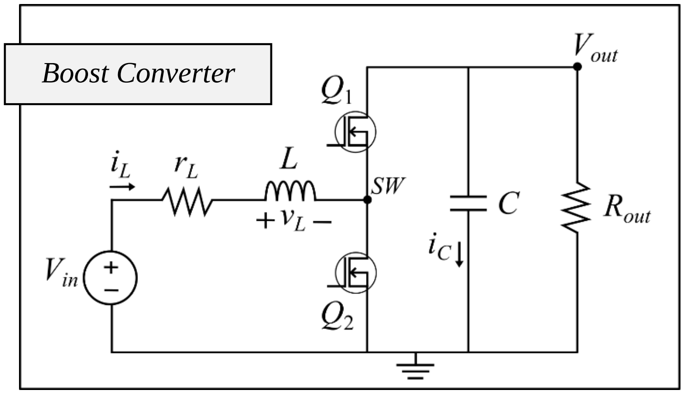
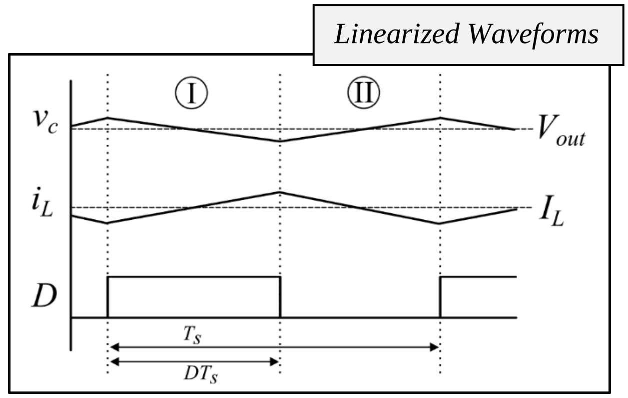
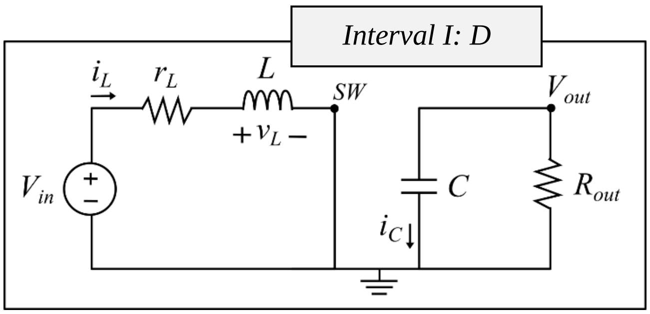
Boost Converter Example Derivation:

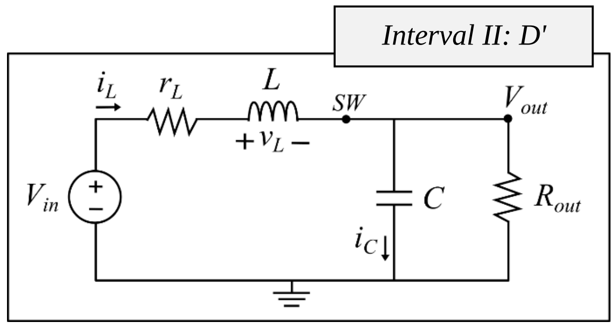
Average Model & Small Signal Circuits →

State Space, Sampled Time, and Discrete Time Models.









Averaged, Small Signal Circuit Description

D (Interval I)	$v_L(t) = V_{in} - i_L(t) r_L \quad \vert \quad i_C(t) = -V_{out}/R_{out}$
D' (Interval II)	$v_L(t) = V_{in} - i_L(t) r_L - V_{out} \quad \vert \quad i_C(t) = i_L(t) - V_{out}/R_{out}$

Cap. Charge & Volt. Second Balancing Equations:

$$L \left\langle \frac{di_L(t)}{dt} \right\rangle = D(V_{in} - i_L(t) r_L) + D'(V_{in} - i_L(t) r_L - V_{out})$$

$$C \left\langle \frac{dv_C(t)}{dt} \right\rangle = D\left(-\frac{V_{out}}{R_{out}}\right) + D'\left(i_L(t) - \frac{V_{out}}{R_{out}}\right)$$

