Trigonometry

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Co-function identities:

$\sin(\pi/2-\alpha) = \cos\alpha$; $\cos(\pi/2-\alpha) = \sin\alpha$; $\tan(\pi/2-\alpha) = \cot\alpha$; $\cot(\pi/2-\alpha) = \tan\alpha$.

Pythagorean identities:

$\sin^2\alpha + \cos^2\alpha = 1$; $\sec^2\theta - \tan^2\theta = 1$; $\csc^2\theta - \cot^2\theta = 1$.

Double-angle formulas:

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$; $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$.

Angle sum and difference identities:

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$; $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.

Sum to product identities:

$\sin\alpha \pm \sin\beta = 2\sin[(\alpha\pm\beta)/2]\cos[(\alpha\mp\beta)/2]$;

$\cos\alpha + \cos\beta = 2\cos[(\alpha+\beta)/2]\cos[(\alpha-\beta)/2]$; $\cos\alpha - \cos\beta = -2\sin[(\alpha+\beta)/2]\sin[(\alpha-\beta)/2]$.

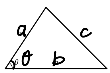
Product to sum identities:

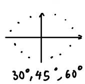
$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$; $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$;

$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$; $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$.

$$\boxed{A = \tfrac{1}{2}ab\sin\theta.}$$

$$\boxed{c^2 = a^2 + b^2 - 2ab\cos\theta.}$$





L $\frac{\sqrt{3}}{2}$ M $\frac{\sqrt{2}}{2}$ S $\frac{1}{2}$

$x : \cos\theta$

$y : \sin\theta$

Logarithm

$\log_a x + \log_a y = \log_a(xy)$; $\log_a(x^r) = r\log_a x$;

$\log_a x = \log_b x / \log_b a$; $\log_a b = r \Leftrightarrow a^r = b$ $(\ln b = r \Leftrightarrow b = e^r)$;

$\log_a(a^r) = r$; $a^{\log_a x} = x$.

Integral

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha\int f(x)\,dx + \beta\int g(x)\,dx.$$

(a) $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C\ (r \neq -1)$; (b) $\int \frac{1}{x}\,dx = \ln|x| + C$;

(c) $\int a^x\,dx = \frac{a^x}{\ln a} + C$; (d) $\int e^x\,dx = e^x + C$;

(e) $\int \sin x\,dx = -\cos x + C$; (f) $\int \cos x\,dx = \sin x + C$;

(g) $\int \sec^2 x\,dx = \tan x + C$; (h) $\int \csc^2 x\,dx = -\cot x + C$;

(i) $\int \sec x\tan x\,dx = \sec x + C$; (j) $\int \csc x\cot x\,dx = -\csc x + C$;

(k) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$; (l) $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$;

(m) $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C$.

Techniques of Integration

The 1st substitution rule

Let $u = u(x)$ be a function in x.

eg. Find $\int x^3\cos(x^4+2)\,dx$. Let $u = x^4 + 2$. Then $du/dx = 4x^3$

$$\int x^3\cos(x^4+2)\,dx = \frac{1}{4}\int 4x^3\cos(x^4+2)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C.$$

When to use? when you see part of integrant = derivative of another integrand

2nd substitution rule

Let $x = x(t)$ be a function, sub t in, integrate with dt

Find $\int \frac{1}{\sqrt{x}(x+1)}\,dx$. Let $t = \sqrt{x}$, i.e., $x = t^2$. Then $dx/dt = 2t$.

$$\int \frac{1}{\sqrt{x}(x+1)}\,dx = \int \frac{1}{t(t^2+1)}2t\,dt = 2\int \frac{1}{t^2+1}\,dt = 2\tan^{-1}t + C = 2\tan^{-1}(\sqrt{x}) + C.$$

In particular, for quadratic forms with square roots, we can use **trigonometric substitution**:

(i) $\sqrt{a^2-x^2}$. Use $x = a\sin t$. Then $\sqrt{a^2-x^2} = a\cos t$ and $dx/dt = a\cos t$.

(ii) $\sqrt{x^2-a^2}$. Use $x = a\sec t$. Then $\sqrt{x^2-a^2} = a\tan t$ and $dx/dt = a\sec t\tan t$.

