$

(b) $P(X_1 = 0) = (1-\theta)$

(c) $P(X_2 = 0) = P(X_2 = 0 | X_1 = 0) P(X_1 = 0) + P(X_2 = 0 | X_1 = 1) P(X_1 = 1)$
$= (1-\theta) + \lambda^0 (1-\lambda)^n \theta$

(d) $P(X_1 = 0 | X_2 = 0) = \dfrac{P(X_1 = 0, X_2 = 0)}{P(X_2 = 0)} = \dfrac{1-\theta}{(1-\theta) + (1-\lambda)^n \theta}$

(e) $P(X_2 = 0 | X_1 = 0) = 1$ since $X_2 | X_1 = 0 \sim \text{Binomial}(n, 0)$

(f) $P(X_1 = 0 | X_2 = 1) = \dfrac{P(X_2 = 1 | X_1 = 0) P(X_1 = 0)}{P(X_2 = 1)} = 0$

(g) $P(X_2 = 0 | X_1 = 1) = \lambda^0 (1-\lambda)^n$

(h) Give intuition about changes in (a) - (g) based on θ, n, λ

(i) Link to (d): Solve the following for n:

$$\frac{0.5}{0.5 + .9^n (0.5)} = 0.95$$

$n = 28$ (found w/ uniroot in R)

16)

$X_1 \sim \text{Bern}\left(\frac{1}{2}\right)$

$X_2 \sim \text{Bern}\left(\frac{1}{3}\right)$

$$Y_1 = \begin{cases} 1 & \text{if } X_1 = 1, X_2 = 1 \\ 0 & \text{o.w.} \end{cases}$$

$$Y_2 = \begin{cases} 0 & \text{if } X_1 = 0, X_2 = 0 \\ 1 & \text{o.w.} \end{cases}$$

(a) $P(X_1 = 1, X_2 = 1 \mid Y_1 = 1) = 1$ since $Y_1 = 1$ iff $X_1 = X_2 = 1$

(b) $$P(X_1 = 1, X_2 = 1 \mid Y_2 = 1) = \frac{P(Y_2 = 1 \mid X_1 = 1, X_2 = 1)\, P(X_1 = 1) P(X_2 = 1)}{P(Y_2 = 1)}$$

$$= \frac{1\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{1 - P(Y_2 = 0)} = \frac{\frac{1}{6}}{1 - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)} = \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$

(c) $P(X_1 = 1 \mid Y_1 = 1) = 1$

(d) $$P(X_1 = 1 \mid Y_2 = 1) = \frac{P(Y_2 = 1 \mid X_1 = 1)\, P(X_1 = 1)}{P(Y_2 = 1)}$$

$$= \frac{1\left(\frac{1}{2}\right)}{\frac{2}{3}} = \frac{3}{4}$$