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Q1, Q2, Q3, Q4 Solutions:

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1) $\sigma = 2$, $n = 6$

$P(a < S^2 < b) = 0{,}90$

S^2 is sample variance of 6 observations

If data is coming from Normal Distribution $\Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(\text{dof}=n-1)$

$\frac{5S^2}{4} \sim \chi^2_5$ } Since equal tail $\Rightarrow \alpha = 1 - 0{,}90 \Leftrightarrow \alpha = 0{,}10 \Leftrightarrow \frac{\alpha}{2} = 0{,}05$

$P(a<S^2<b) = 0{,}90 \Rightarrow P\left(\chi^2_{5,\frac{\alpha}{2}} < \frac{5S^2}{4} < \chi^2_{5,1-\frac{\alpha}{2}}\right) = 0{,}90$

$\Rightarrow P\left(\chi^2_{5,0{,}05} < \frac{5S^2}{4} < \chi^2_{5,0{,}95}\right) = 0{,}90$ (**)

$$\left.\begin{aligned}\chi^2_{5,0{,}05} &= 1{,}145 \\ \chi^2_{5,0{,}95} &= 11{,}070\end{aligned}\right\} (*)$$

(*) $\wedge$ (**) $\Rightarrow P\left(1{,}145 \cdot \frac{4}{5} < S^2 < 11{,}070 \cdot \frac{4}{5}\right) = 0{,}90$

$\Rightarrow a = 1{,}145 \cdot \frac{4}{5} = \boxed{0.916}$

$b = 11.070 \cdot \frac{4}{5} = \boxed{8.856}$

2) $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent unbiased estimators of a given parameter θ and $Var(\hat{\theta}_1) = 3Var(\hat{\theta}_2)$.

(*) $\hat{\theta}_3 = \alpha_1\hat{\theta}_1 + \alpha_2\hat{\theta}_2$ is unbiased estimator with minimum variance.

$\alpha_1 = ?$, $\alpha_2 = ?$

$E(\hat{\theta}_3) = E(\hat{\theta}_1) = E(\hat{\theta}_2) = \theta$ (Unbiased estimator property) (**)

(*) $\wedge$ (**) $\Rightarrow \theta = E(\hat{\theta}_3) = E(\alpha_1\hat{\theta}_1 + \alpha_2\hat{\theta}_2) = \alpha_1 E(\hat{\theta}_1) + \alpha_2 E(\hat{\theta}_2) = (\alpha_1+\alpha_2)\theta$

$\Leftrightarrow \alpha_1 + \alpha_2 = 1$

$$Var(\hat{\theta}_3) = Var(\alpha_1 \hat{\theta}_1 + \alpha_2 \hat{\theta}_2) = \alpha_1^2 Var(\hat{\theta}_1) + \alpha_2^2 Var(\hat{\theta}_2)$$

$$\Leftrightarrow \alpha_1^2\, 3 Var(\hat{\theta}_2) + \alpha_2^2 Var(\hat{\theta}_2) = Var(\hat{\theta}_2)\left[3\alpha_1^2 + \alpha_2^2\right] = Var(\hat{\theta}_2)(4\alpha_1^2 - 2\alpha_1 + 1)$$

$\Rightarrow$ For minimum variance:

$$\frac{d\, Var(\hat{\theta}_3)}{d\alpha_1} = 0 \Rightarrow Var(\hat{\theta}_2)(8\alpha_1 - 2) = 0 \Leftrightarrow \boxed{\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{3}{4}}$$

3) mean = 5000 psi

st. deviation = 400 psi

$P(4900 < \bar{X} < 5100) = 0.95$

$n = ?$

$$n = \left(\frac{z\sigma}{E}\right)^2 \quad (*)$$

Confidence interval level = 0.95

$\Rightarrow z = 1.96$

E = Width of the confidence interval

$$E = \frac{5100 - 4900}{2} = 100$$

$$\wedge (*) \Rightarrow n = \left(\frac{1.96 \cdot 400}{100}\right)^2 = \boxed{(7.84)^2 \approx 61.4656}$$

4)

a) $$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-\left(\frac{x_i - 1}{\theta}\right)}$$

$$l(\theta) = \ln(L(\theta)) = \sum_{i=1}^{n} \ln\left(\frac{1}{\theta} e^{-\left(\frac{x_i-1}{\theta}\right)}\right) = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n}(x_i - 1)$$

$$\frac{\partial l(\theta)}{\partial \theta} = \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}(x_i - 1) \quad \left(\text{For maximum } \frac{\partial l(\theta)}{\partial \theta} = 0\right)$$

$$\Rightarrow 0 = \frac{\partial l(\theta)}{\partial \theta} = \frac{-n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}(x_i - 1)$$

$$\Leftrightarrow n\theta = \sum_{i=1}^{n}(x_i - 1) \Leftrightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(x_i - 1) = \bar{X} - \frac{n}{n} = \boxed{\bar{X} - 1}$$

b) $Y_i = 1 \Rightarrow X_i > 3$

$P(Y_i=1) = P(X_i > 3) = \int_3^{\infty} \frac{1}{\theta} e^{-\left(\frac{x-1}{\theta}\right)} dx \qquad u = \frac{x-1}{\theta} \Rightarrow dx = \theta\, du$