(iii) $\sqrt{a^2+x^2}$ or a^2+x^2. Use $x = a\tan t$. Then $\sqrt{a^2+x^2} = a\sec t$ and $dx/dt = a\sec^2 t$.

general form $\sqrt{ax^2+bx+c}$, we shall first complete the square

$$\boxed{ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right].}$$

Arithmetic Sequences

$n = 1, 5, 9 \ldots$

generic $\boxed{a_n = a_m + (n-m)d.}$

$\boxed{a_n = a + (n-1)d}$ $d = a_{n+1} - a_n$

Let a be the 1st term and d be the common difference

$$\boxed{S_n = \sum_{i=1}^{n} a_i = (a_1 + a_n) \times n \div 2.}$$

$\frac{(\text{initial} + \text{last}) \times \text{total no. of terms}}{2}$

Geometric Sequences

$n = 81, 27, 9, 3, 1 \ldots$

$\boxed{a_n = ar^{n-1}.}$ (multiple diff)

$$\boxed{S_n = \sum_{i=1}^{n} a_i = \begin{cases} \frac{a}{1-r}(1-r^n) & \text{if } r \neq 1, \\ na & \text{if } r = 1. \end{cases}}$$

Sum of Geometric Sequence

$$\boxed{S_\infty = \sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } -1 < r < 1, \\ \text{does not exist} & \text{if } r \leq -1 \text{ or } r \geq 1. \end{cases}}$$

$$1.123232323\cdots = 1.1 + (0.023 + 0.00023 + 0.0000023 + 0.000000023 + \cdots)$$
$$= 1.1 + \frac{0.023}{1-0.01} = \frac{11}{10} + \frac{23}{990} = \frac{556}{495}.$$

Telescoping Sum

A **telescoping sum** is a sum which has a fixed number of terms after cancelation.

- Expand out, try to cancel terms
- Find − + − + terms

eg. $S = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} \ldots + \frac{1}{2017\times2018}$

$\frac{1}{1\times2} = \frac{1}{1} - \frac{1}{2}$

$S = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) \ldots + \left(\frac{1}{2017} - \frac{1}{2018}\right)$

$= \frac{1}{1} - \frac{1}{2018} = \frac{2017}{2018}$ //

Binomial Theorem

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)(n-2)\cdots[n-(r-1)]}{r!}, \quad \text{where } n! = 1\times2\times\cdots\times n.$$

binomial coefficient

$$\boxed{(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.}$$

generalized binomial coefficient

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots[n-(r-1)]}{r!}. \qquad \binom{1/2}{3} = \frac{(1/2)(-1/2)(-3/2)}{3!} = \frac{1}{16}.$$

Mathematical Induction

Let $P(n)$ be a statement on integers n

(i) *Initial step*: Verify that $P(n_0)$ is true

check $n = 1$

(ii) *Inductive step*: Verify that $P(k) \Rightarrow P(k+1)$ is true for any integer $k \geq n_0$, i.e., assume that $P(k)$ is true for some $k \geq n_0$, and then use this fact to verify that $P(k+1)$ is also true.

- Assume true $n = k$
- show true $n = k+1$

Differentiation

$$\frac{d}{dx}(cy) = c\frac{dy}{dx}.$$
$$\frac{d}{dx}(y+z) = \frac{dy}{dx} + \frac{dz}{dx}.$$
$$\frac{d}{dx}(yz) = z\frac{dy}{dx} + y\frac{dz}{dx}.$$
$$\frac{d}{dx}\left(\frac{y}{z}\right) = \frac{y(dz/dx) - z(dy/dx)}{z^2}, \text{ where } z \neq 0.$$

(i) $\frac{d}{dx}(x^r) = rx^{r-1}$, where r is any fixed real number.

(ii) $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a}\cdot\frac{1}{x}$, where $a > 0$ and $a \neq 1$. In particular, $\frac{d}{dx}(\ln x) = \frac{1}{x}$

(iii) $\frac{d}{dx}(a^x) = a^x\ln a$, where $a > 0$. In particular, $\frac{d}{dx}(e^x) = e^x$.

(iv) $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$.

