# Hierarchical beta-binomial model

Stat 340: Bayesian Statistics

Motivating experiment

- Imagine a single toxicity experiment performed on rats. Lots of those are carried out before drugs are approved for use in humans.

- θ is the probability that a rat receiving no treatment in an experiment develops a tumor.

- Current experiment: $n = 14$ rats in the study, and $y = 4$ rats develop a tumor.

- Previous experiments: the same experiment has been performed with 70 other groups of rats

  - $y_j = \#$ rats with tumors in jth experiment
  - $n_j = $ sample size in jth experiment
  - $\theta_j = $ probability that a rat receiving no treatment in an experiment develops a tumor in jth experiment

Structure of hierarchical model
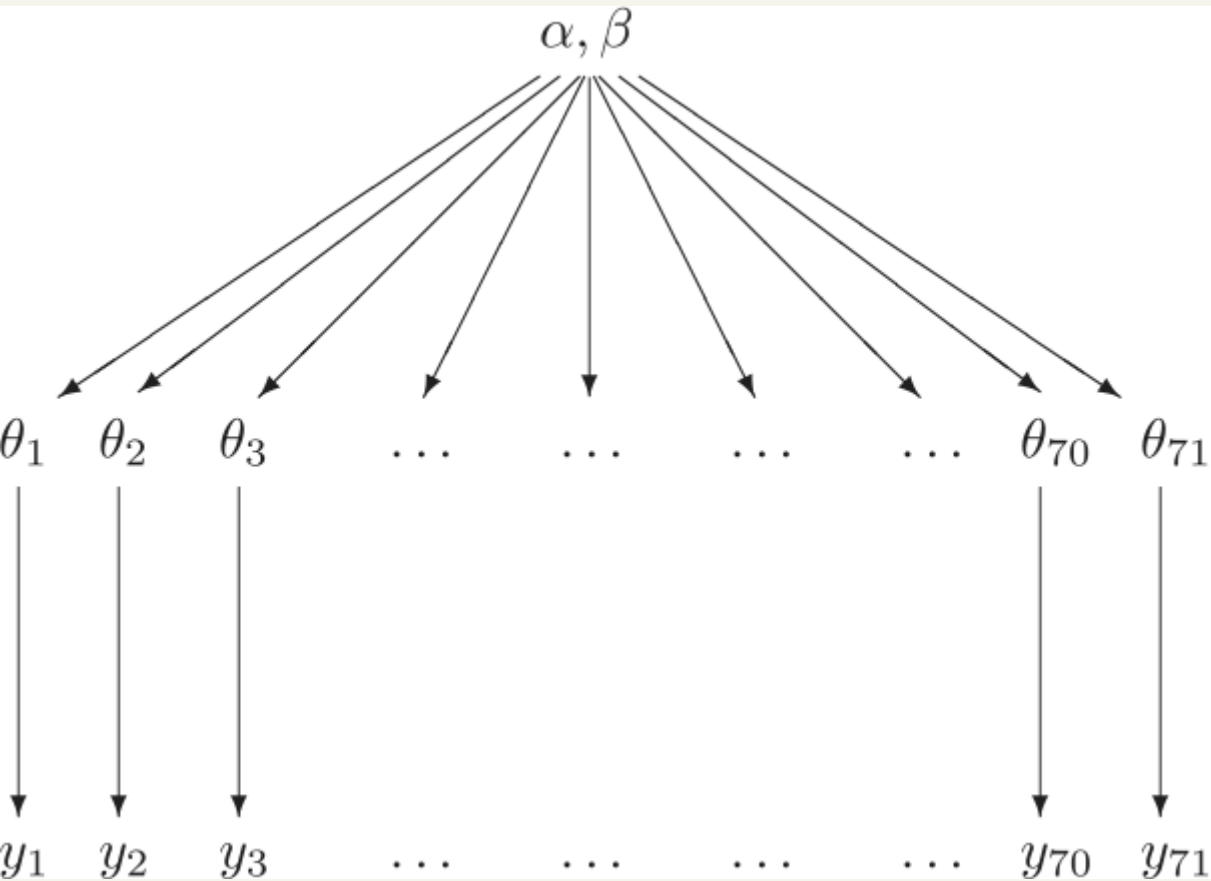
Assumption:

current tumor risk, θ_{71}, and the 70 historical risks, $\theta_1, \ldots, \theta_{70}$, are a random sample from a common distribution

Fig. 5.1 in Gelman et al. (2004)

Posterior derivations

Before moving on to hyper-prior selection and tuning, let's explore how to derive out posterior distributions

Hyper priors (second stage)

- In hierarchical models, the choice of the hyper-prior is important because it is possible to end up with an improper posterior distribution.

- For example, in this beta-binomial example a uniform prior on α, β does not work because the posterior distribution of α, β is non-integrable.

- An approach that often makes the choice of hyper prior easier is to **think of functions of the parameters that are more intuitive.**

Beta-binomial hyperprior

What are functions of α and β that are more intuitive?

Prior mean:
$$\mu = \frac{\alpha}{\alpha + \beta}$$

Prior sample size: $\eta = \alpha + \beta$

Approx. prior SD: $\eta^* = (\alpha + \beta)^{-1/2}$

Noninformative hyperprior - option 1

One way to develop a noninformative prior would be to place uniform densities on (μ, η)

$$\mu = \frac{\alpha}{\alpha + \beta} \sim \text{Beta}(1, 1)$$

$$\eta = \alpha + \beta \sim 1$$

What does this imply about the prior of α, β on the original scale?

