Do you use "AI" (ChatGPT, Gemini,...) for your math coursework?

“Calculus 3”

Multi-Variable Calculus

Instructor: Álvaro Lozano-Robledo

Day 2

Any Reminders? Any Questions?

- Slides will be posted on GitHub! https://github.com/alozanoroble/MATH-2110Q-Spring-2026
- Videos will be posted on YouTube… but they may lag!
- First two worksheets are available in HuskyCT (extra credit)
- First quiz (Friday) will be on derivatives and integrals

Questions?

ALVARO: Start the recording!

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More on Vectors

Today – Finish Vectors!

- Vectors
  - Vector addition and scalar multiplication
  - Components and length
  - Properties
  - Applications

Vectors

A **vector** is a mathematical object with both magnitude (size) and direction, represented as a directed line segment (arrow).

Vector Addition and Scalar Multiplication

Example: Let u = (2,3) and v = (-1,1). Find u+2v and u-2v.

Example: Let $u = (2,3,0)$ and $v = (-1,1,2)$.
Find $u+v$ and $u-v$.

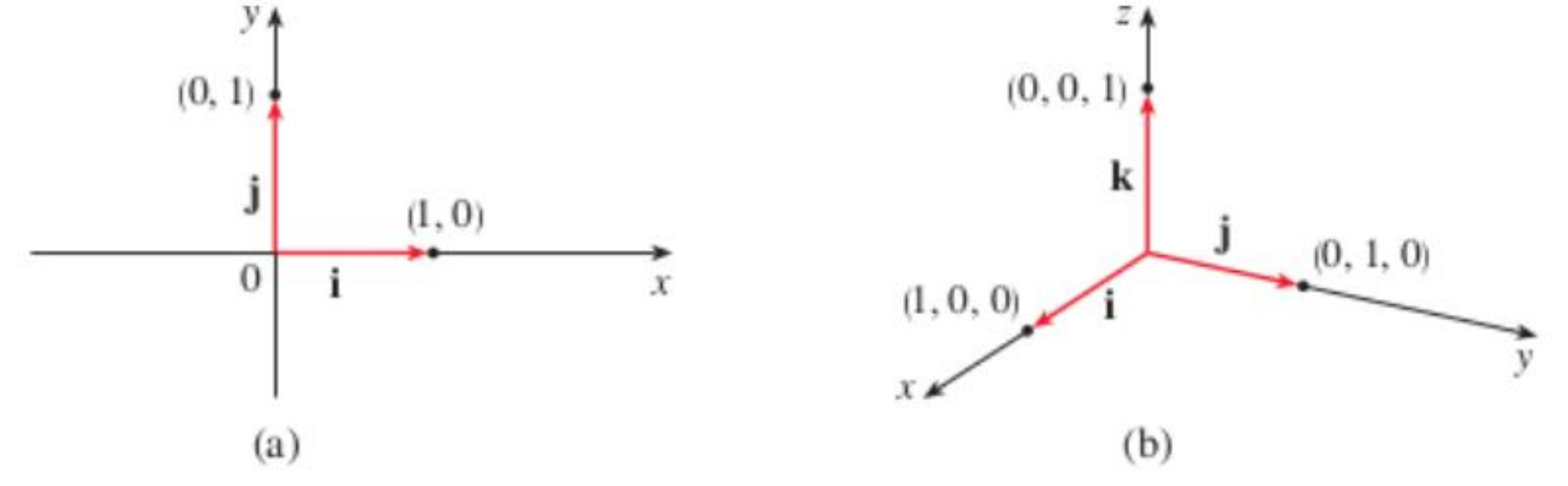
Length of a Vector

Example: Let a = (4, 0, 3) and b = (-2, 1, 5).
Find the lengths of a+b and a-b.

