Imperial College London

Department of AERONAUTICS

Computing and Numerical Methods 2

Coursework Part 1 Report

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Question 1

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} x_r \\ y_r \\ \dot{x}_r \\ \dot{y}_r \end{bmatrix} \tag{1}$$

If v is defined as to equation (1), the system of ODEs can then be defined as:

$$\dot{v} = \begin{bmatrix} v_3 \\ v_4 \\ -G\left(\frac{m_1+m_2}{(v_1^2+v_2^2)^{\frac{3}{2}}}\right)v_1 \\ -G\left(\frac{m_1+m_2}{(v_1^2+v_2^2)^{\frac{3}{2}}}\right)v_2 \end{bmatrix} \text{ where } v(0) = \begin{bmatrix} x_r(0) \\ y_r(0) \\ \dot{x}_r(0) \\ \dot{y}_r(0) \end{bmatrix} \tag{2}$$

Question 2

Part (a)

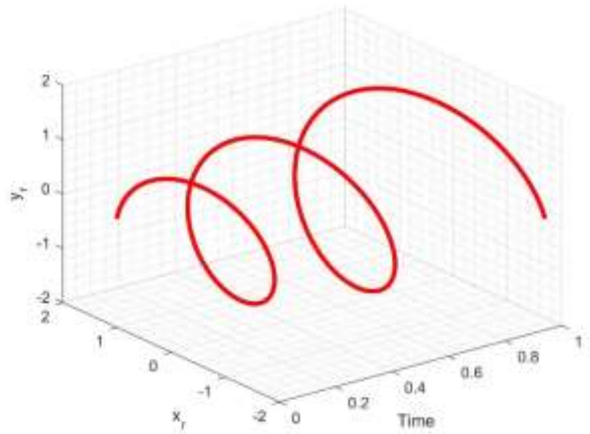


Figure 1: Explicit Euler with $\Delta t = 0.001$ over $t \in [0, 1]$

Part (b)

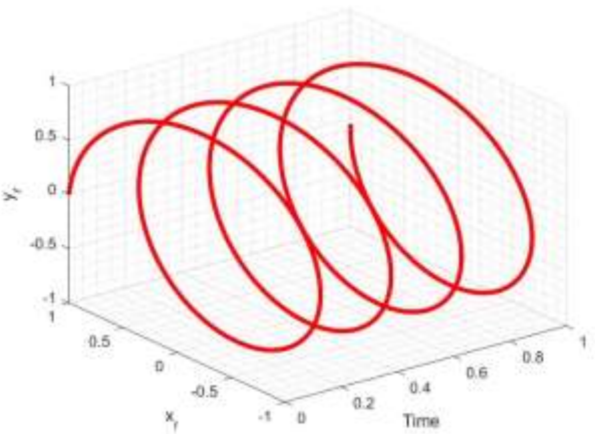


Figure 2: 4th Order Runge Kutta with $\Delta t = 0.001$ over $t \in [0, 1]$

Part (c)

i) By using the equations:

$$v = \omega \, r \; , \; \omega = \frac{2\pi}{T} \quad \text{we can obtain:} \quad T = \frac{2\pi \sqrt{x_r^2 + y_r^2}}{\sqrt{\dot{x}_r^2 + \dot{y}_r^2}} \tag{3}$$

Under initial conditions, T = 0.25 seconds meaning that four orbits should take place every second, which corresponds to the results shown in Figure 2's 4th Order Runge Kutta while the results in Figure 1's Explicit Euler only produces three orbits.
Furthermore, when a smaller body is orbiting around a larger body, it is expected to have a circular motion and the solved numerical solution should be bounded. However, the result in Figure 1 which used Explicit Euler showed an unbounded result, whereby the orbit of the smaller body was growing, which does not correspond to theory. Figure 2 on the other hand showed a bounded result with a constant orbit, which corresponds ideally as during such a short time period of 1 second, the orbit should not be expected to have any change.

ii) 4th Order Runge Kutta is more computationally complex. It took around 40% more time to run due to method's requirement of having to interpolate points between time steps before calculation. However, it produced a far more accurate result.

Question 3

Part (a)