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$, $\frac{d}{dx}(\cot x) = -\csc^2 x$.

(vi) $\frac{d}{dx}(\sec x) = \sec x\tan x$, $\frac{d}{dx}(\csc x) = -\csc x\cot x$.

Implicit Differentiation

eg.

$x^2 + y^2 = 25$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$

$2x + \frac{dy}{dx}\cdot\frac{d}{dy}(y^2) = 0$

$2x + \frac{dy}{dx}(2y) = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

Chain Rule

Let $y = f(x)$ and $z = g(y)$. Then $z = g(f(x))$ and

$$\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}.$$

Find $\frac{dz}{dx}$ if $z = (x^2+1)^5$ $\frac{dz}{dx} = 10x(x^2+1)^4$

$\frac{d}{dx}(\sin^{-1}x) = 1/\sqrt{1-x^2}$.

$\frac{d}{dx}(\cos^{-1}x) = -1/\sqrt{1-x^2}$.

$\frac{d}{dx}(\tan^{-1}x) = 1/(1+x^2)$.

$\frac{d}{dx}(\cot^{-1}x) = -1/(1+x^2)$.

$\frac{d}{dx}(\sec^{-1}x) = 1/(x\sqrt{x^2-1})$.

$\frac{d}{dx}(\csc^{-1}x) = -1/(x\sqrt{x^2-1})$.

Logarithmic Differentiation

(i) Take logarithmic functions both sides:
natural log (ln)

(ii) Differentiate both sides with respect to x:

Differentiation of products and quotients

$\frac{dy}{dx}$ if $y = [f_1(x)]^{a_1}[f_2(x)]^{a_2}\cdots[f_n(x)]^{a_n}$

$$\boxed{\frac{dy}{dx} = \left[a_1\frac{f_1'(x)}{f_1(x)} + \cdots + a_n\frac{f_n'(x)}{f_n(x)}\right][f_1(x)]^{a_1}\cdots[f_n(x)]^{a_n}.}$$

ed **logarithmic differentiation**.

Differentiation of exponentials

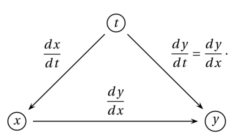
find $\frac{dy}{dx}$ if $y = [f(x)]^{g(x)}$

$$\boxed{\frac{dy}{dx} = \left[\ln f(x)\cdot g'(x) + g(x)\frac{f'(x)}{f(x)}\right][f(x)]^{g(x)}.}$$

Connected Rate of Change

e parameterized equations: Suppose $x = x(t)$ and $y = y(t)$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{or equivalently} \quad \frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}.$$



Parametric Equations

$x = x(t)$ and $y = y(t)$

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}} \quad \text{and} \quad \boxed{\frac{d^2y}{dx^2} = \frac{1}{dx/dt}\frac{d}{dt}\left(\frac{dy}{dx}\right)}.$$

Partial Derivatives

Let $z = x^2 + 3xy + y - 1$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}x^2 + 3y\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y = 2x + 3y$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}x^2 + 3x\frac{\partial}{\partial y}y + \frac{\partial}{\partial y}y = 3x + 1$$
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx},$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy},$$
$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy},$$
$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}.$$

2.3 Graphs of Functions

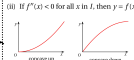
Increasing and Decreasing Functions

$f'(x) > 0$, increasing
$f'(x) < 0$, decreasing
$f'(x) = 0$, constant

Concavity

(i) If $f''(x) > 0$ for all x in I, then $y = f(x)$ is concave up on I.

(ii) If $f''(x) < 0$ for all x in I, then $y = f(x)$ is concave down on I.



Local Extreme Values

1st Derivative Test)
check left to right
+ve to −ve or vice versa

2nd Derivative Test ✓

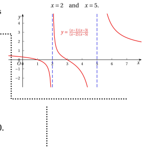
If $f'(c) = 0$ and $f''(c) > 0$, then $y = f(x)$ has a local minimum at $x = c$.

If $f'(c) = 0$ and $f''(c) < 0$, then $y = f(x)$ has a local maximum at $x = c$.

