Homework 2

Math 315, Fall 2019

Due 24 September by 4:00 p.m.

Instructions: Complete the following problems and submit them by 4 p.m. on the due date. Please make sure that your solution is neatly written, clearly organized, and stapled (if there are multiple pages). For this homework, you may handwrite the answers obtained from R and attached your code (which I recommend writing in R Markdown) to your assignment.

1. Bob claims to have ESP (extrasensory perception). To test this claim, you propose the following experiment. You will select one from four large cards with different geometric figures and Bob will try to identify it. Let θ denote the probability that Bob is correct in identifying the figure for a single card. You believe that Bob has no ESP ability ($\theta = .25$), but there is a small chance that θ is either larger or smaller than .25. After some thought, you place the following prior distribution on θ:

θ	0	.125	.25	.375	.5	.625	.75	.875	1
$\pi(\theta)$	.001	.001	.95	.008	.008	.008	.008	.008	.008

Suppose that the experiment is repeated ten times and Bob is correct six times and incorrect four times.

(a) Derive the posterior distribution of θ and display it as a table.

(b) What is your posterior probability that Bob has no ability (and is thus simply guessing)?

2. A study reported on the long-term effects of exposure to low levels of lead in childhood. Researchers analyzed children's shed primary teeth for lead content. Of the children whose teeth had a lead content of more than 22.22 parts per million (ppm), 22 eventually graduated from high school and 7 did not. Suppose your prior density for θ, the proportion of all such children who will graduate from high school is Beta(1, 1).

(a) Derive the posterior distribution of θ.

(b) Calculate the MAP estimate for θ.

(c) Use the R function `qbeta()` to find a 93% equal-tailed credible interval for θ.

(d) To calculate the HPDI for θ we can use the `hdi()` function found in the **HDInterval** R package (be sure to install this if you are working on your own computer). To do this, first use the `rbeta()` function to simulate 10000 draws (i.e. samples) from the posterior distribution, store these objects in the `draws` object. Then run `hdi(draws, credMass = 0.93)` to calculate a 93% HPDI.

(e) Use the function `pbeta()` to find the probability that θ exceeds 0.6.

3. The Weibull distribution is often used as a model for survival times in biomedical, demographic, and engineering analyses. A random variable Y has a Weibull distribution if has PDF

$$f(y|\alpha, \lambda) = \lambda \alpha y^{\alpha-1} \exp(-\lambda y^{\alpha}), \text{ where } y > 0,\ \alpha > 0,\ \lambda > 0.$$

For this problem, assume that α is known, but λ is unknown.

Assume $Y_1, Y_2, \ldots, Y_n$ are i.i.d. Weibull random variables with known α and that λ has prior $\lambda \sim \text{Gamma}(a, b)$. Derive the resulting posterior distribution. If it is a member of a named family of distributions, be sure to specify this, along with its parameter values.