“Calculus 3”

Multi-Variable Calculus

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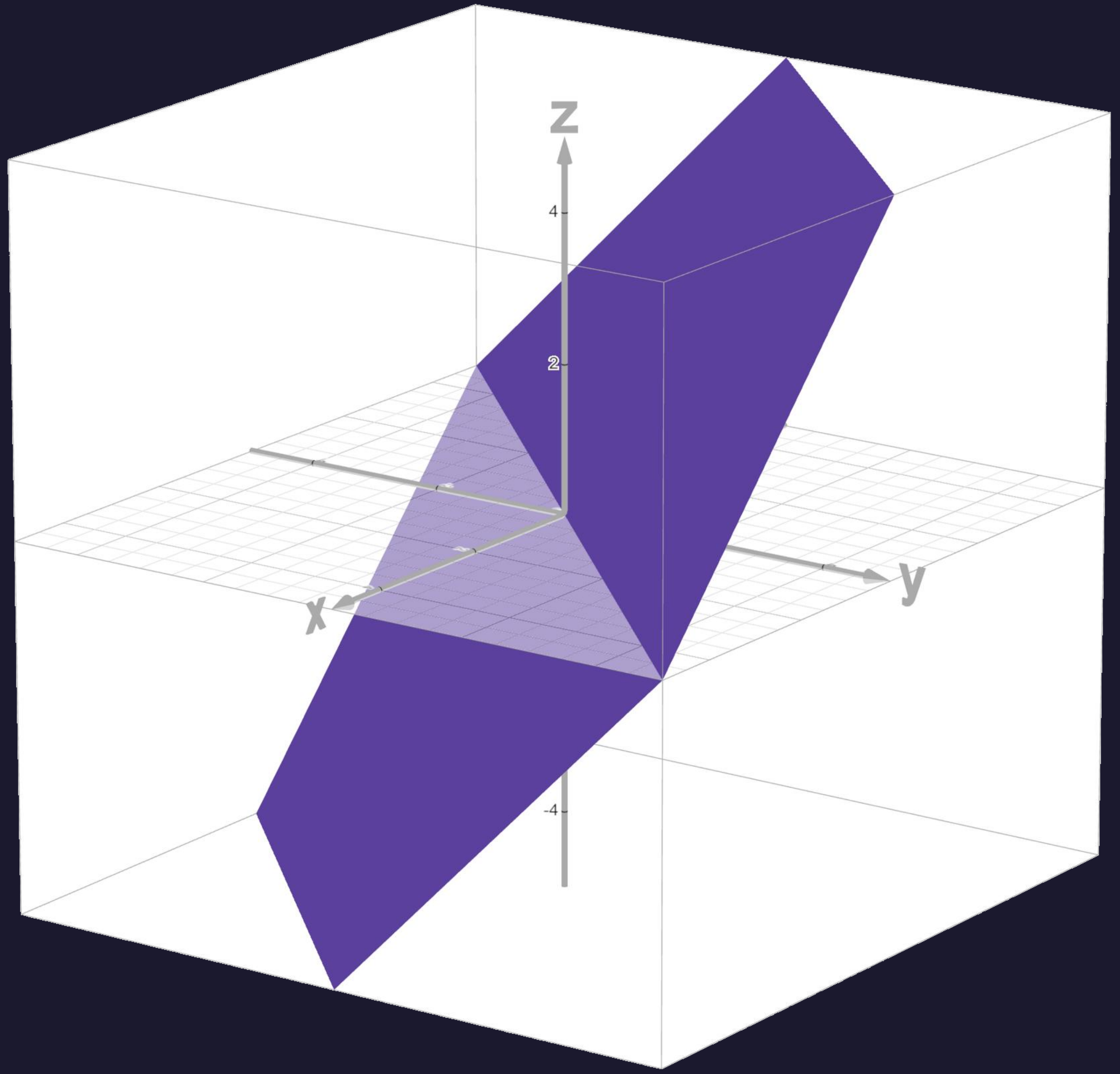
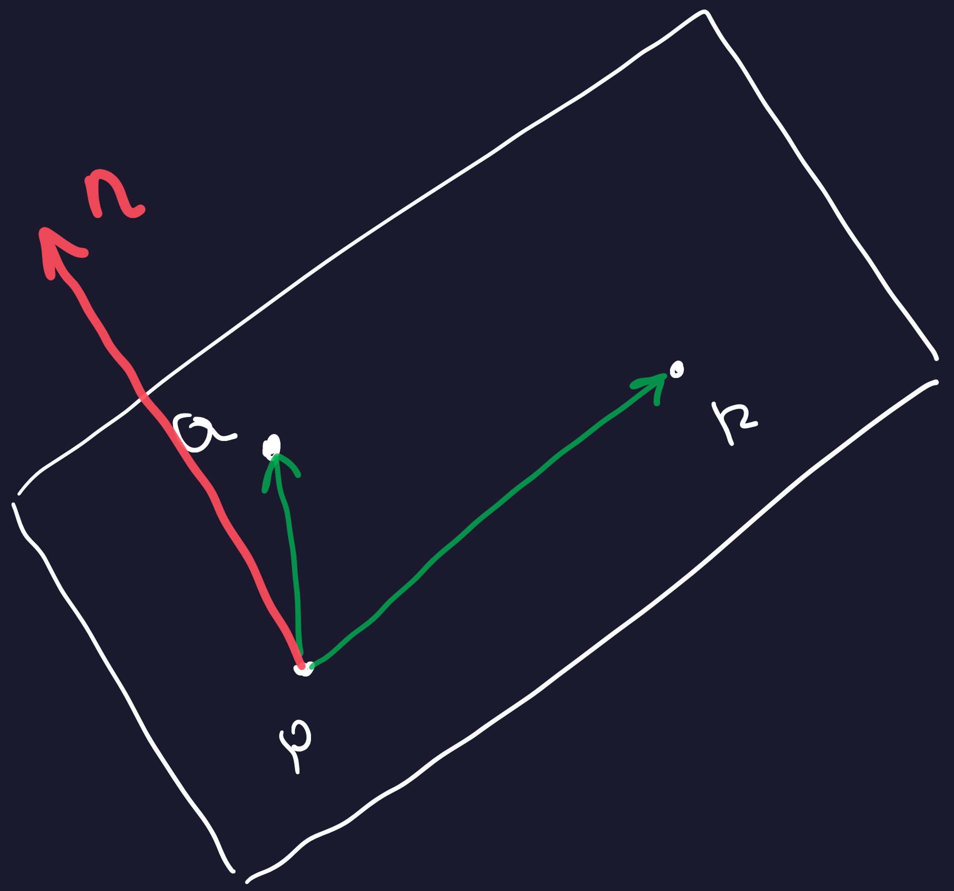
Equations of Planes

