Homework 5 Solution

Math 315, Fall 2019

BSM 2.10

Recall: If $p \sim \text{Beta}(a, b)$, then $E(p) = \frac{a}{a+b}$ and $Var(p) = \frac{ab}{(a+b)^2(a+b+1)}$. Since we are given the expectation and SD (and thus variance) from the experts, we can solve this as a two equation, two unknown problem. This can be done via a pencil and paper, search, or by numeric optimization.

Below is the pencil and paper solution. Here I let μ denote the expectation and σ denote the SD.

(a)

Expert 1: alarm probability 0.95 with SD 0.05

```
mu   <- 0.95
sig2 <- 0.05^2
a1   <- ((mu^2 * (1 - mu)) / sig2) - 1
b1   <- ((mu * (1 - mu)^2) / sig2) - ((1 - mu) / mu)

# Check the mean
a1 / (a1 + b1)
```

```
## [1] 0.95
```

```
# Check the SD
sqrt((a1 * b1) / ((a1 + b1)^2 * (a1 + b1 + 1)))
```

```
## [1] 0.0500694
```

So expert 1 has a Beta($a = 17.1$, $b = 0.9$) prior.

Alternatively, you could use numeric optimization:

```
library(BB)
beta_solver <- function(x, .mean, .sd) {
  obj1 <- x[1] / (x[1] + x[2]) - .mean
  obj2 <- (x[1] * x[2]) / ((x[1] + x[2])^2 * (x[1] + x[2] + 1)) - .sd^2
  return(c(a = obj1, b = obj2))
}

BBsolve(par = c(1, 1), fn = beta_solver, .mean = 0.95, .sd = 0.05, quiet = TRUE)$par
```

```
##          a          b
## 17.0998940  0.8999921
```

Expert 2: alarm probability 0.80 with SD 0.20

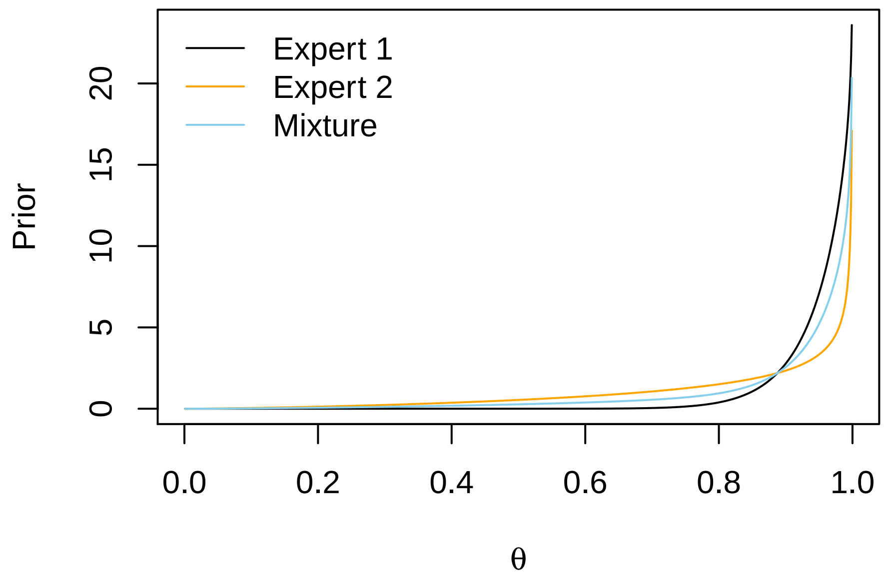
```
BBsolve(par = c(1, 1), fn = beta_solver, .mean = 0.8, .sd = 0.2, quiet = TRUE)$par
```

```
##   a   b
## 2.4 0.6
```

So expert 2 has a Beta(\$a = 2.4, $b = 0.6$) prior.

