Quasibinomial Logistic Regression

Logistic regression – Stat 230

Support for railroad referenda in 1870s Alabama

Research question:

Was voting on railroad referenda during the Reconstruction Era related to distance from the proposed railroad line and the racial composition of a community?

Hypotheses:

- Positive votes were inversely proportional to the distance a voter is from the proposed railroad
- racial composition of a community is hypothesized to be associated with voting behavior

Data

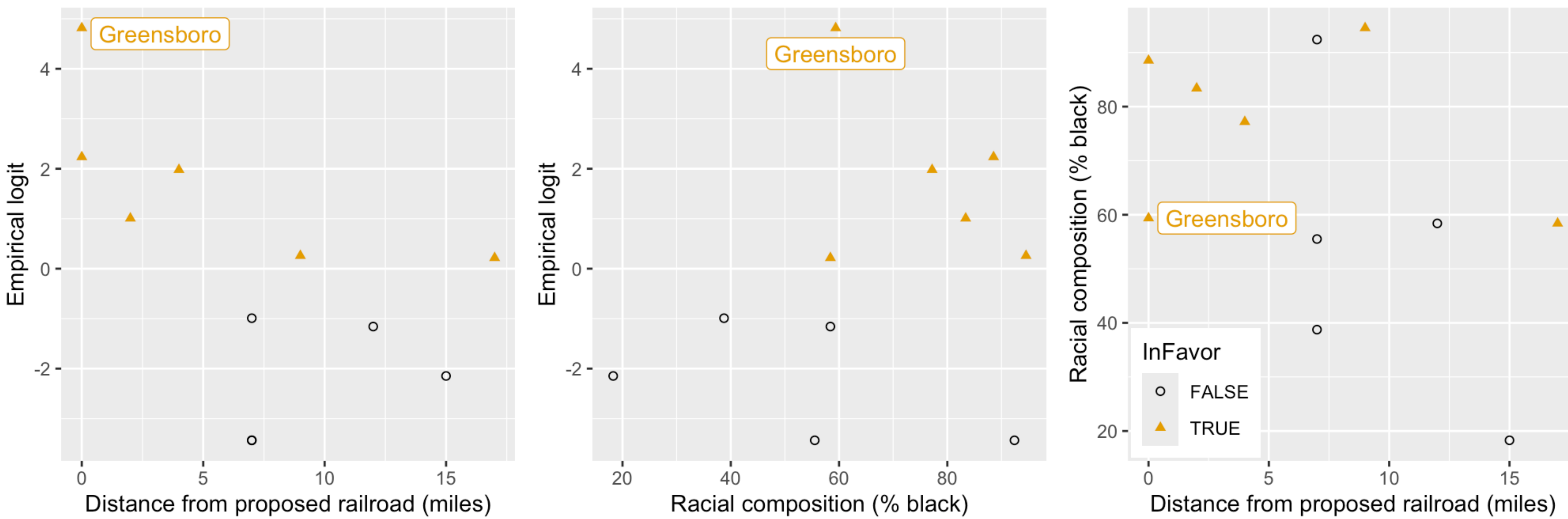
Michael Fitzgerald obtained data from the 1870 U.S. Census from communities in Hale County, Alabama

- `YesVotes` = the number of “Yes” votes in favor of the proposed railroad line (primary response variable)
- `NumVotes` = total number of votes cast in the election
- `pctBlack` = racial composition (% black)
- `distance` = the distance from the proposed railroad (in miles)

community	pctBlack	distance	YesVotes	NumVotes	propYes	InFavor
Carthage	58.40	17	61	110	0.555	TRUE
Cederville	92.40	7	0	15	0.000	FALSE
Five Mile Creek	18.28	15	4	42	0.095	FALSE
Greensboro	59.38	0	1790	1804	0.992	TRUE

EDA

Was voting on railroad referenda during the Reconstruction Era related to distance from the proposed railroad line and the racial composition of a community?



Is there evidence of lack of fit?

```
glm(formula = YesVotes/NumVotes ~ distance * pctBlack,
    family = binomial, data = rrdata, weights = NumVotes)

Coefficients:
                   Estimate Std. Error z value Pr(>|z|)
(Intercept)       7.5509017  0.6383697  11.828  < 2e-16
distance         -0.6140052  0.0573808 -10.701  < 2e-16
pctBlack         -0.0647308  0.0091723  -7.057 1.70e-12
distance:pctBlack  0.0053665  0.0008984   5.974 2.32e-09

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 988.45  on 10  degrees of freedom
Residual deviance: 274.23  on  7  degrees of freedom
```

Possible causes of lack of fit

Outliers

- Deviance in logistic regression is analogous to SSE in linear regression
- Outliers can inflate the deviance

Detection

- Deviance residual plots

Why do we care?