$P(X_i > 3) = \int_{\frac{2}{\theta}}^{\infty} e^{-u} du = -e^{-u} \Big|_{\frac{2}{\theta}}^{\infty} = e^{-\frac{2}{\theta}}$

So, $P(Y_i=1) = P(X_i>3) = e^{-\frac{2}{\theta}}$ $\qquad \left(P(Y_i=0) = 1 - P(Y_i=1)\right) \circledast$

$\circledast \wedge \Rightarrow P(Y_i=0) = 1 - e^{-\frac{2}{\theta}}$

$L(\theta) = \prod_{i=1}^{n} \left(e^{-\frac{2}{\theta}}\right)^{Y_i} \left(1 - e^{-\frac{2}{\theta}}\right)^{1-Y_i} \Leftrightarrow \ell(\theta) = \sum_{i=1}^{n} Y_i \ln\left(e^{-\frac{2}{\theta}}\right) + \sum_{i=1}^{n} (1-Y_i) \ln\left(1 - e^{-\frac{2}{\theta}}\right)$

$\Rightarrow \frac{-2}{\theta} \sum_{i=1}^{n} Y_i + \left(n - \sum_{i=1}^{n} Y_i\right) \ln\left(1 - e^{-\frac{2}{\theta}}\right) \quad \circledast\circledast$

since $\sum_{i=1}^{n} Y_i = 20 \quad \circledast\circledast\circledast$

$\circledast\circledast \wedge \circledast\circledast\circledast \Rightarrow \frac{-40}{\theta} + (n-20) \ln\left(1 - e^{-\frac{2}{\theta}}\right) = \ell(\theta)$

$\frac{\partial \ell(\theta)}{\partial \theta} = \frac{40}{\theta^2} - (n-20) \frac{-2e^{-\frac{2}{\theta}}}{\theta^2\left(1 - e^{-\frac{2}{\theta}}\right)} = 0 \quad \left(\text{To maximise } \frac{\partial \ell(\theta)}{\partial \theta} = 0\right)$

$\frac{40}{\cancel{\theta^2}} = (n-20) \frac{-2e^{-\frac{2}{\theta}}}{\cancel{\theta^2}\left(1 - e^{-\frac{2}{\theta}}\right)}$

Since $n=100$ is given:

$20 = \frac{80 e^{-\frac{2}{\theta}}}{1 - e^{-\frac{2}{\theta}}} \Rightarrow 20 - 20e^{-\frac{2}{\theta}} = 80 e^{-\frac{2}{\theta}} \Leftrightarrow e^{-\frac{2}{\theta}} = \frac{1}{5}$

$\Leftrightarrow \frac{2}{\theta} = \ln(5)$

$\Leftrightarrow \theta = \boxed{\frac{2}{\ln(5)} \approx 1.243}$

Q5.Solution:

5) $\bar{X} = 65$ ms

$\sigma^2 = 36 ms^2$

a)

$\bar{X} \pm t_{\frac{\alpha}{2}} \left(\frac{\sigma}{\sqrt{n}}\right)$ = Confidence interval for population mean when the variance is unknown.

$\Rightarrow 65 \pm 2.947\left(\frac{6}{\sqrt{16}}\right) = CI$

$\Leftrightarrow 65 \pm 2.947 \cdot \left(\frac{6}{4}\right) = CI$

$\Leftrightarrow 65 \pm 4.4205 = CI$

$\Rightarrow \boxed{[60.5795, 69.4205]_{ms} = CI}$

$1-\alpha = 0.99$

$\frac{\alpha}{2} = 0.005$

$t_{\frac{\alpha}{2}} = 2.947 \left(\frac{\alpha}{2} = 0.005, df = 15\right)$

b)

$$CI = \left[\sqrt{\frac{(n-1)\sigma^2}{\chi^2_{\frac{\alpha}{2}}}}, \sqrt{\frac{(n-1)\sigma^2}{\chi^2_{1-\frac{\alpha}{2}}}}\right]$$

$$\Leftrightarrow CI = \left[\sqrt{\frac{15.36}{27.488}}, \sqrt{\frac{15.36}{6.262}}\right]$$

$$\Leftrightarrow CI = \left[\sqrt{19.64}, \sqrt{86.2}\right]$$

$$\Leftrightarrow \boxed{CI \approx [4.43, 9.29]_{ms}}$$

$1-\alpha = 0.95$

$\frac{\alpha}{2} = 0.025$

$\left.\begin{array}{l}\chi^2_{0.025} = 6.262 \\ \chi^2_{0.975} = 27.488\end{array}\right\}$ for $(df = 15)$