Properties of Vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars, then

1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$
2. $\mathbf{a}+(\mathbf{b}+\mathbf{c})=(\mathbf{a}+\mathbf{b})+\mathbf{c}$
3. $\mathbf{a}+\mathbf{0}=\mathbf{a}$
4. $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$
5. $c\,(\mathbf{a}+\mathbf{b})=c\mathbf{a}+c\mathbf{b}$
6. $(c+d)\,\mathbf{a}=c\mathbf{a}+d\mathbf{a}$
7. $(cd)\,\mathbf{a}=c\,(d\mathbf{a})$
8. $\mathrm{l}\mathbf{a}=\mathbf{a}$



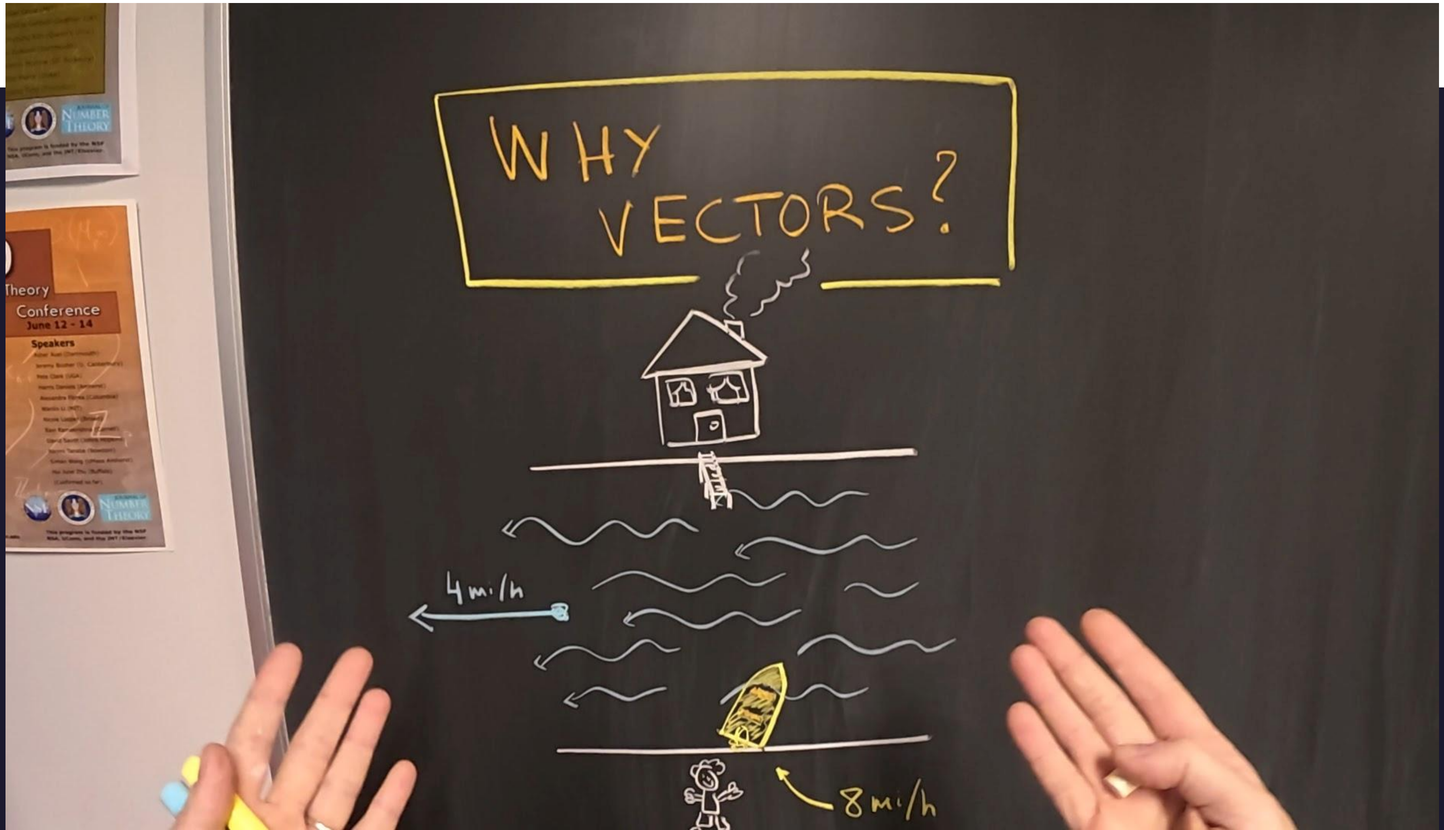
Standard Basis Vectors

- i = (1, 0, 0)
- j = (0, 1, 0)
- k = (0, 0, 1)

Example: Let a = 4i+3k and b = -2i + j + 5k.
Find the length of c = 2a - b in terms of i, j, and k.