Insert Small Signal "hat" Terms: $x(t) = X + \hat{x}$

$$L \frac{d(I_L + \hat{\imath}_L)}{dt} = (D + \hat{d})(V_{in} + \hat{v}_{in} - (I_L + \hat{\imath}_L) r_L) +$$

$$(1 - (D + \hat{d}))(V_{in} + \hat{v}_{in} - (I_L + \hat{\imath}_L) r_L - (V_{out} + \hat{v}_{out})) \rightarrow$$

$$L \frac{d\hat{\imath}_L}{dt} = \hat{d} V_{in} - \hat{d} I_L r_L + D\hat{v}_{in} - D\hat{\imath}_L r_L + \hat{v}_{in} - \hat{\imath}_L r_L - \hat{v}_{out} - \hat{d} V_{in} + \hat{d} I_L r_L + \hat{d} V_{out} - D\hat{v}_{in} + D\hat{\imath}_L r_L + D\hat{v}_{out} \rightarrow$$

Implementation Notes:

- The "(t)" is dropped for notation simplicity.
- Eliminate 2^{nd} order terms: $\hat{x} \cdot \hat{x}$
- Separate 1^{st} order ($\hat{x} \cdot X$) and DC (X) terms.

$$L\frac{d\hat{i}_L}{dt} = \hat{v}_{in} - \hat{i}_L r_L - \hat{v}_{out} + \hat{d}V_{out} + D\hat{v}_{out} \rightarrow$$

$$L\frac{d\hat{i}_L}{dt} = \hat{v}_{in} - \hat{i}_L r_L + \hat{d}V_{out} - D'\hat{v}_{out}$$ Small Signal Equation

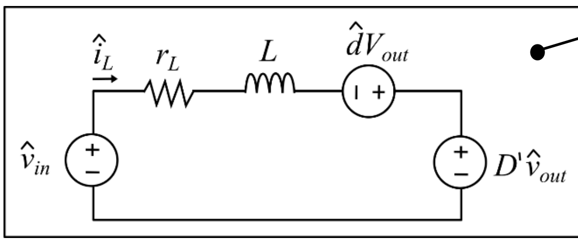
$$0 = V_{in} - I_L r_L - D'V_{out}$$ Averaged Steady State DC Equation

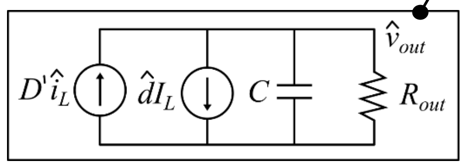
$$C\frac{d(V_C+\hat{v}_C)}{dt} = (D+\hat{d})\left(-\frac{V_{out}}{R_{out}} - \frac{\hat{v}_{out}}{R_{out}}\right) + (1-D-\hat{d})\left(I_L + \hat{i}_L - \frac{V_{out}}{R_{out}} - \frac{\hat{v}_{out}}{R_{out}}\right) \rightarrow$$

$$C\frac{d\hat{v}_C}{dt} = -\frac{D\hat{v}_{out}}{R_{out}} - \frac{\hat{d}V_{out}}{R_{out}} + \hat{i}_L - \frac{\hat{v}_{out}}{R_{out}} - D\hat{i}_L + \frac{D\hat{v}_{out}}{R_{out}} - \hat{d}I_L + \frac{\hat{d}V_{out}}{R_{out}} \rightarrow$$

$$C\frac{d\hat{v}_C}{dt} = D'\,\hat{i}_L - \frac{\hat{v}_{out}}{R_{out}} - \hat{d}I_L$$ Small Signal Equation

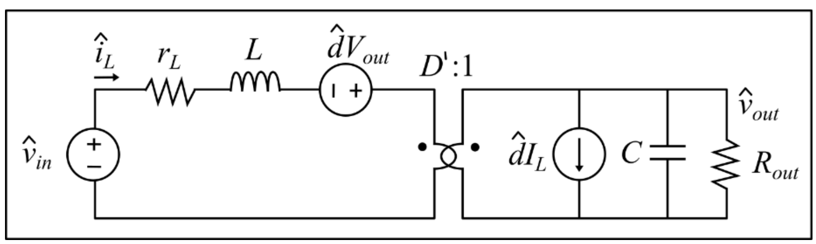
$$0 = -\frac{V_{out}}{R_{out}} + D'I_L$$ Averaged Steady State DC Equation