Asymptote of Rational Functions

Suppose $Q(a) = 0$ and $P(a) \neq 0$. Then $y = P(x)/Q(x)$ has a **vertical asymptote** $x = a$

Let $y = \frac{(x-1)(x-3)}{(x-2)(x-5)}$. Then the curve has 2 vertical asymptotes $x = 2$ and $x = 5$.



Horizontal asymptote

(i) $y = 0$, if $\deg P(x) < \deg Q(x)$.

(ii) $y = a_n/b_n$, if $\deg P(x) = \deg Q(x) = n$, and

eg.

$y = \frac{3x^4+4x^2-3x+1}{6x^6+3x^4+x^3+1}$ has a horizontal asymptote $y = 0$.

$y = \frac{3x^6+4x^2-3x+1}{6x^6+3x^4+x^3+1}$ has a horizontal asymptote $y = 3/6 = 1/2$.

Oblique asymptote

Suppose that $\deg P(x) = \deg Q(x) + 1$.

Find the oblique asymptote of $y = (x^3+x^2+1)/(x^2+1)$.

First find $(x^3+x^2+1) \div (x^2+1)$ using long division:

$$\begin{array}{r} x+1 \\ x^2+0x+1\,\overline{)\,x^3+x^2+0x+1} \\ \underline{x^3+0x^2+x} \\ x^2-x+1 \\ \underline{x^2+0x+1} \\ -x \end{array}$$

Partial fraction

(i) $\frac{P(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$, where $\deg P(x) < 2$.

(ii) $\frac{P(x)}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$, where $\deg P(x) < 3$.

(iii) $\frac{P(x)}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$, where $\deg P(x) < 3$.

(iv) $\frac{P(x)}{(ax^2+bx+c)(dx+e)} = \frac{Ax+B}{ax^2+bx+c} + \frac{C}{dx+e}$, where $\deg P(x) < 3$ and $b^2 < 4ac$.

Integration by Parts

$$\boxed{\int \left(u\frac{dv}{dx}\right)dx = uv - \int \left(v\frac{du}{dx}\right)\cancel{dx}}$$

When there is a product of 2 functions

Find $\int x \ln x \, dx$.

, let $u = x$ and $dv/dx = e^x$. Then $du/dx = 1$ and $v = e^x$. Then

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.$$

Easy to differentiate be u

Easy to integrate be $\frac{dv}{dx}$

Order of which to choose as u ⟶

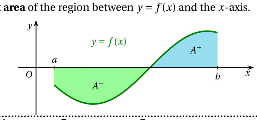
- **L**: Logarithmic functions (e.g., $\ln(x)$),
- **I**: Inverse trigonometric functions (e.g., $\arctan(x)$),
- **A**: Algebraic functions (e.g., x^2),
- **T**: Trigonometric functions (e.g., $\sin(x)$),
- **E**: Exponential functions (e.g., e^x).

Definite Integrals

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}.}$$

$$\int_a^b f(x)\,dx = A^+ - A^-,$$

which is the **net area** of the region between $y = f(x)$ and the x-axis.



Applications of Integral

Area between two curves

Suppose $f(x) \geq g(x)$ for all x in $[a,b]$.

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Volume of Revolution

$$V = \int_a^b \pi[f(x)]^2\,dx.$$, can move π outside

Volume difference

Suppose $|f(x)| \geq |g(x)|$ for all x in $[a,b]$.

$$V = \int_a^b \pi\left\{[f(x)]^2 - [g(x)]^2\right\}dx.$$

Rotation about the horizontal line $y = k$

$$\int_a^b \pi\left[f(x) - k\right]^2 dx.$$

Rotation about the y-axis

$$V = \int_a^b \pi[f(y)]^2\,dy.$$

3.5 Ordinary Differential Equations

Separable Differential Equations is of the form $\frac{dy}{dx} = f(x)g(y)$.

Solution = $\boxed{\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.}$

Solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$ $(x, y > 0)$ such that $y = 2$ when $x = 1$.