Example (an application of vectors):

A woman launches a boat from the south shore of a straight river that flows directly west at 4 $\mathrm{mi/h}$. She wants to land at the point directly across on the opposite shore. If the speed of the boat (relative to the water) is 8 $\mathrm{mi/h}$, in what direction should she steer the boat in order to arrive at the desired landing point?



Questions?

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The Dot Product

Today!

- The Dot Product
  - Definition and Properties
  - Direction Angles
  - Projections

What are we trying to do?

The Dot Product

1 Definition of the Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

The Dot Product

Example: Find the dot product of the vectors

$a = (1,0)$ and $b = (2,3)$.

The Dot Product

Example: Find the dot product of the vectors

$a = (1,0,-1)$ and $b = (2,5,2)$.

Properties of the Dot Product

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Theorem. Two vectors u and v are perpendicular **if and only if** their dot product $u \cdot v = 0$.

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Theorem. If the angle between the vectors u and v is θ then

$$u \cdot v = |u||v|\cos(\theta)$$

where $|u|$ and $|v|$ are their lengths.

Properties of the Dot Product

Theorem. Two vectors u and v are perpendicular **if and only if** their dot product $u \cdot v = 0$.

Theorem. If the angle between the vectors u and v is θ then

$$\cos(\theta) = \frac{u \cdot v}{|u||v|}$$

where $|u|$ and $|v|$ are their lengths.

The Dot Product

Example: Find the angle between the two vectors $a = (1,0,-1)$ and $b = (2,5,2)$.

The Dot Product

Example: The angle between the two vectors $a = (1,1,0)$ and $b = (\sqrt{(3/2)}, \sqrt{(3/2)}, 1)$.

Are the vectors 2i+2j-k and 5i-4j+2k perpendicular?

The Dot Product

Example: Are the vectors 2i+2j-k and 5i-4j+2k perpendicular?

Questions?