- Influential outliers result in biased estimates of the $\widehat{\beta}_i$s

Possible causes of lack of fit

Incorrect logit (mean) function

- We fit a line to a curve
- We omitted important predictor variables

Detection

- Empirical logit plots
- Deviance residual plots
- GOF test

Why do we care?

- Biased estimates of the $\hat{\beta}_i$s

Possible causes of lack of fit

Binomial model for Y is wrong

- Trials are not independent
- Probability of success is not the same across trials
- Important predictors might be omitted

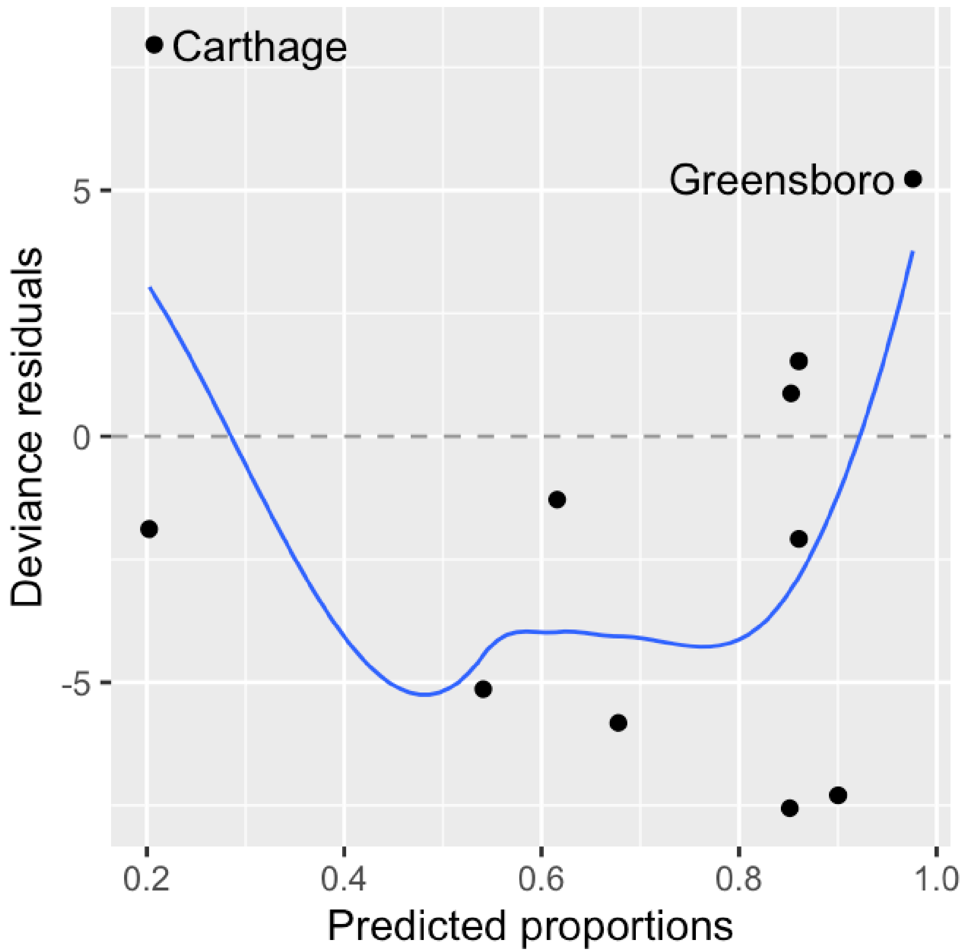
Detection

- Think
- GOF test
- Deviance residuals

Why do we care?

