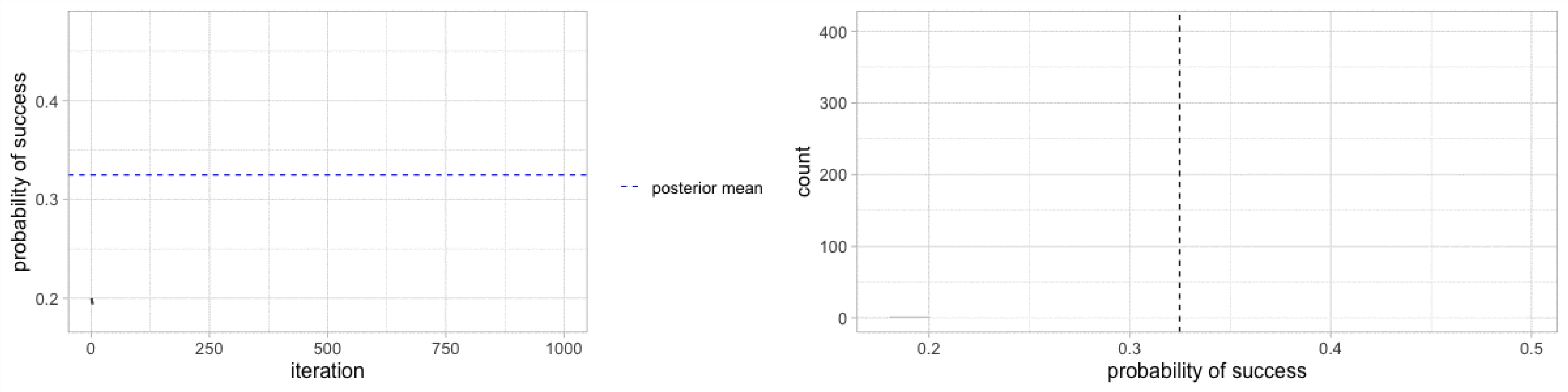
The Gibbs sampler

Stat 340: Bayesian Statistics

1. Gibbs sampler
2. Convergence checks
3. Inference using MCMC draws

(Problem topics 8, 10, 11)

Metropolis algorithm



Your turn

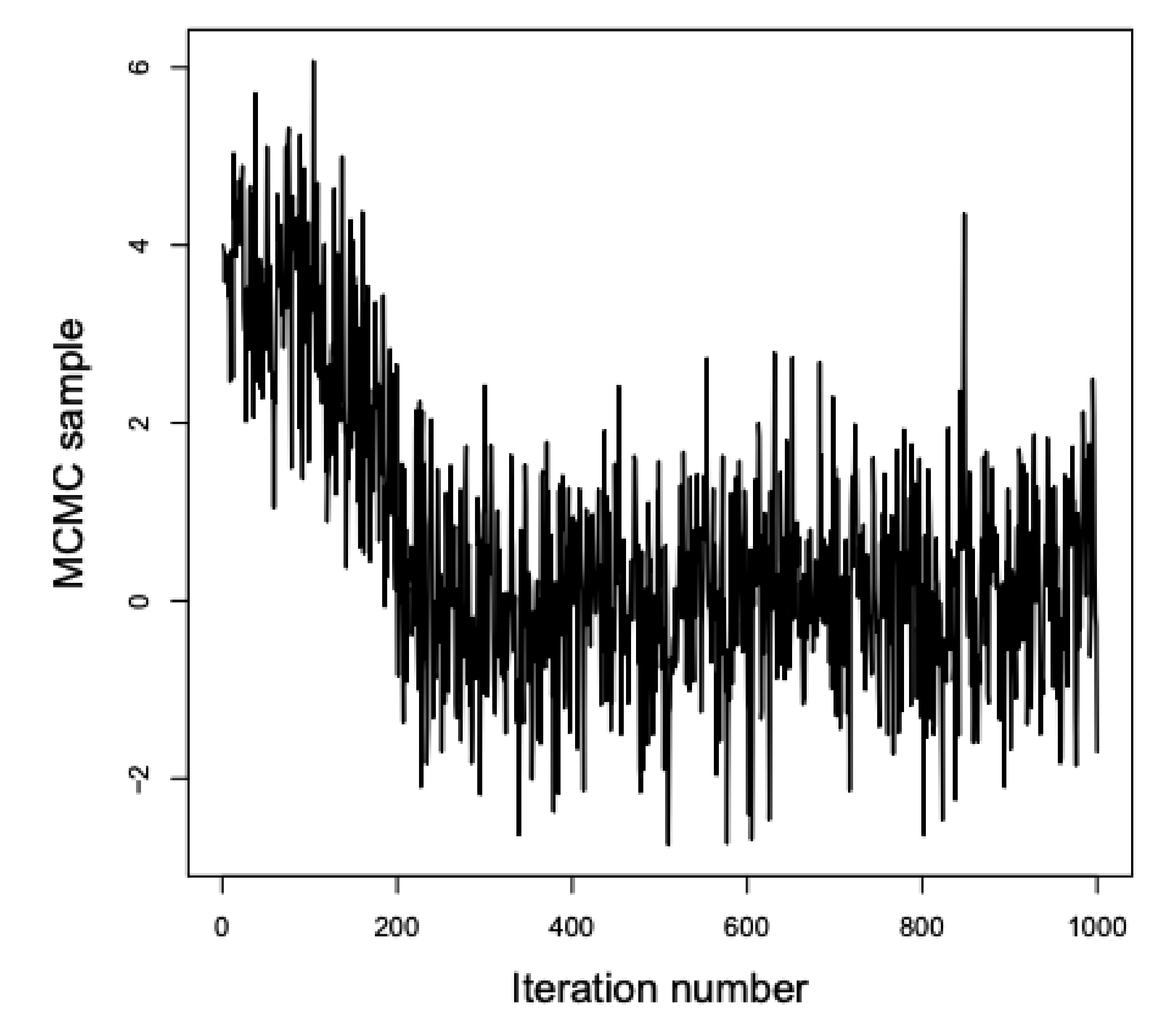
Take a look at the four trace plots provided as an example. For each, determine

1. whether the chain converged
2. roughly how many iterations it took to converge