We need to find the Jacobian of the inverse transformation to answer this.

Noninformative hyperprior - option 2

An alternative approach is to place uniform densities on prior the mean and approximate SD

$$\mu = \frac{\alpha}{\alpha + \beta} \sim \mathrm{Beta}(1, 1)$$

$$\eta^* = (\alpha + \beta)^{-1/2} \sim 1$$

The implied prior for (α, β) is given by

$$\pi(\alpha, \beta) \propto (\alpha + \beta)^{-5/2}$$

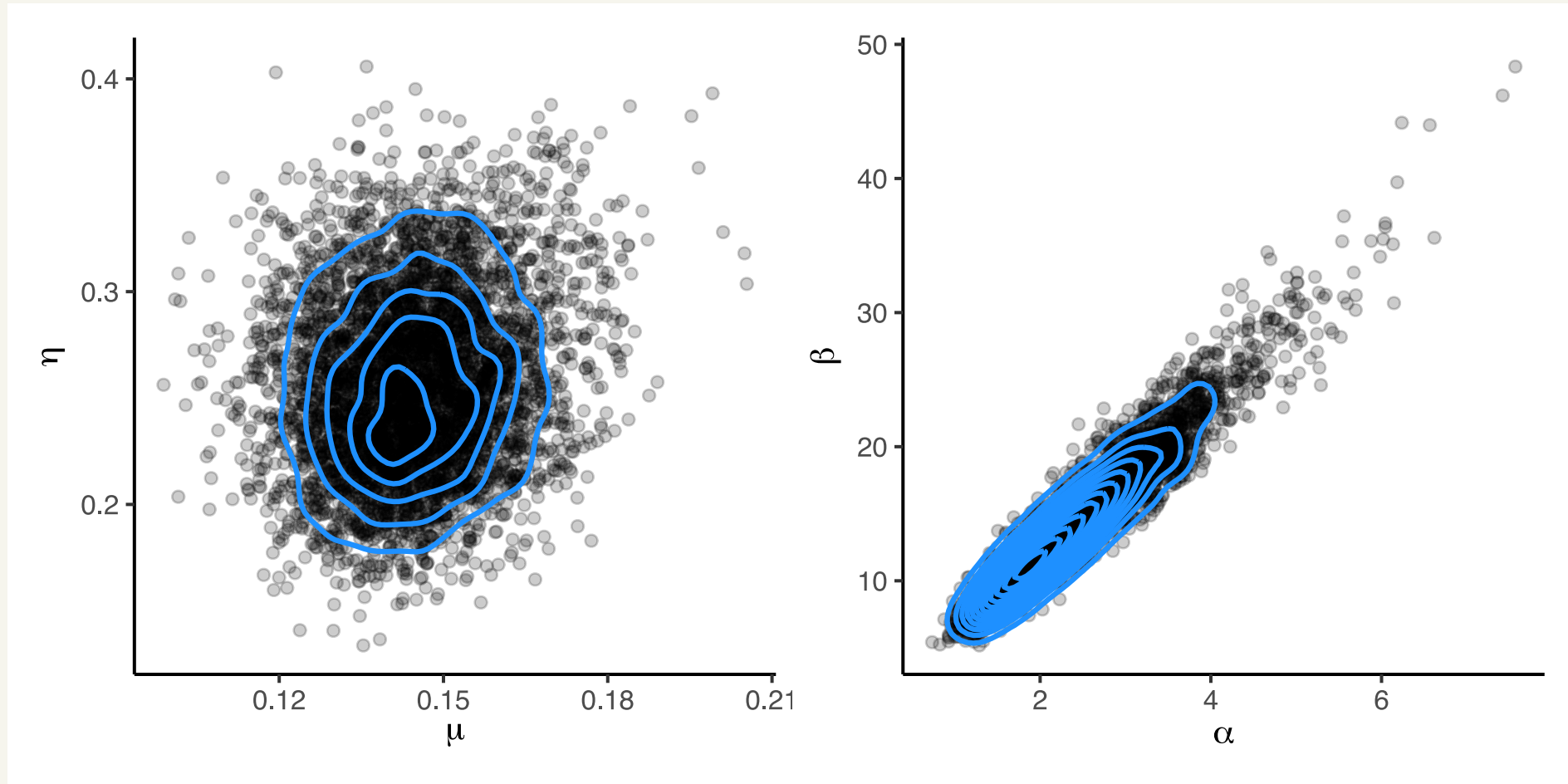which is a proper prior distribution (should be more efficient)

Noninformative priors in JAGS

```
noninform_model<-"
model {
## sampling
for (i in 1:N){
    y[i] ~ dbin(theta[i], n[i])
}

## priors
for (i in 1:N){
    theta[i] ~ dbeta(alpha, beta)
}

## noninformative hyperpriors
alpha <- mu / pow(eta, 2)
beta  <- (1 - mu) / pow(eta, 2)
mu ~ dbeta(1, 1)
eta ~ dbeta(1, 1)
}"
```

Comparing posterior parameterizations

Estimated vs observed tumor rates

Comparing posterior parameterizations