- Variance is greater than $n_i \pi(X_i)(1 - \pi(X_i))$
- SEs are likely too small $\implies$ p-values too small and CIs too narrow

Exploring lack of fit



- Since we have lack of fit, don't treat residuals as normal
- Greensboro not really an outlier
- Possible nonlinearity

Quadratic model

$$\text{logit}(\pi) = \beta_0 + \beta_1 \texttt{distance} + \beta_2 \texttt{pctBlack} + \beta_3 \texttt{distance}^2 \\ + \beta_4 \texttt{distance} \times \texttt{pctBlack}$$

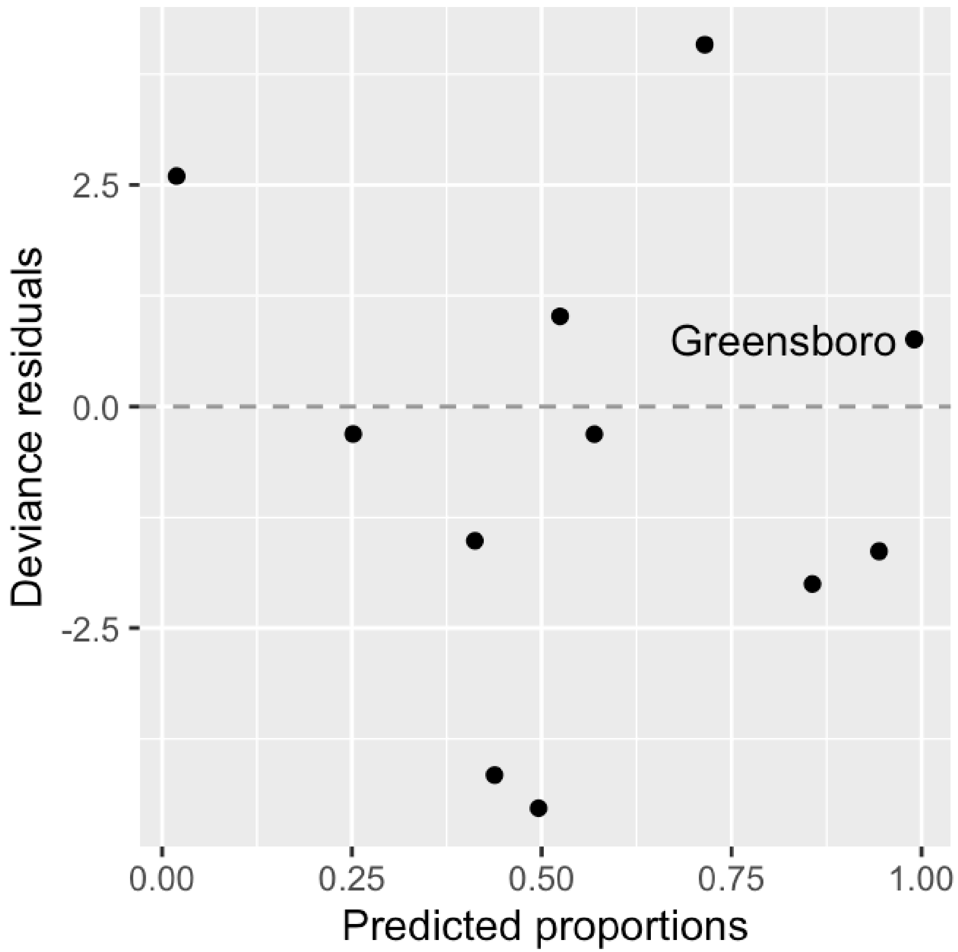
```
                   Estimate Std. Error z value Pr(>|z|)
(Intercept)        8.365538   0.919710   9.096  < 2e-16 ***
distance          -1.592867   0.131070 -12.153  < 2e-16 ***
pctBlack          -0.062498   0.012845  -4.866 1.14e-06 ***
I(distance^2)      0.044576   0.003388  13.156  < 2e-16 ***
distance:pctBlack  0.009830   0.001514   6.493 8.43e-11 ***

    Null deviance: 988.450  on 10  degrees of freedom
Residual deviance:  72.018  on  6  degrees of freedom
```

Is there still evidence of lack of fit?

```
1 - pchisq(72.018, df = 6)
```

```
[1] 1.575406e-13
```

Exploring lack of fit



Do you see anything concerning here?

Overdispersion

(extra-binomial variation)

Model assumptions

We **assume** that Y_i is binomial

$$Y|X_i \sim \text{Binomial}(n_i, \pi_i)$$

This implies that within the same subpopulation (combo of x_i s)

- trials are independent
- trials have the same probability of success
- $\text{E}(Y|X_i) = n_i \pi_i$
- $\text{Var}(Y|X_i) = n_i \pi_i (1 - \pi_i)$

Overdispersion

If the binomial assumptions are not met, then the variance of the Y_i will usually be larger than what is expected for a binomial distribution:

$$\text{Var}(Y|X_i) > n_i \pi_i (1 - \pi_i)$$

Overdispersion

Let $Z_1, \ldots, Z_m$ be iid Bernoulli (S/F) trials with probability of success π.

$$\begin{aligned}
Y &= Z_1 + \ldots + Z_m \\
\mathrm{Var}(Y) &= \mathrm{Var}(Z_1 + \ldots + Z_m) \\
&= \mathrm{Var}(Z_1) + \ldots + \mathrm{Var}(Z_m) + 2\sum_{i<j}\mathrm{Cov}(Z_i, Z_j) \\
&= \psi m\pi(1-\pi)
\end{aligned}$$

So what?