03:00

1. Did the chain converge?

2. If so, how many iterations did it take?



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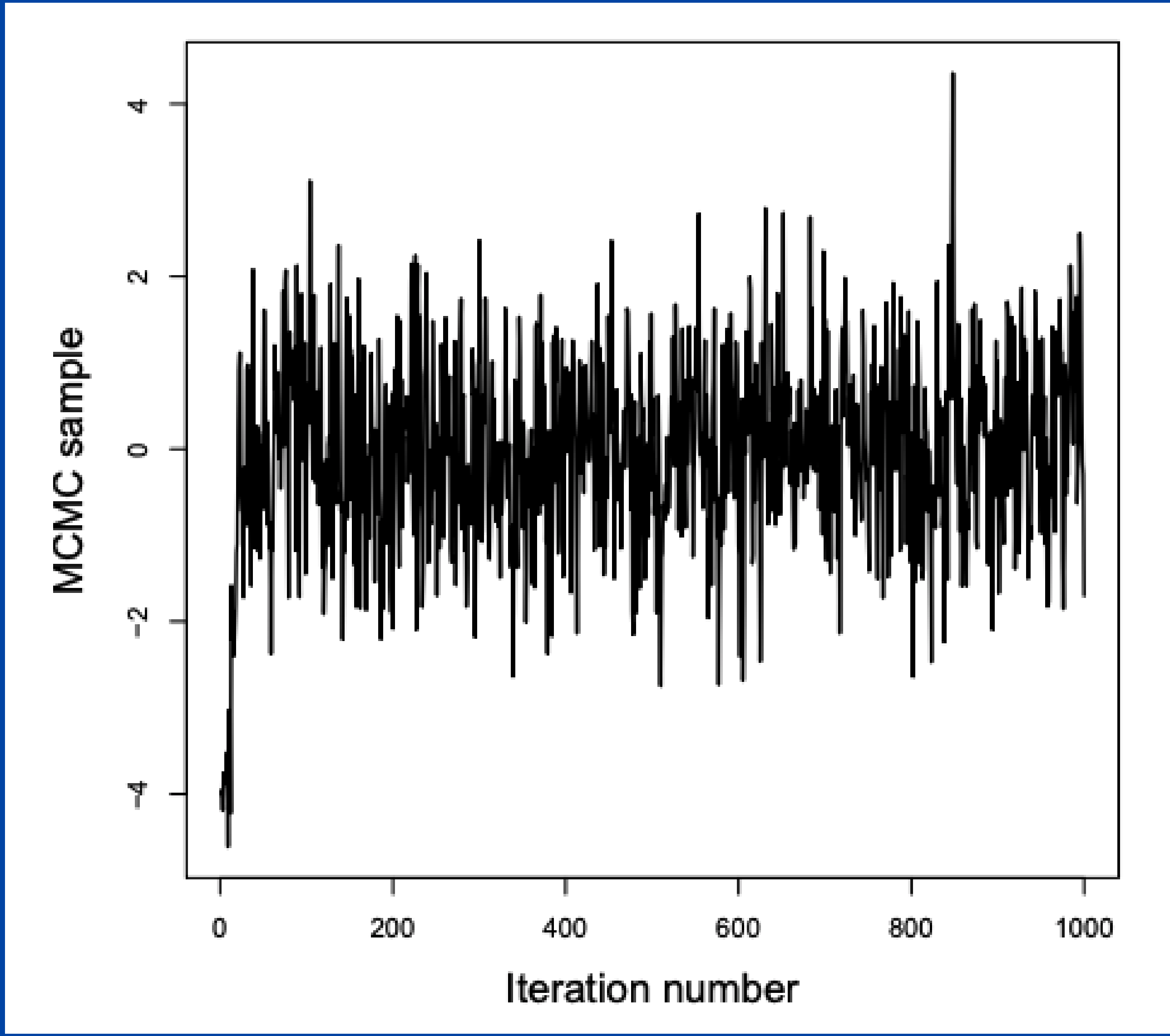

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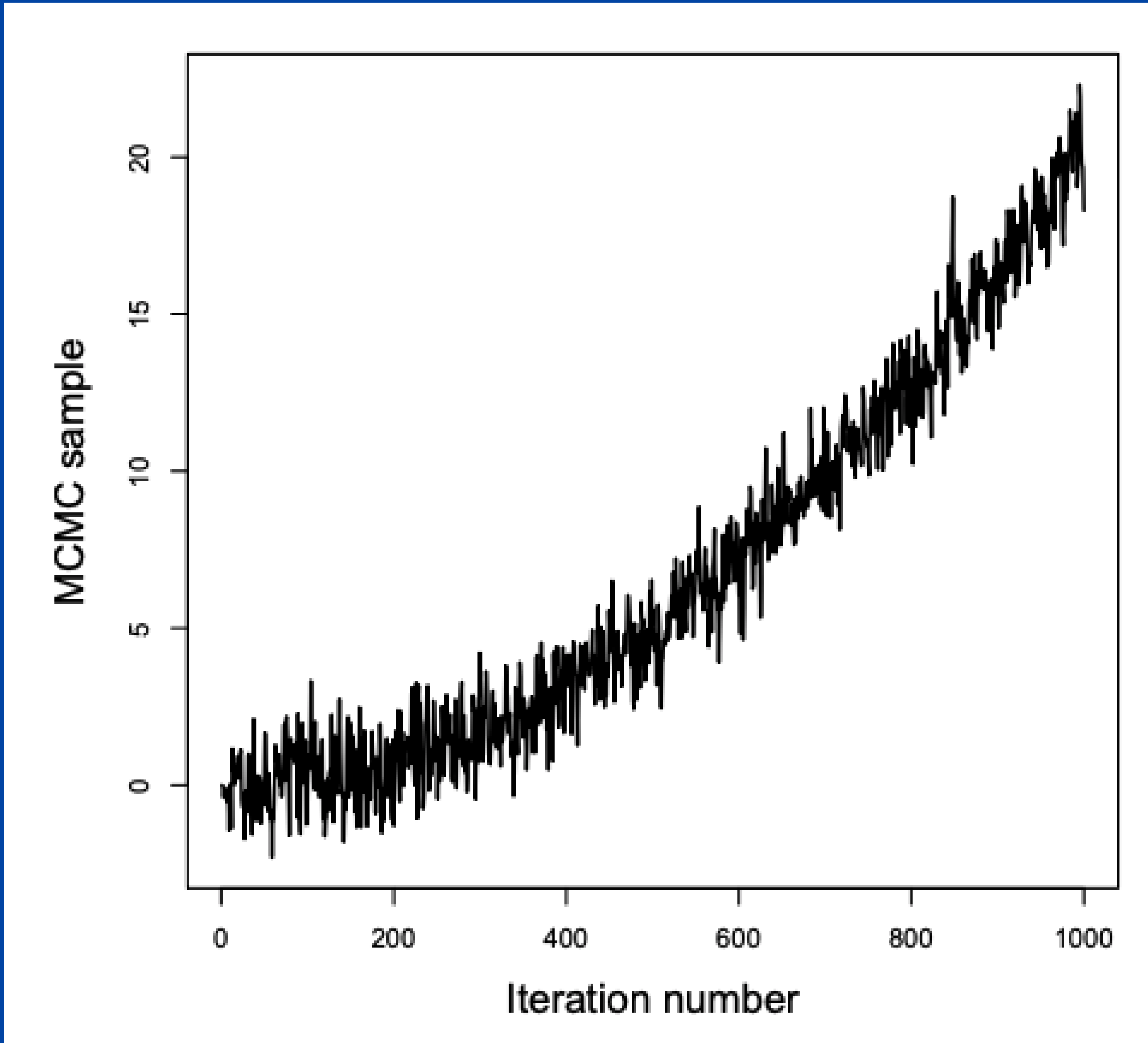