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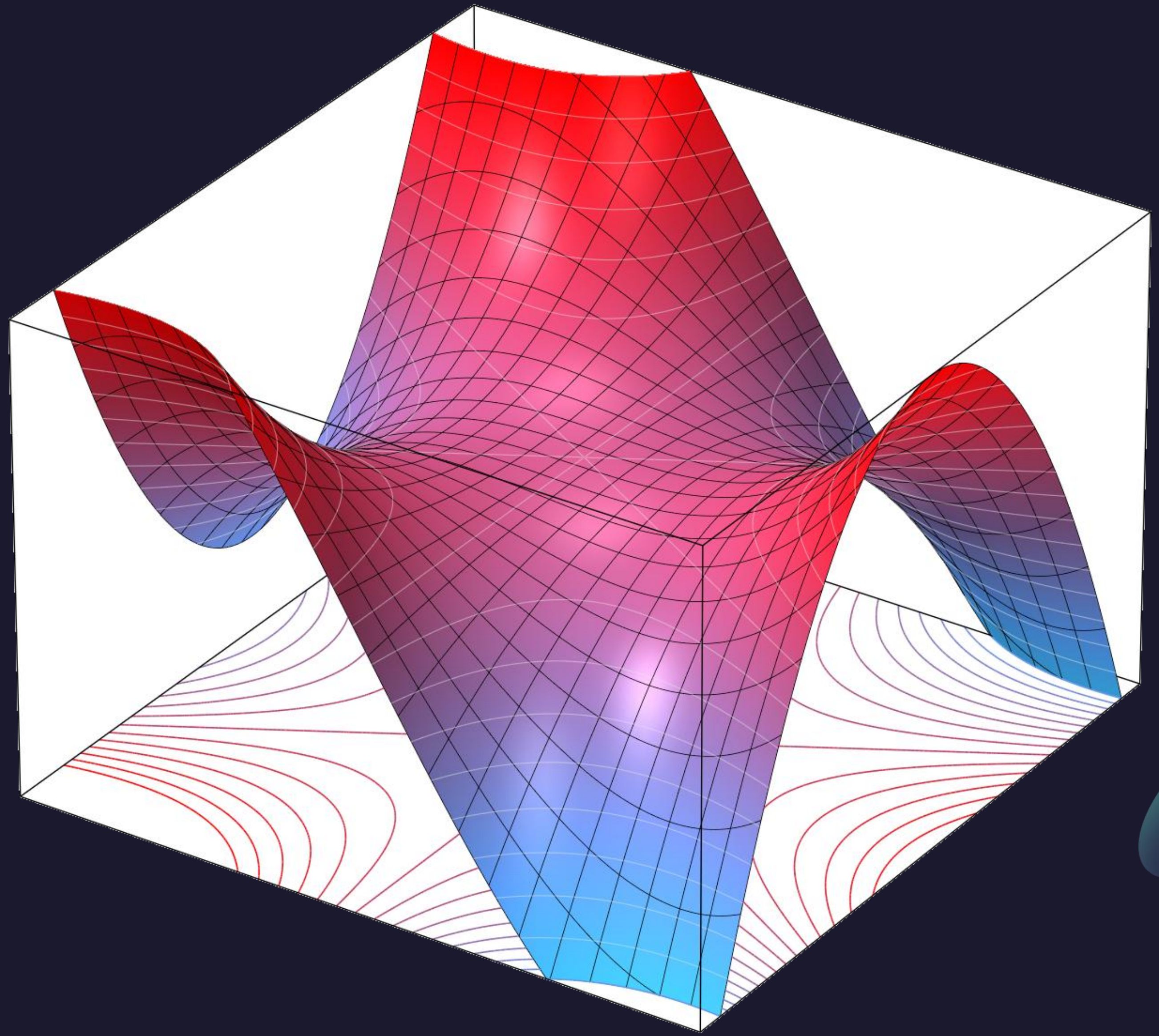
The Cross Product

Why Vectors?



Why Vectors?

Given two vectors, find a perpendicular one

Given two vectors, find a perpendicular one

4 Definition of the Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

A Slight Digression: Determinants

Given two vectors, find a perpendicular one

4 Definition of the Cross Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2,\, a_3 b_1 - a_1 b_3,\, a_1 b_2 - a_2 b_1 \rangle$$

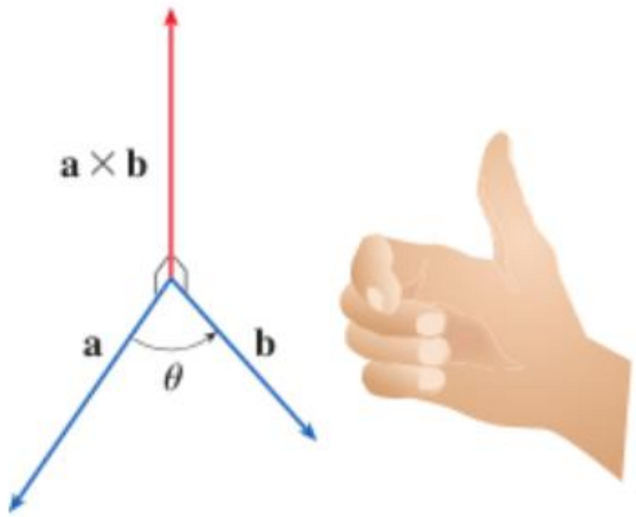
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Find the cross product of $(1, 1, 0)$ and $(0, 1, 0)$.

Example: Find the cross product of (1, 1, 1) and (1, 0, -1).

Properties of the Cross Product

The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.