- p-values too small
- CIs too narrow

Ad hoc test of overdispersion

Recall:

$$D^2 = 2\sum \left[Y_i \log\left(\frac{Y_i}{n_i\hat{\pi}_i}\right) + (n_i - Y_i)\log\left(\frac{n_i - Y_i}{n_i - n_i\hat{\pi}_i}\right)\right]$$

If the model is correct and all *n_i* are large enough, $D^2 \dot{\sim} \chi^2$ with $\text{df} = n - (p+1)$

$$E\left(D^2\right) = \text{df} \implies \frac{D^2}{\text{df}} \approx 1$$

Red flag if $D^2/\text{df} \gg 1$

Model checking

If over-dispersion exists, check whether the assumptions are met

1. Do we have independence?
2. If assumptions met, then check for outliers.
3. If assumptions met, do we need to include an interaction term, some other term(s)?
4. If assumptions met, then maybe an incorrect model (binomial model not appropriate)?

Quasi-likelihood approach

1. Estimate β_i s using ML estimation, as before
2. Estimate ψ using $\widehat{\psi} = \dfrac{\text{deviance}}{\text{df}}$
3. Use $\text{SE}_{\text{quasibinomial}}(\widehat{\beta}_i) = \sqrt{\widehat{\psi}} \cdot \text{SE}_{\text{binomial}}(\widehat{\beta}_i)$ and use the t-distribution with $n - (p + 1)$ degrees of freedom
4. You can do a “drop-in-deviance” test using an F test:

$$\text{F} = \frac{[\text{deviance(reduced)} - \text{deviance(full)}]\,/d}{\widehat{\psi}}$$

where $\text{F} \sim F_{d,\, n-(p+1)}$

Ad-hoc adjustment

```
Coefficients:
                   Estimate Std. Error z value Pr(>|z|)
(Intercept)        8.365538   0.919710   9.096  < 2e-16 ***
distance          -1.592867   0.131070 -12.153  < 2e-16 ***
pctBlack          -0.062498   0.012845  -4.866 1.14e-06 ***
I(distance^2)      0.044576   0.003388  13.156  < 2e-16 ***
distance:pctBlack  0.009830   0.001514   6.493 8.43e-11 ***

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 988.45  on 10  degrees of freedom
Residual deviance: 72.018  on  6  degrees of freedom
```

Let's correct the test for whether $\beta_{\text{distance}} = 0$

Quasi-likelihood in R ("proactive" approach)

```r
glm(YesVotes/NumVotes ~ distance * pctBlack + I(distance^2),
    data = rrdata, weights = NumVotes, family = quasibinomial)
```

```
Coefficients:
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)        8.365538   3.038714   2.753  0.03316 *
distance          -1.592867   0.433054  -3.678  0.01035 *
pctBlack          -0.062498   0.042440  -1.473  0.19127
I(distance^2)      0.044576   0.011195   3.982  0.00727 **
distance:pctBlack  0.009830   0.005003   1.965  0.09701 .

(Dispersion parameter for quasibinomial family taken to be 10.91635)

    Null deviance: 988.450  on 10  degrees of freedom
Residual deviance:  72.018  on  6  degrees of freedom
```

R estimates the dispersion parameter differently than I showed you, that's why the results

Comparing models in R

You have to tell R to use an F-test to conduct the quasi-binomial drop-in-deviance test

```
full <- glm(YesVotes/NumVotes ~ distance * pctBlack + I(distance^2),
    data = rrdata, weights = NumVotes, family = quasibinomial)

reduced <- update(full, . ~ . - distance:pctBlack)

anova(reduced, full, test = 'F')
```

```
Analysis of Deviance Table

Model 1: YesVotes/NumVotes ~ distance + pctBlack + I(distance^2)
Model 2: YesVotes/NumVotes ~ distance * pctBlack + I(distance^2)
  Resid. Df Resid. Dev Df Deviance      F  Pr(>F)
1         7    118.302
2         6     72.018  1   46.284 4.2399 0.08516 .
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```