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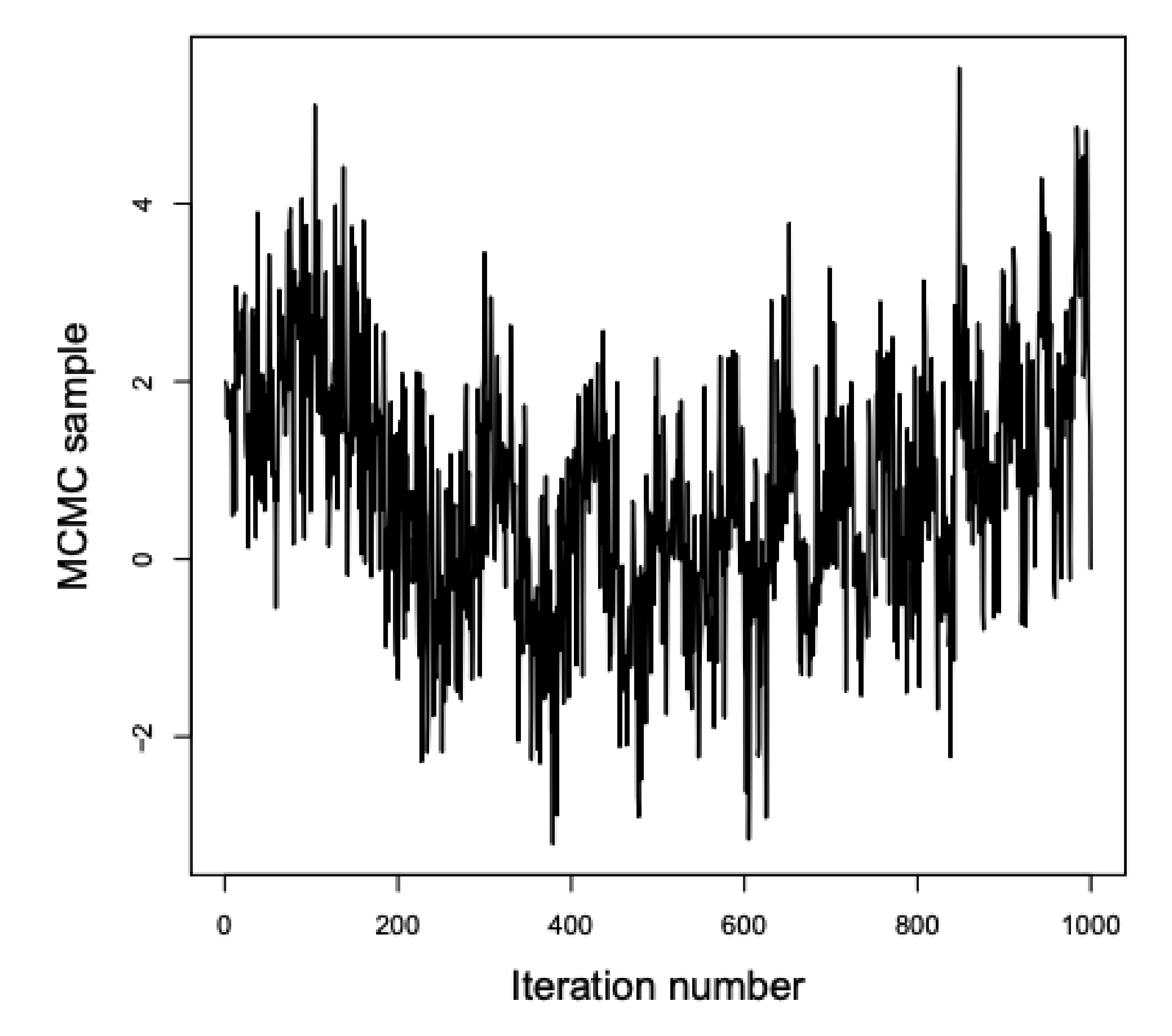


Simulating from a discrete bivariate distribution

Example: Traffic

The joint PMF of the number of cars (X) and the number of buses (Y) per signal cycle at a proposed left-turn lane is given below

X=0	Y=0	1	2
0	0.025	0.015	0.01
1	0.050	0.030	0.02
2	0.125	0.075	0.05
3	0.150	0.090	0.06
4	0.100	0.060	0.04
5	0.050	0.030	0.02

Simulation process

Condition on $X = 1$, simulate a Y

X=0	Y=0	1	2
0	0.025	0.015	0.01
1	0.050	0.030	0.02
2	0.125	0.075	0.05
3	0.150	0.090	0.06
4	0.100	0.060	0.04
5	0.050	0.030	0.02

```
sample(0:5, size = 1, prob = bivariate[[2]])
```

```
## [1] 3
```

Condition on $Y = 3$, simulate an X

X=0	Y=0	1	2
0	0.025	0.015	0.01
1	0.050	0.030	0.02
2	0.125	0.075	0.05
3	0.150	0.090	0.06
4	0.100	0.060	0.04
5	0.050	0.030	0.02

```
sample(0:2, size = 1, prob = bivariate[4,2:4])
```

```
## [1] 1
```

Simulating from a bivariate continuous distribution

Change in the democratic vote

- How did democratic share of the two party vote change from 2016 to 2020?
- MIT Election Data and Science Lab has county-level election results
- We'll look at the percent change in the two-party vote