Today –Planes!

- Planes
  - Vector equation
  - Scalar equation
  - Distance to a plane



How to describe planes in 3D space?

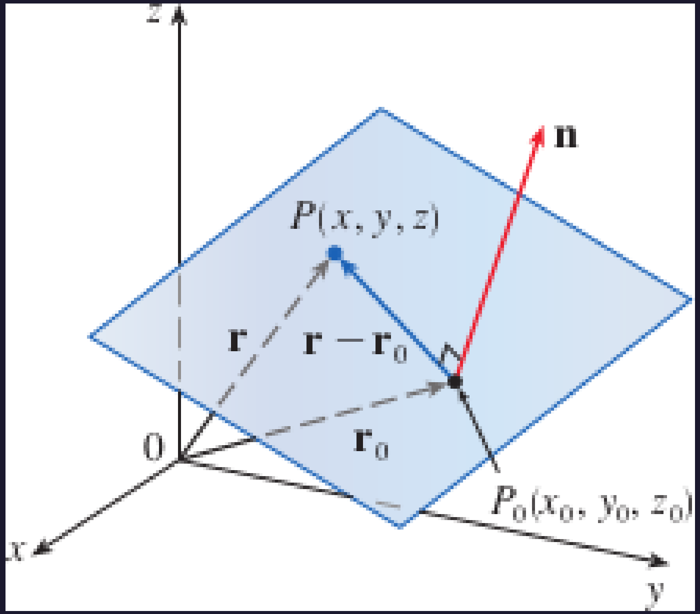


$$n \perp \overrightarrow{PQ}$$

$$n \cdot \overrightarrow{PQ} = 0$$

Vector and Scalar Equation of a Plane

The equation of a plane M that passes through $P_0 = (x_0, y_0, z_0)$ and is perpendicular to the direction of the vector $n = (a, b, c)$.