$f(x) = x$ and $g(y) = 1/y$. Hence,

$$\int \frac{1}{1/y}\,dy = \int x\,dx \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + C \Rightarrow y^2 - x^2 = 2C, \quad \Longrightarrow \quad y^2 - x^2 = 3.$$

Method of Substitution

$$\frac{dy}{dx} = \frac{6-x-y}{x+y} \qquad \frac{dz}{dx} - 1 = \frac{6-z}{z} = \frac{6}{z} - 1 \Rightarrow \frac{dz}{dx} = \frac{6}{z}.$$

(i) Show that the substitution $z = x + y$ reduces the differential equation

$$\int \frac{z}{6}\,dz = \int 1\,dx \Rightarrow \frac{1}{12}z^2 = x + C \Rightarrow \frac{1}{12}(x+y)^2 = x + C.$$

Vectors in $\mathbb{R}^2$

$$\boxed{\overrightarrow{PQ} = (a_2 - a_1)\boldsymbol{i} + (b_2 - b_1)\boldsymbol{j}.}$$

length (magnitude, modulus) $|\overrightarrow{OP}| = |a\boldsymbol{i} + b\boldsymbol{j}| = \sqrt{a^2 + b^2}$.

multiplication $\boxed{\lambda\boldsymbol{u} = (\lambda a)\boldsymbol{i} + (\lambda b)\boldsymbol{j}.}$

Geometrically, $\lambda\boldsymbol{u}$ is parallel to $\boldsymbol{u}$, and its length is $|\lambda|$ times $|\boldsymbol{u}|$, i.e., $|\lambda\boldsymbol{u}| = |\lambda||\boldsymbol{u}|$.

(i) If $\lambda > 0$, then $\lambda\boldsymbol{u}$ and $\boldsymbol{u}$ have the same direction.

(ii) If $\lambda < 0$, then $\lambda\boldsymbol{u}$ and $\boldsymbol{u}$ have the opposite direction.

(iii) If $\lambda = 0$, then $0\boldsymbol{u} = \boldsymbol{0}$, the **zero vector**.

If $|\boldsymbol{u}| = 1$, then $\boldsymbol{u}$ is called a **unit vector**.
Indicate direction without scale

If $\boldsymbol{u} \neq \boldsymbol{0}$, then the **normalized vector** of $\boldsymbol{u}$, ⇒ $\boxed{(1/|\boldsymbol{u}|)\boldsymbol{u}}$ or $\frac{u}{|u|}$

is the unit vector with the same direction as $\boldsymbol{u}$.
Vector scaled to become unit vector

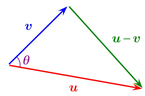
Addition and subtraction

$\boxed{\boldsymbol{u} \pm \boldsymbol{v} = (a_1 \pm a_2)\boldsymbol{i} + (b_1 \pm b_2)\boldsymbol{j}.}$

Scalar (dot) product

$\boxed{\boldsymbol{u} \bullet \boldsymbol{v} = a_1a_2 + b_1b_2.}$

$$\boxed{\boldsymbol{u} \bullet \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta.}$$



$\boldsymbol{u}$ and $\boldsymbol{v}$ are **perpendicular** $(\boldsymbol{u} \perp \boldsymbol{v}) \Leftrightarrow \boldsymbol{u} \bullet \boldsymbol{v} = 0$.

Vectors in $\mathbb{R}^3$ $\quad \overrightarrow{OP} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}.$

$P(a_1, b_1, c_1)$ and $Q(a_2, b_2, c_2)$

$\boxed{\overrightarrow{PQ} = (a_2 - a_1)\boldsymbol{i} + (b_2 - b_1)\boldsymbol{j} + (c_2 - c_1)\boldsymbol{k}.}$

$|\overrightarrow{OP}| = |a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}| = \sqrt{a^2 + b^2 + c^2}$.

$\lambda\boldsymbol{u} = (\lambda a)\boldsymbol{i} + (\lambda b)\boldsymbol{j} + (\lambda c)\boldsymbol{k}$.

normalized vector of $\boldsymbol{u}$ is $(1/|\boldsymbol{u}|)\boldsymbol{u}$

$\boldsymbol{u} \pm \boldsymbol{v} = (a_1 \pm a_2)\boldsymbol{i} + (b_1 \pm b_2)\boldsymbol{j} + (c_1 \pm c_2)\boldsymbol{k}$.