$$Y_i = 100\,(A_i/B_i - 1)$$

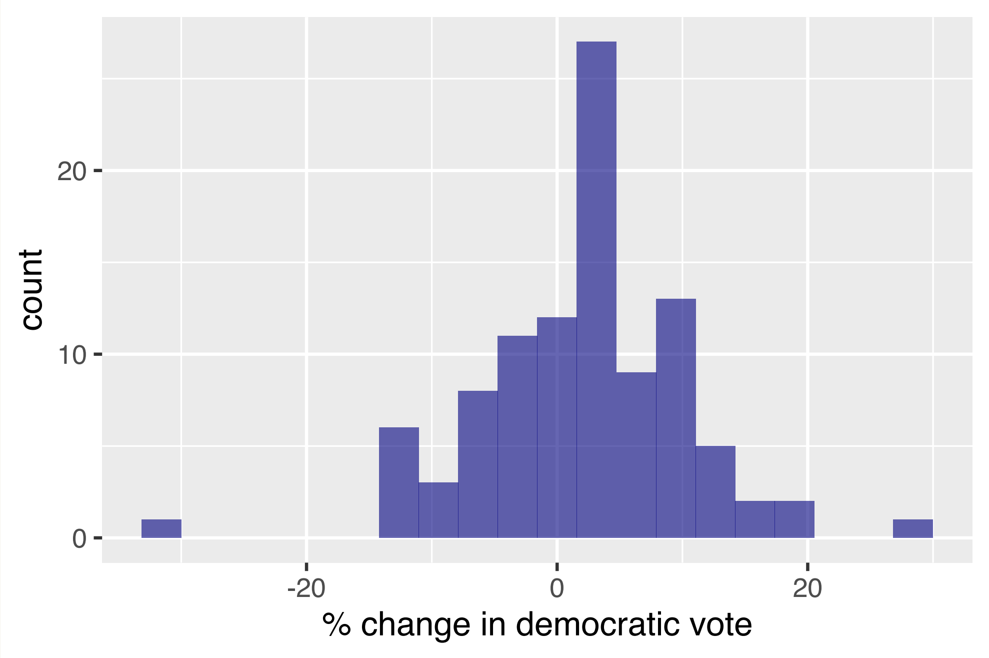
$A_i =$ % of two-party vote cast for democrats in 2020

$B_i =$ % of two-party vote cast for democrats in 2016

MIT Election Data and Science Lab, "County Presidential Election Returns 2000-2020", https://doi.org/10.7910/DVN/VOQCHQ, Harvard Dataverse, V9

Hypothetical sample

We'll work with a hypothetical sample of 100 counties



$\bar{y} = 1.96$

$n = 100$

Model

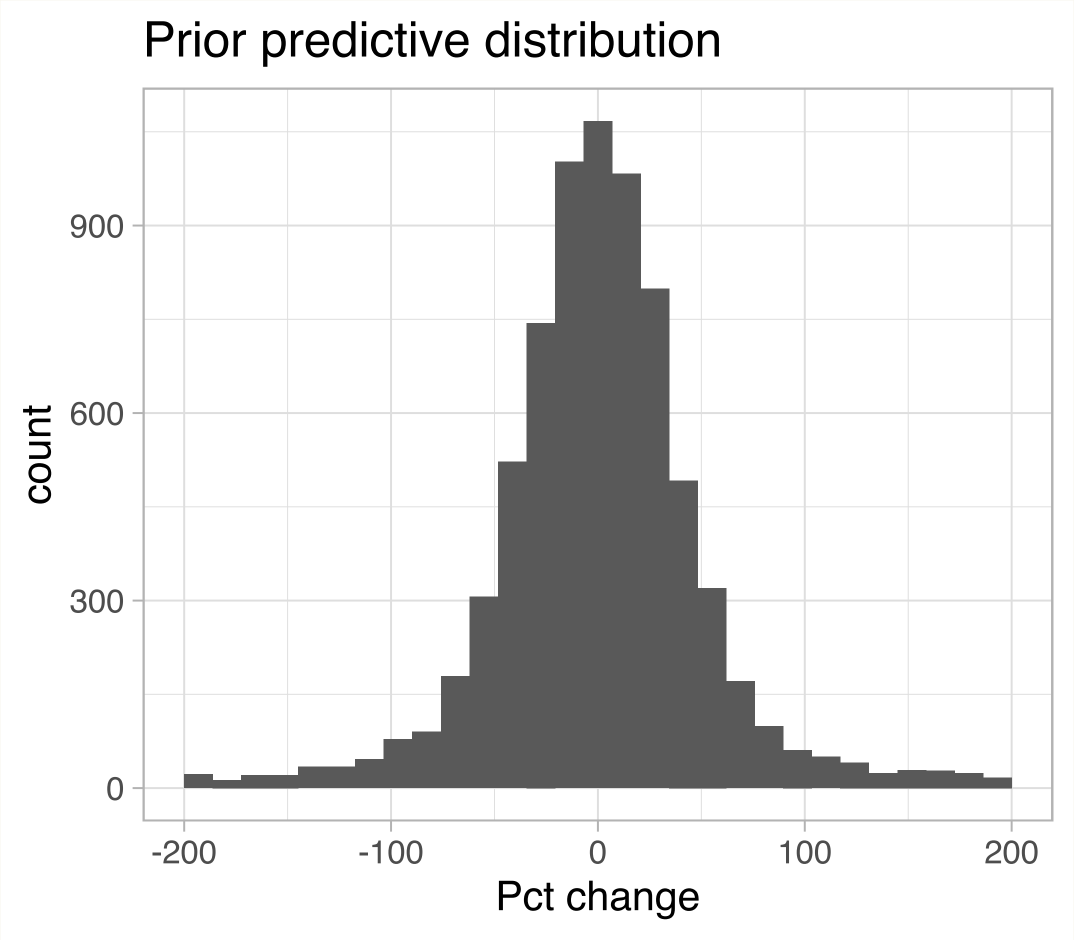
$Y_i = 100(A_i/B_i - 1)$, the percent change in support

$$\begin{aligned} Y_i|\mu,\sigma &\overset{\text{iid}}{\sim} \mathcal{N}(\mu,\sigma) \\ \mu &\sim \mathcal{N}(\mu_0, \sqrt{1/\phi_0}) \\ \phi = 1/\sigma^2 &\sim \text{Gamma}(a,b) \\ \mu &\perp \phi \end{aligned}$$

$$\begin{aligned} \mu_0 &= 0 \\ \phi_0 &= 1/1000 \\ a &= 0.1 \\ b &= 0.1 \end{aligned}$$

These are very weak priors

Prior predictive check



If you don't think the priors induce a reasonable distribution on Y, then tweak the parameters (e.g. inflate σ_0)

Posterior

$$\pi(\mu, \phi | y_1, \ldots, y_n) \propto \pi(\mu)\pi(\phi) \cdot \prod_{i=1}^{n} f(y_i | \mu, \sigma^2)$$
$$\propto \exp\left[-\frac{\phi_0}{2}(\mu - \mu_0)^2\right] \cdot \phi^{a-1} \exp[-b\phi] \cdot$$
$$\prod_{i=1}^{n} \phi^{1/2} \exp\left[-\frac{\phi}{2}(y_i - \mu)^2\right]$$

How can we do this *efficiently* sample from this 2D posterior?