$$r_0 = (x_0, y_0, z_0)$$

$$r = (x, y, z)$$

$$r - r_0 = \overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$$

$$\boxed{(r - r_0) \cdot n = 0}$$ Vector eq'n

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

Scalar eq'n.
$$\boxed{a \cdot (x - x_0) + b \cdot (y - y_0) + c \cdot (z - z_0) = 0}$$

Example: Find the scalar equation of a plane that goes through $P_o = (1, 0, 0)$ and it is perpendicular to $n = (0, 1, -1)$. [Desmos]

$P = (x, y, z)$

$P_o = (1, 0, 0)$

$\overrightarrow{P_oP} = \vec{P} - \vec{P_o} = (x, y, z) - (1, 0, 0)$

$= (x-1, y-0, z-0)$

$\overrightarrow{P_oP} \perp n$ or $\overrightarrow{P_oP} \cdot n = 0$

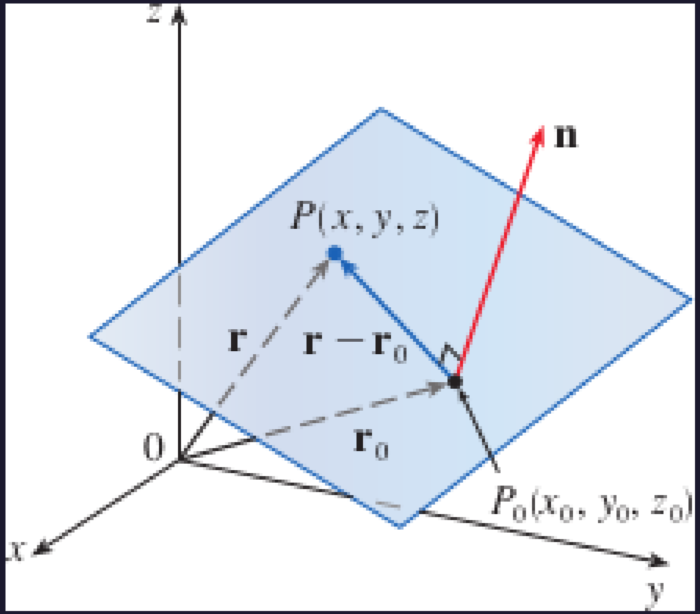
$(x-1, y-0, z-0) \cdot (0, 1, -1) = 0$

$\boxed{0 \cdot (x-1) + 1 \cdot (y-0) - 1 \cdot (z-0) = 0}$ Scalar eq'n

$\boxed{y - z = 0}$ Linear eq'n.

Vector, Scalar, and Linear Equation of a Plane

The equation of a plane M that passes through $P_0 = (x_0, y_0, z_0)$ and is perpendicular to the direction of the vector $n = (a, b, c)$.



$$\boxed{(\vec{p} - \vec{p_0}) \cdot n = 0}$$ Vector

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

$$\boxed{a \cdot (x - x_0) + b \cdot (y - y_0) + c \cdot (z - z_0) = 0}$$

Scalar

$$\boxed{ax + by + cz + d = 0}$$ Linear eqn

Example: Find the linear equation of a plane that goes through P = (2, 4, -1) and it is perpendicular to n = (2, 3, 4). [Desmos]

$$2 \cdot (x-2) + 3 \cdot (y-4) + 4 \cdot (z-(-1)) = 0$$

$$2x - 4 + 3y - 12 + 4z + 4 = 0$$

$$\boxed{2x + 3y + 4z - 12 = 0}$$ Linear!