Now that we have the two priors in hand, we can plot the two along with the 50:50 mixture:

```r
theta <- seq(0.001,0.999,0.001)
d1    <- dbeta(theta, 17.1, 0.9)
d2    <- dbeta(theta, 2.4, 0.6)

plot(NA, xlim=0:1, ylim = range(c(d1,d2)), xlab = expression(theta), ylab="Prior")
lines(theta, d1, col = "black")
lines(theta, d2, col = "orange")
lines(theta, d1/2 + d2/2, col = "skyblue")
legend("topleft", c("Expert 1", "Expert 2", "Mixture"), lty = 1,
       col = c("black", "orange", "skyblue"), bty = "n")
```



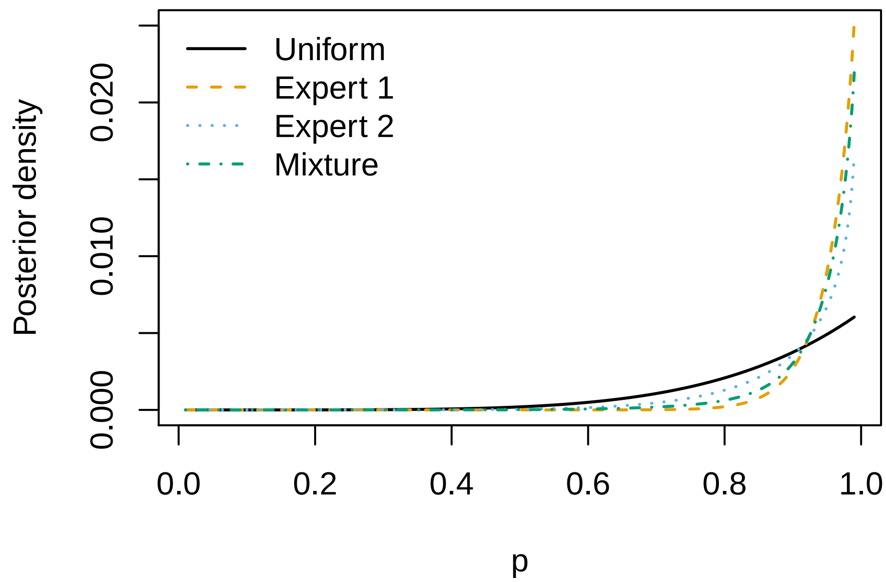
(b)

$$
\begin{aligned}
Y|p &\sim \text{Binomial}(n, p) \\
p &\sim \text{Beta}(1, 1) && \text{(Uniform)} \\
&\sim \text{Beta}(17.1, 0.9) && \text{(Expert 1)} \\
&\sim \text{Beta}(2.4, 0.6) && \text{(Expert 2)} \\
&\sim 0.5\text{Beta}(17.1, 0.9) + 0.5\text{Beta}(2.4, 0.6) && \text{(50:50 mixture)}
\end{aligned}
$$

It is straightforward to plot the posterior densities for the first three models (priors), but the posterior mixture weights are tedious, so we'll derive the normalized prior via grid approximation for all of the cases:

```r
# Grid up the parameter space
p_grid     <-  seq(0.01, 0.99, by = .001)

# Evaluate the priors on the grid
unif_prior <- dbeta(p_grid, 1, 1)
exp1_prior <- dbeta(p_grid, 17.1, 0.9)
exp2_prior <- dbeta(p_grid, 2.4, 0.6)
mix_prior  <- 0.5 * dbeta(p_grid, 17.1, 0.9) + 0.5 * dbeta(p_grid, 2.4, 0.6)

# Evaluate the likelihood on the grid
likelihood <- dbinom(5, size = 5, prob = p_grid)
```

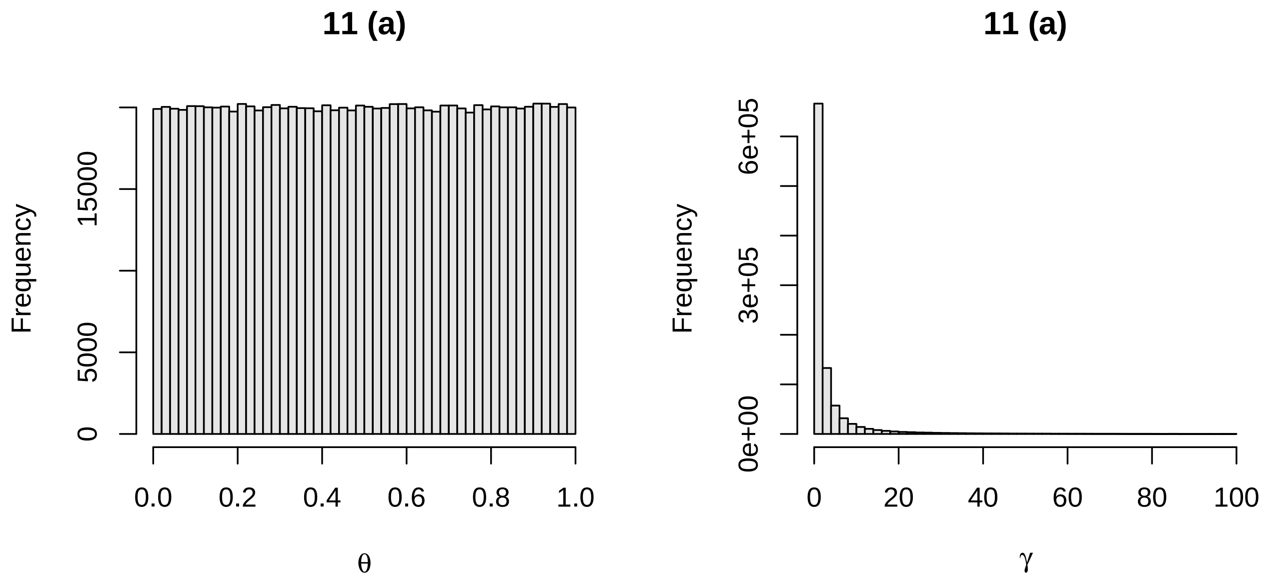
```r
# Approx posteriors
unif_post <- likelihood * unif_prior / sum(likelihood * unif_prior)
exp1_post <- likelihood * exp1_prior / sum(likelihood * exp1_prior)
exp2_post <- likelihood * exp2_prior / sum(likelihood * exp2_prior)
mix_post  <- likelihood * mix_prior / sum(likelihood * mix_prior)

# Plot the posteriors and compare
pal <- ggthemes::colorblind_pal()(4)

plot(p_grid, unif_post, type = "l", ylim = c(0, 0.025), ylab = "Posterior density", xlab = "p", lwd = 1
lines(p_grid, exp1_post, type = "l", lty = 2, col = pal[2], lwd = 1.5)
lines(p_grid, exp2_post, type = "l", lty = 3, col = pal[3], lwd = 1.5)
lines(p_grid, mix_post, type = "l", lty = 4, col = pal[4], lwd = 1.5)
legend("topleft", lty = 1:4, col = pal, legend = c("Uniform", "Expert 1", "Expert 2", "Mixture"), lwd =
```



BSM 2.11

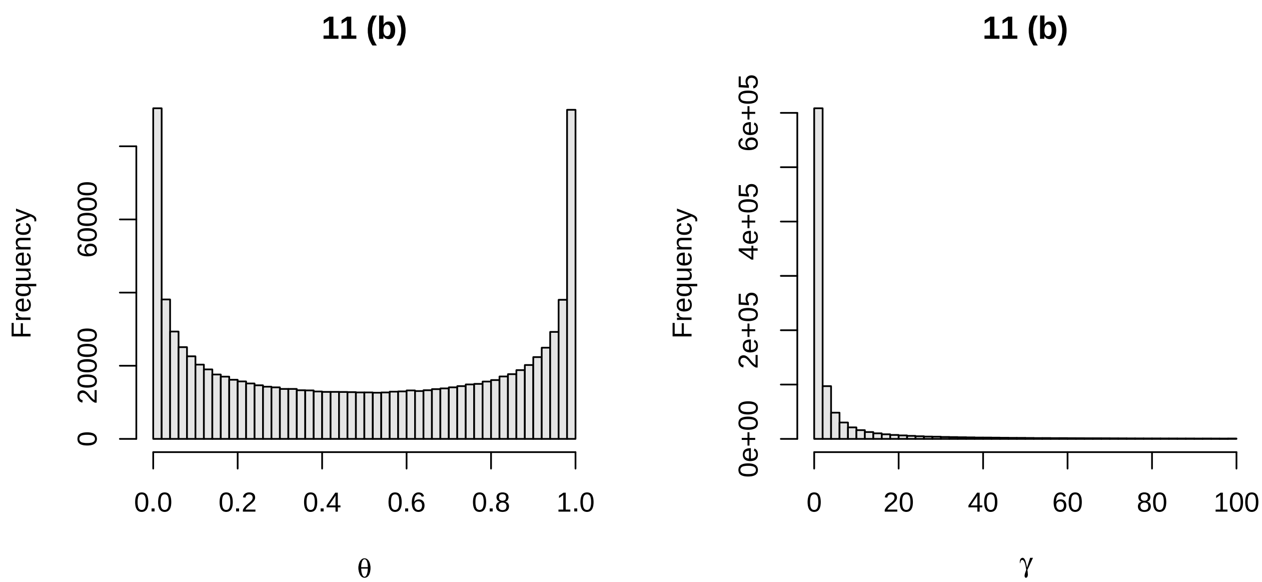
(a)

```r
S <- 1e6
theta_draws <- rbeta(S, 1, 1)
gamma_draws <- theta_draws / (1 - theta_draws)
gamma_draws <- gamma_draws[gamma_draws < 100]

par(mfrow = c(1, 2))
hist(theta_draws, breaks = 50, main = "11 (a)", xlab = expression(theta), col = "gray90")
hist(gamma_draws, breaks = 50, main = "11 (a)",  xlab = expression(gamma), col = "gray90")
```



(b)

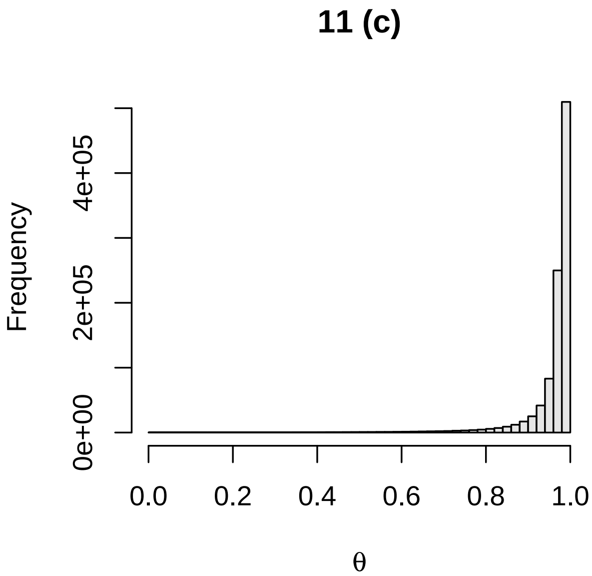
```
theta_draws <- rbeta(S, 0.5, 0.5)
gamma_draws <- theta_draws / (1 - theta_draws)
gamma_draws <- gamma_draws[gamma_draws < 100]

par(mfrow = c(1, 2))
hist(theta_draws, breaks = 50, main = "11 (b)", xlab = expression(theta), col = "gray90")
hist(gamma_draws, breaks = 50, main = "11 (b)",  xlab = expression(gamma), col = "gray90")
```

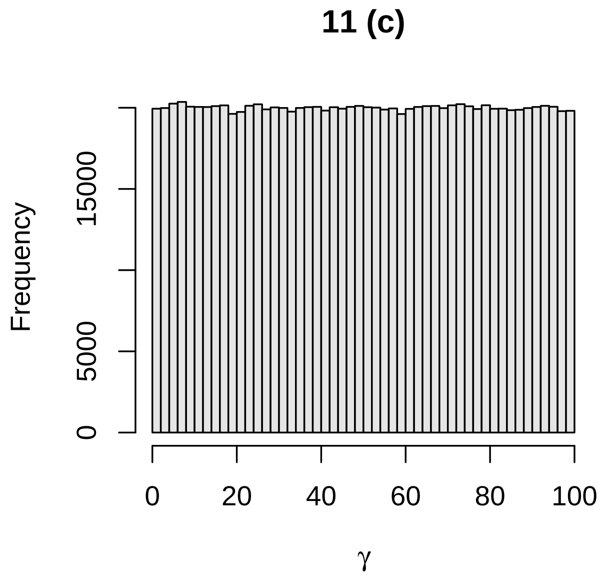


(c)

```
gamma_draws  <- runif(S, 0, 100)
theta_draws <- gamma_draws / (gamma_draws + 1)
```

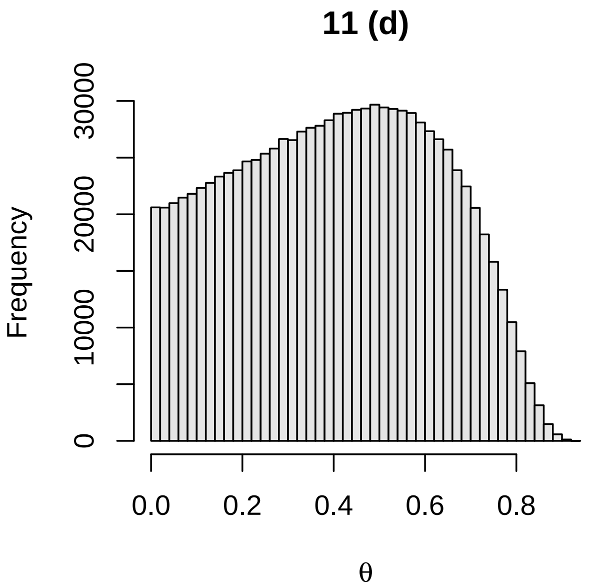
```r
par(mfrow = c(1, 2))
hist(theta_draws, breaks = 50, main = "11 (c)", xlab = expression(theta), col = "gray90")
hist(gamma_draws, breaks = 50, main = "11 (c)",  xlab = expression(gamma), col = "gray90")
```

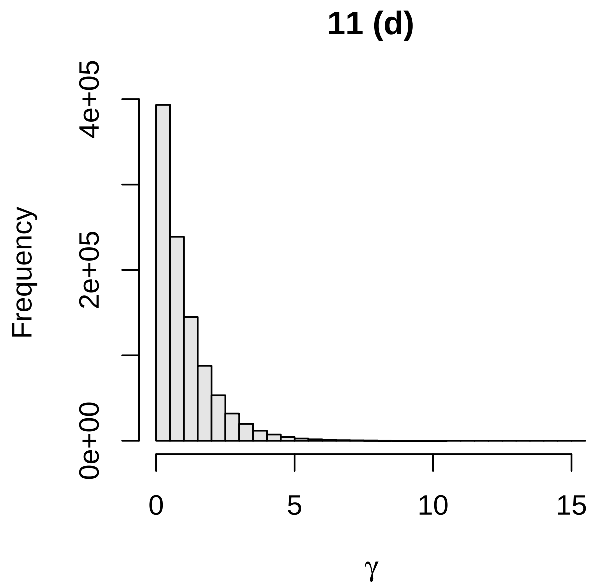




(d)

```r
gamma_draws  <- rgamma(S, 1, 1)
theta_draws <- gamma_draws / (gamma_draws + 1)

par(mfrow = c(1, 2))
hist(theta_draws, breaks = 50, main = "11 (d)", xlab = expression(theta), col = "gray90")
hist(gamma_draws, breaks = 50, main = "11 (d)",  xlab = expression(gamma), col = "gray90")
```





(e)

None of the priors are uninformative for both variables, but the prior in (b) is the Jeffreys' prior so this is as close as you can get.

Natural conjugate problem

Part 1. Let $Y_1, \ldots, Y_n \sim \text{Gamma}(\alpha, \beta)$, where α is fixed and known.

$$\pi(\beta, y_1^o, \ldots, y_m^o, m) \propto \prod_{j=1}^{m} \frac{\beta^\alpha}{\Gamma(\alpha)} (y_j^o)^{\alpha-1} e^{-\beta y_j^o}$$

$$\propto \beta^{m\alpha} e^{-\beta \sum y_j^o}$$

so the natural conjugate prior is $\text{Gamma}\left(m\alpha + 1, \sum_{j=1}^{m} y_j^o\right)$

Part 2. $Y_1, \ldots, Y_n \sim \text{Normal}(\mu, \sigma^2)$, where μ is fixed and known.

$$\pi(\mu \mid y_1^o, \ldots, y_m^o, m) \propto \prod_{j=1}^{m} f(y_j^o \mid \mu)$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{m/2} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{m} (y_i^o - \mu)^2}$$

so the natural conjugate prior is $\text{Inv Gamma}\left(\frac{m}{2} - 1, \frac{SSE}{2}\right)$

where $SSE = \sum_{i=1}^{m} (y_i^o - \mu)^2$