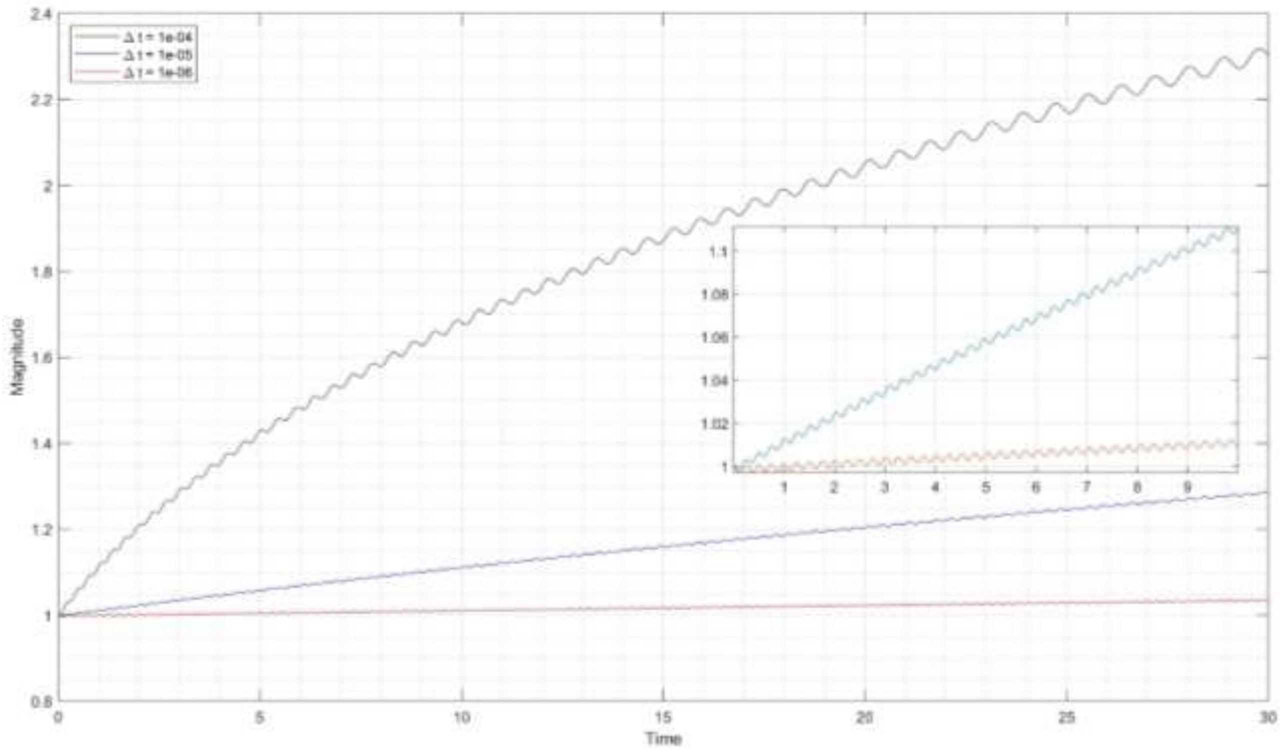


Figure 3: Explicit Euler with varying Δt over $t \in [0, 30]$

In a bounded solution, the magnitude is not expected to grow and the magnitude of 1 is expected to remain constant. However, even by using smaller step sizes down to $\Delta t =$ 0.000001, the solution still remains unbounded when using Explicit Euler.

Part (b)

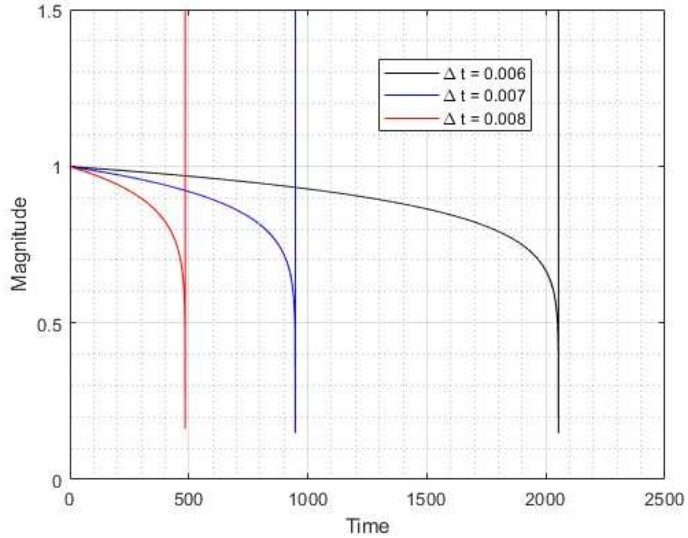


Figure 4: 4th Order Runge Kutta with varying Δt over $t \in [0, 2500]$

When using 4th Order Runge Kutta, the solution remains bounded at varying Δt and the bounded range is shown to grow exponentially as Δt increases. This means that as long as the scheme remains in a valid *T*, the solution would be bounded.

Question 4

Part (a)