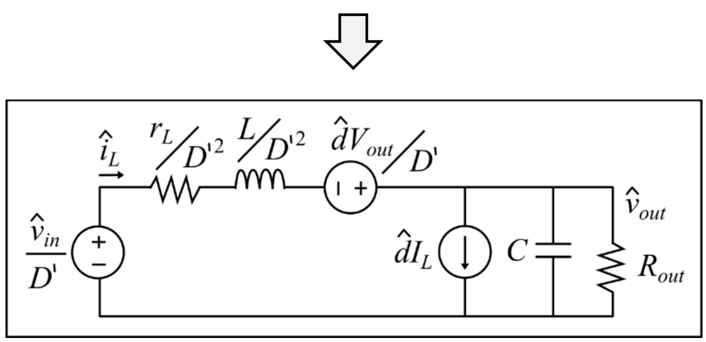




Implementation Note:

- Use each small signal equation to construct equivalent circuits.
- Combine the circuits, using duty cycle as a “transformer.”
- Reduce the circuit by reflecting impedances through the transformation ratio (here: D')
- Find transfer functions by zeroing sources (superposition), calculated below.





Find Control to Output Transfer Function, $G_{vd}(s)|_{\hat{v}_{in}=0} = \frac{\hat{v}_{out}}{\hat{d}}$

$V_{out} = 0,\ \hat{v}_{in} = 0$ $$\hat{v}_{out} = -\left(\left(\frac{r_L + sL}{D'^2}\right) || \frac{1}{sC} || R_{out}\right)\hat{d}I_L$$

$\boxed{I_L = 0,\ \hat{v}_{in} = 0}$ $$\hat{v}_{out} = \left(\frac{\frac{1}{sC} || R_{out}}{\frac{r_L + sL}{D'^2} + \frac{1}{sC} || R_{out}} \right) \frac{\hat{d} V_{out}}{D'}$$

Control (d) to Output (v_c) Transfer Function:

Add the terms and simplify →

$$G_{vd}(s) = \frac{I_L r_L R_{out} - s I_L L R_{out} - R_{out} D' V_{out}}{R_{out} D'^2 + r_L + s(L + r_L C R_{out}) + s^2 C L R_{out}}$$

State Space Description

Fundamental State Space Equations:

$$\frac{dx(t)}{dt} = \boldsymbol{x}'(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \qquad \text{and} \qquad \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$$

$\boldsymbol{x}(t)$: state vector – typically comprised of "history dependent" signals (capacitor voltages, inductor currents).

$\boldsymbol{x}'(t)$: derivative of state vector – each element of $\boldsymbol{x}(t)$ is differentiated with respect to time.

$\boldsymbol{u}(t)$: input vector – typically comprised of independent sources (voltage, current). To achieve linearity (make the state space useful/solvable in a reasonable time), this vector is approximate to be ideal: $\boldsymbol{u}(t) = \boldsymbol{U}$.

$\boldsymbol{y}(t)$: output vector – dependent signals (not states or inputs) that the user would like to extract from the state space.

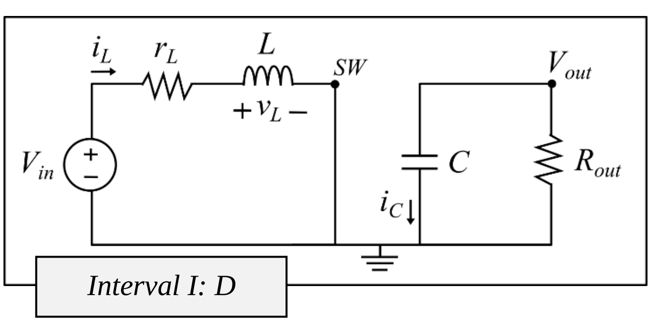
Boost Converter:

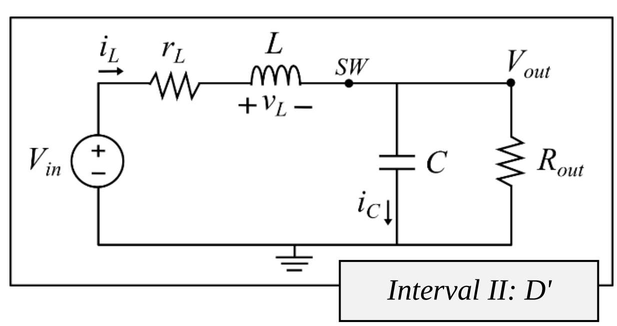
Let $\boldsymbol{x}(t) = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$ such that $\boldsymbol{x}'(t) = \begin{bmatrix} \frac{di_L}{dt} \\ \frac{dv_c}{dt} \end{bmatrix} = \begin{bmatrix} i_L' \\ v_c' \end{bmatrix}$, and let $\boldsymbol{U} = [V_{in}]$ and $\boldsymbol{y}(t) = [v_{rL}]$.

$\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of proportionality such that the elements of $\boldsymbol{x}'(t)$ are described in terms of $\boldsymbol{x}(t)$ and $\boldsymbol{U}$.

Similarly, $\boldsymbol{C}$ and $\boldsymbol{D}$ are used to describe elements of $\boldsymbol{y}(t)$ in terms of $\boldsymbol{x}(t)$ and $\boldsymbol{U}$.

First, we will derive $\boldsymbol{A}$ and $\boldsymbol{B}$ for each interval of the periods: $\boldsymbol{A_1}$, $\boldsymbol{B_1}$ for interval I (duty cycle: D) and $\boldsymbol{A_2}$, $\boldsymbol{B_2}$ for interval II (duty cycle: D'). The original differential equations of each interval are leveraged once again:





D (Interval I)

$$v_L = V_{in} - i_L r_L \quad \rightarrow \quad L\frac{di_L}{dt} = V_{in} - i_L r_L \quad \rightarrow \quad i_L' = \frac{V_{in}}{L} - \frac{i_L r_L}{L}$$

$$i_C = -I_{out} \quad \rightarrow \quad C\frac{dv_c}{dt} = -\frac{V_{out}}{R_{out}} \quad \rightarrow \quad v_c' = -\frac{v_c}{CR_{out}}$$

$$\boldsymbol{A_1} = \begin{bmatrix} -{r_L}/{L} & 0 \\ 0 & -1/{CR_{out}} \end{bmatrix}, \quad \boldsymbol{B_1} = \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix}$$

Implementation Note:

- Again, the "(t)" is dropped.

D' (Interval II)

$$v_L = V_{in} - i_L r_L - V_{out} \quad \rightarrow \quad L\frac{di_L}{dt} = V_{in} - i_L r_L - v_c \quad \rightarrow \quad i_L' = \frac{V_{in}}{L} - \frac{i_L r_L}{L} - \frac{v_c}{L}$$

$$i_C = i_L - I_{out} \quad \rightarrow \quad C\frac{dv_c}{dt} = i_L - \frac{V_{out}}{R_{out}} \quad \rightarrow \quad v_c' = \frac{i_L}{C} - \frac{v_c}{CR_{out}}$$

$$\boldsymbol{A_2} = \begin{bmatrix} -{r_L}/{L} & -1/L \\ 1/C & -1/{CR_{out}} \end{bmatrix}, \quad \boldsymbol{B_2} = \begin{bmatrix} 1/L & 0 \\ 0 & 0 \end{bmatrix}$$

These $\boldsymbol{A}$ and $\boldsymbol{B}$ matrices are the state space descriptions that include the full differential equations. Through iteration with a small timestep (Δt), the state variables (elements of $\boldsymbol{x}$) can be found at any point in time.

To extract a signal that does not exist in the state vector, the equation $\boldsymbol{y}(t) = \boldsymbol{Cx}(t) + \boldsymbol{DU}$ must be leveraged. Here, setting $\boldsymbol{y}(t) = v_{rL}$ means that we want to extract v_{rL} from the circuit.