Two-stage Gibbs sampler

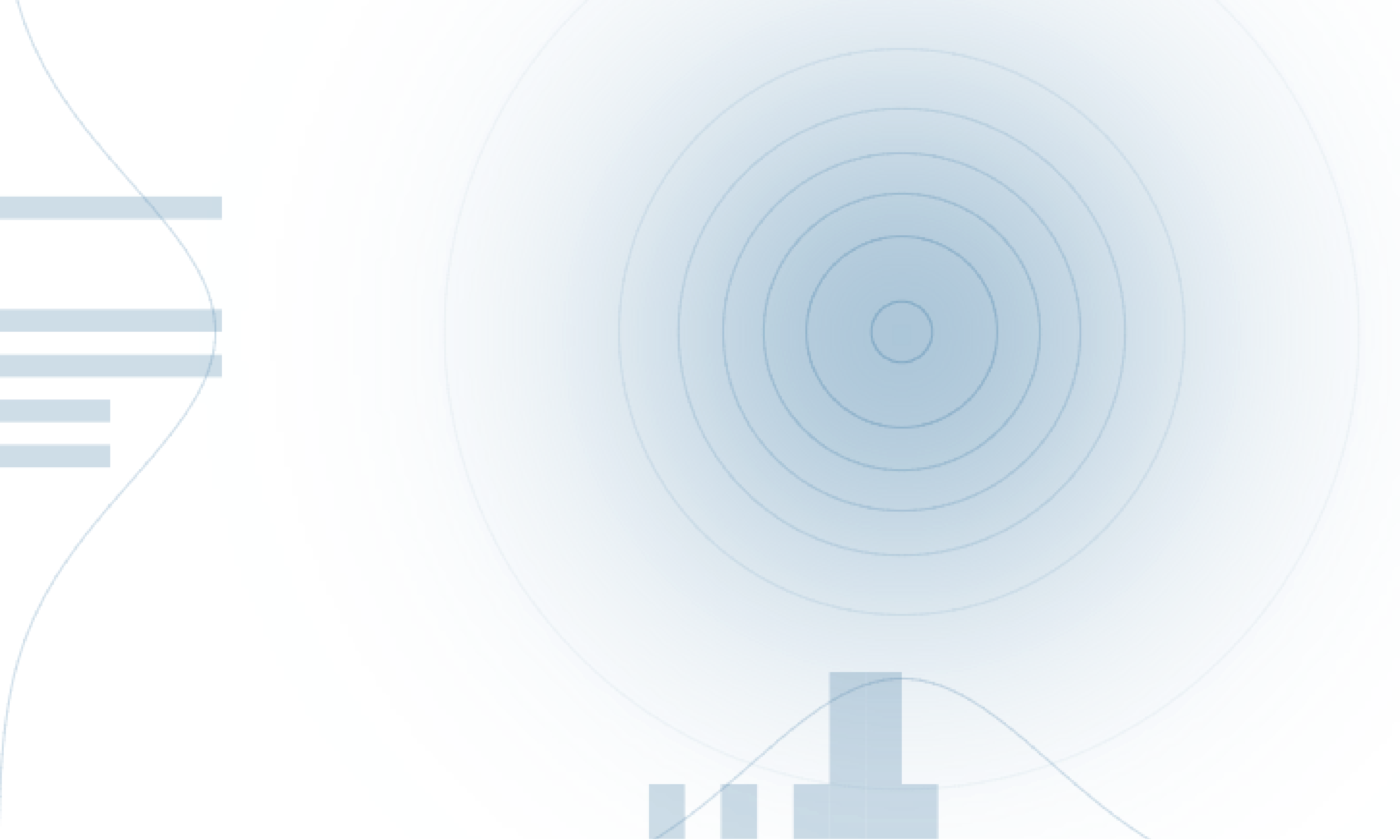
Target: samples from $\pi(\theta_1, \theta_2 | y_1, \ldots, y_n)$

Algorithm:

1. Set initial values for parameter values, $\boldsymbol{\theta}^{(0)} = \left(\theta_1^{(0)}, \theta_2^{(0)}\right)$
2. Draw $\theta_1^{(1)}$ from $\pi(\theta_1 | \theta_2, y_1, \ldots, y_n)$
3. Draw $\theta_2^{(1)}$ from $\pi(\theta_2 | \theta_1, y_1, \ldots, y_n)$
4. Repeat steps 2-3 S times

After convergence, draws $\left(\theta_1^{(k)}, \theta_2^{(k)}\right)$ are from the posterior distribution

Animated Gibbs sampler



Your turn

In our example, $\boldsymbol{\theta} = (\mu, \sigma^2)$

Discuss with your neighbor **how** you would find the following conditional posterior distributions from the joint posterior:

1. $\pi(\mu|\phi, y_1, \ldots, y_n)$

2. $\pi(\phi|\mu, y_1, \ldots, y_n)$

02:00

Full conditional distributions

$$
\begin{aligned}
\pi(\phi|y_1,\ldots,y_n,\mu) &\propto \pi(\phi) f(y_1,\ldots,y_n|\phi,\mu) \\
&\propto \phi^{a-1}\exp[-b\phi]\cdot\prod_{i=1}^{n}\phi^{1/2}\exp\left[-\frac{\phi}{2}(y_i-\mu)^2\right] \\
&= \phi^{a-1}\exp[-b\phi]\cdot\phi^{n/2}\exp\left[-\frac{\phi}{2}\sum_{i=1}^{n}(y_i-\mu)^2\right] \\
&= \phi^{(n/2+a)-1}\exp\left[-\phi\left\{\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2+b\right\}\right]
\end{aligned}
$$

Is this a distribution we have seen before?

Full conditional distributions

$$
\begin{aligned}
\pi(\mu|\phi, y_1, \ldots, y_n) &\propto \pi(\mu) \prod_{i=1}^{n} f(y_i|\mu, \sigma^2) \\
&\propto \exp\left[-\frac{\phi_0}{2}(\mu-\mu_0)^2\right] \cdot \prod_{i=1}^{n} \phi^{1/2} \exp\left[-\frac{\phi}{2}(y_i-\mu)^2\right] \\
&= \exp\left[-\frac{\phi_0}{2}(\mu-\mu_0)^2\right] \cdot \phi^{n/2} \exp\left[-\phi\left\{\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^2\right\}\right] \\
&\propto \exp\left[-\frac{\phi_0+n\phi}{2}\left\{\mu - \frac{\mu_0\phi_0+n\bar{y}\phi}{\phi_0+n\phi}\right\}^2\right]
\end{aligned}
$$

Is this a distribution we know?