$\boldsymbol{u} \bullet \boldsymbol{v} = a_1a_2 + b_1b_2 + c_1c_2$.

Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in $\mathbb{R}^3$.

(i) $\boldsymbol{u} \bullet \boldsymbol{v} = \boldsymbol{v} \bullet \boldsymbol{u}$.

(ii) $(\boldsymbol{u} + \boldsymbol{v}) \bullet \boldsymbol{w} = \boldsymbol{u} \bullet \boldsymbol{w} + \boldsymbol{v} \bullet \boldsymbol{w}$.

(iii) $|\boldsymbol{u}|^2 = \boldsymbol{u} \bullet \boldsymbol{u}$.

(iv) $|\lambda\boldsymbol{u}| = |\lambda||\boldsymbol{u}|$. In particular, if $\boldsymbol{u} \neq \boldsymbol{0}$, then $(1/|\boldsymbol{u}|)\boldsymbol{u}$ is a unit vector.

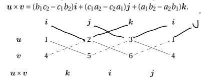
(v) If θ be the angel between vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\boldsymbol{u} \bullet \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta$

(vi) $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular $(\boldsymbol{u} \perp \boldsymbol{v}) \Leftrightarrow \boldsymbol{u} \bullet \boldsymbol{v} = 0$.

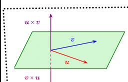
Vector Product

Let $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{v} = 4\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k}$.

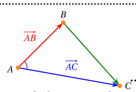
$\boldsymbol{u} \times \boldsymbol{v} = (b_1c_2 - c_1b_2)\boldsymbol{i} + (c_1a_2 - c_2a_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\boldsymbol{k}$.



So $\boldsymbol{u} \times \boldsymbol{v} = (2 \cdot 6 - 3 \cdot 5)\boldsymbol{i} + (3 \cdot 4 - 1 \cdot 6)\boldsymbol{j} + (1 \cdot 5 - 2 \cdot 4)\boldsymbol{k} = -3\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.



u · v ⊥ to plane u & v is on



Area from vector

Method 1: Scalar product:

(i) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(ii) Evaluate $|\overrightarrow{AB}|$ and $|\overrightarrow{AC}|$.

(iii) Evaluate $\cos\theta = (\overrightarrow{AB} \bullet \overrightarrow{AC})/(|\overrightarrow{AB}||\overrightarrow{AC}|)$

(iv) $\sin\angle A = \sqrt{1 - \cos^2\angle A}$.

(v) Area of $\triangle ABC$ is $\frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}|\sin\angle A$.

Method 2: Vector product:

(i) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(ii) Evaluate $\overrightarrow{AB} \times \overrightarrow{AC}$.

(iii) Area of $\triangle ABC$ is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

Vector Equation of Lines $\boxed{\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u}}$

L passes through the points $(-2, 3, 1)$ and $(1, 4, -2)$. (position vector)

$(1 - (-2))\boldsymbol{i} + (4 - 3)\boldsymbol{j} + (-2 - 1)\boldsymbol{k} = 3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$. (direction vector u)

$\boldsymbol{r} = (\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) + \mu(3\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}), \quad \mu \in \mathbb{R}$.

Angles between two lines $\boxed{\cos\theta = |\boldsymbol{u} \bullet \boldsymbol{v}|/(|\boldsymbol{u}||\boldsymbol{v}|)}$ $\quad 0 \leq \theta \leq \pi/2$

$L_1 : \boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u} \quad \lambda \in \mathbb{R}$, when u & v are scalar multiples

$L_2 : \boldsymbol{r} = \boldsymbol{b} + \mu\boldsymbol{v} \quad \mu \in \mathbb{R}$.

sub both into a equation, solve for λ & μ

(i) They are **parallel** $(L_1 \parallel L_2) \Leftrightarrow \boldsymbol{u} \parallel \boldsymbol{v} \Leftrightarrow \boldsymbol{u} = c\boldsymbol{v}$ for some $c \in \mathbb{R}$.

(ii) They **intersect** at a point if there exist $\lambda, \mu \in \mathbb{R}$ such that $\boldsymbol{a} + \lambda\boldsymbol{u} = \boldsymbol{b} + \mu\boldsymbol{v}$.