Therefore, $\boldsymbol{C}$ and $\boldsymbol{D}$ are built to describe $\boldsymbol{y}(t)$ in terms of $\boldsymbol{x}(t)$ and $\boldsymbol{U}$: $v_{rL} = i_L r_L$ [both intervals]

$$\boldsymbol{C_{1,2}} = [r_L \quad 0], \qquad D_{1,2} = [0]$$

Sampled Time Description

For a given interval, the states at time t can be found by using the solution to the state space equation:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(t) + \boldsymbol{A}^{-1}(e^{\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{BU}$$

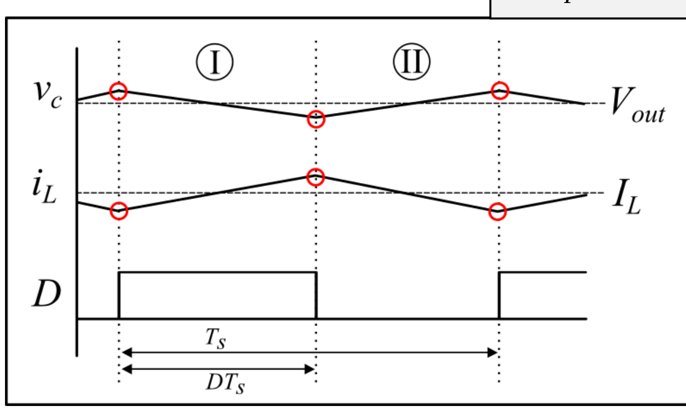
This equation is powerful and can be used for finding steady state iteratively (faster than iterating through the state space because the timestep does not need to be small). This equation can also be leveraged to find a closed form equation for the steady state (where k is the number of intervals in one period):

$$\boldsymbol{X_{SS}} = \left(\boldsymbol{I} - \prod_{i=1}^{k} e^{\boldsymbol{A}_i t_i}\right)^{-1} \cdot \sum_{i=1}^{k}\left(\prod_{m=i+1}^{k} e^{\boldsymbol{A}_m t_m}\right)\boldsymbol{A_i}^{-1}(\boldsymbol{e}^{\boldsymbol{A}_i t_i} - I)\boldsymbol{B_i U}$$

Clearly this is the fastest way to find stead state… a closed form solution!

Sampled Time modeling is essentially using the function $\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}(t) + \boldsymbol{A}^{-1}(e^{\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{BU}$ to quickly propagate through the states – one calculation will allow you to traverse from the beginning of an interval to the end of that interval. Then, run through the next interval with the solution from the last. Example shown in MATLAB!

Sampled Time



The closed form steady state solution of the Boost converter:

$$\boldsymbol{X_{SS}} = \left(I - e^{\boldsymbol{A_1}DT_s}e^{\boldsymbol{A_2}D'T_s}\right)^{-1}\left(e^{\boldsymbol{A_2}D'T_s}\left(\boldsymbol{A_1^{-1}}\left(e^{\boldsymbol{A_1}D'T_s} - I\right)\boldsymbol{B_1U}\right) + \left(\boldsymbol{A_2^{-1}}\left(e^{\boldsymbol{A_2}D'T_s} - I\right)\boldsymbol{B_2U}\right)\right)$$

Note that this is the solution for the state matrix $\boldsymbol{x}$ at the *beginning of the interval* (**not** the averaged values).

Discrete Time Description

The small signal state space equations of the discrete time system are:

$$\widehat{\boldsymbol{x}}[n+1] = \boldsymbol{\Phi}\widehat{\boldsymbol{x}}[n] + \boldsymbol{\Gamma}\hat{d}[n] \quad \text{and} \quad \widehat{\boldsymbol{y}}[n] = \boldsymbol{\Psi}\widehat{\boldsymbol{x}}[n]$$

where n is the current sample and $n+1$ is the next sample. Note that these are small signal equations ("hat" terms only) and are meant to be relevant only around a steady state operating point. The first equation can be manipulated to yield the matrix of discrete time control-to-state transfer functions:

$$\boldsymbol{G_{xd}}(z) = \frac{\widehat{\boldsymbol{x}}}{\hat{d}} = (z\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}$$