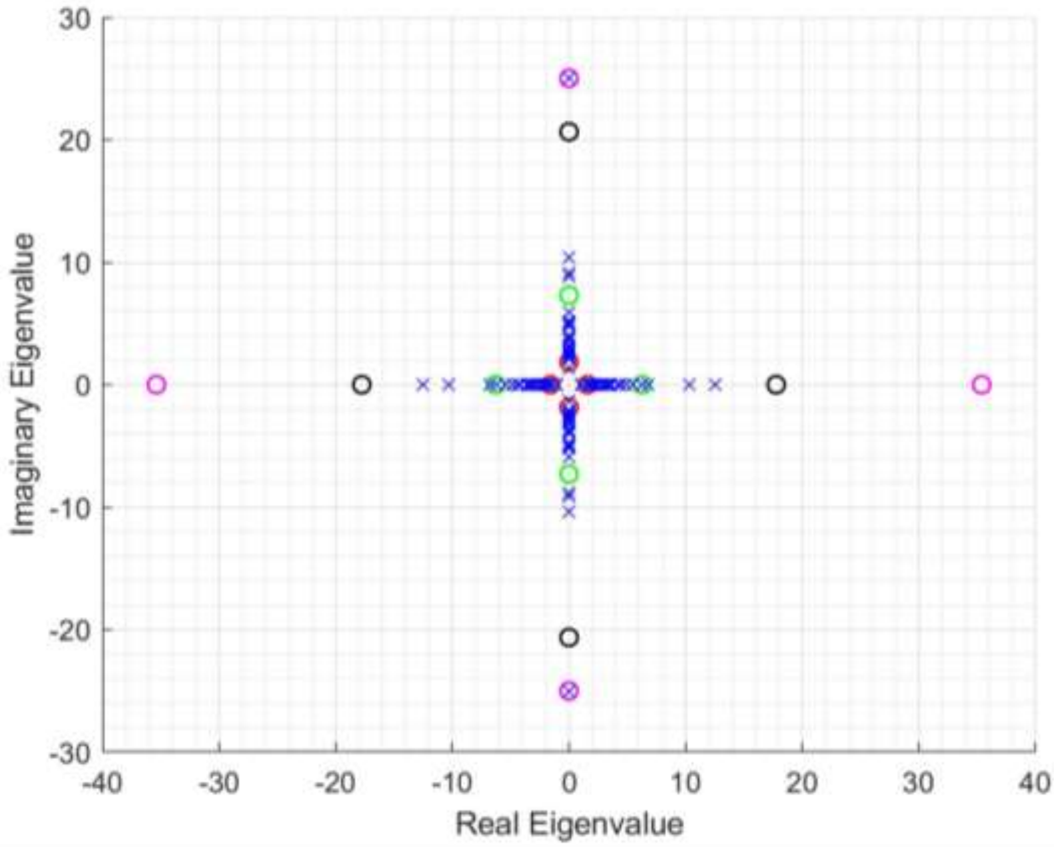
$$\dot{v} = Av + c$$

is the linearisation of the ODEs defined in equation (1) and (2), where:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -G(m1+m2)\left(\frac{-2\check{x}_r{}^2+\check{y}_r{}^2}{(\check{x}_r{}^2+\check{y}_r{}^2)^{\frac{5}{2}}}\right) & -G(m1+m2)\left(\frac{-3\check{x}_r\check{y}_r}{(\check{x}_r{}^2+\check{y}_r{}^2)^{\frac{5}{2}}}\right) & 0 & 0 \\ -G(m1+m2)\left(\frac{-3\check{x}_r\check{y}_r}{(\check{x}_r{}^2+\check{y}_r{}^2)^{\frac{5}{2}}}\right) & -G(m1+m2)\left(\frac{\check{x}_r{}^2-2\check{y}_r{}^2}{(\check{x}_r{}^2+\check{y}_r{}^2)^{\frac{5}{2}}}\right) & 0 & 0 \end{bmatrix} \tag{4}$$

$$c = \begin{bmatrix} 0 \\ 0 \\ -G(m1+m2)\left(\frac{3\check{x}_r{}^2+3\check{x}_r\check{y}_r{}^2}{(\check{x}_r{}^2+\check{y}_r{}^2)^{\frac{5}{2}}}\right) \\ -G(m1+m2)\left(\frac{3\check{y}_r{}^2+3\check{x}_r\check{y}_r{}^2}{(\check{x}_r{}^2+\check{y}_r{}^2)^{\frac{5}{2}}}\right) \end{bmatrix} \tag{5}$$

Part (b)



At each starting grid point, the eigenvalues of the problem are either real or imaginary. As the values of $|x_r|$ and $|y_r|$ decrease, the magnitude of the eigenvalues increase.

Part (c)

Two of the four eigenvalues are always imaginary regardless of the x_r and y_r values. For Explicit Euler, the stability region follows the equation:

$$(1 + h\lambda_R)^2 + (h\lambda_I)^2 \le 1 \tag{6}$$

The equation shows that having only imaginary values would yield an unbounded solution, which is evident in Figure 3 of Q3(a) where all solutions were unbounded.

In the 4th Order Runge Kutta case, three of the four eigenvalues with $\lambda_R \le 0$ fall within the stability region of 4th Order Runge Kutta. At a given initial condition, a suitable h can be found so that $h\lambda_R$ would fall in the stability region. However as shown in Figure 4, the solution is not unconditionally bounded and will become unbounded at a certain point. Considering the negative real eigenvalues only, the equation becomes:

$$|h_0\lambda_p| \le X \tag{7}$$

Where h_0 is the initial step size and X is the unstable value on the real eigenvalue axis. As we know that when the values of $|x_r|$ and $|y_r|$ decrease, the magnitude of the eigenvalues increase, the solution cannot be satisfied anymore at a later point due to the fixed h_0 and eventually reach a point of instability.