Properties of the Cross Product

9 Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin \theta$$

9 Theorem

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

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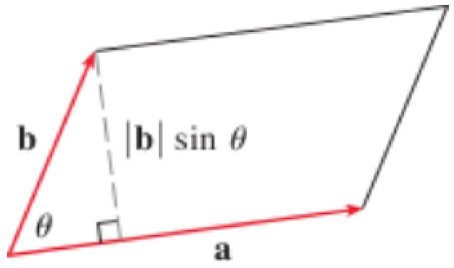
Figure 2

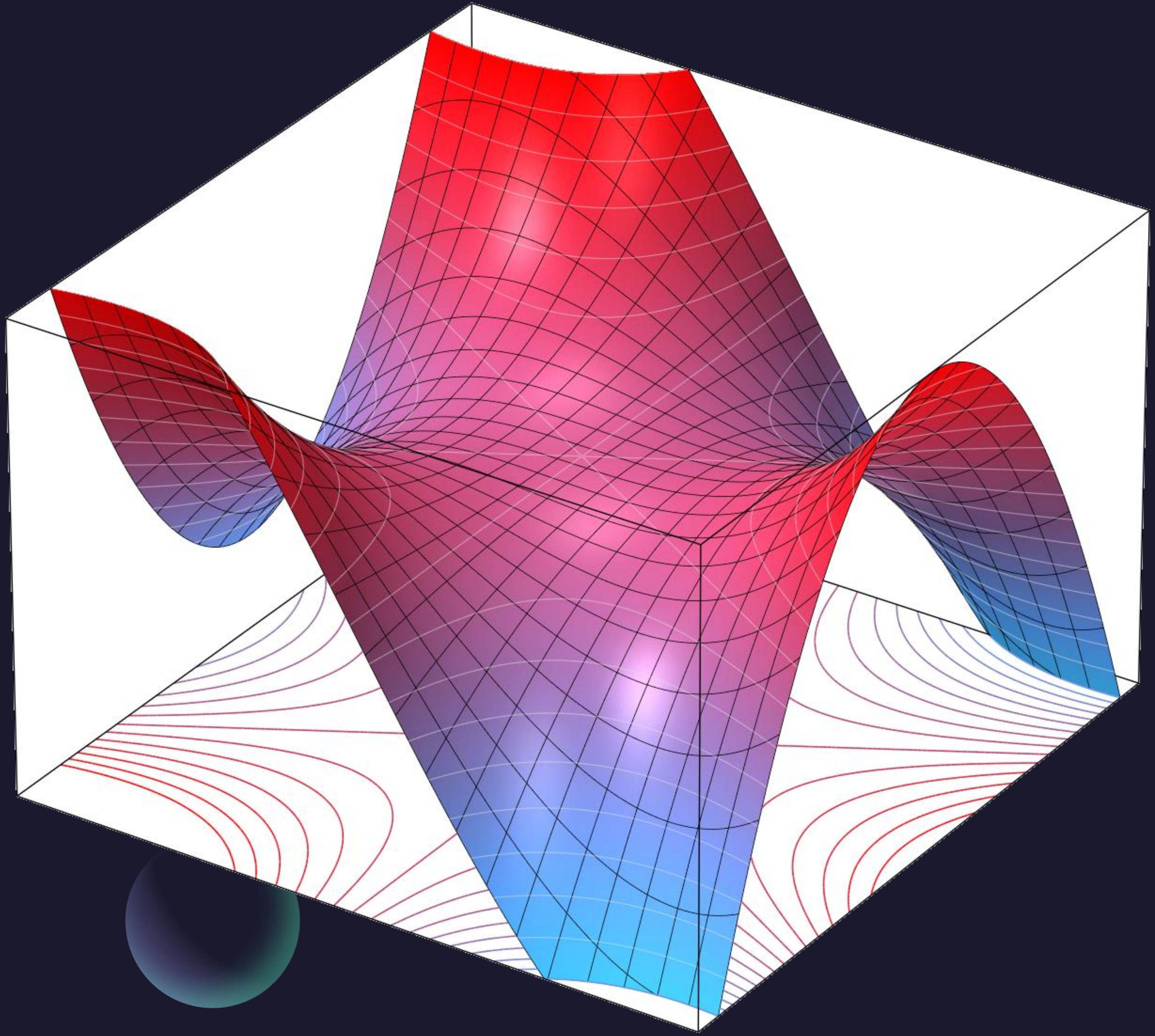


The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Example: Find a vector perpendicular to the plane that passes through the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Thank you

Until next time.



Questions?