(iii) They are **skew lines** if they do not parallel and do not intersect.

Projection of a Point onto a Line

The following is the procedure to find the projection P and the distance $|PQ|$:

(i) Since Q is a point on L, it has position vector $\boldsymbol{q} = \boldsymbol{a} + \lambda\boldsymbol{u}$ for some $\lambda \in \mathbb{R}$.

(ii) $\overrightarrow{PQ} \perp L \Leftrightarrow \overrightarrow{PQ} \perp \boldsymbol{u} \Leftrightarrow (\boldsymbol{a} + \lambda\boldsymbol{u} - \boldsymbol{p}) \bullet \boldsymbol{u} = 0$. Solve this equation to get the value of λ.

(iii) Using the value of λ found in (ii) to evaluate the position vector of Q: $\boldsymbol{q} = \boldsymbol{a} + \lambda\boldsymbol{u}$.

(iv) Find the vector $\overrightarrow{PQ} = \boldsymbol{a} + \lambda\boldsymbol{u} - \boldsymbol{p}$ and hence the distance $|\overrightarrow{PQ}|$.

Complex Numbers $\boxed{z = x + iy}$ $\boxed{i^2 = -1}$ or $i = \sqrt{-1}$

$\boxed{z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2).}$ add/subtract

$\boxed{(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)}$ multiplication

Let $z = x + iy$ be a complex number, Its **complex conjugate** is $x - iy$ $(z^*$ or $\bar{z}$.)

$z\bar{z} = x^2 + y^2$

division $\boxed{\frac{z_1}{z_2} = \frac{z_1z_2^*}{z_2z_2^*} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}.}$

Square Roots of Complex Numbers procedure to find the square roots of $a + bi$:

$(x + iy)^2 = a + bi$ ← given

$(x+iy)^2 = x^2 + y^2 + i2xy$

$x^2 - y^2 = a$ and $2xy = b$

(ii) $x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + (2xy)^2} = \sqrt{a^2 + b^2}$

Use $x^2 + y^2 = \sqrt{a^2 + b^2}$ and $x^2 - y^2 = a$ to evaluate x^2 and y^2,

(iv) If $b > 0$, then $z = A + iB$ or $z = -A - iB$; if $b < 0$, then $z = A - iB$ or $z = -A + iB$

Complex Equations

Linear equations

Solve same as real numbers

Quadratic equations

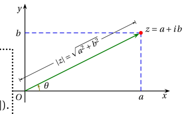
form $az^2 + bz + c = 0$ $\qquad z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$2z^2 - (9 + 8i)z + (-11 + 7i) = 0$

5.2 Geometric Representation can represent as vectors

absolute value (modulus)

$|z| = \sqrt{a^2 + b^2}$.



Argument

(i) If $\operatorname{Im} z \geq 0$, then $\arg z = \cos^{-1}(\operatorname{Re} z/|z|)$.

(ii) If $\operatorname{Im} z < 0$, then $\arg z = -\cos^{-1}(\operatorname{Re} z/|z|)$.

Polar form

$r = |z|$ and $\theta = \arg z$

$\boxed{z = r(\cos\theta + i\sin\theta).}$

Exponential Form

$\boxed{z = re^{i\theta}.}$

Euler's identity

$\boxed{e^{i\pi} + 1 = 0.}$

Geometric Representation of Multiplication

Suppose that $z = r(\cos\theta + i\sin\theta)$ is of the polar form, i.e., $r = |z|$ and $\theta = \arg z$.

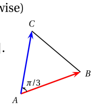
$|zw| = r|w|$ and $\arg(zw) = \theta + \arg w$.

length of the vector w is **dilated** by r times,

direction of w is **rotated** by angle θ (anti-clockwise)

the resulting vector is zw.

eg. $z_C - z_A = (z_B - z_A) \cdot 1[\cos(\pi/3) + i\sin(\pi/3)]$.



De Moirve's Formula

$\boxed{(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).}$

Roots of Complex Numbers

Let n be a positive integer. If $w^n = z$, then w is called an n^{th} **root** of z.

$\boxed{w_0 = \sqrt[n]{r}[\cos(\theta/n) + i\sin(\theta/n)]}$ and $\boxed{\zeta = \cos(2\pi/n) + i\sin(2\pi/n)}$.

$w_1 = w_0\zeta, \; w_2 = w_0\zeta^2, \; \ldots, \; w_{n-1} = w_0\zeta^{n-1}$

EXAMPLE 3 roots

Find the cubic roots of $z = -1 - i$

have seen that $z = \sqrt{2}[\cos(-3\pi/4) + i\sin(-3\pi/4)]$

$$w_0 = \sqrt[3]{\sqrt{2}}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = \sqrt[6]{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt[3]{2}}(1 - i),$$

$$\zeta = \cos\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

$$w_1 = w_0\zeta = \frac{1}{\sqrt[3]{2}}\left[\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)\right],$$

$$w_2 = w_0\zeta^2 = \frac{1}{\sqrt[3]{2}}\left[\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)\right].$$

Projection of a Point onto a Plane

(i) The position vector Q is of the form $\boldsymbol{p} + \lambda\boldsymbol{n}$ for some λ. Substitute it into the equation of Π to evaluate the value of λ: $(\boldsymbol{p} + \lambda\boldsymbol{n}) \bullet \boldsymbol{n} = \boldsymbol{a} \bullet \boldsymbol{n}$

(ii) Evaluate the position vector of Q: $\boldsymbol{p} + \lambda\boldsymbol{n}$.

(iii) Evaluate the distance $|\overrightarrow{PQ}| = |\lambda||\boldsymbol{n}|$.

Relations of Line and Plane

Intersection of line and plane

(i) If there is exactly one solution, then L and Π intersects at exactly one point.

(ii) If there is no solution, then L and Π are parallel, denoted by $L \parallel \Pi$.

(iii) If every $\lambda \in \mathbb{R}$ satisfies the equation, then L is contained in Π, denoted by $L \subset \Pi$.

Acute angle between line and plane

$\boxed{\sin\theta = |\boldsymbol{u} \bullet \boldsymbol{n}|/(|\boldsymbol{u}||\boldsymbol{n}|).}$ $\quad \boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{u}$, $\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{b} \bullet \boldsymbol{n}$

Relations of Planes $\quad \boldsymbol{r} \bullet \boldsymbol{n}_1 = \boldsymbol{a}_1 \bullet \boldsymbol{n}_1$ and $\boldsymbol{r} \bullet \boldsymbol{n}_2 = \boldsymbol{a}_2 \bullet \boldsymbol{n}_2$

Let Π_1 and Π_2 be the planes with normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$.

(i) If $\boldsymbol{n}_1$ is a a multiple of $\boldsymbol{n}_2$, $\boldsymbol{n}_1 = \alpha\boldsymbol{n}_2$ for some α, then the planes are parellel.

(ii) Otherwise, suppose the planes intersect along a line L:

Then the direction of L is given by $\boldsymbol{n}_1 \times \boldsymbol{n}_2$. Let $\boldsymbol{a}$ be a point of the intersection of the planes. Then

$L: \quad \boldsymbol{r} = \boldsymbol{a} + \lambda(\boldsymbol{n}_1 \times \boldsymbol{n}_2), \quad \lambda \in \mathbb{R}$.

$\boxed{\cos\theta = |\boldsymbol{n}_1 \bullet \boldsymbol{n}_2|/(|\boldsymbol{n}_1||\boldsymbol{n}_2|).}$ (**acute**) **angle** θ between Π_1 and Π_2

Planes

Equation of Planes

Let Π be a plane and $\boldsymbol{n}$ be a vector **perpendicular** to Π, denoted by $\boldsymbol{n} \perp \Pi$. (normal vector)

$\boxed{\boldsymbol{r} \bullet \boldsymbol{n} = \boldsymbol{a} \bullet \boldsymbol{n}.}$ This equation is called the **normal form** of the plane Π.

Suppose $\boldsymbol{n} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, and let $d = \boldsymbol{a} \bullet \boldsymbol{n}$.

$\boxed{ax + by + cz = d,}$ called the **Cartesian equation** of the plane Π.