For the boost converter with the states defined as $\boldsymbol{x}(t) = \begin{bmatrix} i_L \\ v_c \end{bmatrix}$, the previous equation yields

$$\boldsymbol{G_{xd}}(z) = \begin{bmatrix} G_{id}(z) \\ G_{vd}(z) \end{bmatrix},$$

where the discrete time transfer function $G_{vd}(z)$ is comparable to the continuous time $G_{vd}(s)$ that was previously derived by the averaging methods.

Now, $\boldsymbol{\Phi}$ and $\boldsymbol{\Gamma}$ must be derived.

The matrix $\boldsymbol{\Phi}$ describes the next state ($\widehat{\boldsymbol{x}}[n+1]$) in terms of the current state ($\widehat{\boldsymbol{x}}[n]$). This is the *natural response* of the system.

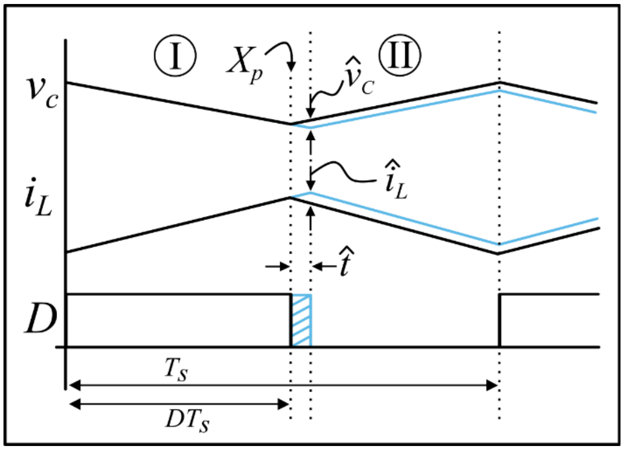
The matrix $\boldsymbol{\Gamma}$ describes the next state ($\widehat{\boldsymbol{x}}[n+1]$) in terms of the current state control perturbation ($\hat{d}[n]$), this is the *forced response* of the system.

The natural response matrix can be described by propagating the states through each interval with $e^{\boldsymbol{A}t}$:

$$\boldsymbol{\Phi} = \prod_{i=k}^{1} e^{\boldsymbol{A_i}t_i}$$

And specifically, for the boost circuit:

$$\boldsymbol{\Phi} = e^{\boldsymbol{A_2}t_2}e^{\boldsymbol{A_1}t_1}.$$



The forced response matrix is more complicated. The control perturbation $\hat{d}$ causes some change in the states, denoted by $\widehat{\boldsymbol{x}} = [\widehat{\boldsymbol{v}}_{\boldsymbol{C}} \ \ \hat{\boldsymbol{i}}_{\boldsymbol{L}}]^T$. This change due to the perturbation is approximated with the linearized affect:

$$\begin{aligned} \widehat{\boldsymbol{x}} &= \left((I + \boldsymbol{A_1})\boldsymbol{X_p} + \boldsymbol{B_1U_1}\right)\hat{t} - \left((I + \boldsymbol{A_2})\boldsymbol{X_p} + \boldsymbol{B_2U_2}\right)\hat{t} \\ &= (\boldsymbol{A_1} - \boldsymbol{A_2})\hat{t} \end{aligned}$$

This change propagates through the end of the period using $e^{A_r t_r}$ using r to represent each interval after the perturbation. Using $\hat{t} = \hat{d}T_s$ to translate from time to duty cycle, the final forced response matrix is represented by multiplying the change in the states by the propagation through the end of the period:

$$\boldsymbol{\Gamma} = \mathrm{e}^{\mathbf{A_2 t_2}}(\boldsymbol{A_1} - \boldsymbol{A_2})\boldsymbol{X_p}T_s.$$

Finally, the discrete time transfer function $G_{vd}(z)$ can be compared with the continuous time transfer function $G_{vd}(s)$ (for arbitrarily chosen circuit values) in the graph below: