Constraining Distributions

Math 315, Fall 2019

In Case Study I, you will need to constrain the support of a PDF. We will not cover this in class, but I want to elaborate on how this is a direct application of Bayes' rule from probability theory so that you can fully understand the required approach.

Example

Suppose you are trying to estimate the probability that Barry Bonds hit a home run in 2001. You observe 73 home runs in 476 at bats. Your prior distribution for θ is Beta(1,5), but you want to add the constraint that $\theta \in (0, 0.15)$. Using a Binomial likelihood, your model can be written as

$$\begin{aligned}
Y &\sim \text{Binomial}(n = 476, \theta) \\
\theta &\sim \text{Beta}(1, 5), \text{ for } \theta \in (0, 0.15)
\end{aligned}$$

To solve this problem using a constrained prior distribution, first do the unconstrained problem, then truncate and renormalize.

Step 1: The unconstrained problem

You know from class that the posterior of p is Beta(74, 404), so our posterior is given by

$$p(\theta|y) \propto \theta^{74}(1-\theta)^{404}, \text{ for } \theta \in (0, 1).$$

Step 2: Truncate

If the prior distribution is constrained to $\theta \in (0, 0.15)$, then the posterior density will be 0 anywhere outside of this range. To reflect this, we can reexpress the posterior:

$$p(\theta|y) \propto \theta^{74}(1-\theta)^{404}, \text{ for } \theta \in (0, 0.15).$$

Step 3: Renormalize

The truncated posterior is not a proper PDF at this point, since it will not integrate to 1. To remedy this, you need to renormalize the posterior. The normalizing constant is then the reciprocal of

$$\int_0^{0.15} \theta^{74}(1-\theta)^{404} d\theta.$$

Summary: General strategy

In the previous example, we applied Bayes' Theorem to derive a constrained posterior where the constraint was induced by the prior. Below is a derivation that proves we can use the approach illustrated above.

Let $\pi^C(\theta)$ denote the prior distribution constrained to $\theta \in \Theta_C$. This prior is formally defined by renormalizing the prior, $\pi(\theta)$:

$$\pi^C(\theta) = \frac{\pi(\theta)}{\int_{\Theta_C} \pi(\theta) d\theta}, \ \theta \in \Theta_C.$$

Next, apply Bayes' Theorem to derive the posterior:

$$\begin{aligned}
p(\theta|\mathbf{y}, \theta \in \Theta_C) &= \frac{f(\mathbf{y}|\theta)\pi^C(\theta)}{\int_{\Theta_C} f(\mathbf{y}|\theta)\pi^C(\theta)d\theta}, \theta \in \Theta_C \\
&= \frac{f(\mathbf{y}|\theta)\frac{\pi(\theta)}{\int_{\Theta_C}\pi(\theta)d\theta}}{\int_{\Theta_C} f(\mathbf{y}|\theta)\frac{\pi(\theta)}{\int_{\Theta_C}\pi(\theta)d\theta}}, \theta \in \Theta_C \\
&= \frac{\frac{f(\mathbf{y}|\theta)\pi(\theta)}{\int_{\Theta} f(\mathbf{y}|\theta)\pi(\theta)d\theta}}{\int_{\Theta_C}\frac{f(\mathbf{y}|\theta)\pi(\theta)}{\int_{\Theta} f(\mathbf{y}|\theta)\pi(\theta)d\theta}}, \theta \in \Theta_C \\
&= \frac{p(\theta|\mathbf{y})}{\int_{\Theta_C} p(\theta|\mathbf{y})d\theta}, \theta \in \Theta_C
\end{aligned}$$