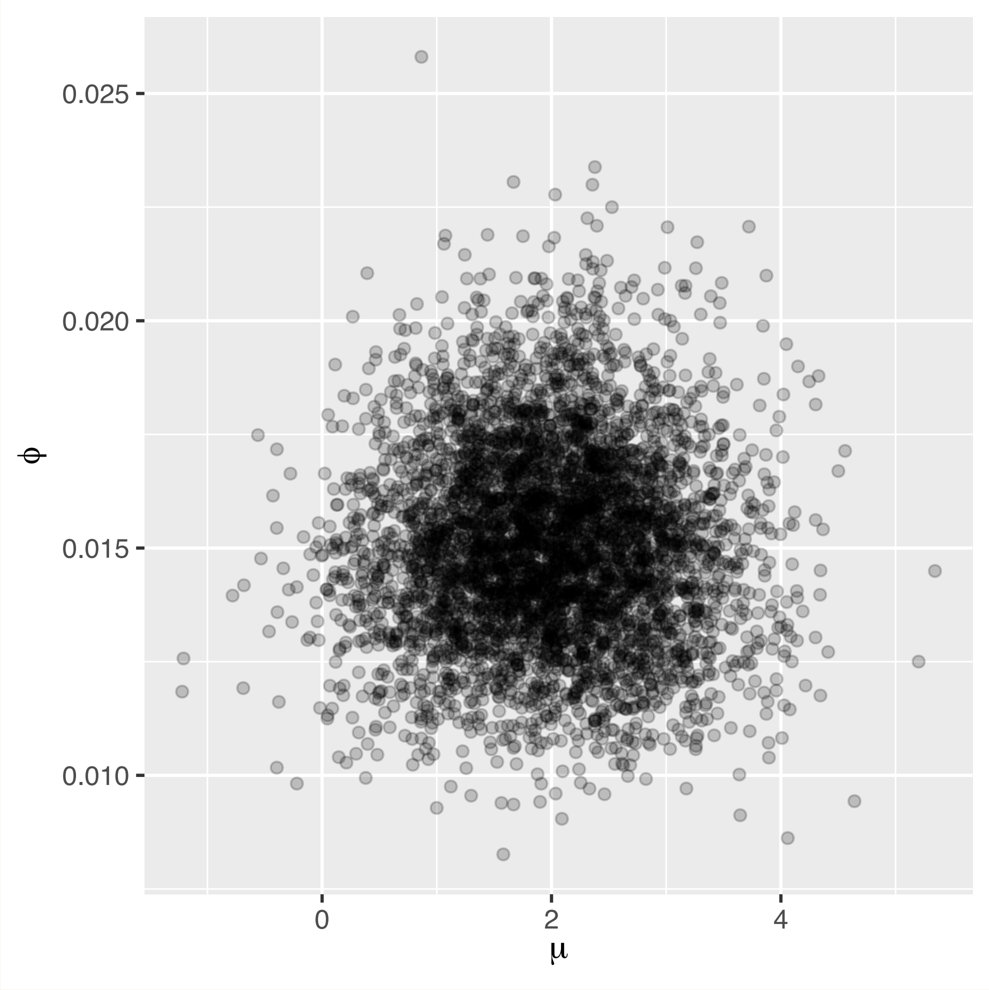
Getting ready to sample

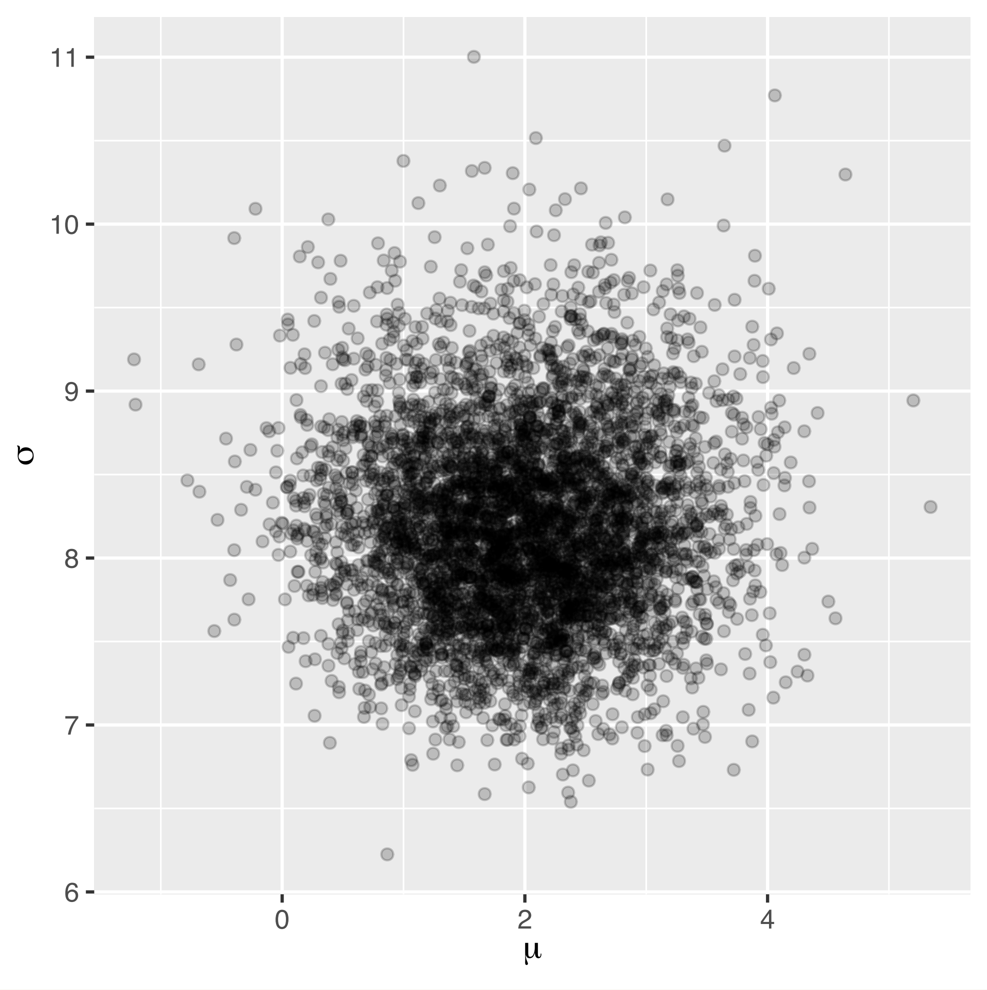
```r
# Data
y <- select_county$pct_change_dem
n <- length(y)

# Prior specification
mu0  <- 0
phi0 <- 1/1000
a    <- 0.1
b    <- 0.1

# Initial parameter values
mu <- mean(y)
s2 <- var(y)
phi <- 1 / s2


# Create empty S x p matrix for MCMC draws
S                    <- 5000
mcmc_draws           <- matrix(NA, nrow = S, ncol = 2)
colnames(mcmc_draws) <- c("mu", "phi")
```

Gibbs sampler

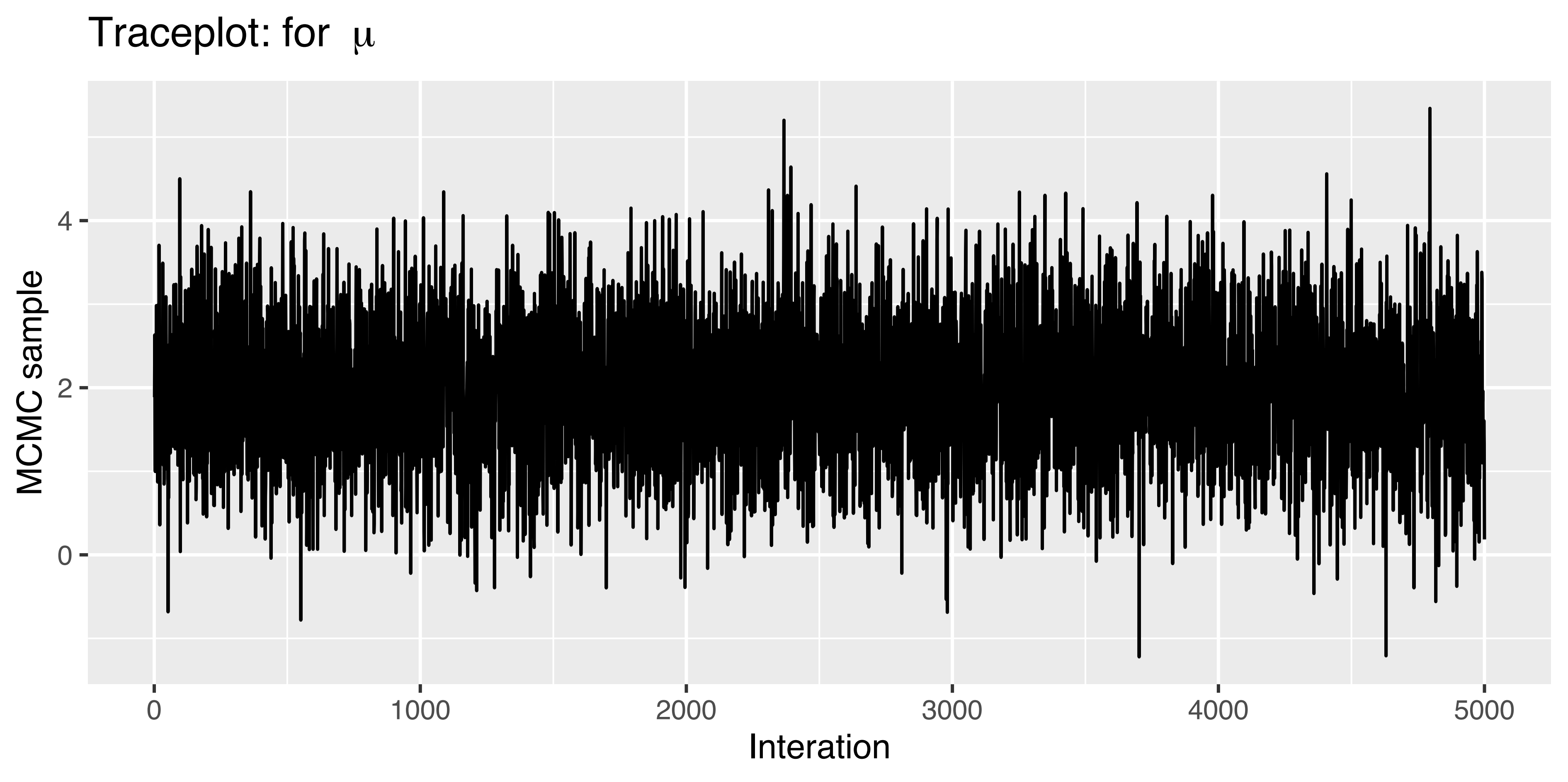
```r
for(i in 1:S) {

  # sample from mu | s2, y
  A    <- sum(y) * phi + mu0 * phi0
  B    <- n * phi + 1 * phi0
  mu   <- rnorm(1, A/B, 1/sqrt(B))

  # sample from s2 | mu, y
  shape  <- n / 2 + a
  scale  <- (sum((y - mu)^2) / 2) + b
  phi    <- rgamma(1, shape, scale)

  # Store the draws
  mcmc_draws[i, ] <- c(mu, phi)
}
```

To get the joint posterior of interest, $\pi(\mu, \sigma | y_1, \ldots, y_n)$, transform ϕ





Did the chain converge?



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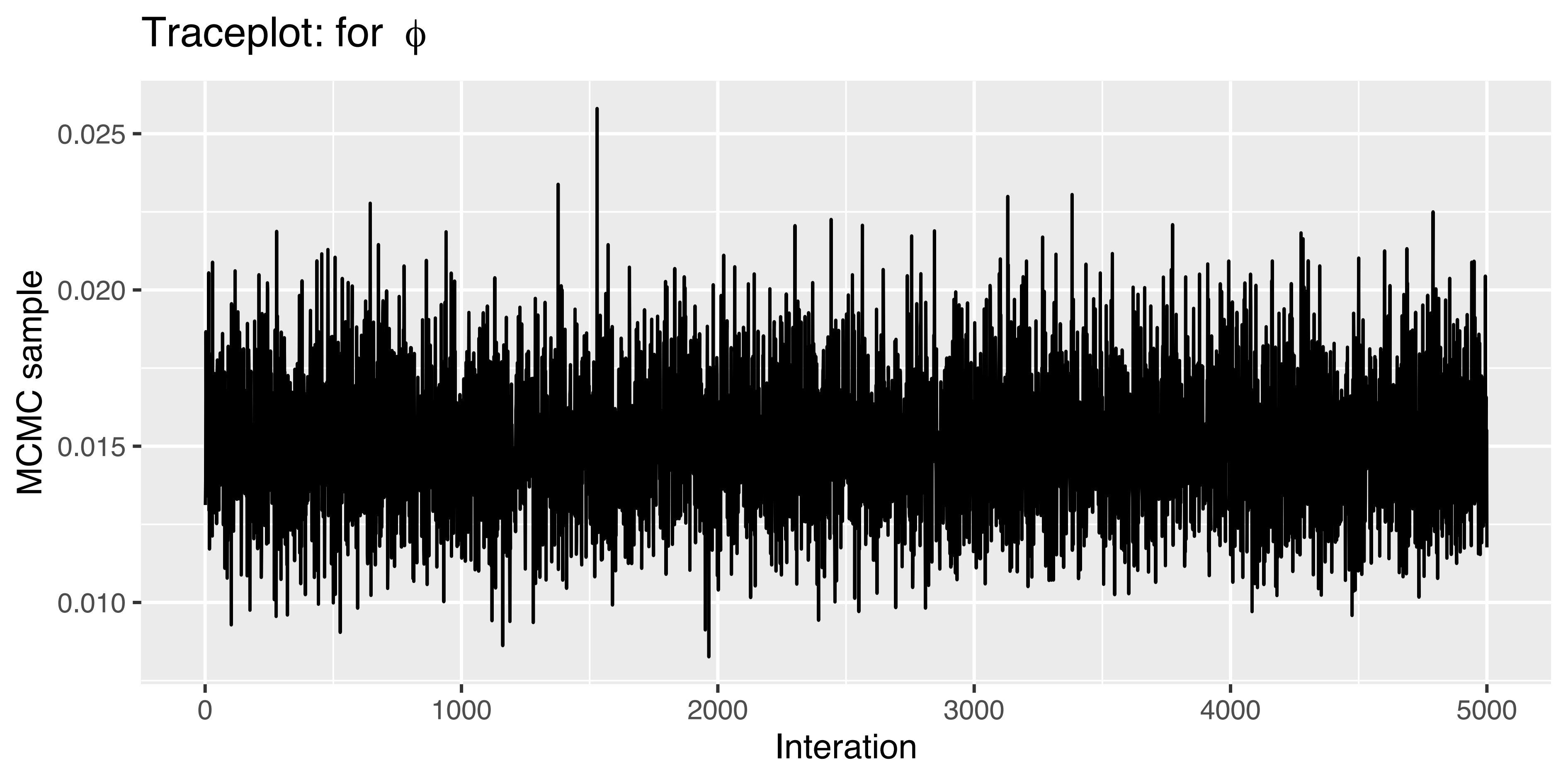
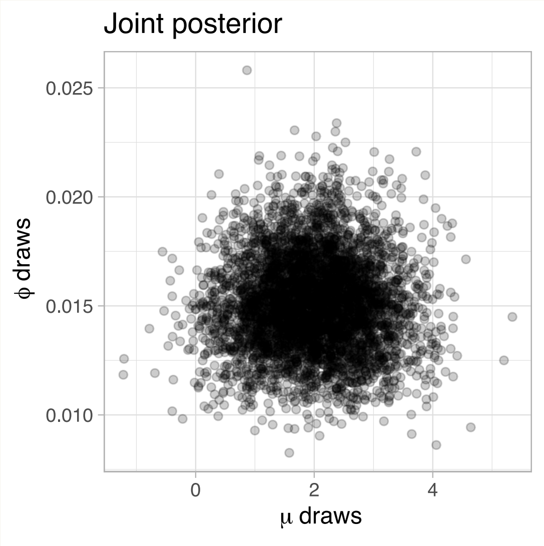


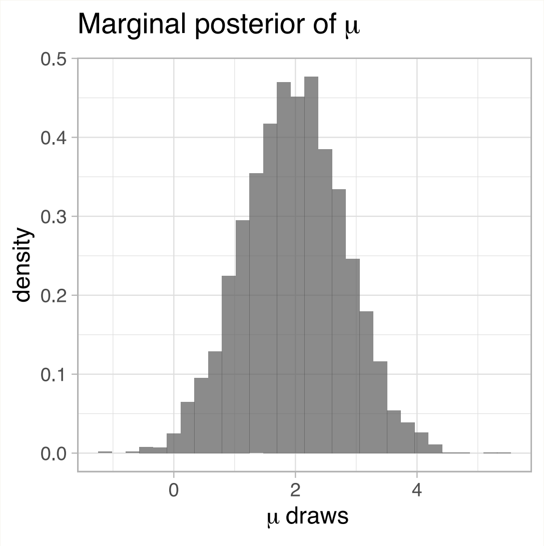
Posterior analysis

Toss out samples prior to convergence (this is called the *burn in* period)

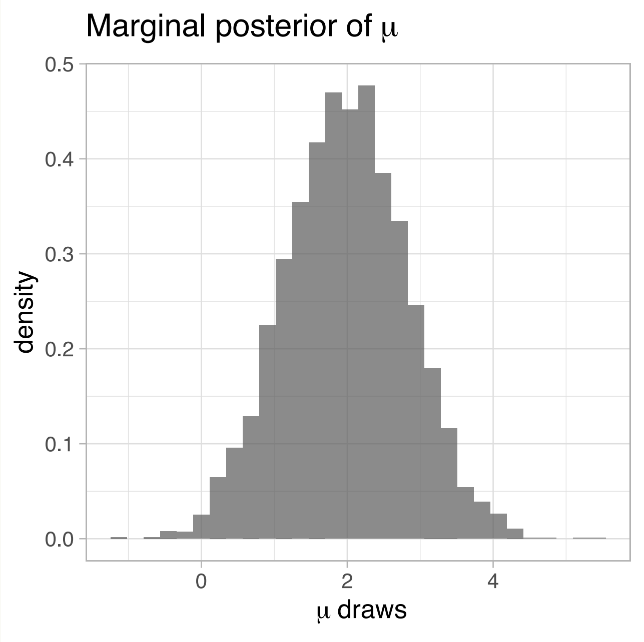
Draw inferences using the remaining MCMC samples just like we have all term

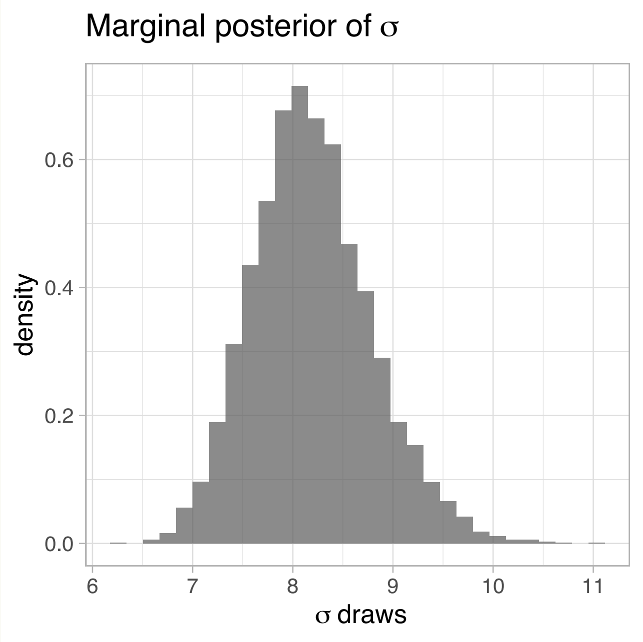
```
no_burn <- mcmc_draws[-c(1:100),]
```





Posterior analysis





	Mean	SD	Q025	Q975
mu	1.98	0.83	0.35	3.61
sigma	8.19	0.59	7.13	9.48

p-stage Gibbs sampler

Target: samples from $\pi(\theta_1, \theta_2, \ldots, \theta_p | y_1, \ldots, y_n)$

1. Set initial values for parameter values, $\boldsymbol{\theta}^{(0)} = \left(\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_p^{(0)}\right)$

2. Draw $\theta_1^{(1)}$ from $\pi(\theta_1 | \theta_2, \ldots, \theta_p, y_1, \ldots, y_n)$

3. Draw $\theta_2^{(1)}$ from $\pi(\theta_2 | \theta_1, \theta_3, \ldots, \theta_p, y_1, \ldots, y_n)$

$\vdots$

p. Draw $\theta_2^{(1)}$ from $\pi(\theta_p | \theta_1, \ldots, \theta_{p-1}, y_1, \ldots, y_n)$

Repeat steps *2-p* S times