Check. $P \in$ plane $\quad 2 \cdot 2 + 3 \cdot 4 + 4 \cdot (-1) - 12 = 0$ ✓

Example: Find the equation of the plane that passes through the points (1,0,0), (0,1,0), and (0,0,1).

$$P \qquad Q \qquad R$$

$$\overrightarrow{PQ} = \vec{Q} - \vec{P} = (0,1,0) - (1,0,0) = (-1,1,0)$$

$$\overrightarrow{PR} = \vec{R} - \vec{P} = (0,0,1) - (1,0,0) = (-1,0,1)$$

$$n = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = i \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + k \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}$$

$$= i \cdot 1 - j \cdot (-1) + k \cdot (1) = i + j + k$$

$$n = (1,1,1)$$

$$1 \cdot (x-1) + 1 \cdot (y-0) + 1 \cdot (z-0) = 0$$

$$x - 1 + y + z = 0 \quad \text{or} \quad \boxed{x + y + z = 1} \checkmark$$

Extra space just in case

Example: Find the equation of the plane that passes through the points (2,0,1), (0,-1,1), and (0,0,-1).

"P" "Q" "R"

$\overrightarrow{PQ} = (-2, -1, 0)$

$\overrightarrow{PR} = (-2, 0, -2)$

$$n = \begin{vmatrix} i & j & k \\ -2 & -1 & 0 \\ -2 & 0 & -2 \end{vmatrix} = i \cdot 2 - j \cdot 4 + k(-2) = 2i - 4j - 2k$$

$= (2, -4, -2)$

or $(1, -2, -1)$ n

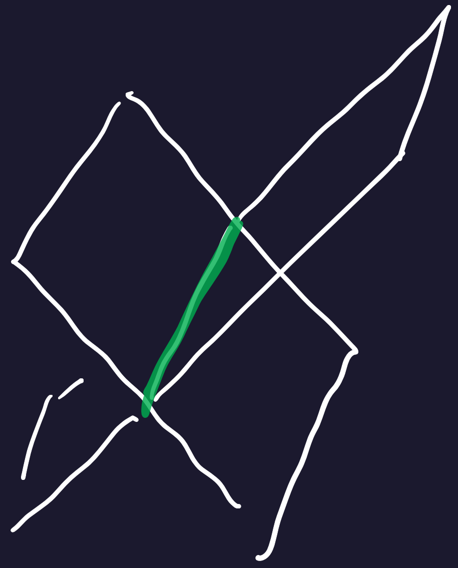
$$1 \cdot (x-2) - 2 \cdot (y-0) - 1 \cdot (z-1) = 0$$

$$x - 2 - 2y - z + 1 = 0$$

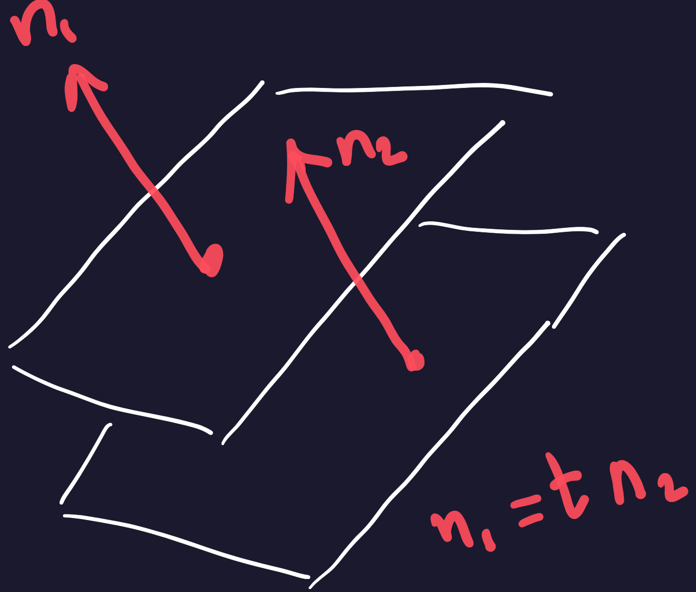
$$\boxed{x - 2y - z = 1}$$ ✓

Extra space just in case

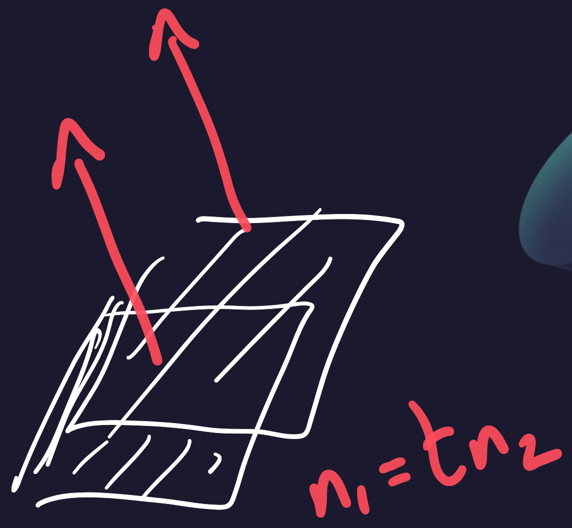
Relative position of two planes



intersecting.



parallel



same

Parallel planes

Example: Investigate the relative positions of the planes

$$\overset{M_1}{\underbrace{x - z = 2}_{(1,0,-1)}},\ \overset{M_2}{\underbrace{2x - 2z = 5}_{(2,0,-2)}},\ \text{and } \overset{M_3}{\underbrace{2x - 2y = 5}_{(2,-2,0)}}.$$

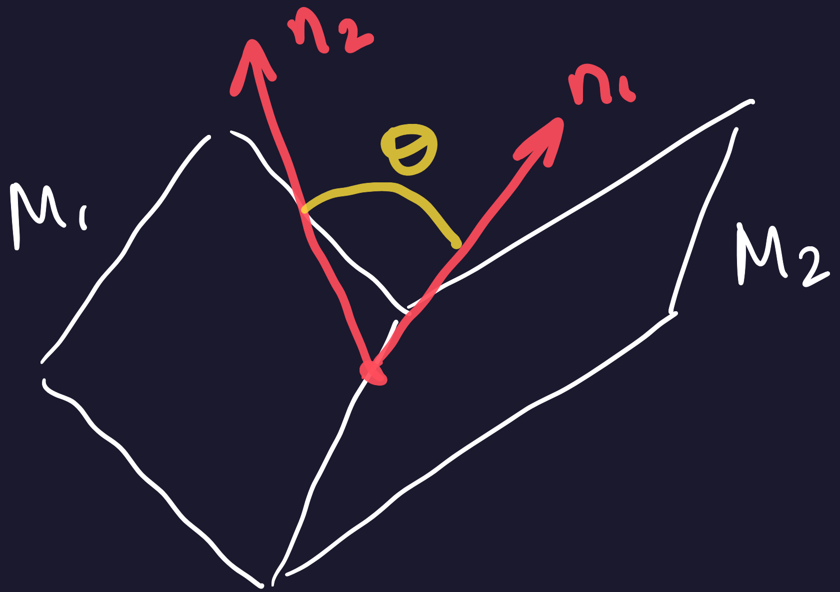
M_3 intersects M_1, M_2

M_1 has a pt $(2,0,0)$

$(2,0,0)$ $\quad 2 \cdot 2 - 0 = 4 \neq 5$ so $(2,0,0)$ is NOT in M_2

$\Rightarrow$ M_1 and M_2 are parallel but NOT same!

Angle between two planes



$$n_1 \cdot n_2 = |n_1| \cdot |n_2| \cdot \cos\theta$$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|}$$

Example: Find the angle between the planes

$$\underbrace{x + y + z = 1}_{n_1 = (1,1,1)} \quad \text{and} \quad \underbrace{x - z = 3.}_{n_2 = (1,0,-1)}$$

[Desmos]

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|} = \frac{1\cdot 1 + 1\cdot 0 + 1\cdot(-1)}{\sqrt{1^2+1^2+1^2} \cdot \sqrt{1^2+(-1)^2}} = \frac{0}{\sqrt{3}\cdot\sqrt{2}} = 0$$

$$\Rightarrow \theta = \frac{\pi}{2} \quad !$$

Planes are perpendicular!

Example: Find the line of intersection between the planes

$$x + y + z = 1 \quad \text{and} \quad x - z = 3.$$

$$L: \begin{cases} x + y + z = 1 \longrightarrow 3 + z + y + z = 1 \\ x - z = 3 \rightarrow x = 3 + z \end{cases}$$

$$y = -2 - 2z$$

$$\boxed{L: \begin{cases} x = 3 + t \\ y = -2 - 2t \\ z = t \end{cases}}$$

Example: Find the angle and the line of intersection between the planes $\underbrace{x + z = 2}_{(1,0,1)}$ and $\underbrace{x - y = 3}_{(1,-1,0)}$.

$$\cos\theta = \frac{1}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \quad (\text{or } 60^\circ)$$

$$L: \begin{cases} x+z=2 \\ x-y=3 \end{cases}$$

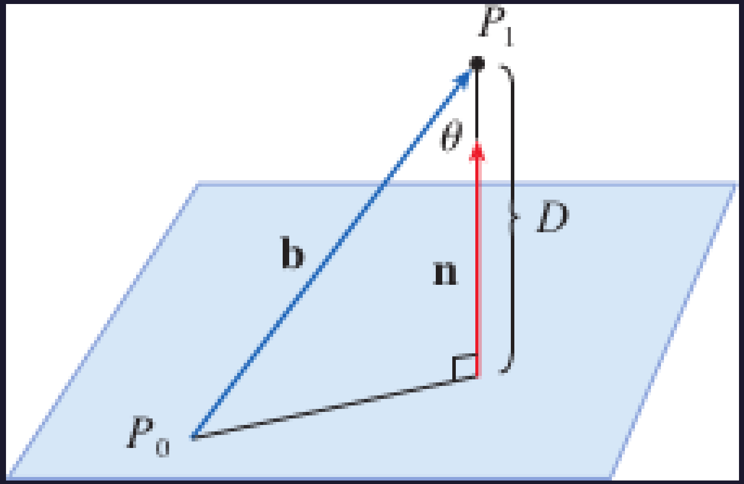
$x = t$

$x+z=2 \Rightarrow z = 2-x = 2-t$

$\rightarrow y = x-3 \Rightarrow y = t-3$

$$\boxed{L: \begin{cases} x=t \\ y=t-3 \\ z=2-t \end{cases}}$$

Distance from a point to a plane



The distance D from the point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Find the distance from the point $P = (1, 2, 3)$ to the plane $x + z = 2$.

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

(x_1, y_1, z_1)

$n = (1, 0, 1) = (a, b, c)$

$x + z - 2 = 0$

$$D = \frac{|1 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 - 2|}{\sqrt{1^2 + 0^2 + 1^2}}$$

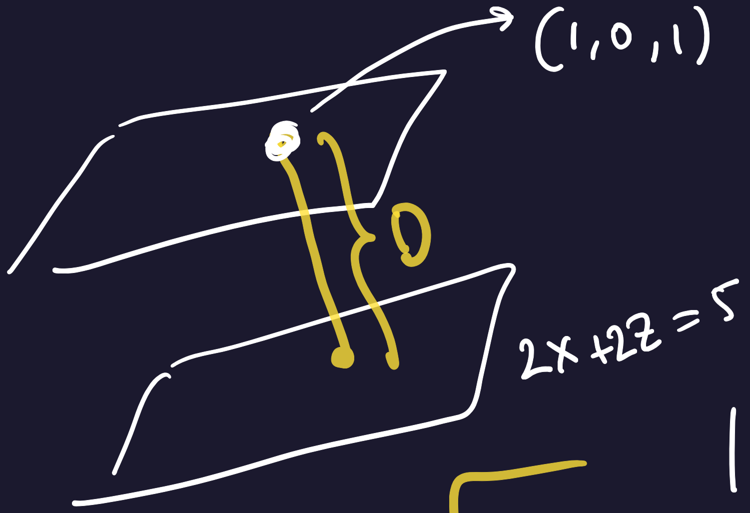
$$= \frac{|2|}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}} \checkmark$$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example: Find the distance between the parallel planes $x + z = 2$ and $\cancel{2x + 2y = 3}$.

$2x + 2z = 5$

$2x + 2z - 5 = 0$



$$\boxed{D} = \frac{|2\cdot 1 + 2\cdot 1 - 5|}{\sqrt{2^2 + 0^2 + 2^2}} = \frac{1}{\sqrt{8}} = \boxed{\frac{1}{2\sqrt{2}}}$$

Questions?

Thank you

Until next time.